**Abstract**—The conventional point-kinematic model of fixtures has only treated point geometry of the contacts between locators and workpiece. However, this model, which ignores the underlying surface properties of the locators-plus-workpiece system, is inherently incapable of capturing the effects of the geometric properties important to accurate positioning of the workpiece. In this paper, we present a fixture kinematic analysis based on the full kinematics of locator-workpiece contact. This model incorporates a “virtual” kinematic chain with meshing parameters of contact kinematics in a velocity formulation. Conditions of a deterministic fixture are derived. It is shown that the workpiece position and orientation are completely characterized by the kinematic properties of the locators contacts with the workpiece, including not only the arbitrary locator location errors but also the surface properties at non-prismatic locator-workpiece contacts. This is illustrated with numerical examples. The fixture kinematic analysis developed here has a strong implication for designing fixtures with high locating precision requirements.

**Keywords**—Fixture model, contact and grasp, fixturing, grasping

**I. INTRODUCTION**

Proper fixture design is crucial to product quality in terms of precision and accuracy in part fabrication and assembly. Fixturing systems, usually consisting of clamps and manufacturing over many years [1], [2], [3], [4], [5], these studies are almost entirely based on a simplified kinematic model for the geometric or kinematic analysis of conditions of workpiece positioning and total constraint (or form closure). In the model, each locator (or a robot finger) and workpiece contact is considered as a theoretical point without involving surface properties of the locators and the workpiece [6], [7]. This model has been widely used for the studies of fixture synthesis [7], [8], fixture contact type and friction effects [9], modular fixtures [10], [11], machining fixtures [9], [12], and traditional or computer-based fixture design and planning [13], [14]. The conventional model is referred to as point-kinematic model in the paper.

In a recent development of methods for fixture diagnosis [15], [16] and fixture tolerancing schemes [7], [8], [17], it raised an issue of the accuracy of the conventional linearized point-kinematic model for high precision applications such as the airfoil manufacturing. A quadratic sensitivity analysis is recently presented in [17]. It is also based on the same simplifying assumptions and is a second order Taylor approximation of the nonlinear geometric constraints defined by the point-kinematic assumptions. Aimed for a more accurate description of a locating scheme the quadratic theory deals with the effects of the workpiece curvature around the contact points and interaction effects of locator positioning errors. However, the effect of locator geometry is not included in the quadratic analysis and it is quite complicated for the use in fixture layout and setup design.

In either a linear or a nonlinear analysis, the point-kinematic model will underestimate the positioning errors of a non-prismatic workpiece in the presence of locator errors. In neglecting the geometric properties of the locator and workpiece surfaces, the insufficient prediction of the positioning errors could have a significant impact in industrial practice. A good example is a turbine airfoil which has highly complex geometry. In the process of laser drilling of air cooling holes, the positioning accuracy of the fixtured airfoil is critical for producing the cooling holes with high precision in their locations. Another case is in dealing with small parts in assembly, for instance, of electronics devices. The positioning errors of the locators may be relatively large compared to the dimension of the part. Therefore, it is important to fully take into account of the locator dimensional errors in fixture analysis and design, especially when the fixture structure may involve long tolerance chains.

This paper presents a fixture model considering the surface geometry of locators and the workpiece. The fixture model is based on a differential description of the full kinematics of two rigid objects and the motion of their contact point over the surfaces of their bodies [18]. The velocity based formulation yields a full consideration of the effects of curvature in the conditions of kinematic localization of the workpiece by the locators. Thus, this model is referred to as the full-kinematic model. The development of this model is aimed to provide the foundation not only for precision fixture design but also for tolerance budgeting and specification in fixture manufacturing and assembly. Furthermore, when including the curvature effects, the fixture model offers the potential for a more realistic analysis of the stiffness
properties of a locator-workpiece fixture system based on a contact mechanics description. Another advantage of the velocity based formulation is to provide a highly accurate, strictly continuous tracing between contact bodies. This capability may have a potential for applications in haptic simulation and planning of contact tasks such as workpiece loading/unloading in fixturing.

The paper is organized as follows. We first present a discussion of the contact kinematics of Montana [18] describing the relationship between the relative motion of two objects and the motion of their point of contact on their surfaces (Section II). Based on the contact kinematics, a description of the full-kinematic fixture model is presented in Section III. We show that the fixture kinematics can be represented by virtual kinematic chains of the locator contacts and the workpiece. Conditions for a deterministic contact fixture are then described and compared to those of the conventional point-kinematic model in Section IV. Section V describes an analysis of the fixture localization accuracy within the framework of the fixture model presented here. Numerical examples are presented next to illustrate some effects of locator errors on the localization error which are captured by the full-kinematic model, especially the locator tangential errors.

II. THE CONTACT KINEMATICS

In this section we discuss the concept of contact kinematics concerning two rigid bodies that move while maintaining contact with each other. The contact kinematics were fully described in [18], [19]. As the foundation for the fixture model to be presented, it is necessary to describe the relevant information here largely following that of [18].

A. The Meshing Contact

Consider two rigid bodies $O$ and $i$ with smooth surfaces in contact at a point $c$ as shown in Fig. 1. Choose body frames $C_o$ and $C_i$ fixed relative to body $O$ and $i$ respectively. At the contact point $c$, we define a contact frame $C_{co}$ fixed on body $O$ with $z$ axis aligning with the outward normal of the surface of $O$. Similarly, another contact frame $C_{ci}$ is fixed on body $i$. These coordinate systems are given in Fig. 1. There exists an angle between the $x$ axes of these two body frames, which is called contact angle $\psi$ and can be defined such that a rotation of $C_{co}$ through angle $-\psi$ around its $z$ axis would align the $x$ axes [18].

![Fig. 1. The coordinate frames for contact kinematics.](image)

Furthermore, in a mathematical rigor a surface is defined to be a continuous mapping from $R^2$ to $R^3$, $r(\xi)$, with surface parameters $\xi = (u, v)$. The unit normal vector of the surface is given as

$$n = \frac{r_u \times r_v}{\|r_u \times r_v\|} \tag{1}$$

In the so called Gaussian frame, three properties of curvature $K$, torsion $T$, and scale $M$ of the surface are defined as

$$K = \begin{bmatrix} n_u \cdot r_u & n_u \cdot r_v & n_u \cdot r_w \\ n_v \cdot r_u & n_v \cdot r_v & n_v \cdot r_w \\ n_w \cdot r_u & n_w \cdot r_v & n_w \cdot r_w \end{bmatrix} \tag{2}$$

$$T = \begin{bmatrix} \phi_u & \phi_v & \phi_w \\ \phi_v & \phi_w & \phi_u \\ \phi_w & \phi_u & \phi_v \end{bmatrix} \tag{3}$$

$$M = \begin{bmatrix} |r_u| & 0 \\ 0 & |r_v| \end{bmatrix} \tag{4}$$

As clearly described in [18] the relative motion of the two bodies has five degrees of freedom, when they maintain a single point of contact between their surfaces with only sliding or rolling motions allowed. Among these five degrees of freedom, two are for the position of the point of contact on the surface of body $O$ and, similarly, another two degrees of freedom are for body $i$. The last degree of freedom is for the relative rotation $\psi$ around the common surface normal. We shall refer to these internal motion parameters as meshing parameters and denote them in a vector form by

$$\varphi = \begin{bmatrix} M_i \xi_i \\ M_i \xi_o \\ \psi_i \end{bmatrix} \tag{5}$$

For a more direct and natural description the relative velocity between the two bodies can be expressed in a contact frame. We shall define the velocity of body $i$ relative to body $O$ in contact frame $C_{ci}$ as

$$u_i = [u_x, u_y, u_z, \omega_x, \omega_y, \omega_z]^T_i \tag{6}$$

This is also called the contact velocity [18].

B. The Contact Equation

With the internal and external descriptions of the contact motion using the meshing parameters $\varphi$ (or the meshing velocity $\dot{\varphi}$) and the contact velocity $u_i$ respectively, the kinematics of contact are defined as the relationship between these two velocity descriptions. A linear relation has been derived by Montana [18] as

$$u_i = J_i \varphi_i \tag{7}$$

where the matrix $J_i$ is given by

$$J_i = \begin{bmatrix} -I & S_i & 0 \\ 0 & 0 & 0 \\ \Delta K_i & \Delta S_i K_o & 0 \\ -T_i & -T_o & 1 \end{bmatrix} \tag{8}$$

and

$$S_i = \begin{bmatrix} \cos \psi_i & -\sin \psi_i \\ -\sin \psi_i & -\cos \psi_i \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{9}$$
This linear relation is called the kinematics of full contact in [18].

C. Regular Contact

In the expression of the contact equation (7) there exists a direct condition for the contact velocity

$$u_{zi} = 0 \quad (10)$$

This explicitly expresses the kinematic condition that the two bodies maintain contact.

Thus, we may define the non-zero components of the contact velocity as

$$u_{xi} = [u_x, u_y, \omega_x, \omega_y, \omega_z]^T \quad (11)$$

and the contact equation becomes

$$u_{xi} = H_i \varphi_i \quad (12)$$

where $H$ is called the contact matrix and is given by

$$H_i = \begin{bmatrix} -I & S_i & 0 \\ \Delta K_i & \Delta S_i & K_o & 0 \\ -T_i & -T_o & 1 \end{bmatrix} \quad (13)$$

The contact equation (12) is meaningful only when the contact matrix $H$ is of full rank, or $(\det H) \neq 0$. It is straightforward to show that this is equivalent to

$$\det(K_o + S_i K_i S_i) \neq 0 \quad (14)$$

Contacts that meet this condition are said to be regular. Obviously, a theoretical point-point contact is not regular (i.e., singular). When two contacting bodies have an equal but opposite curvature at their contacting point, the contact is also singular. In the kinematic model presented here, singular contacts are excluded since they are not practical for fixtures.

III. The Fixture Model

We now present the fixture model based on the contact kinematics of two bodies. We assume that the workpiece is sufficiently rigid with piecewise differentiable surfaces. Each locator is considered to be in contact with the workpiece without friction. At each locator contact, the surface properties of the locator and the workpiece are assumed to be well-defined and regular. For a unique localization in 3D, six and only six locators are used. The kinematic model of this paper describes the kinematics of fixture localization only.

A. The Kinematics of Locators-Plus-Workpiece System

In the contact equation above, let body $O$ be the workpiece and body $i$ be a locator. Similar to the case of a multi-fingered robotic grasp [20], the fixture system of the locators plus workpiece can be represented as a kinematic “chain” shown in Fig. 2. The workpiece body frame $C_o$ is defined with respect to the global coordinate frame $C_g$. This transformation can also be defined by a chain of transformations from $C_o$ to the contact frame $C_{oi}$ on the workpiece, the contact frame $C_{ci}$ on the locator $i$, the locator body frame $C_i$, and the final global frame $C_g$. Each locator-workpiece contact represents a virtual kinematic subchain with equivalent “joints” defined by the meshing parameters $\varphi_i$. The contact equation (Eq. 7) defines the forward kinematics of each subchain, while the six locator-workpiece contacts form six parallel subchains.

However, the contact velocity $u_i$ in Eq. 7 is defined in the contact frame $C_{ci}$. In order to represent the full kinematics of the locators plus workpiece system, $u_i$ should be transformed into the body frame $C_i$ of its locator. For further convenience we shall let each $C_i$ and the global frame $C_g$ be coincident to the workpiece body frame $C_o$, i.e., $C_i = C_o = C_g$. Therefore, the velocity $v_i$ of locator $i$ relative to that $v_o$ of the workpiece $O$ is expressed in the body frame $C_i$ as

$$v_i - v_o = A_i u_i \quad (15)$$

Here, $A_i$ is the velocity transformation for the locator $i$. In the well known mathematics of rigid-body motions [20], [21], when a body frame transform from $C_i$ to $C_{ci}$ is given as

$$Q_i = \begin{bmatrix} R & r \\ 0 & 1 \end{bmatrix}_i \quad (16)$$

then the velocity transform $A_i$ is given by

$$A_i = \begin{bmatrix} R & \hat{R} R \\ 0 & R \end{bmatrix}_i \text{ and } A_i^{-1} = \left[ \begin{array}{c} R^T \\ 0 \\ \hat{R} R^T \end{array} \right]_i \quad (17)$$

where $\hat{R}$ is defined to be equivalent to $r \times$ (see [21]).

Finally, applying Eq. 7 to Eq. 15 yields the following

$$v_i - v_o = A_i J_i \varphi_i \quad (18)$$

for each locator-workpiece contact $i = 1, \ldots, 6$. Therefore, the closure conditions of the six locator subchains can be succinctly expressed by

$$v_o = -A_1 J_1 \varphi_1 + v_1 = \ldots = -A_6 J_6 \varphi_6 + v_6 \quad (19)$$

Now, the kinematics of the full locators-plus-workpiece fixture system are expressed as a linear system of equation

$$\begin{bmatrix} I & A_1 J_1 & 0 & \cdots & 0 \\ I & 0 & A_2 J_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I & 0 & 0 & \cdots & A_6 J_6 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_6 \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_6 \end{bmatrix} \quad (20)$$

Fig. 2. The kinematic chain of locators-plus-workpiece fixture system.
This equation defines the full kinematics of the fixture system in terms of the body velocities of the locators and the workpiece as well as the internal meshing velocities of each locator contact. The state vector of the full system is an “extended” set of a combination of the workpiece velocity and the contact meshing velocities of all locators. The workpiece velocity \( \mathbf{v}_w \) has 6 components, while each of the 6 locator contacts has 5 “internal” meshing velocities. Thus, the full fixture system has a total of 36 degrees of freedom.

**B. The Full-Kinematic Model**

It is often convenient to express the locator velocity \( \mathbf{v}_i \) in its contact frame \( C_i \). Given the velocity transformation matrix \( \mathbf{A}_i \), the locator velocity is expressed in the contact frame as

\[
\hat{\mathbf{v}}_i = \mathbf{A}_i^{-1} \mathbf{v}_i \tag{21}
\]

Thus, the locator sub-chain kinematics are described by

\[
\hat{\mathbf{v}}_i - \mathbf{A}_i^{-1} \mathbf{v}_o = \mathbf{J}_i \mathbf{\phi}_i \quad (i = 1, \cdots, 6) \tag{22}
\]

As described in Section II-C, it is convenient to separate the normal linear component from the velocity expressed in its contact frame. The third equation in Eq. 22 is the essential contact constraint condition \( u_{ij} = 0 \) (Eq. 10). If the locator is in contact with the workpiece at position \( \mathbf{r}_i \), the orientation of the contact frame \( C_i \) at the point is represented by

\[
\mathbf{R}_i = [t_i \ b_i \ \mathbf{n}_i]_i \tag{23}
\]

in terms of the locator surface normal \( \mathbf{n}_i \) and bi-normals \( t_i \) and \( b_i \). Thus, the contact constraint is expressed as, using Eq. 17,

\[
\hat{\mathbf{v}}_{z_i} - h_i^T \mathbf{v}_o = 0 \tag{24}
\]

where \( \hat{\mathbf{v}}_{z_i} \) represents the locator linear velocity along the normal and \( h_i^T = [\mathbf{n}_i^T \ (r_i \times \mathbf{n}_i)^T] \). The other two linear velocity components and three rotational components of \( \hat{\mathbf{v}}_i \) are denoted by \( \hat{\mathbf{v}}_{xi} \), similar to \( \mathbf{n}_i \) defined in Eq. 11. The remaining five equations in Eq. 22 are written as

\[
\hat{\mathbf{v}}_{xi} - \mathbf{E}_i \mathbf{v}_o = \mathbf{H}_i \mathbf{\phi}_i \tag{25}
\]

with the contact matrix \( \mathbf{H}_i \) defined in Eq. 13 and \( \mathbf{E}_i \) being \( \mathbf{A}_i^{-1} \) (Eq. 17) without its third row.

Now, collecting equations (24) and (25) respectively for all of 6 locators \( (i = 1, \cdots, 6) \) would yield the following

\[
\begin{align*}
\mathbf{G} \mathbf{v}_o &= \hat{\mathbf{v}}_n \tag{26} \\
\hat{\mathbf{E}} \mathbf{v}_o + \hat{\mathbf{H}} \mathbf{\phi} &= \hat{\mathbf{v}}_s
\end{align*} \tag{27}
\]

where

\[
\mathbf{G} = \begin{bmatrix}
\mathbf{h}_1^T \\
\mathbf{h}_2^T \\
\vdots \\
\mathbf{h}_6^T
\end{bmatrix}, \quad \hat{\mathbf{E}} = \begin{bmatrix}
\mathbf{E}_1 \\
\mathbf{E}_2 \\
\vdots \\
\mathbf{E}_6
\end{bmatrix}, \quad \hat{\mathbf{H}} = \begin{bmatrix}
\mathbf{H}_1 & 0 & \cdots & 0 \\
0 & \mathbf{H}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{H}_6
\end{bmatrix}.
\]

Thus, the kinematics of the full locators-plus-workpiece fixture system are expressed with the locator velocities in their contact frames as

\[
\begin{bmatrix}
\mathbf{G} & 0 \\
\hat{\mathbf{E}} & \hat{\mathbf{H}}
\end{bmatrix} \begin{bmatrix}
\mathbf{v}_o \\
\mathbf{\phi}
\end{bmatrix} = \begin{bmatrix}
\hat{\mathbf{v}}_n \\
\hat{\mathbf{v}}_s
\end{bmatrix} \tag{28}
\]

**IV. PROPERTIES OF DETERMINISTIC CONTACT SYSTEM**

The velocity equation (28) of the full fixture system allows us to study its kinematic properties required for proper fixturing, including deterministic positioning \([6]\) and the role of regular contacts described in Section II-C.

**A. Deterministic Localization**

The six equations (26) concerning the normal velocity of each locator involve the workpiece velocity \( \mathbf{v}_o \) only, but without any of the internal meshing velocities \( \mathbf{\phi} \) of the locator contacts. In fact, these equations represent the conventional point-kinematic model of fixtures derived with the pure point geometry assumption \([6], [8]\). Therefore, the conventional fixture model is subsumed by the full kinematics of the locators plus workpiece fixture system.

As discussed in \([6]\) a fundamental function of the fixture is deterministic localization which requires the location (including the position and orientation) of the workpiece to be uniquely determined when all 6 locators are made to contact the workpiece surface. In other words, for any \( \mathbf{\hat{v}}_n \) in Eq. 26, there should exist a unique solution for \( \mathbf{v}_o \) of the workpiece velocity. This is true if and only if matrix \( \mathbf{G} \) has full rank, i.e., rank \( (\mathbf{G}) = 6 \).

**B. Deterministic Contact System**

Further, it is important for the whole locators-plus-workpiece system to be uniquely defined. This can be stated such that for any set of locator velocities \( \hat{\mathbf{v}}_n \) and \( \hat{\mathbf{v}}_s \) in Eq. 28 there exists a unique solution for the workpiece velocity \( \mathbf{v}_o \) and all of the meshing velocities \( \mathbf{\phi} \). This is true if and only if

\[
\left| \begin{bmatrix}
\mathbf{G} & 0 \\
\hat{\mathbf{E}} & \hat{\mathbf{H}}
\end{bmatrix}
\right| = |\mathbf{G}| \left| \hat{\mathbf{H}} \right| = |\mathbf{G}| \left| \mathbf{H}_1 \right| \left| \mathbf{H}_2 \right| \cdots \left| \mathbf{H}_6 \right| \neq 0 \tag{29}
\]

with | | denotes the determinant. While condition \(|\mathbf{G}| \neq 0\) represents the deterministic localization condition and \(|\hat{\mathbf{H}}| \neq 0\) represents a regular contact (Section II-C), the whole locators-plus-workpiece system shall be referred to as a deterministic contact system if every locator-workpiece contact is regular and the system is of deterministic localization.

A fixture system of deterministic contact allows for a direct separation of the “internal” meshing parameters \( \mathbf{\phi} \) from the “external” variables \( \mathbf{v}_o \) such that its kinematic solution is given by

\[
\begin{bmatrix}
\mathbf{v}_o \\
\mathbf{\phi}
\end{bmatrix} = \mathbf{G}^{-1} \begin{bmatrix}
\hat{\mathbf{v}}_n \\
\hat{\mathbf{v}}_s
\end{bmatrix} \tag{30}
\]

Thus, this solution shows that for a fixture of deterministic contact the deterministic localization of the workpiece will be maintained for any small perturbation in the location and orientation of any locator. The full kinematic model presents a solution to determine the change of the meshing points of all contacts as well as the location of the workpiece, in contrast to the conventional point-contact model which provides the deterministic localization solution only.
C. Two Special Systems

There exist two special cases of the full-kinematic model worthy mentioning:

1. \( K_i \neq 0 \) or \( K_i = 0 \) for all locators. This is the case of a prismatic workpiece or of all prismatic locators. The surface properties of the locators (or workpiece) will have no effects on the fixture kinematics. Thus, the full-kinematic model would not yield any more information than the conventional model.

2. \( K_i = \infty \) for \( i = 1, 2, \ldots, 6 \). This is the case that all of the locators are represented as theoretical points. However, it should be noted that the full-kinematic model would not degenerate into the conventional fixture model. The geometric effects of the workpiece surface remain to be captured by the full-kinematic model as opposed to the point-kinematic model which neglects surface properties.

V. Accuracy of Localization

The full-kinematic fixture model of Eq. 28 is a velocity formulation of the fixture-workpiece kinematics. The velocity space formulation offers two advantages over the conventional fixture model (Eq. 26 only). First, the new fixture model is an exact relation at any instant of contact between the workpiece and the locators; it is not a linear approximation as the conventional model. The full-kinematic model can be used to provide a more accurate and deeper analysis of the fixture performance, such as localization accuracy [22] and fixture robustness [23]. Another advantage to the velocity formulation is that the relation of parametric differentials with the relative linear and angular velocities of the workpiece-locator contacts can provide a highly accurate, strictly continuous tracing of the contacts. This can be used to facilitate efficient computation in haptic simulation of contact tasks [24], such as loading (unloading) the workpiece into (out) the fixture. In this section, we shall discuss the accuracy of localization.

In general, each locator may have certain location error, \( \delta q_i \), in its position and rotation, depending on the dimensioning and tolerancing scheme assigned to the fixture assembly and its components. One approach to an analysis of the resulting workpiece localization error \( \delta q_o \), is to consider the locator errors as statistical variables. The resulting \( \delta q_o \) can be characterized in statistical measures [25], [23]. This is especially useful in the early stage of fixture design, where only variations or tolerances of the error sources may be known.

The point-kinematic model of fixture (Eq. 26 only) is capable of representing only the locator errors \( \delta q_i \) \((i = 1, 2, \ldots, 6)\) projected along the surface normal \( n_i \) at the contact, i.e., the locator normal errors. It fails to capture two other important error sources. (1) The locator position generally exhibits equally significant errors in the tangent plane of contact defined by the bi-normals \( t_i \) and \( b_i \), referred to as the locator tangential errors. (2) Further, the surface geometries of the workpiece-locators system and the locator position error interactions will result in additional workpiece localization error. Thus, the point-kinematic model would underestimate the positioning error of the workpiece in the worst case, predicting only a subset of the resultant localization error.

As reported in [17], the effects of these two additional error causes are complex and difficult to characterize, especially in using a perturbation method of the conventional algebraic formulation. With the velocity formulation of the full-kinematic fixture model (Eq. 28), however, the whole set of localization error can be captured completely, accurately and efficiently, using numerical integration.

From Eq. 26 we define the locator normal errors as

\[
\delta q_n = \int_0^T \dot{\bar{v}}_n \, dt
\]

(32)

where \( T \) is specified by the locator positional tolerance specification. Similarly, other components of the locator error (including the tangent errors) are given as

\[
\delta q_s = \int_0^T \dot{\bar{v}}_s \, dt
\]

(33)

Thus, the resulting position and orientation errors of the workpiece are described as, following Eqs. 30 and 31,

\[
\delta q_o = \int_0^T G^{-1} \dot{\bar{v}}_n \, dt
\]

\[
\delta \bar{q} = \int_0^T \dot{H}^{-1} (\dot{\bar{v}}_s - \bar{E} G^{-1} \dot{\bar{v}}_n) \, dt
\]

(34)

(35)

This technique of numerical integration is simple and practical. It provides an incremental tracing of the parametric contact positions between the workpiece and the locator surfaces. Therefore, the resulting localization error from any specified locator positioning error is fully determined, unlike a perturbation method which provides an approximate prediction only [17]. If the workpiece-plus-locator system is a deterministic contact system, numerical integration of the localization errors may be done with a simple method, for example, the standard Euler’s method. Even higher order integration methods can be employed for a higher level of numerical precision without any excessive computational cost due to the efficiency of the numerical techniques available, as we have experienced for the examples shown next.

VI. Numerical Examples

Numerical examples are given in this section for different locating schemes with locator errors. The goal of the numerical examples is to illustrate the gain of localization accuracy predicted with the full-kinematic model as opposed to the conventional point-kinematic model. An elliptical workpiece in two dimensions is chosen to simplify the illustration and discussion and it is not due to any limitation of the method.

A. Example 1

The workpiece considered is an ellipse with a major half axis of 2 mm and a minor half axis of 1 mm. The three locators are circular with a radius of 0.1 mm. The locating scheme of this example is shown in Fig. 3.

In the first case, we illustrate the effects of locator tangential errors. Each of the three locator is assumed to have a position error \( \Delta_i \), along the tangential direction \( t_i \) \((i = 1, 2, 3)\) at its
contact with the workpiece. Therefore, in the velocity formulation, the locator error velocity and its position error are given respectively as

$$v_i = \Delta_i t_i \text{ and } \delta q_i = \Delta_i t_i t$$

$$\text{ (0} \leq t \leq T, \ T = 1, \ i = 1, 2, 3)$$

Then, using the numerical integration of Eqs. 34 and 35, we obtain the resulting position and orientation errors of the workpiece. Here, we let $\Delta_i = 0.2$ mm.

The initial and final positions of the locators and the workpiece are shown in Fig. 3. The full-kinematic model predicts a localization error of $\delta x = 5.73 \times 10^{-3}, \delta y = -4.87 \times 10^{-3}$ and $\delta \theta_2 = 1.21^\circ$. These localization errors would fail to be captured by the point-kinematic model. The trajectory of the center of the workpiece is shown in Fig. 4 from its initial location to the final location. The linear and angular displacements of the workpiece are plotted in Fig. 5 for the locator displacement.

Further, the workpiece localization errors due to the locator tangential errors are examined for locators of different sizes. As the locator radius increases, the resulting localization errors become less significant. As shown in Fig. 6, the localization errors tend to vanish for a very large locator radius. The numerical results confirm with the assessment of the special case discussed in Section IV-C that in the case of a prismatic workpiece or of all prismatic locators the full-kinematic model is degenerated into the point-kinematic model.

Next, the locator position errors are specified arbitrarily with $v_1 = [-0.1, -0.05]^T$, $v_2 = [0.05, -0.2]^T$, and $v_3 = [0.1, 0.1]^T$. The analysis with the velocity formulation yields the resulting localization errors for $\Delta_i = 0.2$. The initial and final positions of the locators and the workpiece are shown in Fig. 7, and the trajectory of the center of the workpiece is shown in Fig. 8. The linear and angular displacements of the workpiece are plotted in Fig. 9 as a function of $t$ representing the locator displacement, for the predictions made with full-kinematic model and the conventional point-kinematic model respectively.

**B. Example 2**

Another locating scheme is used as shown in Fig. 10. Again, the effects of locator tangential errors are examined for $\Delta_i = 0.2$ mm. The initial and final positions of the locators and the workpiece are shown in Fig. 10. The trajectory of the center of the workpiece is shown in Fig. 11 from its initial location to the final location. The linear and angular displacements of the workpiece are plotted in Fig. 12. In this case, the full-kinematic model predicts a localization error of $\delta x = 1.79 \times 10^{-3}, \delta y = 6.57 \times 10^{-2}$ and $\delta \theta_2 = -4.61^\circ$. These localization errors would fail to be captured by the point-kinematic model.

Next, the locator position errors are specified as in the second case of Example 1. The initial and final positions of the locators and the workpiece are shown in Fig. 13, and the tra-
VII. CONCLUSIONS

In this article we present a kinematic analysis of fixtures based on the full kinematics of locator-workpiece contact. This model incorporates the surface properties of both the workpiece and the locators, including the surface curvature, torsion and scale factors, as opposed to the conventional point-kinematic model. The necessary and sufficient conditions for deterministic localization and for a deterministic fixture system are given.

In an analysis of the positioning errors of the workpiece as a result of locator positioning errors, it is shown that the workpiece position and orientation are completely characterized by the kinematic properties of the locator contacts with the workpiece, including not only the locator locations but also the surface properties at all locator-workpiece contacts. The velocity formulation of the errors allows for an accurate evaluation of the effects of non-prismatic surface contact, locator error interactions and locator errors in arbitrary directions. This is illustrated with numerical examples. This analysis could be used to determine if the surface properties should be included when analyzing the precision of a specific locating scheme including locator shape and size.

This velocity formulation of fixture model could have a strong implication in analysis of fixturing precision. In general, it is understood that the full-kinematic model will provide a more accurate estimate of the localization error than the conventional model. It would be of practical relevance to further determine if a theoretic upper bond can be derived on the precision gain over the conventional model. Ongoing and future research also includes the development of fixture synthesis methods using the full fixture model to minimize the influence of fixture tolerances on fixturing precision.

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REFERENCES

Fig. 9. The linear and angular localization errors for Example 1 with given locator errors: – for the full-kinematic analysis; − for the conventional analysis.

Fig. 10. The locator and workpiece locations for Example 2 with locator tangential errors.

Fig. 11. The resulting workpiece center displacement for Example 2 with locator tangential errors.


Fig. 12. The linear and angular localization errors for Example 2 with locator
tangential errors.

Fig. 13. The locator and workpiece locations for Example 2 with given locator
errors.

Fig. 14. The resulting workpiece center displacement for Example 2 with given
locator errors.

Fig. 15. The linear and angular localization errors for Example 2 with given
locator errors: \( \cdots \) for the full-kinematic analysis; \( \cdots \) for the conventional
analysis.