Optimizing Fixture Layout in a Point-Set Domain

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Abstract—This paper describes an approach to optimal design of a fixture layout with the minimum required number of elements, i.e., six locators and a clamp. The approach applies to parts with arbitrary 3-D geometry and is restricted to be within a discrete domain of locations for placing the fixture elements of nonfrictional contacts. The paper addresses two major issues in the fixture layout design: 1) to develop an efficient algorithm for fixture synthesis in the point set domain and 2) to evaluate the acceptable fixture designs based on several performance criteria and to select the optimal fixture according to practical requirements. The performance objectives considered include the workpiece localization accuracy, and the norm and dispersion of the locator contact forces, respectively, for high precision in workpiece localization and force balance among the fixture contacts.

Based on a concept of optimum experiment design, an interchange algorithm with random initiation is developed for the purpose. A study of the interrelationship between locators and clamps leads to two different design strategies. Furthermore, the fixture performance characteristics are evaluated to understand their tradeoffs. Conclusions on the importance of the accurate localization and the contact force balance are made based on empirical observations in the study of several examples.

Index Terms—Fixture layout, fixtures, fixture synthesis, force-closure, localization, optimal design.

I. INTRODUCTION

FIXTURES are devices used in manufacturing to immobilize and localize workpieces. A fixturing system, usually consisting of clamps and locators, must satisfy a set of performance requirements including localization accuracy and force-closure conditions. Fixture layout is the primary task of fixture design and consists of the determination the number and locations of locators and clamps. Traditionally, an appropriate fixturing scheme is determined by relying on design experience or by using trial-and-error methods. These practices do not lend themselves well to flexible manufacturing.

This paper describes an approach to layout design of a class of fixtures for 3-D workpieces. There are three major considerations. First, a fixture is modeled as a collection of nonfrictional point contacts of locators and clamps. These contact elements are further restricted to be applied to the workpiece within a given collection of discrete candidate locations. This given collection of candidates is referred to as the point-set domain [21]. In general, the point set is assumed to be a potentially large collection. For example, the candidate locations might be generated by discretizing the exterior surfaces of the workpiece. The part to be fixtured may have an arbitrarily complex geometry. Second, the goal of the fixture layout design is to determine feasible fixture configurations that satisfy several fixturing requirements and constraints, including the essential kinematic localization and total fixturing (i.e., force-closure). The task is to evaluate a set of feasible configurations based on several performance criteria and to select an optimal or suboptimal fixture layout. The performance measures considered in this paper are workpiece localization accuracy and norm and dispersion of the locator contact forces. These objectives are the most critical considerations in a fixture design. Third, for simplicity in presentation, the fixtures considered are limited to the minimum number of elements required, i.e., six locators and a single clamp. The approach presented is also applicable to a general case of more locators and/or clamps.

In the point-set domain with a given number $N$ of candidates, the fixture layout design becomes a problem of discrete optimization with a combinatorial complexity. For six locators and one clamp, the complexity is of $O(N^7)$, which is computationally prohibitive even for a modest number of candidates, e.g., $N = 100$. One needs to use an efficient method for a practical solution.

Another issue is the interrelationship between the locators and the clamp and their roles in affecting the performance measures of localization accuracy and contact force dispersion. While the localization accuracy is determined by the locators only, the contact forces depend on the locators and the clamp as well. Thus, an optimal locator scheme may have to be considered in conjunction with an optimal clamping scheme. Furthermore, there may exist conflicts between the performance objectives, and their tradeoffs have to be evaluated in the process of a multicriteria optimal design.

The approach presented in this paper is based on a concept of optimum design of experiments. The essential technique employed is an interchange algorithm, in which locator locations are first selected randomly and are further improved in a subsequent set of interchange processes. In the interchange algorithm, a set of recursive updates for the localization and contact analyzes are exploited, allowing for an efficient computation of optimal or suboptimal fixture layout without resorting to an exhaustive search. Moreover, tradeoffs between the two performance objectives could be incorporated in the interchange algorithm such that the final fixture design would be satisfactory with a balanced overall performance quality.

The remainder of the paper is organized as follows. Section II presents a background overview of the previous research in fix-
ture design and in the related fields. In Section III, the problem of fixture layout design is defined. A fixture model with a force-closure analysis is described and various performance criteria are defined. The interchange algorithm is derived in Section IV with details of its performance evaluation. In Section V, we conduct a thorough analysis of different optimal fixture designs based on different optimization objectives with examples. Tradeoffs between the multiple criteria are highlighted, especially for finding the best locators and clamps to achieve accurate localization and balanced force-closure. An empirical analysis of the computational effort is given in Section VII. Finally, conclusions and suggestions for future work are made.

II. RELATED WORK

There is a substantial literature on fixturing and the related topic of robotic grasping [1], [2]. Although there are some important differences between the requirements of workpiece fixturing and object grasping, the main concept is similarly defined by the century-old concept of force closure [3], which has been extensively studied in the robotics field in recent years.

This condition of force-closure or total restraint has been formulated by using the popular screw theory [4]–[7]. Sufficient and necessary conditions for force-closure are extensively discussed in [6]–[10], including the concept of exceptional surfaces such as cylinders and spheres. A number of techniques have been developed for automatically synthesizing grasp configurations [9], [16]–[18].

A different approach for total constraint analysis based on a geometric perturbation technique was reported by Asada and By [11]. Extending this analysis, Brost and Goldberg developed an automatic modular fixture design procedure including geometric access constraints in addition to the kinematic closure [12]. The problem of designing modular fixtures has gained more attention recently [12], [13], [29]. However, modular fixtures are difficult to implement for complex shaped 3-D parts, and this fact is evident in industrial production of turbine airfoils.

In the context of fixture planning, Chou et al. developed a procedure that designs fixtures for prismatic parts using screw algebra and linear programming methods [18]. DeMeter presented an algorithm that uses the min–max loading criteria as basis for determining the optimal layout of locators and clamps as well as clamp actuation intensities [19]. There have also been extensive research activities in machining fixture designs, with a focus on workpiece and fixture structural rigidity, tool accessibility, and path clearance [1].

The grasp synthesis or the fixture design problem is typically considered such that a large portion of continuous surfaces of the workpiece is assumed to be available for fixture element placement. However, a different situation may arise that the fixture elements are allowed to contact with the workpiece only at a set of discrete point locations instead. This situation is referred to as the point-set domain, and this constraint might be imposed by practical conditions related to the functional and/or manufacturing requirements. In [21], Wang presented an example of laser drilling of small holes on a turbine airfoil. In this case, the airfoil is required to be fixtured at some of a discrete set of point locations on the part surface in its manufacture and inspection processes. The concept of point-set domain has been a topic of interest in multiple body fixturing as well [20], [26]. A greedy algorithm for efficient fixture synthesis is presented in [21] with a focus on minimization of workpiece positional errors only. In this paper, a broader set of issues involving contact force dispersion and clamp optimization are considered. In addition, a variation of the design method based on the concept of optimal design of experiments is presented here with an aim to increase the computational efficiency of the approach.

III. FIXTURE MODEL AND PERFORMANCE CRITERIA

A. Fixture Model

This section describes briefly the mathematical model of the class of fixtures considered. The main assumptions are the following: 1) the workpiece and the fixture elements are sufficiently rigid; 2) each fixture element is considered as a point-contact without friction; and 3) the surface of the workpiece body is smooth, with piecewise differentiable regions. The edges and the vertices existing in the workpiece are not considered as feasible places for fixturing. Unique surface normal is defined at each location.

Following the geometric perturbation analysis presented in [11] and [12], we suppose that the $i$th locator of position vector $\mathbf{r}_i$ with respect to the workpiece has a small perturbation $\delta \mathbf{r}_i$. This perturbation will result in a small displacement of the workpiece $\delta \mathbf{q} = [\delta \mathbf{h}^T, \delta \mathbf{r}]^T$, including translational and rotational components, such that

$$\delta \mathbf{y}_i = \mathbf{h}_i^T \delta \mathbf{q}$$

where $\mathbf{h}_i^T = -[\mathbf{n}_i^T, (\mathbf{r}_i \times \mathbf{n}_i)^T]$, $\mathbf{y}_i = \mathbf{n}_i^T \delta \mathbf{r}_i$, and $\mathbf{n}_i$ represents the outward unit normal vector at contact point. For a fixture with $n$ locators, the individual perturbation equations can be collected into a single formula describing the whole fixturing system as

$$\delta \mathbf{y} = \mathbf{G}^T \delta \mathbf{q}$$

where $\delta \mathbf{y} = [\delta y_1, \delta y_2, \ldots, \delta y_n]^T$ and $\mathbf{G} = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n]$. Matrix $\mathbf{G}$ is called the locator matrix and it completely characterizes the kinematics of the fixturing system [21].

The functional requirements of a fixture are determined by the kinematic constraints imposed on the workpiece being held by the fixture. Deterministic localization is a fundamental requirement indicating that the workpiece cannot make any infinitesimal motion without losing contact with at least one locator [11]. This is true if and only if the locator matrix $\mathbf{G}$ has a full rank of six [11], i.e., rank $\mathbf{G} = 6$. Force-closure is another major concern in fixture layout design [11], [18], [20]. A derivation for the force-closure requirement in the case of a fixture consisting of $n$ locators and one clamp has been presented in [21], [22]. Let’s consider the clamp location completely defined by the position vector $\mathbf{r}_c$ and unit surface normal $\mathbf{n}_c$. For the point contact, the
generalized clamping force applied on the workpiece is given as:

\[ F_c = h_c \lambda_c = -\left[n_c^T (r_c \times n_c)\right]^T \lambda_c \]  

(3)

where \( \lambda_c > 0 \) represents the clamping force magnitude. Let \( Q \) represent all external forces applied on the workpiece. Then, the workpiece equilibrium equation is given as

\[ G \alpha + h_c \lambda_c + Q = 0 \]  

(4)

where the nonnegative vector \( \alpha = [\alpha_1 \alpha_2 \ldots \alpha_n]^T \) is a collection of the magnitudes of each generalized locator contact forces \( f_i \)

\[ f_i = \alpha_i h_i \quad (i = 1, 2, \ldots, n). \]

The general form for the solution of this equation for the intensities of the locator contact forces \( \alpha \) can be expressed as

\[ \alpha = -G^+ Q - G^+ h_c \lambda_c + (I - P) \gamma \]

(5)

where

\[ G^+ = G^T (GG^T)^{-1}, \quad P = G^+ G = G^T (GG^T)^{-1} G. \]

Here \( G^+ \) is the pseudo-inverse of \( G \), \( P \) is the prediction matrix and \( \gamma \) is an arbitrary \( n \times 1 \) vector. The first two terms are the particular solutions of the equation and represent the magnitudes of the locator contact forces in response to the external forces \( Q \) and the clamping action, respectively. The last term represents the homogeneous solution and it corresponds to the internal forces among locators. Because the locators are passive elements, these internal forces should not arise physically. Therefore, the last term must be ignored [21], and the force-closure requirement becomes

\[ \alpha_i = -h_i^T M^{-1} (Q + h_c \lambda_c) > 0 \quad (i = 1, 2, \ldots, n) \]

(6)

where \( M = GG^T \) and is called the Fisher information matrix of the fixture.

The analysis shows that the force-closure condition depends simultaneously on locations \( \langle h_i \rangle \) of each locator as well as that of the clamp \( h_c \). Moreover, if the force-closure condition is satisfied for each clamp separately, then any combination of the clamps shall satisfy the condition also. Therefore, force-closure analysis for a multiclamp fixture can be reduced to a successive one-clamp fixture analysis.

### B. Fixture Performance Criteria

The fixture analysis leads us to the conclusion that the fixture layout design consists of two critical aspects: locator position determination for deterministic localization and clamp configuration design for force-closure. It is obvious that for a given part there may exist a large number of fixturing schemes to satisfy the fundamental conditions. Therefore, it becomes imperative to study performance quality of a fixture. This would allow the designer to select the most appropriate fixture solution for a specific application. For example, in a machining operation, an essential aspect of the fixture performance quality is the accuracy of the workpiece localization. Other important objectives include minimizing the support forces at the locator contact regions and balancing the reaction forces, distributing them as uniformly as possible among the locator contacts. In the following, measures will be presented to quantify these performance factors. They will serve as objectives for optimal fixture layout design to be discussed later.

1) **Accurate Localization:** In general, the workpiece positional errors are due to the geometric variability of its surface and the positional variability of the locators. The locator positional variability depends on the dimensioning and tolerancing scheme assigned to the fixture assembly and its components. A complete model of the variability is usually not available at the early stage of fixture layout design. However, the impact of a locating scheme can be predicted based on a statistical characterization as presented in [21]. Essentially, positional errors of locators are represented by \( \delta_y \) in the fixture model (2). Then, the resulting workpiece positional accuracy can be described by a norm of \( \delta q \) as

\[ ||\delta q||^2 = \delta q^T \delta q = \delta y^T (GG^T)^{-1} \delta y. \]

(7)

It has been shown that a suitable criterion for accurate localization is to maximize the determinant of the information matrix \( M = GG^T \) [23], [30], i.e.,

\[ \max \det(M) = \max \det(GG^T). \]

(8)

Clearly, the localization accuracy depends only on the locator locations. This optimal design criterion is called D-optimality. Compared to other criteria such as the trace of \( M \) (A-optimality), the condition number, and the minimum eigenvalue of \( M \) (E-optimality) [23], D-optimality criterion has been chosen for its evident advantages. For example, D-optimal designs are invariant to linear transformations of coordinate systems. Another advantage is that this criterion can be easily implemented for an efficient construction of fixture layout to be discussed in the next section.

2) **Minimal Locator Contact Forces:** Another objective in fixture layout design might be to minimize the support forces at the locator contact regions while fulfilling the force-closure requirement. This objective is crucial especially for fragile objects when overloading is not desirable [12].

For simplicity in presentation, the locator contact forces in response to the clamping action alone without considering the external force are given as

\[ \alpha_c = -G^T M^{-1} h_c \lambda_c. \]

(9)

Indeed, the inequalities can also be used to represent force-closure, since an arbitrary value of \( \lambda_c \) as large as necessary can be found to compensate for any exterior forces. Normalizing these forces with respect to the clamping intensity, we obtain [22]

\[ p_c = \frac{\alpha_c}{\lambda_c} = -G^T M^{-1} h_c \quad \text{with} \quad p_{ci} = -h_i^T M^{-1} h_c \quad (i = 1, 2, \ldots, n). \]

(10)

The force-closure condition specifies that these forces are always positive for each locator \( i \) of the set of \( n \) locators

\[ p_{ci} = -h_i^T M^{-1} h_c > 0 \quad (i = 1, 2, \ldots, n). \]
Hence, an appropriate criterion for the minimal contact force objective is

$$\min(\|p_c\|)$$

(11)

using the Euclidean norm of the locator contact forces,

$$\|p_c\| = p_c^T p_c = \sum_{i=1}^{n} f_{ci}^2 = h_i^T M^{-1} h_i$$

It should be noted that both locator (through $M$) and clamp ($h_c$) positions would affect this objective.

3) Balanced Locator Contact Forces: While the minimal locator contact forces are often considered to be important, it might be more desirable to have the contact forces distributed as uniformly as possible. This would help avoid unnecessarily high local stresses and distortions in the workpiece. Let $\bar{p}$ represent the mean reactive locator forces in response to the clamp action [22], [27], [28]. Then, a balance of the contact forces is represented by the dispersion as defined as

$$d = \frac{1}{n} \sum_{i=1}^{n} (p_{ci} - \bar{p})^2$$

where $\bar{p} = \frac{1}{n} \sum_{i=1}^{n} p_{ci}$.

Therefore, an optimal criterion for balanced force-closure condition is given as

$$\min(d).$$

(12)

IV. OPTIMAL LAYOUT DESIGN ALGORITHMS

A. Discrete Optimization in the Point-Set Domain

As mentioned earlier, the major difficulty of fixture layout design in the point-set domain is the combinatorial complexity in a complete search of the global optimal fixture layout. Clearly, exhaustive searches are not practical and an efficient technique is required for optimal or sub-optimal fixture synthesis. For practical reasons, we have to resort to an efficient method with a trade-off to obtain a suboptimal solution, as long as the solution meets the requirements in practice. Fig. 1 shows an example of the point-set domain. In this paper, the point set available for fixture layout design is considered to be feasible as a result of a feasibility analysis in an earlier design stage.

In earlier work on fixture synthesis presented in [21], an efficient technique is found possible by exploiting an incremental change of the locator matrix with respect to a sequence of deletions of locators from the candidate set. Owing to some special properties of matrix $M$ and its inverse, a set of recursive formulas could be obtained for the construction of algorithms for fixture layout design. For an initial set of $n$ locators, their information matrix can be written as

$$M = GG^T = \sum_{i=1}^{n} h_i h_i^T.$$  (13)

Now, consider the $j$th location to be added to (or deleted from) the initial set. With some straightforward linear algebra manipulations, it is found that the resulting information matrix $M_{(j)}$ and its inverse are given as

$$M_{(j)} = GG^T \pm h_j h_j^T$$

(14)

$$M_{(j)}^{-1} = M^{-1} \mp \frac{(M^{-1} h_j) (M^{-1} h_j)^T}{(1 \pm p_{jj})}$$

(15)

where

$$p_{jj} = h_j^T M^{-1} h_j$$

with $\pm$ for addition or deletion accordingly.

These recursive relations naturally lead to a numerical procedure of a greedy algorithm [21]. Starting from all initial candidate locations, an iterative process is involved to sequentially delete one candidate location at a time, leading to a final set of six locators and one clamp [21]. In each deletion cycle, the location with the least contribution to the objective pursued will be removed, resulting in the best (or locally best) set of locators within each cycle of optimization.

The greedy algorithm of [21] is a “top–down” approach. While it reduces the combinatorial complexity to $O(N^2)$,
its efficiency is still quite limited, especially in the case of a large number of candidates. It seems logical to take an alternative approach of “bottom-up” [20], [22]. In an initial step, one may select six locators from the candidate set. Then an interchange process can be performed as a sequence of additions and deletions of locator elements to further improve the layout design. The interchange process focuses on iterative improvement according to a specified objective, while the force-closure condition is eventually fulfilled. When combined with a random generation of the initial locator set, such a bottom-up approach has the potential for an efficient finding of a fixture layout design with high-performance quality.

B. Interchange Algorithms

Now we describe the concept of interchange algorithms in detail. The accurate localization objective is used first, and formulas for other objectives are given later. Input to an interchange algorithm is the collection of \( N \) feasible candidate locations represented by \( h_i \) \((i = 1, 2, \ldots, N)\). Based on (13)–(15), an incremental change in the determinant of the information matrix \( M \) is easily shown as follows [21]–[23]:

\[
\det M_{(j)} = (1 \pm p_{jj}) \det M
\]

where

\[
p_{jj} = h_j^T M^{-1} h_j
\]

and the symbol \( \pm \) implies addition or deletion, respectively. The term \( p_{jj} \) is known to satisfy \( 0 \leq p_{jj} \leq 1 \). At limit, \( p_{jj} = 1 \) when \( G \) is a square matrix. Based on these equations, an interchange algorithm is constructed with the following two phases.

1) Phase 1: Generation of Initial Locators: In the initial phase of an interchange algorithm, six initial locators must be generated such that the information matrix \( M \) is not singular. These initial locators serve as the base for the next step of exchanging between a current locator \( j \) \((j = 1, 2, \ldots, N - 6)\) and a candidate location \( k \) \((k = 1, 2, \ldots, N - 6)\) to improve the objective function.

There are two different ways to generate the initial locators [22], [28]. The first method is a sequential generation to give maximum increase to \( \det M \). At any step of the addition process when \( n < 6 \), matrix \( M \) will be singular and thus the inverse of \( M \) does not exist. This problem is avoided by regularizing the information matrix through an addition of a small multiple of the identity matrix. That is, we use \( M = \sum_{i=1}^{n} h_i h_i^T + \varepsilon I \), where \( \varepsilon \) is a small number, typically less than \( 10^{-6} \). At each addition, the \( j \)th locator is added such that \( \max p_{jk} \). The second way is random selection to use a random number generator. Six locators are randomly and simultaneously selected from the \( N \) candidates such that the symmetric matrix \( M \) is not singular, i.e., \( \det M > 0 \).

2) Phase 2: Improvement of Layout by Interchange: The interchange step is intended for the improvement of the initial locators. This is achieved by, at each iteration, selecting a locator from the current locator aggregate and exchanging it with another locator from the candidate list so that a maximum increase in \( \det M \) is obtained. The process would terminate when the interchange yields no further increase in the determinant.

In the case of accurate localization, it is easy to show, using (13)–(16), that the interchange of a current locator \( j \) and a candidate location \( k \) results in a change in \( \det M \) as

\[
\det M_{(jk)} = p_{jk}^2 \det M
\]

where \( p_{jk} = h_j^T M^{-1} h_k \), defining the correlation of the interchange pair with respect to the objective.

Figs. 2 and 3 illustrate the determinant improvement through this procedure, for a sequential initial generation and a random initial generation, respectively, for the example shown in Fig. 1. It is noted that the sequential initialization might result in a set of locators near an optimal solution, and the following interchange phase may gain only modest improvement. On the other hand, however, the random generation of the initial locator set usually results in a lower determinant value. The subsequent interchange phase could substantially improve the initial selection [22]. The initial locator set of random generation and the final locator set after the interchange improvements are shown in Fig. 4.

C. Multiple Interchange Procedure

The interchange algorithm has a limitation. Starting from a different set of locators, the interchange procedure may not end
up as a unique optimal fixture configuration, but as several improved designs. These solutions are related to the local optima of the objective function. It is likely that an interchange process will reach a local optimal solution. Fig. 5 illustrates the results for three interchange processes that start from three different initial sets of locators, respectively. For a larger collection of random initial sets of locators, the interchange algorithm obtains several different final solutions as shown in Fig. 6. It is noticed that the objective function values for the initial locators vary substantially. The objective values of the suboptimal solutions obtained with the interchange algorithm become relatively close. Clearly, the maximum value corresponds to the best optimal fixture configuration with respect to the selected performance objective.

Thus, given the dependency on the initial choice of locator set, it is necessary to generate a large set of different initial locators and to perform the interchange procedure accordingly. This will increase the chance to find the global or near-global optimum. The nature of this algorithm is similar to that of a hill climbing algorithm.

The recursive formulas of (13)–(15) suggest many variations to the interchange algorithm proposed above. One variation is the seesaw algorithm. Similar to the interchange algorithm, it involves a sequential procedure of alternate additions and deletions based on a given initial locator set. It differs in the number of locators to be added and deleted in the subsequent forward and backward processes to improve the initial design. Instead of a single locator to be interchanged in each cycle, multiple locator elements are added sequentially such that each addition would make the most contribution to the objective function defined by (8). A sequential deletion procedure will follow to bring the size of the locators back to exact number of 6. The locators are deleted in sequence with the one of the least contribution to the required objective to be selected for deletion. The resulting set of six locators represent an intermediate optimal solution after a single seesaw process. Repeating this whole procedure until no improvement in the objective is obtained, the optimal design process terminates. It is obvious that the seesaw algorithm is less effective than the single-pair interchange [27].

V. OPTIMAL LOCATORS AND OPTIMAL CLAMPS

A fixture layout design implies a simultaneous selection of the locators as well of the clamps. For the performance objective of accurate localization, it allows us to divide the design process into two distinct phases: 1) the selection of the optimal locators and 2) the selection of the optimal clamp. We shall describe two different strategies to accomplish the total task.

A. Optimal Locators Without Closure Considerations

The mathematical analysis of the previous sections shows that the localization accuracy of a fixture is dependent only on the locator positions. Therefore, an optimal locator design may simply focus on achieving the best localization first without considering clamping in relation to the force-closure constraint. With the interchange algorithm with randomly generated initial locator sets, the process objective is to maximize the determinant of the information matrix, i.e., max(\(\det M\)).
Fig. 7. Four families of symmetric, optimal, or suboptimal locator solutions for precise localization.

Fig. 7 shows the results of optimized locator schemes for the example illustrated in Fig. 1. Starting from different sets of random locator sets, the interchange process yields several distinct solutions (see Fig. 6.) It is interesting to note that among these suboptimal solutions four of the six locators in each solution set are identical. For the other two locators, the different local optimal solutions are clustered around their best solutions which are marked in the figure with thicker arrows. This part has certain symmetry in the geometry, and the symmetry will be reflected also in the obtained optimal solutions. The figure illustrates four families of symmetric solutions corresponding to the two symmetry planes of the part.

B. Feasible Clamping Set

Once the optimal locator positions are determined, the next step of the layout design is to identify the possible clamp positions that would satisfy the force-closure requirement. Based on the required conditions of (6), the entire feasible set of clamp locations can be identified [24], [25]. Fig. 8 illustrates the feasible clamping set for three different examples. It is noticed that the feasible clamps, if there exists any, may gather in compact regions. They are highly dependent on the given locator schemes as concluded in other analyses [24], [25]. It is possible that an optimal locator set may result in no feasible clamp solutions for force-closure as shown in Fig. 8(f). Thus, the corresponding locator set must be excluded to form a feasible fixture. On the other hand, when a feasible clamping set exists, we may choose a clamp that would optimize the fixture design with respect to another performance requirement.

C. Optimal Locators with Force-Closure

In the algorithm described above, attention is given only to the optimization of locators. The force-closure requirement is evaluated based on the resulting optimal locating scheme. A drawback of this approach is that the clamping requirements may be impossible to satisfy as in the case of Fig. 8(f). This could be a common problem of fixture design schemes that focus on locators alone. Therefore, it is more suitable to consider the clamping requirements simultaneously with an optimal locating scheme.

In many applications, the clamp could be predetermined given some practical considerations. Then the optimal fixture design could be constructed by selecting a suitable set of locators with respect to a performance criterion such that the force-closure requirement is always satisfied. In the case of accurate localization, the objective is still to maximize the determinant of the fixture information matrix, i.e., \( \max(\det \mathbf{M}) \). In the meantime, a set of constraints defined by (6) must be fully maintained. These constraints can be incorporated into the interchange algorithm described above. Thus, for each interchange improvement, the interchange pair is selected to maximize \( \det \mathbf{M} \) such that the contact force for every locator is always strictly positive, i.e., \( p_{ij} = -h_i \mathbf{M}^{-1} h_j > 0 \) \( (i = 1, 2, \ldots, 6) \).

Fig. 9 illustrates such an interchange process on a set of randomly generated initial locators. At each iteration, the interchange process first targets the locator pair that would result in positive locator reaction forces in the presence of the clamp. Then the D-optimality is sought while selecting the best pair for the interchange. The interchange optimization process converges to several improved solutions, which usually cluster around their best solution. As shown in the previous section, this strong tendency is believed to be related to local optimal solutions that define distinct optimal fixture configurations. Among these local optima, the best solution is found as shown in Fig. 10 with thick arrows. It is further noticed that the symmetrical solutions reflect the symmetry of the part and of the clamp position.
VI. MULTIPLE PERFORMANCE CHARACTERISTICS TRADEOFFS

In this section, we present a study of performance tradeoffs in fixture layout design, emphasizing the relationships between the different fixture optimization characteristics. Knowing the general behavior of a fixture with respect to the key performance objectives would allow the designer to predict the way in which the synthesis can be conducted. This would help the designer to determine an appropriate criterion to balance multiple requirements. The tradeoffs described below are based on the empirical observations of several examples. Unfortunately, no analytical verifications are available presently given the complexity of the objective functions considered.

A. Accurate Localization Objective

Accurate localization objective is defined by (8). In the previous sections, this objective is demonstrated for several examples with the interchange algorithm. While the determinant \(|\det(M)|\) is maximized, the other two force conditions of force normal and dispersion are analyzed. We have made the following empirical observations regarding the relations between these different fixturing characteristics. While maximizing the determinant, it has a general effect of decreasing the norm and dispersion of the contact forces, with a few cases of exceptions. This indicates that there seldom exists a conflict between these characteristics. The contact forces are minimized and become more balanced as a result of achieving precise localization. Therefore, accurate localization would usually make a significant improvement in the overall performance quality of the fixture. Fig. 11 shows the results of multiple interchange processes with randomly generated initial locator sets.

B. Minimal Locator Contact Force Objective

The minimization of the locator contact forces as the response to the clamping action, i.e., \(\min(|\mathbf{p}_c|)\), is another important objective in fixture design. For an interchange of a current locator \(j\) with a candidate location \(k\), an analysis similar to (17) can be derived as follows. After adding the \(k\)th location, the contact force norm becomes

\[
|\mathbf{p}_{(++k)}| = \mathbf{h}_c^T M_{(++)}^{-1} \mathbf{h}_c = |\mathbf{p}_c|^2 - \frac{r_k^2}{1 + p_{kk}}.
\]

Then, deleting the \(j\)th locator yields

\[
|\mathbf{p}_{(-j+k)}| = \mathbf{h}_c^T M_{(-j+k)}^{-1} \mathbf{h}_c = |\mathbf{p}_c|^2 + \frac{r_k^2}{1 - p_{jj}(+k)}
\]

where \(r_{cj}(+k) = r_{cj} - (r_{ck}p_{jk})/(1 + p_{kk})\) and \(r_{jj}(+k) = 1 - (r_{jj}^2)/(1 + p_{kk})\). Thus, the change in the locator contact force norm as a result of the interchange is described by

\[
|\mathbf{p}_{(+k)}| = |\mathbf{p}_{c}|^2 + \frac{r_{cj}^2}{p_{jk}} [p_{cj}(1 + p_{kk}) - 2r_{ck}p_{jk}]
\]

where \(r_{ck} = -h_c^T M^{-1} h_c\), \(r_{cj} = -h_c^T M^{-1} h_c\), \(p_{j+k} = -h_j^T M^{-1} h_k\), and \(p_{k+k} = -h_k^T M^{-1} h_k\). For this objective function, the force-closure condition must be evaluated at each interchange step in the interchange process.

Fig. 12 shows an example of the process of \(\min(|\mathbf{p}_c|)\), with a random initial configuration and the optimized configuration.
When comparing to Fig. 9 of the same part, it is interesting to find that the optimal locators for this objective are no longer distributed near the surface boundaries but are moved toward the centers of the surfaces. For the same example, the multiple interchange process reveals another interesting finding. While the locator contact force norm is minimized, the determinant $\det(M)$ would often deteriorate significantly, affecting adversely the localization quality (Fig. 13). This side effect is not desirable. On the other hand, it is noticed that the dispersion of the contact forces are generally decreased. This seems to suggest that there is no conflict between the locator contact force norm and dispersion objectives.

Another useful observation is related to the range of the locator contact forces. Fig. 14 shows the ranges of the contact forces among the six locators for the different trials. The lowest contact force for many cases is near zero after the interchange optimization, meaning that one or more locators of the resulting fixture could have reactive forces close to zero and be almost inactive. This implies that a large acting force may have to be imposed on the clamp in order to ensure the force-closure condition. This would lead to excessive loads on the other locators and on the workpiece.

Fig. 15 shows the local optimal solutions obtained for the three different examples used earlier. Again, there are several distinct suboptimal fixture configurations, and the best solutions obtained are shown in thick arrows, respectively. When comparing with the case of accurate localization objective (Fig. 10), it is found that these local optimal locator sets are more scattered. The best solution is also quite different from that for accurate localization, respectively, for all three examples.

### C. Balanced Locator Contact Force Objective

The objective of balanced reactive forces of the locators does not focus on the magnitudes of the contact forces, but on the dispersion between the forces. When the dispersion is minimized, i.e., $\min(d)$, it is said that the locator reaction forces are near balanced. Fig. 16 illustrates an example of the interchange process for this objective, with a random initial configuration and the optimized configuration. In this case, the force-closure condition is required during the interchange process. For the same example, the results of multiple interchange processes with initial configurations randomly generated are shown in Fig. 17. While the locator contact force dispersion is minimized, the change direction of determinant $\det(M)$ is not predictable. However, the fluctuations in the determinant value are relatively small compared with those obtained with force norm.
Fig. 16. Interchange for $\min(d)$ objective: (a) initial fixture and (b) optimal fixture.

Fig. 17. Changes in the three performance objectives for multiple trials in $\min(d)$. Arrows indicate the change of direction.

optimization previously (see Fig. 13). Also, as expected from the previous case (Fig. 13), the contact force norm in general is decreased. More importantly, as Fig. 18 illustrates, the minimum value of the contact forces among the locators is usually not near zero, representing an improvement comparing to the case of force norm objective (see Fig. 14). Clearly, this is a desired behavior toward force balancing. On the other hand, it is noticed that the dispersion of the contact forces are generally decreased. This seems to suggest that there is no conflict between the locator contact force norm and dispersion objectives.

The multiple local optimal solutions obtained for the three examples are shown in Fig. 19. It is interesting to note that the best locators in this case almost evenly cover all the surface area, tending to migrate toward the surface interior. This is in contrast to the cases of the determinant and force norm objectives previously shown in Figs. 10 and 15. Therefore, due to these beneficial facts, we recommend the minimization of the contact force dispersion instead of the force norm as the force objective in fixture design.

VII. COMPUTATION TIME

Before concluding, we present an empirical analysis of the computational efficiency of the interchange algorithm. In general, the efficiency depends on a number of factors, including: 1) the number of candidate locations; 2) the number of repeated trials with randomly generated starting designs; and 3) the complexity of the objective function.

Figs. 20 and 21 show the run time (seconds) for different cases of optimization for the example shown in Fig. 1. The interchange algorithm is implemented with MATLAB and is executed on a Pentium II PC. In Fig. 20 the accurate localization objective is used, and the three variations of the interchange algorithm are compared, i.e., (a) interchange with sequential initialization of locators, (b) interchange with random initialization and without force-closure constraint, and (c) interchange with random initialization and with force-closure constraint. The run times are compared for different numbers of trials in the case of random initialization of locators. It is observed that the run time is approximately proportional to the number of trials as expected. For the sequential initialization, its computational complexity is equivalent to a single trial of random initialization process. Fig. 21 illustrates the effects of the three different objective functions on the computational time with respect to different number of trials in the case of random initialization. The minimal force dispersion objective requires the most computational effort, mainly due to the lack of a recursive formula for its update in the interchange algorithm. The accurate localization objective requires the least computational effort.
VIII. CONCLUSION

This paper describes a research approach to optimizing fixture layout for 3-D workpieces. The approach applies to parts with arbitrarily complex geometry, but the fixture elements are restricted in a point set domain within a collection of discrete locations. The paper addresses two major issues in fixture layout design: 1) multiple performance criteria to define optimal fitturing schemes appropriate for practice and 2) efficient algorithms to overcome the combinatorial complexity for fixture synthesis with multiple characteristics tradeoffs.

The performance measures presented include maximization of the workpiece localization accuracy, and minimization of the norm and dispersion of the locator contact forces. Based on a concept of optimum experiment design, an interchange algorithm is devised for generating and improving optimal locators and clamps. With randomly generated initial locator sets, this interchange algorithm would usually result in a set of distinct local optimal solutions. Examples are presented to show the effectiveness of the proposed approach.

The interrelationship between the locators and the clamps plays a significant role on the kinematic closure, as well as on the fixture quality measures. Two different design strategies are presented in dealing with the interactions between the locators and clamps. A hierarchical approach solves for optimal locators first, then an optimal clamp. It is particularly suited for precision localization designs. Another approach is to consider both localization and force-closure conditions. This would allow for evaluations of tradeoffs between different performance objectives. Empirical observations were made with respect to the tradeoffs between the accurate localization objective and the contact force norm and dispersion objectives, through the studies of several examples. It is concluded that the accurate localization and the contact force dispersion should be considered as the most important objectives in fixture design.

A more coherent and complete approach to the study of simultaneous locator and clamp optimization is a subject of further studies. Moreover, the approaches described here could be extended to include friction at the locator and clamp contacts, although the frictionless assumption yields a conservative solution for force-closure. Recently, a more accurate kinematic model of fixtures is developed [30] based on the full kinematics of contact. It would be certainly interesting to generalize the approach of this paper to incorporate the full-kinematic model of fixture.

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