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Analysis and design for discrete-time linear systems subject to actuator saturation ☆

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Abstract

We present a method to estimate the domain of attraction for a discrete-time linear system under a saturated linear feedback. A simple condition is derived in terms of an auxiliary feedback matrix for determining if a given ellipsoid is contractively invariant. Moreover, the condition can be expressed as linear matrix inequalities (LMIs) in terms of all the varying parameters and hence can easily be used for controller synthesis. The following surprising result is revealed for systems with single input: suppose that an ellipsoid is made invariant with a linear feedback, then it is invariant under the saturated linear feedback if and only if there exists a saturated (nonlinear) feedback which makes the ellipsoid invariant. Finally, the set invariance condition is extended to determine invariant sets for systems with persistent disturbances. LMI based methods are developed for constructing feedback laws that achieve disturbance rejection with guaranteed stability requirements. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper, we are interested in the control of linear systems subject to actuator saturation and persistent disturbances,

$$x(k+1) = Ax(k) + B\mathrm{sat}(u(k)) + Ew(k), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ w \in \mathbb{R}^q, \tag{1}$$

where x is the state, u is the control, w is the disturbance and $\mathrm{sat}(\cdot)$ is the standard saturation function. First, we will consider the closed-loop stability under a given linear state feedback $u = Fx$ in the absence of the disturbance. There has been a lot of work on this topic (see, e.g. [3–5,9–14] and the references therein). For the continuous-time case, various simple and general methods for estimating the domain of attraction have been developed by applying the absolute stability analysis tools, such as the circle and Popov criteria (see, e.g. [5,9,10,12], where the saturation is treated as a locally sector bounded nonlinearity and the domain of attraction is estimated by use of quadratic and Lur'e type Lyapunov functions). The multivariable circle criterion in [9]

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was restated in [12], in terms of (nonlinear) matrix inequalities in controller parameters and other auxiliary optimization parameters, such as the positive definite matrix P in the Lyapunov function $V(x)=x^{\mathrm{T}}Px$ and the saturation levels. By fixing some of the parameters, these matrix inequalities simplify to linear matrix inequalities (LMIs) and can be treated with the LMI software. A nice feature of these analysis tools is that they can be adapted for controller synthesis by simply considering the feedback gain matrix as an additional optimization parameter.

In [8], a simpler criterion is derived in terms of an auxiliary feedback matrix for determining if a given ellipsoid is contractively invariant under a given feedback law. This condition is shown to be less conservative than the existing conditions which are based on the circle criterion or the vertex analysis. The most important feature of this new condition is that it can be expressed as LMIs in terms of all the varying parameters and hence can easily be used for controller synthesis.

In this paper, the set invariance criterion in [8] will be extended to discrete-time systems although the approach has to be quite different. In [8], the set invariance criterion is proven by expanding the derivative of the Lyapunov function and examining the terms that include the saturated feedback $\mathrm{sat}(Fx)$. However, for the discrete-time case, the terms of the increment of the Lyapunov function cannot be examined separately. A new approach by placing the saturated control $\mathrm{sat}(Fx)$ in the convex hull of a group of linear controls is derived to establish the main results. By further exploiting the idea, we will reveal a surprising fact for the single input systems: Given a feedback matrix F, assume that an ellipsoid is invariant for the linear system

$$x(k+1)=Ax(k)+BFx(k).$$

Then, it is invariant for the system

$$x(k+1)=Ax(k)+B\mathrm{sat}(Fx(k))$$

if and only if there exists a feedback law $u=h(x)$, $|h(x)|\leqslant 1$, such that the ellipsoid is invariant for the system

$$x(k+1)=Ax(k)+Bh(x(k)).$$

This means that the set invariance property under a group of saturated linear feedback laws is in some sense independent of a particular feedback in this group as long as all the corresponding linear feedback laws make the ellipsoid invariant.

Based on the stability analysis result, some disturbance rejection problems will be considered, such as, set invariance property in the presence of disturbance, invariant set enlargement, disturbance rejection and disturbance rejection with guaranteed stability region.

This paper is organized as follows. Section 2 addresses the analysis of and design for closed-loop stability. Section 3 addresses the issues related to disturbance rejection. In particular, Section 2.1 presents conditions for an ellipsoid to be invariant. Section 2.2 derives a necessary and sufficient condition for an ellipsoid to be invariant in the case of single input. Section 2.3 proposes an optimization approach to estimating the domain of attraction. Section 2.4 presents a controller design approach to enlarging the domain of attraction. A brief concluding remark is given in Section 4.

2. Estimation of the domain of attraction

2.1. Condition for set invariance — multiple input case

Consider the open-loop system

$$x(k+1)=Ax(k)+B\mathrm{sat}(u(k)),\quad x\in\mathbb{R}^n,\ u\in\mathbb{R}^m, \tag{2}$$

where $\mathrm{sat}(\cdot)$ is the standard saturation function of appropriate dimensions. In the above system, $\mathrm{sat}:\mathbb{R}^m\to\mathbb{R}^m$, and $\mathrm{sat}(u)=[\mathrm{sat}(u_1),\mathrm{sat}(u_2),\ldots,\mathrm{sat}(u_m)]^{\mathrm{T}}$, where $\mathrm{sat}(u_i)=\mathrm{sgn}(u_i)\min\{1,|u_i|\}$. Here we have slightly abused the notation by using sat(·) to denote both the scalar valued and the vector valued saturation functions. Suppose that a state feedback $u=Fx$ has been designed such that $A+BF$ is Schur stable and that the closed-loop linear system satisfies some performance requirement. We would like to know how the closed-loop system behaves in the presence of saturation nonlinearity, in particular, to what extent the stability is preserved. Our first objective of this paper is to obtain an estimate of the domain of attraction of the origin for the closed-loop system

$$x(k+1)=Ax(k)+B\text{sat}(Fx(k)). \tag{3}$$

For a matrix $F\in\mathbb{R}^{m\times n}$, denote the jth row of F as f_j and define

$$\mathscr{L}(F):=\{x\in\mathbb{R}^n: |f_jx|\leqslant 1,\ j\in[1,m]\}.$$

If F is the feedback matrix, then $\mathscr{L}(F)$ is the region where the feedback control $u=\text{sat}(Fx)$ is linear in x.

For $x(0)=x_0\in\mathbb{R}^n$, denote the state trajectory of the system (3) as $\psi(k,x_0)$. The *domain of attraction* of the origin is

$$\mathscr{S}:=\left\{x_0\in\mathbb{R}^n: \lim_{k\to\infty}\psi(k,x_0)=0\right\}.$$

A set is said to be *invariant* if all the trajectories starting from it will remain in it.

Let $P\in\mathbb{R}^{n\times n}$ be a positive-definite matrix. For $\rho>0$, denote

$$\mathscr{E}(P,\rho)=\{x\in\mathbb{R}^n: x^{\mathrm{T}}Px\leqslant\rho\}.$$

Let $V(x)=x^{\mathrm{T}}Px$. The set $\mathscr{E}(P,\rho)$ is said to be *contractively invariant* if

$$\Delta V(x):=(Ax+B\text{sat}(Fx))^{\mathrm{T}}P(Ax+B\text{sat}(Fx))-x^{\mathrm{T}}Px<0$$

for all $x\in\mathscr{E}(P,\rho)\setminus\{0\}$. Clearly, if $\mathscr{E}(P,\rho)$ is contractively invariant, then it is inside the domain of attraction.

We will develop conditions under which $\mathscr{E}(P,\rho)$ is contractively invariant and thus obtain an estimate of the domain of attraction.

Let $\mathscr{D}$ be the set of $m\times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathscr{D}$. Suppose that each element of $\mathscr{D}$ is labeled as D_i, $i=1,2,\ldots,2^m$. Then, $\mathscr{D}=\{D_i: i\in[1,2^m]\}$. Denote $D_i^-=I-D_i$. Clearly, D_i^- is also an element of $\mathscr{D}$ if $D_i\in\mathscr{D}$. Given two vectors, $u,v\in\mathbb{R}^m$, $\{D_iu+D_i^-v: i\in[1,2^m]\}$ is the set of vectors formed by choosing some elements from u and the remaining from v. Given two matrices $F,H\in\mathbb{R}^{m\times n}$, $\{D_iF+D_i^-H: i\in[1,2^m]\}$ is the set of matrices formed by choosing some rows from F and the remaining from H.

With these D_i and D_i^- matrices, a discrete-time counterpart of Theorem 10.4 in [9] (when applied to saturation nonlinearities) can be derived with some standard technique in robustness analysis for systems with varying parameters.

Proposition 1. *Given an ellipsoid $\mathscr{E}(P,\rho)$, if there exists a positive diagonal matrix $K\in\mathbb{R}^{m\times m}$, $K<I$ such that*

$$(A+B(D_iF+D_i^-KF))^{\mathrm{T}}P(A+B(D_iF+D_i^-KF))-P<0,\quad \forall i\in[1,2^m], \tag{4}$$

and $\mathscr{E}(P,\rho)\subset\mathscr{L}(KF)$, then $\mathscr{E}(P,\rho)$ is a contractively invariant set.

Here, the varying gain of each control channel (due to saturation) is viewed as an uncertain parameter varying between K_{ii} and 1, and the quadratic stability (within $\mathscr{E}(P,\rho)$) of the systems corresponding to this box of uncertain parameters is guaranteed by those on the vertices of the box, $\{D_iF+D_i^-KF: i\in[1,2^m]\}$. Similar to [8], we have the following less conservative criterion for an ellipsoid to be contractively invariant.

Theorem 1. *Given an ellipsoid $\mathscr{E}(P,\rho)$, if there exists an $H\in\mathbb{R}^{m\times n}$ such that*

$$(A+B(D_iF+D_i^-H))^{\mathrm{T}}P(A+B(D_iF+D_i^-H))-P<0,\quad \forall i\in[1,2^m], \tag{5}$$

and $\mathscr{E}(P,\rho)\subset\mathscr{L}(H)$, then $\mathscr{E}(P,\rho)$ is a contractively invariant set.

Although a natural discrete-time counterpart of Theorem 1 in [8], the above theorem cannot be proven in a similar way. Before starting the proof of Theorem 1, we need some simple facts about the convex hull of a set of points. Recall that for a group of points, $u^1,u^2,\dots,u^{\mathscr{I}}$, the convex hull of these points is defined as,

$$\mathrm{co}\{u^i\colon i\in[1,\mathscr{I}]\}:=\left\{\sum_{i=1}^{\mathscr{I}}\alpha_i u^i\colon \sum_{i=1}^{\mathscr{I}}\alpha_i=1,\ \alpha_i\geqslant 0\right\}.$$

Lemma 1. *Let $u,u^1,u^2,\dots,u^{\mathscr{I}}\in\mathbb{R}^{m_1},\ v,v^1,v^2,\dots,v^{\mathscr{J}}\in\mathbb{R}^{m_2}$. If $u\in\mathrm{co}\{u^i\colon i\in[1,\mathscr{I}]\}$ and $v\in\mathrm{co}\{v^j\colon j\in[1,\mathscr{J}]\}$, then*

$$\begin{bmatrix} u\\ v\end{bmatrix}\in\mathrm{co}\left\{\begin{bmatrix} u^i\\ v^j\end{bmatrix}\colon i\in[1,\mathscr{I}],\ j\in[1,\mathscr{J}]\right\}. \tag{6}$$

Proof. Since $u\in\mathrm{co}\{u^i\colon i\in[1,\mathscr{I}]\}$ and $v\in\mathrm{co}\{v^j\colon j\in[1,\mathscr{J}]\}$, there exist $\alpha_i,\beta_j\geqslant 0$, $i=1,2,\dots,\mathscr{I}$, $j=1,2,\dots,\mathscr{J}$, such that

$$\sum_{i=1}^{\mathscr{I}}\alpha_i=\sum_{j=1}^{\mathscr{J}}\beta_j=1,\quad u=\sum_{i=1}^{\mathscr{I}}\alpha_i u^i,\quad v=\sum_{j=1}^{\mathscr{J}}\beta_j v^j.$$

Therefore,

$$\begin{bmatrix} u\\ v\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{\mathscr{I}}\alpha_i u^i\\ \sum_{j=1}^{\mathscr{J}}\beta_j v^j\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{\mathscr{I}}\alpha_i u^i(\sum_{j=1}^{\mathscr{J}}\beta_j)\\ \sum_{j=1}^{\mathscr{J}}\beta_j v^j(\sum_{i=1}^{\mathscr{I}}\alpha_i)\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{\mathscr{I}}\sum_{j=1}^{\mathscr{J}}\alpha_i\beta_j u^i\\ \sum_{i=1}^{\mathscr{I}}\sum_{j=1}^{\mathscr{J}}\alpha_i\beta_j v^j\end{bmatrix}=\sum_{i=1}^{\mathscr{I}}\sum_{j=1}^{\mathscr{J}}\alpha_i\beta_j\begin{bmatrix} u^i\\ v^j\end{bmatrix}.$$

Noting that

$$\sum_{i=1}^{\mathscr{I}}\sum_{j=1}^{\mathscr{J}}\alpha_i\beta_j=\sum_{i=1}^{\mathscr{I}}\alpha_i\sum_{j=1}^{\mathscr{J}}\beta_j=1,$$

we obtain (6). □

Lemma 2. *Let $u,v\in\mathbb{R}^m$,*

$$u=\begin{bmatrix}u_1\\u_2\\ \vdots\\ u_m\end{bmatrix},\quad v=\begin{bmatrix}v_1\\v_2\\ \vdots\\ v_m\end{bmatrix}.$$

Suppose that $|v_j|\leqslant 1$ for all $j\in[1,m]$, then

$$\mathrm{sat}(u)\in\mathrm{co}\{D_iu+D_i^-v\colon i\in[1,2^m]\}.$$

Proof. Since $|v_j|\leqslant 1$, we have

$$\mathrm{sat}(u_j)\in\mathrm{co}\{u_j,v_j\},\quad \forall j\in[1,m].$$

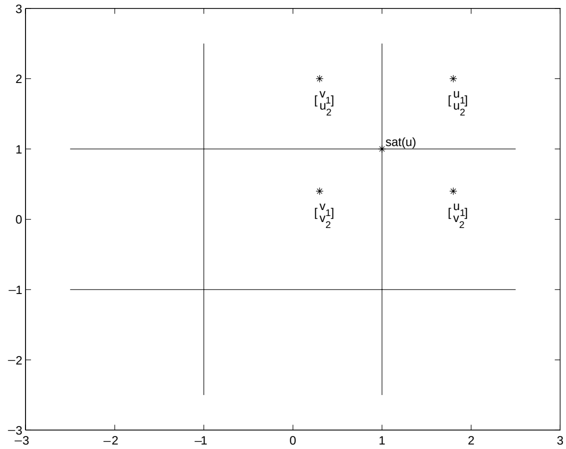

Fig. 1. Illustration for Lemma 2.

By applying Lemma 1 inductively, we have

$$\mathrm{sat}(u_1) \in \mathrm{co}\{u_1, v_1\},$$
$$\mathrm{sat}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) \in \mathrm{co}\left\{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\},$$
$$\mathrm{sat}\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) \in \mathrm{co}\left\{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right\},$$
$$\vdots$$

and finally,

$$\mathrm{sat}(u) \in \mathrm{co}\{D_i u + D_i^- v\colon i \in [1, 2^m]\}. \qquad \square$$

Lemma 2 is illustrated in Fig. 1 for the case where $m = 2$.

Given two feedback matrices $F, H \in \mathbb{R}^{m\times n}$, suppose that $|h_j x| \leqslant 1$ for all $j \in [1, m]$, then by Lemma 2, we have

$$\mathrm{sat}(Fx) \in \mathrm{co}\{D_i Fx + D_i^- Hx\colon i \in [1, 2^m]\}.$$

In this way, we have placed $\mathrm{sat}(Fx)$ into the convex hull of a group of linear feedbacks.

Proof of Theorem 1. Let $V(x) = x^{\mathrm{T}} Px$, we need to show that

$$\Delta V(x) = (Ax + B\mathrm{sat}(Fx))^{\mathrm{T}} P(Ax + B\mathrm{sat}(Fx)) - x^{\mathrm{T}} Px < 0, \quad \forall x \in \mathscr{E}(P, \rho)\backslash\{0\}. \tag{7}$$

Since $\mathscr{E}(P, \rho) \subset \mathscr{L}(H)$, i.e., $|h_j x| \leqslant 1$ for all $j \in [1, m]$ and $x \in \mathscr{E}(P, \rho)$, by Lemma 2, for every $x \in \mathscr{E}(P, \rho)$,

$$\mathrm{sat}(Fx) \in \mathrm{co}\{D_i Fx + D_i^- Hx\colon i \in [1, 2^m]\}.$$

It follows that

$$Ax + B\mathrm{sat}(Fx) \in \mathrm{co}\{Ax + B(D_i F + D_i^- H)x\colon i \in [1, 2^m]\}.$$

By the convexity of the function $V(z) = z^{\mathrm{T}}Pz$, we have

$$(Ax + B\mathrm{sat}(Fx))^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) \leqslant \max_{i\in[1,2^m]} x^{\mathrm{T}}(A + B(D_iF + D_i^- H))^{\mathrm{T}}P(A + B(D_iF + D_i^- H))x$$

for every $x \in \mathscr{E}(P,\rho)$. Since condition (5) is satisfied, we have

$$\max_{i\in[1,2^m]} x^{\mathrm{T}}(A + B(D_iF + D_i^- H))^{\mathrm{T}}P(A + B(D_iF + D_i^- H))x < x^{\mathrm{T}}Px$$

for all $x \neq 0$. Therefore, for every $x \in \mathscr{E}(P,\rho)\backslash\{0\}$,

$$(Ax + B\mathrm{sat}(Fx))^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) < x^{\mathrm{T}}Px.$$

This verifies (7). □

We see that Proposition 1 is a special case of Theorem 1 by setting $H = KF$. Clearly the condition in Theorem 1 is less conservative than that in Proposition 1. This will be illustrated in Example 1. Another important advantage of Theorem 1 is that, when optimization is concerned, it leads to constraints in the form of linear matrix inequality while from Proposition 1 we can only get bilinear matrix inequalities. This will be investigated later.

2.2. The necessary and sufficient condition — single input case

For the single input case ($m = 1$), $\mathscr{D} = \{0, 1\}$. So the condition in Theorem 1 for $\mathscr{E}(P,\rho)$ to be contractively invariant simplifies to: there exists an $H \in \mathbb{R}^{1\times n}$ such that

$$(A + BF)^{\mathrm{T}}P(A + BF) - P < 0, \quad (A + BH)^{\mathrm{T}}P(A + BH) - P < 0$$

and $\mathscr{E}(P,\rho) \in \mathscr{L}(H)$. In fact, we can go one step further to obtain the following surprising result.

Theorem 2. *Assume $m = 1$. Given an ellipsoid $\mathscr{E}(P,\rho)$, suppose that*

$$(A + BF)^{\mathrm{T}}P(A + BF) - P < 0. \tag{8}$$

Then, $\mathscr{E}(P,\rho)$ is contractively invariant under $u = \mathrm{sat}(Fx)$ if and only if there exists a function $h(x) : \mathbb{R}^m \to \mathbb{R}$, $|h(x)| \leqslant 1$ for all $x \in \mathscr{E}(P,\rho)$, such that $\mathscr{E}(P,\rho)$ is contractively invariant under the control $u = h(x)$, i.e.,

$$(Ax + Bh(x))^{\mathrm{T}}P(Ax + Bh(x)) - x^{\mathrm{T}}Px < 0, \quad \forall x \in \mathscr{E}(P,\rho) \setminus \{0\}. \tag{9}$$

Proof. The "only if" part is obvious. Now we show the "if" part. Here we have $|h(x)| \leqslant 1$ for all $x \in \mathscr{E}(P,\rho)$. It follows from Lemma 2 that for every $x \in \mathscr{E}(P,\rho)$, $\mathrm{sat}(Fx) \in \mathrm{co}\{Fx, h(x)\}$.

By the convexity of the function $V(z) = z^{\mathrm{T}}Pz$, we have

$$(Ax + B\mathrm{sat}(Fx))^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) \leqslant \max\{x^{\mathrm{T}}(A + BF)^{\mathrm{T}}P(A + BF)x, (Ax + Bh(x))^{\mathrm{T}}P(Ax + Bh(x))\}.$$

By (8) and (9), we obtain

$$(Ax + B\mathrm{sat}(Fx))^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) - x^{\mathrm{T}}Px < 0, \quad \forall x \in \mathscr{E}(P,\rho)\backslash\{0\}.$$

This shows that $\mathscr{E}(P,\rho)$ is contractively invariant under $u = \mathrm{sat}(Fx)$. □

Theorem 2 implies that, for the single input case, the invariance of an ellipsoid $\mathscr{E}(P,\rho)$ under a saturated linear control $u = \mathrm{sat}(Fx)$ is in some sense independent of F as long as the condition $(A + BF)^{\mathrm{T}}P(A + BF) - P < 0$ is satisfied. In other words, suppose that both F_1 and F_2 satisfy the condition $(A + BF_i)^{\mathrm{T}}P(A + BF_i) - P < 0$, $i = 1, 2$, then the maximal invariant ellipsoid $\mathscr{E}(P,\rho)$ (with ρ maximized under a fixed P) is the same under either $u = \mathrm{sat}(F_1x)$ or $u = \mathrm{sat}(F_2x)$. In [6, Chapter 11], we developed a computational method to determine the largest ρ such that $\mathscr{E}(P,\rho)$ can be made invariant.

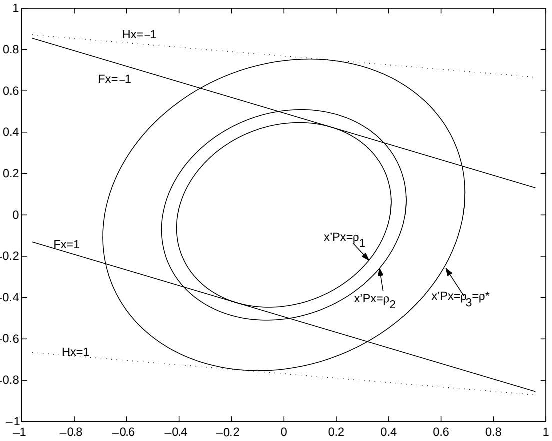

Fig. 2. Invariant ellipsoids determined with different methods.

Example 1. Consider the closed-loop system (3) with

$$A=\begin{bmatrix}0.8876 & -0.5555\\ 0.5555 & 1.5542\end{bmatrix},\quad B=\begin{bmatrix}-0.1124\\ 0.5555\end{bmatrix},\quad F=[\,-0.7651\quad -2.0299\,].$$

Given

$$P=\begin{bmatrix}5.0127 & -0.6475\\ -0.6475 & 4.2135\end{bmatrix}.$$

By combining Theorem 2 and Hu and Lin's method [6], the maximal $\mathscr{E}(P,\rho)$ is $\mathscr{E}(P,\rho^*)$ with $\rho^*=2.3490$.
Let's compare the largest invariant ellipsoid, $\mathscr{E}(P,\rho^*)$, with those obtained by other methods.

(1) The maximal ρ such that $\mathscr{E}(P,\rho)\subset\mathscr{L}(F)$ is $\rho_1=0.8237$;
(2) The maximal ρ satisfying the condition in Proposition 1 is $\rho_2=1.0710$;
(3) The maximal ρ satisfying the condition in Theorem 1 is $\rho_3=\rho^*=2.3490$, with

$$H=[-0.1389\quad -1.3018].$$

Shown in Fig. 2 is a comparison of the invariant ellipsoids obtained with different methods. It is very interesting to note that $\rho_3=\rho^*$. In this case, the largest invariant ellipsoid obtained by Theorem 1 is not conservative at all. In fact, as we have proven for continuous-time systems in [7], if $m=1$, then the condition in Theorem 1 (in its continuous-time form) is also necessary.

2.3. Estimation of the domain of attraction — an LMI approach

With all the ellipsoids satisfying the set invariance condition in Theorem 1, we would like to choose from among them the "largest" one to get a least conservative estimate of the domain of attraction. In the literature (e.g., see [2,3,5]), the largeness of a set is usually measured by its volume. Here, we will follow the idea in [8] and take the shape of a set into consideration. Let $X_R\subset\mathbb{R}^n$ be a prescribed bounded convex set. For a set $S\subset\mathbb{R}^n$, define

$$\alpha_R(S):=\sup\{\alpha>0:\ \alpha X_R\subset S\}.$$

If $\alpha_R(S)\geqslant 1$, then $X_R\subset S$. Two typical types of X_R are the ellipsoids

$$X_R=\{x\in\mathbb{R}^n:\ x^TRx\leqslant 1\},\quad R>0$$

and the polyhedrons

$$X_R = \mathrm{co}\{x_1, x_2, \ldots, x_l\}.$$

We can choose the reference set X_R according to the available information on the initial conditions. For instance, if some possible initial conditions are known, we can choose X_R as a polyhedron containing all these initial conditions. In the extreme case, we may choose X_R to be $\mathrm{co}\{x_0, -x_0\}$ when we want to know if x_0 is in the domain of attraction.

Now we would like to choose from all the $\mathscr{E}(P,\rho)$'s that satisfy the condition in Theorem 1 such that the quantity $\alpha_R(\mathscr{E}(P,\rho))$ is maximized. For this reason, we call X_R the shape reference set. The problem of maximizing the contractively invariant ellipsoid with respect to a shape reference set can be formulated as

$$\begin{aligned} &\sup_{P>0,\rho,H} \alpha \\ &\text{s.t.} \quad \text{(a) } \alpha X_R \subset \mathscr{E}(P,\rho), \\ &\qquad\quad \text{(b) } (A + B(D_iF + D_i^- H))^T P (A + B(D_iF + D_i^- H)) - P < 0, \quad \forall i \in [1, 2^m], \\ &\qquad\quad \text{(c) } |h_j x| \leqslant 1, \quad \forall x \in \mathscr{E}(P,\rho), \ j \in [1, m]. \end{aligned} \tag{10}$$

We will transform the above optimization constraints into LMIs. If X_R is a polyhedron, then by Schur complement, (a) is equivalent to

$$\alpha^2 x_k^T \left(\frac{P}{\rho}\right) x_k \leqslant 1 \quad \Leftrightarrow \quad \begin{bmatrix} 1/\alpha^2 & x_k^T \\ x_k & (P/\rho)^{-1} \end{bmatrix} \geqslant 0, \quad k \in [1, l]. \tag{11}$$

If X_R is an ellipsoid $\{x: x^T R x \leqslant 1\}$, then (a) is equivalent to

$$\frac{R}{\alpha^2} \geqslant \frac{P}{\rho} \quad \Leftrightarrow \quad \begin{bmatrix} 1/\alpha^2 R & I \\ I & (P/\rho)^{-1} \end{bmatrix} \geqslant 0. \tag{12}$$

Also by Schur complement and some manipulation, the constraint (b) is equivalent to

$$\begin{bmatrix} (P/\rho)^{-1} & (P/\rho)^{-1}(A + B(D_iF + D_i^- H))^T \\ (A + B(D_iF + D_i^- H))(P/\rho)^{-1} & (P/\rho)^{-1} \end{bmatrix} > 0, \quad i \in [1, 2^m]. \tag{13}$$

From [8], the constraint (c) is equivalent to

$$\rho h_j P^{-1} h_j^T \leqslant 1 \Leftrightarrow \begin{bmatrix} 1 & h_j (P/\rho)^{-1} \\ (P/\rho)^{-1} h_j^T & (P/\rho)^{-1} \end{bmatrix} \geqslant 0, \quad j \in [1, m]. \tag{14}$$

Let $\gamma = 1/\alpha^2$, $Q = (P/\rho)^{-1}$ and $Z = H(P/\rho)^{-1}$. Also let the jth row of Z be z_j, i.e., $z_j = h_j(P/\rho)^{-1}$. If X_R is a polyhedron, then from (11), (13) and (14), the optimization problem (10) can be rewritten as

$$\begin{aligned} &\inf_{Q,Z} \gamma \\ &\text{s.t.} \quad \text{(a1) } \begin{bmatrix} \gamma & x_k^T \\ x_k & Q \end{bmatrix} \geqslant 0, \quad k \in [1, l], \\ &\qquad\quad \text{(b) } \begin{bmatrix} Q & (AQ + B(D_iFQ + D_i^- Z))^T \\ AQ + B(D_iFQ + D_i^- Z) & Q \end{bmatrix} > 0, \quad i \in [1, 2^m], \\ &\qquad\quad \text{(c) } \begin{bmatrix} 1 & z_j \\ z_j^T & Q \end{bmatrix} \geqslant 0, \quad j \in [1, m], \end{aligned} \tag{15}$$

where all the constraints are given in LMIs.

If X_R is an ellipsoid, we just need to replace (a1) with another LMI,

$$\text{(a2) } \begin{bmatrix} \gamma R & I \\ I & Q \end{bmatrix} \geqslant 0.$$

2.4. Controller design

Our objective in this section is to choose a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that the estimated domain of attraction as obtained by the method of Section 2.3 is maximized with respect to X_R. This can be simply done by taking the F in (15) as an extra optimization parameter. To make the optimization easy, we use a new parameter Y to replace FQ in (15(b)) and the resulting LMI problem is

$$\begin{aligned} &\inf_{Q,Y,Z} \gamma \\ &\text{s.t.} \quad (15(\text{a}1)),\ (15(\text{c})), \\ &\qquad (\text{b}) \begin{bmatrix} Q & (AQ + B(D_iY + D_i^- Z))^{\mathrm{T}} \\ AQ + B(D_iY + D_i^- Z) & Q \end{bmatrix} > 0, \quad i \in [1, 2^m]. \end{aligned} \tag{16}$$

The optimal F will be recovered from YQ^{-1}. Denote the optimal value of the above optimization problem as γ^*.

Let's consider a simpler optimization problem

$$\begin{aligned} &\inf_{Q,Z} \gamma \\ &\text{s.t.} \quad (15(\text{a}1)),\ (15(\text{c})), \\ &\qquad (\text{b}1) \begin{bmatrix} Q & (AQ + BZ)^{\mathrm{T}} \\ AQ + BZ & Q \end{bmatrix} > 0. \end{aligned} \tag{17}$$

Denote its optimal value as γ_1^*. We claim that $\gamma^* = \gamma_1^*$. The argument goes as follows. Since (b1) is only one of the inequality constraints in (b) (when $D_i = 0$), (17) can be viewed as a problem resulting from dropping the other $2^m - 1$ constraints in (16(b)), hence we have $\gamma^* \geqslant \gamma_1^*$. On the other hand, we can also see (b1) as a result of (b) by restricting $Y = Z$ (recall that $D_i + D_i^- = I$). This means that the constraints of (17) are more restrictive than that of (16). Hence (16) admits a less infimum than (17), i.e., $\gamma^* \leqslant \gamma_1^*$. In summary, we must have $\gamma^* = \gamma_1^*$.

In view of the above observation, we might as well solve the simpler optimization problem (17) if our objective is to enlarge the domain of attraction. If we solve (17) and let $H = ZQ^{-1}$, then the resulting invariant ellipsoid is in the linear region of the state feedback $u = \mathrm{sat}(Hx)$, i.e., $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$. In this case, the convergence of trajectories would be generally very slow inside the ellipsoid. Suppose that we have another feedback $u = \mathrm{sat}(Fx)$ such that F and H satisfy (5), then by Theorem 1, the ellipsoid is also invariant under $u = \mathrm{sat}(Fx)$. There could be infinitely many such F's. We may choose among these F's to optimize other performances such as convergence rate.

3. Disturbance rejection

3.1. Problem statement

Consider the open-loop system

$$x(k+1) = Ax(k) + B\mathrm{sat}(u(k)) + Ew(k), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ w \in \mathbb{R}^q, \tag{18}$$

where, without loss of generality, we assume that the bounded disturbance w belongs to the set $\mathscr{W} := \{w\colon w(k)^{\mathrm{T}}w(k) \leqslant 1,\ \forall k \geqslant 0\}$. Let the state feedback be $u = Fx$. The closed-loop system is

$$x(k+1) = Ax(k) + B\mathrm{sat}(Fx(k)) + Ew(k). \tag{19}$$

For an initial state $x(0) = x_0$, denote the state trajectory of the closed-loop system under w as $\psi(k, x_0, w)$.

Our primary concern is the boundedness of the trajectories. A set in $\mathbb{R}^n$ is said to be *invariant* if all the trajectories starting from it will remain in it regardless of $w \in \mathscr{W}$. An ellipsoid $\mathscr{E}(P,\rho)$ is said to be *strictly invariant* if

$$(Ax + B\mathrm{sat}(Fx) + Ew)^{\mathrm{T}} P (Ax + B\mathrm{sat}(Fx) + Ew) < \rho$$

for all $x \in \mathscr{E}(P,\rho)$ and $w, w^{\mathrm{T}}w \leqslant 1$.

The notion of invariant set plays an important role in studying the stability and other performances of a system, see [1,2,9] and the references therein. To keep the state trajectory bounded for a large range of initial conditions, it is desired to have a large invariant set. On the other hand, a small invariant set indicates that the system is insensitive to the disturbance. Suppose that we have an invariant set containing the origin in its interior, then all the trajectories starting from the origin will remain inside the invariant set regardless of the disturbance. Hence for the purpose of disturbance rejection, we would also like to have a small invariant set containing the origin in its interior.

To formally state the objectives of this section, we need to extend the notion of the domain of attraction of an equilibrium to that of an invariant set as follows.

Definition 1. Let $\mathscr{B}$ be a bounded invariant set of (19). The domain of attraction of $\mathscr{B}$ is

$$\mathscr{S}(\mathscr{B}) := \left\{ x_0 \in \mathbb{R}^n \colon \lim_{k \to \infty} d(\psi(k,x_0,w),\mathscr{B}) = 0,\ \forall w \in \mathscr{W} \right\},$$

where $d(\psi(k,x_0,w),\mathscr{B}) = \inf_{x \in \mathscr{B}} \|\psi(k,x_0,w) - x\|$ is the distance from $\psi(k,x_0,w)$ to $\mathscr{B}$.

In the above definition, $\|\cdot\|$ can be any norm. The problems we are to address in this section are formulated as follows.

Problem 1 (*Set invariance analysis*). Let F be known. Given an ellipsoid $\mathscr{E}(P,\rho)$, determine if $\mathscr{E}(P,\rho)$ is (strictly) invariant.

Problem 2 (*Invariant set enlargement*). Given a shape reference set $X_0 \subset \mathbb{R}^n$, design an F such that the closed-loop system has an invariant set $\mathscr{E}(P,\rho) \supset \alpha_2 X_0$ with α_2 maximized.

Problem 3 (*Disturbance rejection*). Given a shape reference set $X_\infty \subset \mathbb{R}^n$, design an F such that the closed-loop system has an invariant set $\mathscr{E}(P,\rho) \subset \alpha_3 X_\infty$ with α_3 minimized. Here we can also take X_∞ to be the (possibly unbounded) polyhedron $\{x \in \mathbb{R}^n \colon |c_i x| \leqslant 1,\ i \in [1,p]\}$. In this case, the minimization of α_3 leads to the minimization of the L_∞-norm of the output $y = Cx \in \mathbb{R}^p$.

Problem 4 (*Disturbance rejection with guaranteed domain of attraction*). Given two shape reference sets, X_∞ and X_0. Design an F such that the closed-loop system has an invariant set $\mathscr{E}(P,1) \supset X_0$, and for all $x_0 \in \mathscr{E}(P,1)$, $\psi(k,x_0,w)$ will enter a smaller invariant set $\mathscr{E}(P,\rho_1) \subset \alpha_4 X_\infty$ with α_4 minimized.

3.2. Condition for set invariance

We consider the closed-loop system (19) with a given F.

Theorem 3. *For a given ellipsoid $\mathscr{E}(P,\rho)$, if there exist an $H \in \mathbb{R}^{m \times n}$ and a positive number η such that*

$$(1+\eta)(A + B(D_iF + D_i^- H))^{\mathrm{T}} P (A + B(D_iF + D_i^- H)) + \left(\frac{1+\eta}{\rho\eta} \lambda_{\max}(E^{\mathrm{T}}PE) - 1 \right) P \leqslant (<) 0 \qquad (20)$$

for all $i \in [1, 2^m]$, and $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$, then $\mathscr{E}(P,\rho)$ is a (strictly) invariant set for system (19).

Proof. We prove the strict invariance. That is, we will show that

$$(Ax + B\mathrm{sat}(Fx) + Ew)^{\mathrm{T}} P (Ax + B\mathrm{sat}(Fx) + Ew) < \rho, \quad \forall x \in \mathscr{E}(P,\rho),\ \ w^{\mathrm{T}}w \leqslant 1.$$

Since $\mathscr{E}(P,\rho) \subset \mathscr{L}(H)$, similar to the proof of Theorem 1, we can show that

$$Ax + B\text{sat}(Fx) + Ew \in \text{co}\{Ax + B(D_iF + D_i^- H)x + Ew\text{: } i \in [1, 2^m]\}$$

for every $w \in \mathbb{R}^q$ and $x \in \mathscr{E}(P,\rho)$. By the convexity of the function $V(z) = z^{\mathrm{T}}Pz$, for every $x \in \mathscr{E}(P,\rho)$ and every w, $w^{\mathrm{T}}w \leqslant 1$,

$$\begin{aligned}&(Ax + B\text{sat}(Fx) + Ew)^{\mathrm{T}}P(Ax + B\text{sat}(Fx) + Ew)\\ &\quad\leqslant \max_{i\in[1,2^m]}(Ax + B(D_iF + D_i^- H)x + Ew)^{\mathrm{T}}P(Ax + B(D_iF + D_i^- H)x + Ew).\end{aligned}$$

Using the fact that $(a+b)^{\mathrm{T}}(a+b) \leqslant (1+\eta)a^{\mathrm{T}}a + (1+1/\eta)b^{\mathrm{T}}b$ for any $\eta > 0$, we have

$$\begin{aligned}&(Ax + B\text{sat}(Fx) + Ew)^{\mathrm{T}}P(Ax + B\text{sat}(Fx) + Ew)\\ &\quad\leqslant \max_{i\in[1,2^m]}(1+\eta)x^{\mathrm{T}}(A + B(D_iF + D_i^- H))^{\mathrm{T}}P(A + B(D_iF + D_i^- H))x + \left(1 + \frac{1}{\eta}\right) w^{\mathrm{T}}E^{\mathrm{T}}PEw\\ &\quad\leqslant \max_{i\in[1,2^m]}(1+\eta)x^{\mathrm{T}}(A + B(D_iF + D_i^- H))^{\mathrm{T}}P(A + B(D_iF + D_i^- H))x + \left(1 + \frac{1}{\eta}\right) \lambda_{\max}(E^{\mathrm{T}}PE).\end{aligned}$$

To prove the strict invariance, it suffices to show that there exists an $\eta > 0$ such that for all $x \in \partial\mathscr{E}(P,\rho)$ and for all $i \in [1, 2^m]$,

$$(1+\eta)x^{\mathrm{T}}(A + B(D_iF + D_i^- H))^{\mathrm{T}}P(A + B(D_iF + D_i^- H))x + \left(1 + \frac{1}{\eta}\right) \lambda_{\max}(E^{\mathrm{T}}PE) < \rho. \tag{21}$$

Noticing that $1 = x^{\mathrm{T}}(P/\rho)x$ on $\partial\mathscr{E}(P,\rho)$, we see that (21) is guaranteed by (20). □

Theorem 3 deals with Problem 1 and can be easily used for controller design in Problem 2 and Problem 3. For Problem 2, we can solve the following optimization problem:

$$\begin{aligned}&\sup_{P>0,\rho,\eta>0,F,H} \alpha_2\\ &\text{s.t.}\quad \text{(a) } \alpha_2 X_0 \subset \mathscr{E}(P,\rho),\\ &\qquad\ \text{(b) } (A + B(D_iF + D_i^- H))^{\mathrm{T}}P(A + B(D_iF + D_i^- H))\\ &\qquad\qquad \frac{1}{1+\eta}\left(\frac{1+\eta}{\eta}\lambda_{\max}\left(\frac{E^{\mathrm{T}}PE}{\rho}\right) - 1\right)P \leqslant 0, \quad i \in [1, 2^m],\\ &\qquad\ \text{(c) } |h_jx| \leqslant 1, \quad \forall x \in \mathscr{E}(P,\rho), \quad j \in [1, m].\end{aligned} \tag{22}$$

Let $\gamma = 1/\alpha_2^2$, $Q = (P/\rho)^{-1}$, $Y = FQ$ and $Z = HQ$. Similar to Section 2.3, the above optimization problem can be transformed into one with LMI constraints (providing two scalars are fixed). Here, constraint (b) is equivalent to the existence of $\lambda \in (0, \eta/(1+\eta))$ such that

$$\begin{bmatrix} \frac{1}{1+\eta}\left(1 - \frac{1+\eta}{\eta}\lambda\right)Q & (AQ + BD_iY + BD_i^- Z)^{\mathrm{T}} \\ AQ + BD_iY + BD_i^- Z & Q \end{bmatrix} \geqslant 0, \quad i \in [1, 2^m] \tag{23}$$

and

$$\begin{bmatrix} \lambda & E^{\mathrm{T}} \\ E & Q \end{bmatrix} \geqslant 0. \tag{24}$$

Hence, the optimization problem (22) is equivalent to

$$\begin{aligned} &\inf_{\eta>0,\lambda,Q,Y,Z} \gamma \\ &\text{s.t.} \qquad (15(\text{a}1)),\ (15(\text{c})),\ (23),\ (24). \end{aligned} \tag{25}$$

We see that (15(a1)) and (15(c)) are LMIs. If we fix η and λ, then (23) and (24) are also LMIs. The global infimum of γ can be obtained by running η from 0 to ∞ and λ from 0 to $\eta/(1+\eta)$.

In fact, the computation can be simplified by reducing the number of parameters fixed beforehand (η and λ) from two to one. Denote

$$g = \frac{1}{1+\eta}\left(1 - \frac{1+\eta}{\eta}\lambda\right).$$

We see that as η varies from 0 to ∞ and λ from 0 to $\eta/(1+\eta)$, g varies from 0 to 1. If we fix g, then

$$\lambda = 1 - \frac{1}{1+\eta} - \eta g.$$

It can be shown with standard analysis that as η varies from 0 to ∞, the maximal value of λ is

$$\lambda^* = (1-\sqrt{g})^2,$$

obtained at $\eta^* = (1/\sqrt{g}) - 1$. Since the constraint (24) is the least restrictive by taking $\lambda = \lambda^*$, the optimal solution to (25) will be obtained with $\lambda = \lambda^*$. In view of these arguments, we can solve (25) by running g from 0 to 1, taking $\lambda = \lambda^* = (1-\sqrt{g})^2$, solving the resulting LMI problems and picking the minimal γ. In this case, we only need to fix the parameter g before solving an LMI problem.

For Problem 3, we have

$$\begin{aligned} &\inf_{P>0,\rho,\eta>0,F,H} \alpha_3 \\ &\text{s.t.} \qquad (\text{a})\ \mathscr{E}(P,\rho) \subset \alpha_3 X_\infty,\ (22(\text{b})),\ (22(\text{c})), \end{aligned} \tag{26}$$

which can be solved similarly as (22).

3.3. Disturbance rejection with guaranteed domain of attraction

Given $X_0 \subset \mathbb{R}^n$, if the optimal solution of Problem 2 is $\alpha_2^* > 1$, then there are infinitely many choices of the feedback matrices F's such that X_0 is contained in some invariant ellipsoid. We will use this extra freedom for disturbance rejection. That is, to construct another invariant set $\mathscr{E}(P,\rho_1)$ which is as small as possible with respect to some X_∞. Moreover, X_0 is inside the domain of attraction of $\mathscr{E}(P,\rho_1)$. In this way, all the trajectories starting from X_0 will enter $\mathscr{E}(P,\rho_1) \subset \alpha_4 X_\infty$ for some $\alpha_4 > 0$. Here the number α_4 is a measure of the degree of disturbance rejection.

Before addressing Problem 4, we need to answer the following question: Suppose that for given F and P, both $\mathscr{E}(P,\rho_1)$ and $\mathscr{E}(P,\rho_2)$, $\rho_1 < \rho_2$, are strictly invariant sets, then under what conditions will the other ellipsoids $\mathscr{E}(P,\rho)$, $\rho \in (\rho_1,\rho_2)$ also be strictly invariant? If they are, then all the trajectories starting from within $\mathscr{E}(P,\rho_2)$ will enter $\mathscr{E}(P,\rho_1)$ and remain inside it.

Theorem 4. *Given two ellipsoids, $\mathscr{E}(P,\rho_1)$ and $\mathscr{E}(P,\rho_2)$, $\rho_2 > \rho_1 > 0$, if there exist $H_1, H_2 \in \mathbb{R}^{m\times n}$ and a positive η such that*

$$(1+\eta)(A + B(D_iF + D_i^- H_1))^{\mathrm{T}} P (A + B(D_iF + D_i^- H_1)) + \left(\frac{1+\eta}{\rho_1\eta}\lambda_{\max}(E^{\mathrm{T}}PE) - 1\right) P < 0, \tag{27}$$

$$(1+\eta)(A + B(D_iF + D_i^- H_2))^{\mathrm{T}} P (A + B(D_iF + D_i^- H_2)) + \left(\frac{1+\eta}{\rho_2\eta}\lambda_{\max}(E^{\mathrm{T}}PE) - 1\right) P < 0, \tag{28}$$

for all $i\in[1,2^m]$, *and* $\mathscr{E}(P,\rho_1)\subset\mathscr{L}(H_1)$, $\mathscr{E}(P,\rho_2)\subset\mathscr{L}(H_2)$, *then for every* $\rho\in[\rho_1,\rho_2]$, *there exists an* $H\in\mathbb{R}^{m\times n}$ *such that*

$$(1+\eta)(A+B(D_iF+D_i^-H))^{\mathrm{T}}P(A+B(D_iF+D_i^-H))+\left(\frac{1+\eta}{\rho\eta}\lambda_{\max}(E^{\mathrm{T}}PE)-1\right)P<0 \tag{29}$$

and $\mathscr{E}(P,\rho)\in\mathscr{L}(H)$. *This implies that* $\mathscr{E}(P,\rho)$ *is also strictly invariant.*

Proof. Let $h_{1,j}$ and $h_{2,j}$ be the jth rows of H_1 and H_2, respectively. The conditions $\mathscr{E}(P,\rho_1)\subset\mathscr{L}(H_1)$ and $\mathscr{E}(P,\rho_2)\subset\mathscr{L}(H_2)$ are equivalent to

$$\begin{bmatrix}1/\rho_1 & h_{1,j}\\ h_{1,j}^{\mathrm{T}} & P\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}1/\rho_2 & h_{2,j}\\ h_{2,j}^{\mathrm{T}} & P\end{bmatrix}\geqslant 0,\quad j\in[1,m].$$

Since $\rho\in[\rho_1,\rho_2]$, there exists an $\alpha\in[0,1]$ such that $1/\rho=\alpha/\rho_1+(1-\alpha)/\rho_2$. Let $H=\alpha H_1+(1-\alpha)H_2$. Clearly

$$\begin{bmatrix}1/\rho & h_j\\ h_j^{\mathrm{T}} & P\end{bmatrix}\geqslant 0,$$

which implies that $\mathscr{E}(P,\rho)\subset\mathscr{L}(H)$. Since (27) and (28) are equivalent to

$$\begin{bmatrix}\dfrac{1}{1+\eta}\left(1-\dfrac{1+\eta}{\rho_1\eta}\lambda_{\max}(E^{\mathrm{T}}PE)\right)P & (A+B(D_iF+D_i^-H_1))^{\mathrm{T}}\\ A+B(D_iF+D_i^-H_1) & P^{-1}\end{bmatrix}>0$$

and

$$\begin{bmatrix}\dfrac{1}{1+\eta}\left(1-\dfrac{1+\eta}{\rho_2\eta}\lambda_{\max}(E^{\mathrm{T}}PE)\right)P & (A+B(D_iF+D_i^-H_2))^{\mathrm{T}}\\ A+B(D_iF+D_i^-H_2) & P^{-1}\end{bmatrix}>0$$

by convexity, we have

$$\begin{bmatrix}\dfrac{1}{1+\eta}\left(1-\dfrac{1+\eta}{\rho\eta}\lambda_{\max}(E^{\mathrm{T}}PE)\right)P & (A+B(D_iF+D_i^-H))^{\mathrm{T}}\\ A+B(D_iF+D_i^-H) & P^{-1}\end{bmatrix}>0,$$

which is equivalent to (29). □

In view of Theorem 4, to solve Problem 4, we can construct two invariant ellipsoids $\mathscr{E}(P,\rho_1)$ and $\mathscr{E}(P,\rho_2)$ satisfying the condition of Theorem 4 such that $X_0\subset\mathscr{E}(P,\rho_2)$ and $\mathscr{E}(P,\rho_1)\subset\alpha_4X_\infty$ with α_4 minimized. Since ρ_2 can be absorbed into other parameters, we assume for simplicity that $\rho_2=1$ and $\rho_1<1$. Problem 4 can then be formulated as

$$\begin{aligned}&\inf_{P>0,0<\rho_1<1,\eta>0,F,H_1,H_2}\alpha_4\\
\text{s.t. }&\text{(a) } X_0\subset\mathscr{E}(P,1),\quad \mathscr{E}(P,\rho_1)\subset\alpha_4X_\infty,\\
&\text{(b) }\begin{bmatrix}\dfrac{1}{1+\eta}\left(1-\dfrac{1+\eta}{\rho_1\eta}\lambda_{\max}(E^{\mathrm{T}}PE)\right)P & (A+B(D_iF+D_i^-H_1))^{\mathrm{T}}\\ A+B(D_iF+D_i^-H_1) & P^{-1}\end{bmatrix}>0,\quad i\in[1,2^m],\\
&\text{(c) }\begin{bmatrix}\dfrac{1}{1+\eta}\left(1-\dfrac{1+\eta}{\eta}\lambda_{\max}(E^{\mathrm{T}}PE)\right)P & (A+B(D_iF+D_i^-H_2))^{\mathrm{T}}\\ A+B(D_iF+D_i^-H_2) & P^{-1}\end{bmatrix}>0,\quad i\in[1,2^m],\\
&\text{(d) } |h_{1,j}x|\leqslant 1,\quad \forall x\in\mathscr{E}(P,\rho_1),\quad j\in[1,m],\\
&\text{(e) } |h_{2,j}x|\leqslant 1,\quad \forall x\in\mathscr{E}(P,1),\quad j\in[1,m].\end{aligned} \tag{30}$$

If we fix ρ_1, λ and η, then the constraints of the optimization problem can be transformed into LMIs. To obtain the global infimum, we may vary ρ_1 from 0 to 1, η from 0 to ∞ and λ from 0 to $\rho_1\eta/(1+\eta)$. Similar

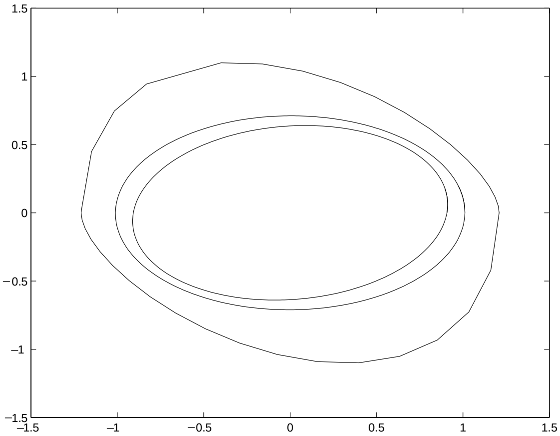

Fig. 3. The invariant ellipsoids and the null controllable region.

to the treatment of the optimization problem (25), we can reduce the number of parameters fixed beforehand (ρ_1, λ and η) from three to two.

3.4. An example

Consider the system (18) with

$$A=\begin{bmatrix}0.9741 & -0.2474\\ 0.2474 & 1.2710\end{bmatrix},\quad B=\begin{bmatrix}-0.0259\\ 0.2474\end{bmatrix},\quad E=\begin{bmatrix}0.0057\\ 0.0082\end{bmatrix}.$$

Let's first consider Problem 2 of enlarging the invariant ellipsoid. Here we choose the shape reference set as a unit ball, i.e., $X_0=\mathscr{E}(I,1)$. By solving (22), we obtain $\alpha_2^*=0.6337$, along with $\eta^*=0.0143$, $\lambda^*=1.9879*10^{-4}$ and

$$P^*=\begin{bmatrix}1.2148 & -0.1667\\ -0.1667 & 2.4681\end{bmatrix},\quad F^*=[-0.5726 \quad -1.2574].$$

The invariant set $\mathscr{E}(P^*,1)$ is the smaller ellipsoid in Fig. 3. The larger ellipsoid is obtained as the maximal invariant ellipsoid in the absence of disturbance ($E=0$). The outmost closed curve is the boundary of the null controllable region of the system in the absence of disturbance, which is the largest possible invariant set that can be achieved with any control law (see [6]).

Next, we consider Problem 3. We take the reference set X_∞ also to be the unit ball. The optimal α_3 is found to be $\alpha_3^*=0.0825$ with $\eta^*=0.2126$, $\lambda^*=0.0307$ and

$$P^*=\begin{bmatrix}1920.2 & -1884.6\\ -1884.6 & 2150.6\end{bmatrix},\quad F^*=[-0.3314 \quad -2.4721].$$

For Problem 4, we take X_∞ to be the unit ball and $X_0=\alpha_0\mathscr{E}(I,1)$. From the solution to Problem 2, we know that α_0 must be less than $\alpha_2^*=0.6337$. From the solution to Problem 3, we know that α_4^* must be greater than $\alpha_3^*=0.0825$. First, we choose $\alpha_0=0.5$. By solving (30), we obtain $\alpha_4^*=0.2960$, along with $\eta^*=0.0431$, $\lambda^*=2.4565*10^{-4}$, $\rho_1^*=0.1440$ and

$$P^*=\begin{bmatrix}1.9000 & -0.7335\\ -0.7335 & 3.7429\end{bmatrix},\quad F^*=[-0.5707 \quad -1.4722].$$

In Fig. 4, the smaller dotted ellipsoid is $\mathscr{E}(P^*,\rho_1)$ and the bigger one is $\mathscr{E}(P^*,1)$. A trajectory starting from the boundary of $\mathscr{E}(P^*,1)$ is plotted in Fig. 4. In the simulation, the disturbance is chosen as

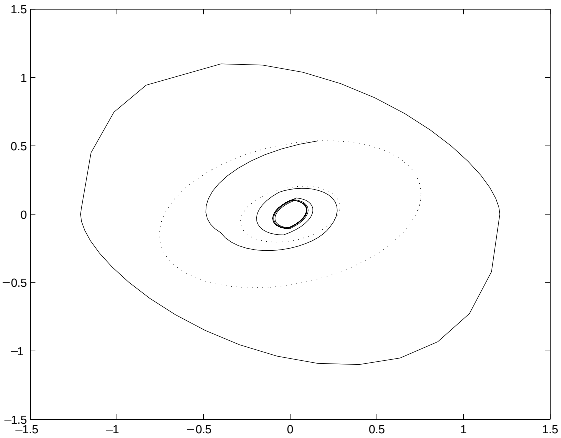

Fig. 4. The invariant ellipsoids and a trajectory, $\alpha_0 = 0.5$, $\alpha_4^* = 0.2960$.

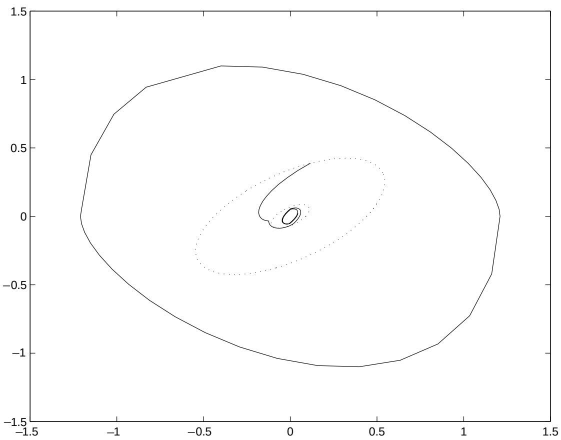

Fig. 5. The invariant ellipsoids and a trajectory, $\alpha_0 = 0.3$, $\alpha_4^* = 0.1262$.

$w(k) = \text{sign}(\sin(0.2k))$. We see that the trajectory enters $\mathscr{E}(P, \rho_1)$ and stays inside of it. However, the disturbance may not be rejected to a satisfactory level. This is because enlarging the outer ellipsoid and reducing the inner ellipsoid are conflicting objectives. To obtain a better disturbance rejection performance, we have to choose smaller X_0. For example, if we choose $\alpha_0 = 0.3$, then we obtain $\alpha_4^* = 0.1262$, along with

$$P^* = \begin{bmatrix} 5.1710 & -3.9277 \\ -3.9277 & 8.5125 \end{bmatrix}, \quad F^* = [-0.5123 \quad -1.8880].$$

Fig. 5 shows the invariant ellipsoids $\mathscr{E}(P^*, 1)$ and $\mathscr{E}(P^*, \rho_1)$, and a trajectory starting from the boundary of $\mathscr{E}(P^*, 1)$.

4. Conclusions

We considered linear systems subject to actuator saturation and persistent disturbance. Simple criteria for determining if a given ellipsoid is contractively invariant have been derived. With the aid of these criteria, we developed analysis and design methods for closed-loop stability and disturbance rejection. Examples were used to demonstrate the effectiveness of these methods.

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