ABSTRACT

Variational Geometry technique, also known as Constraints Satisfaction Problem technique (CSP), is one of the core components in modern CAD systems. One of the known problems of the existing technique is its inability of solving topological changes. In this paper, we propose a new Variational Geometry technique with Algebraic Level Set (ALS) geometric model. Based on half-space model theory and classic Level Set formulation, Algebraic Level Set concept is first described, its merits are also addressed within this context. Using this geometry representation, the topological changing problem caused by constraint variations could be solved in a natural way. This proposed method is capable of performing shape deformation capability that can be provided by conventional Variational Geometry technique, meanwhile, keeps the implicit topological definition of Algebraic Level Set functions which accounts for the topological changes in our proposed framework. Several numerical results demonstrate the preliminary capacity of our proposed approach.

1. INTRODUCTION

In a Computer-Aided Design (CAD) system, two popular approaches are used for representing a solid: the Constructive Solid Geometry (CSG) approach [1] and the Boundary Representation (B-Rep) approach [2].

CSG representation represents a solid as a set-theoretic Boolean expression of primitive solid objects, while B-Rep models are more flexible and has a much richer operation set, and this makes B-Rep a more favorable choice for modern CAD systems.

To meet the requirements of simplifying the modification of geometries during iterative design process, Variational Geometry (VG) technique was officially introduced by Gossard and his students [3-7]. The core concept of VG is the use of generalized dimensional constraints to modify geometry through shape variations, and use of geometric properties to constrain geometry. In conventional VG, the geometry of an object is considered to be determined by a set of $N$ characteristic points in 3D space. The characteristic points could be either points used for constructing a geometry (like the center point of a circle), or the points on a geometry entity (like vertices of a triangle). A complete set of characteristic points is described as a vector $X$,

$$X = (X_1, X_2, \cdots, X_N)$$  \hspace{1cm} (1)

Resembling the subdivision of characteristic points, constraints could also be divided into two main groups: implicit constraints (used for specifying relationship between topological elements) and explicit constraints (used for specifying dimensional value, like linear or angular dimensions), and all types of dimensions that define a specific geometry could be stored in a vector as:

$$d = (d_1, d_2, \cdots, d_k)$$ \hspace{1cm} (2)

where $k$ is the number of dimensions.

The geometry solving process is to resolve the set of nonlinear equations that relate constraints with geometry entities. Each of equations expresses a specific constraint and could be stated in a same form as:
\[ f_i = f_i(X, d) = 0, \ i = 1,2,\cdots m \]  

(3)

where \( m \) is the total number of constraint equations. If written in the matrix form, the collection of all constraint equations could be written as:

\[ f(X, d) = 0 \]  

(4)

Simultaneous solution of the equations produces the target geometry corresponding to the given constraints. Although Graph-based constraint solving techniques are dominate in CAD industry [8, 9], Newton-Raphson method is the most easy-to-implement method to resolve the non-linear set of equations. The typical geometry design procedure using conventional VG is described in Fig. 1.

Figure 1. Typical design procedure using conventional VG

From the designer’s point of view, if using VG method to reconcile CAD geometry, when topological changes occur, conventional VG will lead to an undesirable non-manifold solid. A typical example is demonstrated in Fig. 2. When \( \beta \) decreases, conventional VG could only give a solution as Fig. 2(c) shows, the desired geometry shown in Fig. 2(b) by no means could be obtained using this traditional methodology.

Figure 2. Invalid solid generated using conventional VG (I)

The same issue occurs in a similar problem shown in Fig. 3. As the linear dimension \( B \) in Fig. 3(a) grows, suppose the conventional VG is used, if without upper limit, its value will ultimately over take dimension \( A \) and forms the geometry as shown in Fig. 3(c). Notice that this in turn will lead to another non-manifold solid. Again, how to discard the upper rectangle in dashed line shown in Fig. 3(d) stays undetermined because of the inability of conventional VG, and there is no automatic way to obtain the shape shown in Fig. 3(b). Although this kind of ambiguities could be alleviated by human-software interaction during the design process, it’ll cause disasters for an optimization problem when the topology of a given geometry changes [10].

Figure 3. Invalid solid generated using conventional VG (II)

As in the B-Rep data structure, the topology and geometry are the two core elements. The topological elements and their geometric information together constitute a solid. The conventional VG only establishes a relation between topological elements and constraints, but the topology of a solid is pre-defined. Conventional B-rep based VG only changes the geometric information of topological elements which accounts for shape deformation, but has no effect on re-modifying how these topological elements are connected.

In this paper, we mainly concerned with presenting a new Variational Geometry technique using Algebraic Level Set model that is capable of handling topological changes. The Algebraic Level Set model will be first introduced in Section 2. Section 3 further explains the Variational Geometry with Algebraic Level Set model (VGALS) and related formulations. Numerical examples are presented in Section 4 followed by conclusions in Section 5.

2. ALGEBRAIC LEVEL SET (ALS) MODEL

Implicit representation of geometry has a long history in geometric modeling circle [11, 12]. It defines an arbitrary geometry \( \Omega \subset D \) implicitly in terms of non-negative values of implicit function \( \Phi(\vec{x}) \) as \( \Omega = \{ \vec{x} \in D | \Phi(\vec{x}) \geq 0 \} \), where \( D \) is the predefined admissible reference domain that contains all possible geometry \( \Omega \) of interest. By doing so, one could embed the geometry \( \Omega \) as a level set of a higher dimensional function \( \Phi(\vec{x}) \) [13-15]. The boundary \( \Gamma \) of \( \Omega \) is the zero level set of \( \Phi \) and can be expressed as \( \Gamma = \{ \vec{x} \in D | \Phi(\vec{x}) = 0 \} \). The
geometry could be intuitively explained in Fig. 4 and could be expressed by a level set function as:

\[
\begin{align*}
\Phi(\vec{x}) > 0, & \quad \forall \vec{x} \in \Omega \setminus \Gamma \\
\Phi(\vec{x}) = 0, & \quad \forall \vec{x} \in \Gamma \\
\Phi(\vec{x}) < 0, & \quad \forall \vec{x} \in D \setminus \Omega 
\end{align*}
\]  

(5)

![Level set function](image)

Figure 4. Level set function for defining geometry

### 2.1 LEVEL SET GEOMETRY REPRESENTATION FOR CONSTRUCTIVE SOLID GEOMETRY (CSG)

Suppose we have two independent level set functions \(\phi_1(\vec{x})\) and \(\phi_2(\vec{x})\), the Intersection of interior regions of them can be expressed as: \(\Omega = \{\vec{x} \in D | \Phi(\vec{x}) \geq 0\}\), where \(\Phi(\vec{x}) = \min(\phi_1, \phi_2)\). Also, the Union of interior regions of the said functions could be written as \(\Omega = \{\vec{x} \in D | \Phi(\vec{x}) \geq 0\}\), where \(\Phi(\vec{x}) = \max(\phi_1, \phi_2)\) [13, 16]. The whole idea can be clearly identified by Fig. 5.

![Figure 5](image)

Figure 5. (a) \(\Omega_1\) and \(\Omega_2\) for two different level set functions  
(b) Intersection of interior region and constructed \(\Phi\)  
(c) Union of interior region and constructed \(\Phi\)

### 2.2 HALF-SPACE MODEL AND ALGEBRAIC LEVEL SET GEOMETRY REPRESENTATION

Considering any half-space models in multi-dimensional spaces, say, an unbounded straight line in 2D space that divides \(\mathbb{R}^2\) into two semi-infinite regions. Such a straight line half-space model can be expressed as \(h(\vec{x}) = ax + by + c\) [17, 18]. Observe that this definition is consistent with the level set definition. Therefore, by considering the parameters in the half-space model as design variables, we come up with the concept as Algebraic Level Set (ALS). Based on typical 2D and 3D half-space models, we can have a full set of basic ALS primitives that could represent various kinds of planar and spatial shapes.

In level set method, the same boundary could be represented by infinite numbers of implicit functions (as is also the case in ALS representation), and Signed Distance Function (SDF) is usually chosen as the level set function because of its excellent computational merits. SDF is a special implicit function \(\Phi(\vec{x})\) with \(\|\Phi(\vec{x})\| = d(\vec{x})\) for all \(\vec{x}\) within admissible domain, where \(d(\vec{x}) = \|\vec{x} - \vec{x}_c\|\) and \(\vec{x}_c\) is the point on \(\Gamma\) that is closest to \(\vec{x}\) [12-14]. SDF \(\Phi(\vec{x})\) could be constructed using the following rules:

\[
\begin{align*}
\Phi(\vec{x}) = d(\vec{x}), & \quad \forall \vec{x} \in \Omega \setminus \Gamma \\
\Phi(\vec{x}) = 0, & \quad \forall \vec{x} \in \Gamma \\
\Phi(\vec{x}) = -d(\vec{x}), & \quad \forall \vec{x} \in D \setminus \Omega 
\end{align*}
\]  

(6)

Similar to SDF in conventional level set, here we choose the normalized ALS with SDF property to make it consistent with level set definition. The extra benefit for doing so is \(\|\Phi\| = 1\), which could reduce the computational errors when we do Boolean operations between different ALS primitives in the geometry construction process.

The normalized 2D ALS primitive expressions are:

**Line ALS primitive:**

\(\phi(\vec{x}, \vec{p}) = ax + by + c\)  
\(s.t.\quad a^2 + b^2 = 1\)  

**Circle ALS primitive (inside void):**

\(\phi(\vec{x}, \vec{p}) = \sqrt{(x - x_0)^2 + (y - y_0)^2} - r\)  

The normalized 3D ALS primitive expressions are:

**Plane ALS primitive:**

\(\phi(\vec{x}, \vec{p}) = ax + by + cz + d\)  
\(s.t.\quad a^2 + b^2 + c^2 = 1\)

**Sphere ALS primitive (inside void):**

\(\phi(\vec{x}, \vec{p}) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} - r\)  

**Cylinder ALS primitive (inside void, axis parallel to z):**

\(\phi(\vec{x}, \vec{p}) = \sqrt{(x - x_0)^2 + (y - y_0)^2} - r\)
Of all above-mentioned ALS functions, vector \( p \) refers to corresponding primitive’s design variables. For line primitive, \( p = (a, b, c) \), and for circle primitive, \( p = (x_0, y_0, r) \).

Like half-space models, ALS models can also be combined using the Boolean operators: union, intersection, etc. By doing so, complex geometries could be easily constructed. Fig. 6 shows a rectangle shape with a centered hole produced through the Boolean operations given at the bottom part of the figure.

For any complex geometry defined by multiple ALS primitives, we can obtain all characteristic coordinates using Eq. (14). By substituting coordinates of all characteristic points into Eq. (4), we have the new VG formulation

\[ f(X(P), d) = 0 \]  

The solving process of the new set of equations is the solving process of the new geometry with enabled topological changing property. Therefore, we call this as Variational geometry with Algebraic Level Set model (VGALS).

The solving scheme of the new set of non-linear equations here adopted is still Newton-Raphson method for the easy-to-implement reason. Compared with conventional VG, which updates coordinate value of characteristic points directly at each iteration, the proposed VGALS method does not update geometric information of topological elements directly. Alternatively, it first updates all parameters within all primitives, and then figures out possible locations of all characteristic points. Based on the implicitly defined topology of the geometry using Boolean operations, we can perform standard point membership classification as well as rebuild the B-Rep data structure for the resulted geometry. The algorithm of the proposed VGALS is shown in Fig. 8.

![Figure 6. Complex shape defined by multiple ALS primitives](image)

![Figure 7. Characteristic point definition with ALS models](image)

![Figure 8. Workflow of the proposed VGALS](image)
4. NUMERICAL EXAMPLES

In this section, three typical numerical examples are to be shown with both conventional B-rep based VG and the proposed VGALS to demonstrate the capability of VGALS, the comparisons will be given as well. Conventional B-rep based VG scheme is realized within the commercial CAD software Solidworks because of its powerful built-in constraint solver. As to the proposed VGALS scheme, Matlab is employed for rapid prototyping purpose. To exemplify the capability of the proposed method, all examples given in the following sessions only include 2D line ALS primitive due to its simplicity.

4.1 QUADRILATERAL GEOMETRY CASES

Consider a 2D polygon region bounded by 4 ALS straight line primitives, as shown in Fig. 9. The interior region $\Omega$ can be expressed as:

$$\Omega = \{\vec{x} \in D | \Phi(\vec{x}, \mathbf{P}) \geq 0\}, \quad \Phi(\vec{x}, \mathbf{P}) = \bigcap_{i=1}^{4} \phi_i(\vec{x}, \mathbf{p}_i) \quad (16)$$

Each characteristic point (vertex) can be formed using Eq. (14)

$$\phi_i \big| \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{array} \big| \begin{array}{c}
\Omega \\
\mathbf{P}_{12} \\
\mathbf{P}_{23} \\
\mathbf{P}_{34}
\end{array}$$

Figure 9. Quadrilateral area defined by 4 ALS primitives

4.1.1 QUADRILATERAL GEOMETRY DEFORMATION WITHOUT TOPOLOGICAL CHANGES

The initial design of the geometry is shown in Fig. 10(a), and the constraints are shown in Fig. 10(b), the solved geometry in Solidworks based on conventional VG is also given in Fig. 10(b).

As to the proposed VGALS method, with the same initial geometry guess (Fig. 11(a)), the constructed algebraic level set function is shown in Fig. 11(b), with the blue contour as the zero level set.

Figure 10. (a) Initial guess of the geometry
(b) Geometric constraints and final result of conventional VG

Figure 11. (a) Initial guess of the geometry
(b) Initial constructed ALS function with zero level set

After enforcing the same geometric constraints on corresponding characteristic points, based on the proposed algorithm, the Newton-Raphson iterations converged to the result shown in Fig. 12 within 8 iterations. Fig. 12(a) shows the converged geometry with four characteristic points identified by standard point membership classification. Fig. 12(b) shows the converged ALS function delivered by the proposed VGALS method.

Figure 12. (a) Converged geometry using ALSVG
(b) Converged ALS function with zero level set

Comparing Fig. 10(b) with Fig. 12(a), we can see that both conventional VG and the proposed VGALS method give the same results. That means, when the shape is not deforming
dramatically, the geometries obtained using both methods remains identical.

4.1.2 QUADRILATERAL GEOMETRY DEFORMATION WITH TOPOLOGICAL CHANGES
In this example, only the initial guess of the geometry is different, all constraints are remaining the same as described in the previous section.

The initial guess of the geometry is shown in Fig. 13(a), and the same sets of constraints shown in Fig. 13(b) are added to the 4 topological elements. The converged result based on conventional VG is also shown in Fig. 13(b).

One could clearly identify the problem from the above figure that with conventional VG, the inability of topological change leads to an unacceptable non-manifold geometry.

Using the proposed VGALS method, with the same initial geometry guess shown in Fig. 14(a), the constructed algebraic level set function is shown in Fig. 14(b).

After 6 iterations, the geometry converged with three characteristic points left. Actually, the converged geometry is the lower triangle of the resulting shape shown in Fig. 13(b). Fig. 15(a) shows the converged valid triangle geometry and Fig. 15(b) shows the converged ALS function given by the proposed VGALS method.

As can be easily observed that even with the same set of geometric constraints, the converged results in the above two examples are different geometries. The explanation for this phenomenon is, for the given set of non-linear equations that defines the geometry, there might be more than one reasonable solution, and this is the same situation in both conventional VG and our proposed VGALS.

4.2 SIMPLE CONCAVE NUMERICAL EXAMPLE WITH TOPOLOGICAL CHANGES
Consider another 2D polygon region bounded by 6 ALS line primitives, as shown in Fig. 16. The interior region of the geometry can be defined as

\[ \Phi = \phi_1 \cap \phi_2 \cap \phi_3 \cap \phi_4 \cap (\phi_5 \cup \phi_6) \]  

(17)

Fig. 15. (a) Converged geometry with topological changes (b) Converged ALS function with zero level set

\[ \Phi = \phi_1 \cap \phi_2 \cap \phi_3 \cap \phi_4 \cap (\phi_5 \cup \phi_6) \]  

(17)

Fig. 16. Concave geometry defined by 6 ALS primitives
Fig. 17(a) is the initial geometry and Fig. 17(b) gives the converged geometry using conventional VG with related constraints, which obviously, is an invalid solid with the upper rectangle part as the redundant geometry part.

The initial geometry and the corresponding ALS functions defined in our VGALS framework are shown in Fig. 18(a) and Fig. 18(b) respectively; the converged geometry and its corresponding ALS function obtained using VGALS is shown in Fig. 18(c) and Fig. 18(d). By comparing the initial geometry with the converged geometry, it’s easy to identify that the redundant part caused by conventional VG could be eliminated in the proposed method in a natural way.

The above three examples verified that, with conventional VG, if without side constraints, the topology of a resulted geometry might be invalid due to the reasons we addressed before. On the contrary, with the proposed VGALS, as the topology of the geometry is defined implicitly by the Boolean operations defined upon ALS functions, the topology of the geometry could change naturally and a new B-Rep data structure could be generated accordingly.

5. CONCLUSIONS

In this paper, we first analyzed the existing issues in conventional Variational Geometry with B-rep model, then, based on half-space model and the classic level-set formulation, we present the Algebraic Level Set concept as well as the advantages of this model representation. By combining conventional Variational Geometry formulation and ALS model, Variational Geometry with Algebraic Level Set model is thereafter given in this paper. This proposed method is a natural extension of the conventional VG, it’s not only capable of performing conventional VG functions to realize shape deformation, but also keeps the implicit topological definition of ALS which accounts for the topological changing capability in the preliminary numerical examples.

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