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# Structural decomposition of linear singular systems: the single-input and single-output case

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**Abstract**

In this paper, we present a structural decomposition for general single-input and single-output linear singular systems. Such a decomposition has a distinct feature of capturing and displaying all the structural properties, such as the finite and infinite zero structures and redundant dynamics, of the given system. It is expected to be a powerful tool in solving control problems for singular systems, such as $H_2$ and $H_\infty$ control, model reduction and disturbance decoupling problems, to name a few.


*Keywords:* Singular systems; Structural invariants; Zero structures; System decomposition

## 1. Introduction

Singular systems, also commonly called generalized or descriptor systems in the literature, appear in many practical situations including engineering systems, economic systems, network analysis, and biological systems (see e.g. [4,8,9]). In fact, many systems in the real life are singular in nature. They are usually simplified as or approximated by nonsingular models because there is still lacking of efficient tools to tackle problems related to such systems. The structural analysis of linear singular systems, using either algebraic or geometric approach, has attracted considerable attention from many researchers during the last three decades (see e.g. [5–7,10,12,14,17,18] and the references cited therein). Generally speaking, almost all the research works dealing with singular systems are the natural extension of those results for nonsingular counterparts, although it is much harder in obtaining solutions associated with singular systems.

It has been extensively demonstrated and proven for nonsingular systems that the system structural properties, such as the finite zero and infinite zero structures as well as the invertibility structures, play a very important role in solving related control problems including $H_2, H_\infty$ control and disturbance decoupling (see [2,15]). In this paper, we present a structural decomposition of general single-input and single-output linear singular systems, which is capable of capturing and displaying all the structural properties of the given system. Our method can be regarded as a natural extension of the work of Sannuti and Saberi [16]. However, it will be seen shortly that the structural

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decomposition of a singular system is much more complicated than that of a nonsingular system. Such a decomposition technique is expected to be a powerful tool and play an important role in solving control problems for singular systems, such as $H_2$ and $H_\infty$ control, model reduction and disturbance decoupling, to name just a few.

To be more specific, we consider a linear time-invariant system $\Sigma$ characterized by

$$\begin{cases} E\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad u(0) = u_0, \\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$ and $y \in \mathbb{R}$ are, respectively, the state, input and output of the system, and $E, A, B, C$ and $D$ are constant matrices of appropriate dimension. The system $\Sigma$ is said to be singular if rank $(E) < n$. As usual, in order to avoid any ambiguousness in solutions to the system, we assume that the given singular system $\Sigma$ is regular, i.e., $\det(sE - A) \not\equiv 0$, for all $s \in \mathbb{C}$. In this paper, we will present a constructive algorithm that decomposes the state of the system $x$ into several distinct parts, which are directly associated with the finite zero dynamics and infinite zero dynamics of the given system. It is interesting to note that our decomposition will automatically and explicitly separate the redundant dynamics of the system as well.

The outline of this paper is as follows: In Section 2, we present the main results of our work, i.e., the structural decomposition of single-input and single-output singular systems and its properties. For the clarity of presentation, the detailed proofs of these results are given in Section 3, whereas an illustrative example is given in Section 4. Finally, Section 5 draws some concluding remarks on the work and on the future research along the line.

Throughout this paper, the following notation will be used: $I$ denotes an identity matrix with appropriate dimensions; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}, \mathbb{C}^0, \mathbb{C}^-$ and $\mathbb{C}^+$ represent, respectively, the set of all complex numbers, the imaginary axis, the open left-half plane and the open right-half plane; $\lambda(X)$ is the set of eigenvalues of a real square matrix $X$; and $u^{(v)}$ is the $v$th order derivative of $u$, where $v$ is an integer.

## 2. Structural decomposition and properties

We present in this section the main results of the paper, i.e., the structural decomposition of the singular system (1) and its properties. We first have the following theorem.

**Theorem 2.1.** *Consider the singular system $\Sigma$ of (1) satisfying the regularity assumption, i.e., $\det(sE - A) \not\equiv 0$ for $s \in \mathbb{C}$, and its transfer function is nontrivial, i.e., $H(s) = C(sE - A)^{-1}B \not\equiv 0$ for $s \in \mathbb{C}$. There exist*

1. *coordinate free non-negative integers $n_\mathrm{z}, n_\mathrm{a}, n_\mathrm{d}, n_\mathrm{e}$ and $v$; and*
2. *nonsingular state, input and output transformations $\Gamma_\mathrm{s} \in \mathbb{R}^{n \times n}, \Gamma_\mathrm{i} \in \mathbb{R}$ and $\Gamma_\mathrm{o} \in \mathbb{R}$, and a nonsingular constant matrix $\Gamma_\mathrm{e} \in \mathbb{R}^{n \times n}$, which together give a structural decomposition of $\Sigma$ and display explicitly its finite and infinite zero structures.*

*The structural decomposition of $\Sigma$, or the transformed system, can be described by the following set of equations:*

$$x = \Gamma_\mathrm{s}\tilde{x}, \quad \tilde{x} = \begin{pmatrix} x_\mathrm{e} \\ x_\mathrm{z} \\ x_\mathrm{a} \\ x_\mathrm{d} \end{pmatrix}, \quad x_\mathrm{d} = \begin{pmatrix} x_{\mathrm{d}_1} \\ x_{\mathrm{d}_2} \\ \vdots \\ x_{\mathrm{d}n_\mathrm{d}} \end{pmatrix},$$

$$y = \Gamma_\mathrm{o}\tilde{y}, \quad u = \Gamma_\mathrm{i}\tilde{u}, \tag{2}$$

*where $x_\mathrm{e} \in \mathbb{R}^{n_\mathrm{e}}$, $x_\mathrm{z} \in \mathbb{R}^{n_\mathrm{z}}$, $x_\mathrm{a} \in \mathbb{R}^{n_\mathrm{a}}$, $x_\mathrm{d} \in \mathbb{R}^{n_\mathrm{d}}$, and*

*Case* 1: *If $n_\mathrm{d} = 0$,*

$$\left.\begin{aligned} & x_e = \tilde{u}^{(v)}, \\ & x_\mathrm{z} = 0, \\ & \dot{x}_\mathrm{a} = A_{\mathrm{aa}}x_\mathrm{a} + B_{0\mathrm{a}}\tilde{y}, \ \tilde{y} = \bar{C}x_\mathrm{a} + \bar{D}\tilde{u}^{(v)}. \end{aligned}\right\} \tag{3}$$

*Case* 2: *If* $n_{\mathrm{d}} > 0$,

$$\left.\begin{aligned} &x_e = \tilde{u}^{(v)},\\ &x_{\mathrm{z}} = 0,\\ &\dot{x}_{\mathrm{a}} = A_{\mathrm{aa}}x_{\mathrm{a}} + L_{\mathrm{ad}}y_{\mathrm{d}},\\ &\dot{x}_{\mathrm{d}1} = x_{\mathrm{d}2},\\ &\dot{x}_{\mathrm{d}2} = x_{\mathrm{d}3},\\ &\quad\vdots\\ &\dot{x}_{\mathrm{d}n_{\mathrm{d}}} = M_{\mathrm{da}}x_{\mathrm{a}} + L_{\mathrm{dd}}y_{\mathrm{d}} + \tilde{u}^{(v)},\ \tilde{y} = y_{\mathrm{d}} = x_{\mathrm{d}1} \end{aligned}\right\} \tag{4}$$

*with initial conditions* $\tilde{x}(0) = \tilde{x}_0 = \Gamma_{\mathrm{s}}^{-1}x_0$ *and* $\tilde{u}(0) = \tilde{u}_0 = \Gamma_i^{-1}u_0$.

A constructive proof of the structural decomposition in Theorem 2.1 will be given later in the next section. We note that the impulsive modes, if any, caused by the derivatives of the system input are all preserved under the structural decomposition. Fig. 1 gives a block diagram interpretation of the dynamics of the structurally decomposed system in Case 2 of Theorem 2.1. In the figure, a signal given by a double-edged arrow is some linear combination of output $y_{\mathrm{d}}$, whereas a signal given by the double-edged arrow with a solid dot is some linear combination of all the states.

As mentioned earlier, the structural decomposition of Theorem 2.1 has distinct feature of revealing the structural properties of the given singular system $\Sigma$. In what follows, we will study how the system properties of $\Sigma$ such as the stabilizability, detectability, invertibility, as well as finite zero and infinite zero structures, can be obtained from our decomposition.

We first recall the definitions of stability, stabilizability and detectability of linear singular systems from the literature (see e.g. [4]).

**Definition 2.1** (Stability, stabilizability and detectability) The singular system $\Sigma$ of (1) is said to be stable if its characteristic polynomial $\det(sE - A)$ has all roots in $\mathbb{C}^-$. It is said to be stabilizable if there exists an appropriate dimensional constant matrix $F$ such that the roots of $\det(sE - A - BF)$ are stable. Similarly, it is said to be detectable if there exists an appropriate dimensional constant matrix $K$ such that the roots of $\det(sE - A - KC)$ are stable.

We have the following property.

**Property 2.1** (Stabilizability and detectability)**.** *The given system* $\Sigma$ of (1) *is stabilizable if and only if the pair* $(A_{\mathrm{aa}}, B_{\mathrm{con}})$ *is stabilizable.* $\Sigma$ *is detectable if and only if the pair* $(A_{\mathrm{aa}}, C_{\mathrm{obs}})$ *is detectable. Here* $B_{\mathrm{con}} := B_{0\mathrm{a}}$ *and* $C_{\mathrm{obs}} := \bar{C}$ *in Case* 1, *while* $B_{\mathrm{con}} := L_{\mathrm{ad}}$ *and* $C_{\mathrm{obs}} := M_{\mathrm{da}}$ *in Case* 2.

The definition of invariant zeros of singular systems can be done similarly as that for nonsingular systems (see e.g. [2,12]) or in the Kronecker canonical form associated with $\Sigma$ (see e.g. [13]).

**Definition 2.2** (Invariant zeros). A complex scalar $\alpha \in \mathbb{C}$ is said to be an invariant zero of the singular system $\Sigma$ of (1) if

$$\operatorname{rank}\{P_\Sigma(\alpha)\} < n + \operatorname{normrank}\{H(s)\}, \tag{5}$$

where $\operatorname{normrank}\{H(s)\}$ denotes the normal rank of $H(s) = C(sE - A)^{-1}B$, which is defined as its rank over the field of rational functions with real coefficients, and $P_\Sigma(s)$ is the Rosenbrock system matrix associated with $\Sigma$ and is given by

$$P_\Sigma(s) = \begin{bmatrix} A - sE & B \\ C & 0 \end{bmatrix}. \tag{6}$$

The following property shows that the invariant zeros of $\Sigma$ can be obtained in the structural decomposition in a trivial matter.

**Property 2.2** (Invariant zeros). *The invariant zeros of* $\Sigma$ *are the eigenvalues of* $A_{\mathrm{aa}}$.

The infinite zero structure of $\Sigma$ can be either defined in association with the Kronecker canonical form of $P_\Sigma(s)$ or as Smith–McMillan zeros of the transfer function $H(s)$ at infinity. The indices defined by these two methods are somewhat different, although they are related to each other. This can be seen in Property 2.3. For the sake of simplicity, we only consider the infinite zeros from the point of view of Smith–McMillan theory here. To define the zero structure of $H(s)$ at

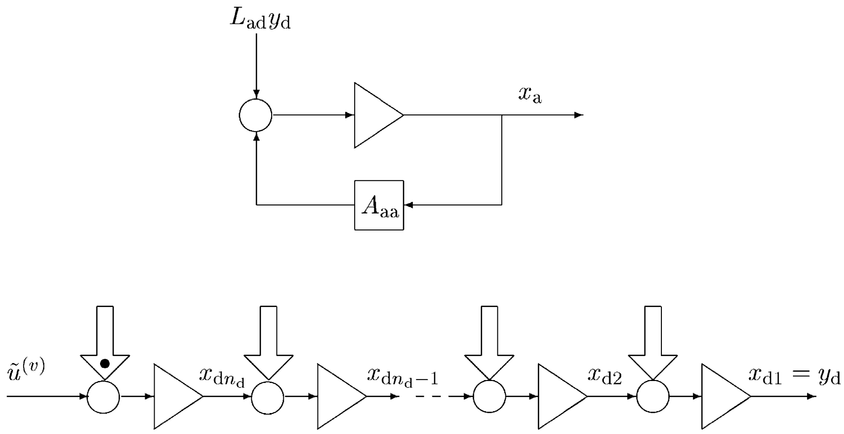


Fig. 1. Block diagram representation of dynamics of the structurally decomposed system.

infinity, one can use the familiar Smith–McMillan description of the zero structure at finite frequencies of $H(s)$. Namely, a rational matrix $H(s)$ possesses an infinite zero of order $k$ when $H(1/z)$ has a finite zero of precisely that order at $z=0$ (see [3,17]). The number of zeros at infinity together with their orders indeed defines an infinite zero structure. For a single-input and single-out system, the infinite zero structure is equivalent to the relative degree of the system.

**Property 2.3** (Infinite zero structure). *The infinite zero structure of the singular system $\Sigma$ is given by $\{n_{\mathrm{d}}-v\}$, i.e., $\Sigma$ has an infinite zero of order or relative degree $n_{\mathrm{d}}-v$. However, $\Sigma$ has an infinite elementary divisor of order $n_{\mathrm{d}}$ in its corresponding Kronecker canonical form.*

Again, the rigorous proofs to all these properties are given in the next section.

## 3. Proofs of main results

We are now ready to give proofs to the main results of our paper, i.e., the structural decomposition of Theorem 2.1 and its properties.

### 3.1. Proof of Theorem 2.1

The following is a step-by-step constructive procedure for the structural decomposition of $\Sigma$.

*Step* 1 (*Preliminary decomposition*): It follows from Dai [4] that there exist two nonsingular matrices $P\in\mathbb{R}^{n\times n}$ and $Q\in\mathbb{R}^{n\times n}$ such that

$$PEQ=\begin{bmatrix} I_{n_1} & 0\\ 0 & N\end{bmatrix},\quad PAQ=\begin{bmatrix} A_1 & 0\\ 0 & I_{n_2}\end{bmatrix},$$

$$PB=\begin{bmatrix} B_1\\ B_2\end{bmatrix},\quad CQ=[C_1\quad C_2], \tag{7}$$

where $A_1, B_1, B_2, C_1$ and $C_2$ are matrices with appropriate dimensions, and $N$ is a nilpotent matrix with an appropriate nilpotent index, say $h$, i.e., $N^{h-1}\neq 0$ and $N^h=0$. Equivalently, $\Sigma$ can be decomposed into the following two subsystems:

$$\Sigma_1:\begin{cases}\dot{x}_1=A_1x_1+B_1\,u,\ x_1(0)=x_{10},\\ y_1=C_1\,x_1\end{cases} \tag{8}$$

and

$$\Sigma_2:\begin{cases}N\,\dot{x}_2=x_2+B_2\,u,\ x_2(0)=x_{20},\\ y_2=C_2\,x_2,\end{cases} \tag{9}$$

where $x_1\in\mathbb{R}^{n_1}$ and $x_2\in\mathbb{R}^{n_2}$ with $n_1+n_2=n$, and $y=y_1+y_2$.

*Step* 2 (*Decomposition of* $\Sigma_2$): If $B_2=0$, we have $x_0=x_2, n_{\mathrm{z}}=n_2, x_{\mathrm{e}}=\emptyset, n_{\mathrm{e}}=0$ and $v=0$.

For this case, the following procedure does not apply. We go directly to Step 3.

For the case when $B_2\neq 0$, it follows from Brunovsky [1] and Luenberger [11] (see also [2]) that there exist a nonsingular transformation $T_2$ and $\alpha \neq 0$ such that

$$x_2 = T_2 \begin{pmatrix} x_{\rm v} \\ x_{\rm z} \end{pmatrix}, \quad x_{\rm z} \in \mathbb{R}^{n_z}, \quad x_{\rm v} \in \mathbb{R}^{v_{\rm d}},$$

$$x_{\rm v} = \begin{pmatrix} x_{\rm v1} \\ \vdots \\ x_{{\rm v}v_d} \end{pmatrix} \tag{10}$$

and

$$T_2^{-1} N T_2 = \begin{bmatrix} J_{c0} & N_{c\bar{c}} \\ 0 & J_{n_z} \end{bmatrix},$$

$$T_2^{-1} B_2 = \begin{bmatrix} B_{2c} \\ 0 \end{bmatrix},$$

$$C_2 T_2 = [C_{2c} \quad C_{2\bar{c}}], \tag{11}$$

where $(J_{c0}, B_{2c})$ is a completely controllable pair. Since $N$ has all its eigenvalues at 0 and $B_{2c}$ is a column vector, $(J_{c0}, B_{2c})$ can actually be written as

$$J_{c0} = \begin{bmatrix} 0 & I_{v_{\rm d}-1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_{2c} = \begin{bmatrix} 0 \\ -1/\alpha \end{bmatrix}. \tag{12}$$

Also note that $J_{n_z}$ has all its eigenvalues at 0. As such, it is simple to verify that $\Sigma_2$ is decomposed into the following two subsystems:

$$J_{n_z} \dot{x}_{\rm z} = x_{\rm z} \Rightarrow x_{\rm z} = 0 \tag{13}$$

and $J_{c0}\dot{x}_{\rm v} + N_{c\bar{c}}\dot{x}_{\rm z} = x_{\rm v} + B_{2c}u$, which is equivalent to $J_{c0}\dot{x}_{\rm v} = x_{\rm v} + B_{2c}u$ or

$$u = \alpha x_{{\rm v}v_{\rm d}}, \quad \dot{x}_{{\rm v}v_d} = x_{{\rm v}v_d-1}, \ldots, \dot{x}_{\rm v2} = x_{\rm v1}, \tag{14}$$

which implies

$$x_{\rm e} := x_{\rm v1} = \frac{1}{\alpha} u^{(v)} \quad \text{and} \quad n_{\rm e} = 1 \tag{15}$$

and where $v = \max(0, v_{\rm d} - 1)$. The output $y_2$ can then be expressed as

$$y_2 = C_{2c} x_{\rm v} + C_{2c} x_{\rm z} = C_{2\bar{c}} x_{\rm v}. \tag{16}$$

*Step* 3 (*Decomposition of the finite and infinite zero structures*): Observing the results in (8), (9), (13)–(16), we can obtain the following regular system:

$$\left.\begin{aligned} \dot{x}_1 &= A_1 x_1 + \alpha B_1 x_{{\rm v}v_{\rm d}}, \\ x_{\rm v1} &= x_{\rm e} = \frac{1}{\alpha} u^{(v)}, \\ \dot{x}_{\rm v2} &= \frac{1}{\alpha} u^{(v)}, \\ &\vdots \\ \dot{x}_{{\rm v}v_{\rm d}} &= x_{{\rm v}v_{\rm d}-1}, \\ x_{\rm z} &= 0, \\ y &= C_1 x_1 + C_{2c} x_{\rm v}. \end{aligned}\right\} \tag{17}$$

Next, let us partition

$$C_{2c} = [c_{\rm v1} \quad c_{\rm v2} \quad \cdots \quad c_{{\rm v}v_{\rm d}}]. \tag{18}$$

Thus, the nonsingular system (17) can be rewritten as

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}, \\ y = \bar{C}\bar{x} = \bar{D}\bar{u}, \end{cases} \tag{19}$$

where

$$\bar{x} = \begin{pmatrix} x_1 \\ x_{\rm v2} \\ \vdots \\ x_{{\rm v}v_{\rm d}-1} \\ x_{{\rm v}v_{\rm d}} \end{pmatrix},$$

$$\bar{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 & \alpha B_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

and

$$\bar{u}=\frac{1}{\alpha}u^{(v)}, \quad \bar{C}=[C_1 \quad c_{v2} \quad \cdots \quad c_{vv_d-1} \quad c_{vv_d}],$$
$$\bar{D}=c_{v1}. \tag{21}$$

Note that $H(s)$ is nontrivial. We have the following two distinct cases.

(1) $\bar{D}=0$ and it is corresponding to Case 2 of Theorem 2.1. It follows from the result of Sannuti and Saberi [16] that there exist nonsingular transformations $\bar{\Gamma}_s$ and $\Gamma_o$ such that when we apply the following changes of coordinates:

$$\bar{x}=\bar{\Gamma}_s\check{x}=\bar{\Gamma}_s\begin{pmatrix} x_a \\ x_d \end{pmatrix}, \qquad y=\Gamma_o\tilde{y} \tag{22}$$

to the system in (19), and in view of (15), we have

$$\dot{\check{x}}=\begin{bmatrix} A_{aa} & L_{ad}C_d \\ B_dM_{da} & A_{dd} \end{bmatrix}\check{x}+\begin{bmatrix} 0 \\ B_d \end{bmatrix}\alpha^{-1}u^{(v)} \tag{23}$$

and

$$\tilde{y}=[0 \quad C_d]\check{x}, \tag{24}$$

where $A_{dd}, B_d$ and $C_d$ have the form as given in (29). Let

$$u=\Gamma_i\tilde{u}=\alpha\tilde{u}\Rightarrow\alpha^{-1}u^{(v)}=\tilde{u}^{(v)}. \tag{25}$$

(2) $\bar{D}\neq 0$ and it is corresponding to Case 1 of Theorem 2.1. In this case, it is simple to obtain $x_d=\emptyset$, $n_d=0, x_a=\bar{x}, n_a=n_1+v$ and

$$\dot{x}_a=(\bar{A}-\bar{B}\bar{D}^{-1}\bar{C})x_a+\bar{B}\bar{D}^{-1}y=A_{aa}x_a+B_{0a}y \tag{26}$$

and

$$y=\bar{C}x_a+\bar{D}\alpha^{-1}u^{(v)}=\bar{C}x_a+\bar{D}\tilde{u}^{(v)}, \tag{27}$$

if we let $u=\Gamma_i\tilde{u}=\alpha\tilde{u}$.

This completes the algorithm for the structural decomposition of $\Sigma$. □

Actually, we can rewrite the structural decomposition of $\Sigma$ in a compact matrix form, which will be handy in proving the properties of the structural decomposition. For simplicity, we will only focus on Case 2 of Theorem 2.1, i.e., $n_d>0$. The compact form for Case 2 of Theorem 2.1 is given by

$$\left.\begin{aligned}
\tilde{E}&=\Gamma_e^{-1}E\Gamma_s \\
&=\begin{bmatrix} 0 & E_{ez} & 0 & 0 \\ 0 & J_{n_z} & 0 & 0 \\ 0 & E_{az} & I_{n_a} & 0 \\ 0 & E_{dz} & 0 & I_{n_d} \end{bmatrix}, \\
\tilde{A}&=\Gamma_e^{-1}A\Gamma_s \\
&=\begin{bmatrix} 0 & A_{ez} & N_{ea} & N_{ed} \\ 0 & I_{n_z} & 0 & 0 \\ 0 & A_{az} & A_{aa} & L_{ad}C_d \\ B_d & A_{dz} & B_dM_{da} & A_{dd} \end{bmatrix}, \\
\tilde{B}&=\Gamma_e^{-1}B\Gamma_i=\begin{bmatrix} B_e \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\tilde{C}&=\Gamma_o^{-1}C\Gamma_s=[0 \quad C_z \quad 0 \quad C_d],
\end{aligned}\right\} \tag{28}$$

where $J_{n_z}$ is in a Jordan canonical form with all its diagonal elements being equal to 0, and $N_{ea}, N_{ed}$ and are sub-matrices with appropriate dimensions, and $B_e\neq 0$. Furthermore, matrices $A_{dd}, B_d, C_d$ are in the following forms:

$$A_{dd}=\begin{bmatrix} 0 & I_{n_d-1} \\ \star & 0 \end{bmatrix}, \quad B_d=\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_d=[1 \quad 0 \quad \cdots \quad 0]. \tag{29}$$

### 3.2. Proof of Property 2.1

It follows from Dai [4] that the singular system $\Sigma$ of (1) is stabilizable if and only if

$$\operatorname{rank}[sE-A \quad B]=n \tag{30}$$

for all $s\in\mathbb{C}^0\cup\mathbb{C}^+$. Let us again focus on Case 2 of Theorem 2.1. In the structural decomposition form

$$\operatorname{rank}[sE-A \quad B]$$
$$=\operatorname{rank}[s\tilde{E}-\tilde{A} \quad \tilde{B}]$$

$$= \text{rank}\begin{bmatrix} 0 & sE_{ez}-A_{ez} & -N_{ea} & -N_{ed} & B_e \\ 0 & sJ_{n_z}-I_{n_z} & 0 & 0 & 0 \\ 0 & sE_{az}-A_{az} & sI_{n_a}-A_{aa} & -L_{ad}C_d & 0 \\ -B_d & sE_{dz}-A_{dz} & -B_dM_{da} & sI_{n_d}-A_{dd} & 0 \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & B_e \\ 0 & sJ_{n_z}-I_{n_z} & 0 & 0 & 0 \\ 0 & 0 & sI_{n_a}-A_{aa} & -L_{ad}C_d & 0 \\ -B_d & 0 & 0 & sI_{n_d}-A_{dd} & 0 \end{bmatrix}. \tag{31}$$

Noting that $B_e \neq 0$ and the special structures of $J_{n_z}, A_{dd}, C_d$ and $B_d$, it is straightforward to show that $\Sigma$ is stabilizable if and only if $(A_{aa}, L_{ad})$ is stabilizable. Results for Cases 1 of Theorem 2.1 can be shown in a similar way.

Similarly, the proof for the detectability can be done in a dual fashion. This completes the proof of Property 2.1. □

### 3.3. Proof of Property 2.2

Again, we prove this property for Case 2 of Theorem 2.1. Observing that for $\alpha \in \mathbb{C}$, we have

$$\text{rank}\{P_\Sigma(\alpha)\}$$

$$= \text{rank}\{P_{\tilde{\Sigma}}(\alpha)\}$$

$$= \text{rank}\begin{bmatrix} 0 & A_{ez}-\alpha E_{ez} & N_{ea} & N_{ed} & B_e \\ 0 & I_{n_z}-\alpha J_{n_z} & 0 & 0 & 0 \\ 0 & A_{az}-\alpha E_{az} & A_{aa}-\alpha I_{n_a} & L_{ad}C_d & 0 \\ B_d & A_{dz}-\alpha E_{dz} & B_dM_{da} & A_{dd}-\alpha I_{n_d} & 0 \\ 0 & C_z & 0 & C_d & 0 \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} 0 & 0 & 0 & 0 & B_e \\ 0 & I_{n_z}-\alpha J_{n_z} & 0 & 0 & 0 \\ 0 & 0 & A_{aa}-\alpha I_{n_a} & 0 & 0 \\ B_d & 0 & 0 & A_{dd}-\alpha I_{n_d} & 0 \\ 0 & 0 & 0 & C_d & 0 \end{bmatrix}$$

$$= n_e + n_z + n_d + 1 + \text{rank}\{A_{aa} - \alpha I_{n_a}\}. \tag{32}$$

Obviously, the rank of $P_\Sigma$ drops if and only if $\alpha \in \lambda(A_{aa})$. Hence, the invariant zeros of $\tilde{\Sigma}$ are given by the eigenvalues of $A_{aa}$. In fact, the eigenstructure of $A_{aa}$ defines the finite zero structure of $\Sigma$. This completes the proof of Property 2.2. □

### 3.4. Proof of Property 2.3

It is well known that the infinite zero structure or relative degree of $\Sigma$ is nothing more than the number of integrators that are inherent in between the system input $u$ and the system output $y$. As all transformations involved in our structural decomposition are nonsingular, the number of inherent integrators remains unchanged under such transformations. It follows from the constructive proof of Theorem 2.1 (see also Fig. 1) that there are $n_d$ integrators in between $\tilde{u}^{(v)}$ and $\tilde{y}$, where $v = \max(0, v_d - 1)$. Thus, the number of inherent integrators in between $u$ and $y$ is $n_d - v$. Hence, the result of Property 2.3 follows.

## 4. An illustrative example

In this section, an example is presented to illustrate the structural decomposition procedure and its properties. We consider a singular system of (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

and

$$C = [2 \quad 0 \quad -2 \quad 1 \quad -1], \quad D = 0. \tag{34}$$

Following the procedure given in the previous section, we obtain $n_a = 2, n_d = 1, n_z = 1, n_e = 1$ and $v = 2$, and all the necessary transformations,

$$\Gamma_e = \begin{bmatrix} 1 & 0 & -0.3333 & -0.7071 & 0 \\ 0 & 0 & 0.6667 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.6667 & -0.7071 & 0 \end{bmatrix},$$

$$\Gamma_s = \begin{bmatrix} 0 & 0 & -0.3333 & -0.7071 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.6667 & -0.7071 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.6667 & 0 & 1 \end{bmatrix} \tag{35}$$

and $\Gamma_o = -1, \Gamma_i = 1$. The transformed system is then given by

$$x_e = \ddot{u}, \quad x_z = 0, \tag{36}$$

$$\dot{x}_a = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x_a + \begin{bmatrix} -3 \\ 1.4142 \end{bmatrix} y_d, \tag{37}$$

$$\dot{x}_{d1} = [0.6667 \quad 0] x_a + 2y_d + \ddot{u},$$

$$\tilde{y} = y_d = x_{d1}. \tag{38}$$

It is simple to see now from the above decomposition that there are two invariant zeros are $s_1 = -1$ and $s_2 = 0$, and the infinite zero structure or relative degree of $\Sigma$ from $\ddot{u}$ to $y$ is equal to 1. Thus, $\Sigma$ has a relative degree of $-1$ from $u$ to $y$. These results can be easily verified from the transfer function of $\Sigma$,

$$H(s) = C(sE - A)^{-1}B = \frac{s(s+1)}{s-1}. \tag{39}$$

Finally, we note that it can be shown that there is an infinite elementary divisor of order $n_d = 1$ in the Kronecker canonical form associated with $\Sigma$.

## 5. Conclusions

We have presented in this paper a structural decomposition technique for general single-input and single-output linear singular systems, which has a distinct feature of explicitly capturing and displaying the structural properties, such as the finite and infinite zero structures, of the given system. As its counterpart in nonsingular systems, the technique is expected to play an important role in solving many control problems related to singular systems. This will actually be the subject of our future research. Our immediately future research will also be focusing on the development of a similar technique for multi-input and multi-output singular systems.

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