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Direct computation of infimum in discrete-time H_∞-optimization using measurement feedback

Ben M. Chen *

Department of Electrical Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260

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Abstract

A direct and non-iterative method for the computation of the infimum for a class of discrete-time H_∞ optimal control problem is considered in this paper. The problem formulation is fairly general and does not place any restrictions on any direct feedthrough terms of the given systems. The method is applicable to systems where (i) the transfer function from the disturbance input to the measurement output is free of unit circle invariant zeros and left invertible, and (ii) the transfer function from the control input to the controlled output of the given system is free of the unit circle invariant zeros and right invertible. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction and Problem Statement

Ever since the original formulation of the H_∞ optimal control problem in [17], a great deal of work has been done on the solution of this problem in both continuous-time setting (see for example [7–12]), and discrete-time setting (see for example [1, 16]), The solution to the discrete-time H_∞ optimal control problem can be obtained from purely time-domain methods based on the γ-dependent discrete-time algebraic Riccati equations (AREs). Typically in ARE approaches to H_∞-optimal control problems, the achieved design solution is suboptimal in the sense that the H_∞-norm of the closed-loop system transfer function from the disturbances to the controlled outputs is less than a prescribed value, say γ. The ARE-based approach to this problem (see for example [16]) provides an iterative scheme of approximating the infimum (denoted here by γ^*) of the H_∞-norm of the closed-loop transfer function. In this paper, we address the problem of computing the infimum in discrete-time H_∞ optimization. We propose a non-iterative method for computing this γ^* for a class of discrete-time H_∞-optimization problems in which the transfer function from the disturbance to the measurement output is left invertible, and the transfer function from the control input to the output to be controlled is right invertible. The work of this paper can be regarded as a counterpart of our earlier work (see [2, 3]) in non-iterative computation of the infimum for the continuous-time H_∞-optimal control problem. It is hoped that the results of this paper will provide new insight into solutions to the discrete-time H_∞ optimal control problem, as the role its counterpart [3] has played in the continuous-time problem.

* Fax: 65 779 1103; e-mail: bmchen@nus.edu.sg; http//www.ee.nus.sg/˜bmchen.

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More specifically, we consider in this paper the following standard linear time-invariant discrete time system Σ characterized by

$$\begin{aligned} x(k+1) &= A\ x(k) + Bu(k) &&+ Ew(k), \\ y(k) &= C_1\ x(k) &&+ D_1\ w(k), \\ z(k) &= C_2\ x(k) + D_2\ u(k) &&+ D_{22}\ w(k), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measurement, $w \in \mathbb{R}^q$ is the unknown disturbance and $z \in \mathbb{R}^p$ is the output to be controlled. A, B, E, C_1, D_1, C_2, D_2 and D_{22} are constant matrices of appropriate dimension. Without loss of generality but for simplicity of presentation, we assume throughout this paper that matrices $[C_1 \quad D_1]$ and $[B' \quad D_2']$ are of maximal rank. This is because if these two matrices are not of maximal rank, one can simply drop the redundant control inputs and measurement outputs to make them maximal rank. The H_∞ optimal control problem is to find an internally stabilizing causal controller such that the H_∞-norm of the overall closed-loop system is minimized. To be more specific, we will investigate dynamic feedback laws of the form

$$\Sigma_c : \begin{cases} x_c(k+1) = K\ x_c(k) + L\ y(k), \\ u(k) = M\ x_c(k) + N\ y(k). \end{cases} \tag{2}$$

We will say that the controller Σ_c of Eq. (2) is internally stabilizing when applied to the system Σ, if the following matrix is asymptotically stable:

$$A_{cl} := \begin{bmatrix} A + BNC_1 & BM \\ LC_1 & K \end{bmatrix}, \tag{3}$$

i.e., all its eigenvalues lie in the open unit disc. Denote by G_{cl} the corresponding closed-loop transfer matrix. Then the H_∞ norm of the transfer matrix G_{cl} is given by

$$\|G_{cl}\|_\infty := \sup_{\omega \in [0, 2\pi]} \sigma_{\max}[G_{cl}(e^{j\omega})], \tag{4}$$

where $\sigma_{\max}[\cdot]$ denotes the largest singular value. The infimum γ^* can now be formally defined as

$$\gamma^* := \inf\{\|G_{cl}\|_\infty \mid \Sigma_c \text{ internally stabilizes } \Sigma\}. \tag{5}$$

Given a $\gamma > \gamma^*$, the H_∞ optimal (or more precisely suboptimal) control problem is to find an internally stabilizing controller Σ_c such that the resulting $\|G_{cl}\|_\infty < \gamma$. Also, Σ_c is said to be a γ suboptimal controller for Σ if the corresponding $\|G_{cl}\|_\infty < \gamma$. The main purpose of this paper is to present a non-iterative method that computes exactly this γ^* for Σ under the following assumptions:

(A1): (A, B) is stabilizable;
(A2): (A, B, C_2, D_2) is free of unit circle invariant zeros;
(A3): (A, B, C_2, D_2) is right invertible;
(A4): (A, C_1) is detectable;
(A5): (A, E, C_1, D_1) is free of unit circle invariant zeros;
(A6): (A, E, C_1, D_1) is left invertible.

Here we should point out that Assumptions (A1) and (A4) are necessary for any control problems, while (A2) and (A5) are fairly standard in H_∞ literature. The assumptions (A3) and (A6) are not essential and can be relaxed as in the continuous time case (see for example [4]). Note that (A3) and (A6) also imply that matrices $[C_2\ D_2]$ and $[E'\ D_1']$ are of maximal rank.

The paper is organized as follows: In Section 2, we recall the special coordinate basis of linear systems, which is instrumental to the development and derivation of the main results. Section 3 gives the main results, namely, non-iterative algorithms for computation of γ^* for three common cases, i.e., the full information, the output feedback and the state feedback cases. Finally, the concluding remarks are drawn in Section 4.

Throughout this paper, X' denotes the transpose of matrix A. I denotes an identity matrix with appropriate dimension. $\mathbb{R}$ is the set of real numbers. $\lambda(X)$ is the set of eigenvalues of a real square matrix X. $\lambda_{\max}(X)$ denotes the maximum eigenvalue of X where $\lambda(X)\subset\mathbb{R}$, and finally $\sigma_{\max}(X)$ denotes the maximum singular value of matrix X.

2. Background materials

In this section, we should recall a theorem of the special coordinate basis of linear systems from [14, 15] which will be instrumental to the main results developed in the next sections. Consider the system described by

$$\begin{aligned} x(k+1) &= A\ x(k) + B\ u(k) + E\ w(k),\\ z(k) &= C_2\ x(k) + D_2\ u(k) + D_{22}\ w(k). \end{aligned} \tag{6}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a non-singular matrix V that puts the direct feedthrough matrix D_2 into the following form:

$$UD_2V = \begin{bmatrix} I_{\rm r} & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

where r is the rank of D_2. Without loss of generality one can assume that the matrix D_2 in Eq. (6) has the form as shown in Eq. (7). Thus the system in Eq. (6) can be rewritten as

$$\begin{aligned} x(k+1) &= A\ x(k) + [\,B_0\ B_1\,]\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix} + E\ w(k),\\ \begin{pmatrix} z_0(k) \\ z_1(k) \end{pmatrix} &= \begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix}\ x(k) + \begin{bmatrix} I_{\rm r} 0 \\ 0 0 \end{bmatrix}\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix} + \begin{bmatrix} D_{22,0} \\ D_{22,1} \end{bmatrix}\ w(k), \end{aligned} \tag{8}$$

where B_0, B_1, $C_{2,0}$, $C_{2,1}$, $D_{22,0}$ and $D_{22,1}$ are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely,

$$u = V\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = Uz. \tag{9}$$

Also, note that the H_∞-norm of the system transfer function from w to z is unchanged when we apply an orthogonal transformation on the output z, and under any non-singular transformations on the states and control inputs. We have the following theorem.

Theorem 2.1. *Consider the linear system as in* Eq. (6). *Assume that* (A,B,C_2,D_2) *is right invertible with no unit circle invariant zeros. Then, there exist non-singular transformations* $\Gamma_{\rm s}$ and $\Gamma_{\rm i}$ *such that*

$$x = \Gamma_{\rm s}\begin{pmatrix} x_{\rm c} \\ x_{\rm a}^- \\ x_{\rm a}^+ \\ x_{\rm d} \end{pmatrix}, \qquad \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \Gamma_{\rm i}\begin{pmatrix} u_0 \\ u_{\rm d} \\ u_{\rm c} \end{pmatrix} \tag{10}$$

and

$$\Gamma_s^{-1}(A-B_0C_{2,0})\Gamma_s=\begin{bmatrix} A_{cc} & B_cE_{ca}^- & B_cE_{ca}^+ & L_{cd}C_d \\ 0 & A_{aa}^- & 0 & L_{ad}^-C_d \\ 0 & 0 & A_{aa}^+ & L_{ad}^+C_d \\ B_dE_{dc} & B_dE_{da}^- & B_dE_{da}^+ & A_{dd} \end{bmatrix}, \qquad \Gamma_s^{-1}E=\begin{bmatrix} E_c \\ E_a^- \\ E_a^+ \\ E_d \end{bmatrix}, \tag{11}$$

$$\Gamma_s^{-1}\begin{bmatrix} B_0 & B_1 \end{bmatrix}\Gamma_i=\begin{bmatrix} B_{0c} & 0 & B_c \\ B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0d} & B_d & 0 \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix}\Gamma_s=\begin{bmatrix} C_{0c} & C_{0a}^- & C_{0a}^+ & C_{0d} \\ 0 & 0 & 0 & C_d \end{bmatrix}, \quad D_2\Gamma_i=\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

where the pair (A_{cc}, B_c) *is completely controllable, while the subsystem* (A_{dd}, B_d, C_d) *is invertible and free of any invariant zeros. Also,* $\lambda(A_{aa}^+)$ *and* $\lambda(A_{aa}^-)$ *are, respectively, the sets of unstable and stable invariant zeros of* (A, B, C_2, D_2)*. Moreover, the pair* (A, B) *is stabilizable if and only if the pair* (A_{con}, B_{con}) *is controllable, where*

$$A_{con}:=\begin{bmatrix} A_{aa}^+ & L_{ad}^+C_d \\ B_dE_{da}^+ & A_{dd} \end{bmatrix} \quad \textit{and} \quad B_{con}:=\begin{bmatrix} B_{0a}^+ & 0 \\ B_{0d} & B_d. \end{bmatrix}. \tag{14}$$

Also, (A, B, C_2, D_2) *is invertible if and only if* x_c *is non-existent. For future use, we define an integer scalar* $n_x := \dim(x_a^+) + \dim(x_d)$.

3. Main results

Now, we are ready to present our main results, i.e., the non-iterative algorithms for computing the infimum, γ^*, of discrete-time H_∞ optimization. This section is naturally divided into two subsections. The first subsection deals with the full information case, while the second subsection deals the general output feedback case. The full state feedback problem is then treated as a special case in the second subsection.

3.1. The full information case

We assume that $y=[x' \ w']'$ and conditions (A1)–(A3) are satisfied. Without loss of generality, but for simplicity of presentation of our results, we also assume that D_2 is in the form of Eq. (7). In what follows, we state a step-by-step algorithm for the computation of the infimum γ^* for the full information problem.

Step 1: Transform the following system:

$$\begin{aligned} x(k+1) &= A\ x(k) + B\ u(k) + E\ w(k), \\ z(k) &= C_2\ x(k) + D_2\ u(k) + D_{22}\ w(k), \end{aligned} \tag{15}$$

into the special coordinate basis as in Section 2 and Theorem 2.1, and define A_x, B_x, E_x, C_x and D_x as follows:

$$A_x := \begin{bmatrix} A_{\rm aa}^+ & L_{\rm ad}^+ C_{\rm d} \\ B_{\rm d} E_{\rm da}^+ & A_{\rm dd} \end{bmatrix}, \quad B_x := \begin{bmatrix} B_{0\rm a}^+ & 0 \\ B_{0\rm d} & B_{\rm d} \end{bmatrix}, \quad E_x := \begin{bmatrix} E_{\rm a}^+ \\ E_{\rm d} \end{bmatrix}, \tag{16}$$

$$C_{\rm x} := \begin{bmatrix} 0 & 0 \\ 0 & C_{\rm d} \end{bmatrix} \quad \text{and} \quad D_x := \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{17}$$

It is simple to see from the special coordinate basis that the quadruple (A_x, B_x, C_x, D_x) is invertible and free of stable invariant zeros. Also, (A_x, B_x) is controllable.

Step 2: Find a matrix F_x such that $A_x + B_x F_x$ has no eigenvalues at -1. Then define $\tilde{A}_x$, $\tilde{B}_x$, $\tilde{E}_x$, $\tilde{C}_x$, $\tilde{D}_x$ and $\tilde{D}_{22}$ as in the following:

$$\begin{aligned}
&\tilde{A}_x := (A_x + B_x F_x + I)^{-1}(A_x + B_x F_x - I), \\
&\tilde{B}_x := 2(A_x + B_x F_x + I)^{-2} B_x, \\
&\tilde{E}_x := 2(A_x + B_x F_x + I)^{-2} E_x, \\
&\tilde{C}_x := C_x + D_x F_x, \\
&\tilde{D}_x := D_x - (C_x + D_x F_x)(A_x + B_x F_x + I)^{-1} B_x, \\
&\tilde{D}_{22} := D_{22} - (C_x + D_x F_x)(A_x + B_x F_x + I)^{-1} E_x.
\end{aligned} \tag{18}$$

Step 3: Solve the following two continuous-time γ-independent algebraic Lyapunov equations:

$$(\tilde{A}_x - \tilde{B}_x \tilde{D}_x^{-1} \tilde{C}_x)\tilde{S}_x + \tilde{S}_x(\tilde{A}_x - \tilde{B}_x \tilde{D}_x^{-1} \tilde{C}_x)' = \tilde{B}_x(\tilde{D}_x' \tilde{D}_x)^{-1} \tilde{B}_x', \tag{19}$$

$$(\tilde{A}_x - \tilde{B}_x \tilde{D}_x^{-1} \tilde{C}_x)\tilde{T}_x + \tilde{T}_x(\tilde{A}_x - \tilde{B}_x \tilde{D}_x^{-1} \tilde{C}_x)' = (\tilde{E}_x - \tilde{B}_x \tilde{D}_x^{-1} \tilde{D}_{22})(\tilde{E}_x - \tilde{B}_x \tilde{D}_x^{-1} \tilde{D}_{22})' \tag{20}$$

for positive definite solution $\tilde{S}_x$ and positive semi-definite solution $\tilde{T}_x$. The existences of such solutions will be justified in the proof of this algorithm. For future use, we define

$$S_x := (A_x + B_x F_x + I)\tilde{S}_x(A_x' + F_x' B_x' + I)/2 \tag{21}$$

and

$$T_x := (A_x + B_x F_x + I)\tilde{T}_x(A_x' + F_x' B_x' + I)/2. \tag{22}$$

Step 4: The infimum, γ^*, is given by

$$\gamma^* = \sqrt{\lambda_{\max}(\tilde{T}_x \tilde{S}_x^{-1})} = \sqrt{\lambda_{\max}(T_x S_x^{-1})}. \tag{23}$$

Proof of the Algorithm. Following the results of [5, 16], it is straightforward to show that the following three statements are equivalent:

1. There exists a γ suboptimal controller for Σ of (1) with $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$ and $D_1 = \begin{pmatrix} 0 \\ I. \end{pmatrix}$
2. There exists a γ suboptimal controller for the following auxiliary system:

$$\begin{aligned}
x_x(k+1) &= A_x\ x_x(k) + B_x\ u_x(k) + E_x\ w_x(k), \\
y_x(k) &= \begin{pmatrix} 0 \\ I \end{pmatrix} x_x(k) + \begin{pmatrix} I \\ 0 \end{pmatrix} w_x(k), \\
z_x(k) &= C_x\ x_x(k) + D_x\ u_x(k) + D_{22}\ w_x(k),
\end{aligned} \tag{24}$$

where A_x, B_x, E_x, C_x and D_x are defined as in Eqs. (16) and (17):

3. There exists a γ suboptimal controller for the following auxiliary system:

$$\begin{aligned} \dot{\tilde{x}}_x &= \tilde{A}_x \ \tilde{x}_x + \tilde{B}_x \ \tilde{u}_x + \tilde{E}_x \ \tilde{w}_x \\ \tilde{y}_x &= \begin{pmatrix} 0 \\ I \end{pmatrix} \tilde{x}_x + \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{w}_x \\ \tilde{z}_x &= \tilde{C}_x \ \tilde{x}_x + \tilde{D}_x \ \tilde{u}_x + \tilde{D}_{22} \ \tilde{w}_x \end{aligned} \tag{25}$$

where $\tilde{A}_x$, $\tilde{B}_x$, $\tilde{E}_x$, $\tilde{C}_x$, $\tilde{D}_x$ and $\tilde{D}_{22}$ are as defined in Eq. (18).

We would like to note that items 2 and 3 above are also equivalent to the following:

1. There exists a $P_x > 0$ to the following discrete-time Riccati equation:

$$P_x = A_x' P_x A_x + C_x' C_x - \begin{bmatrix} B_x' P_x A_x + D_x' C_x \\ E_x' P_x A_x + D_{22}' C_x \end{bmatrix}' G_x(P_x)^{-1} \begin{bmatrix} B_x' P_x A_x + D_x' C_x \\ E_x' P_x A_x + D_{22}' C_x \end{bmatrix}, \tag{26}$$

where

$$G_x(P_x) := \begin{bmatrix} D_x' D_x & D_x' D_{22} \\ D_{22}' D_x & D_{22}' D_{22} - \gamma^2 I \end{bmatrix} + \begin{bmatrix} B_x' \\ E_x' \end{bmatrix} P_x [\, B_x \quad E_x \,], \tag{27}$$

such that the following conditions are satisfied:

$$V_x := B_x' P_x B_x + D_x' D_x > 0, \tag{28}$$

$$R_x := \gamma^2 I - D_{22}' D_{22} - E_x' P_x E_x + (E_x' P_x B_x + D_{22}' D_x) V_x^{-1} (B_x' P_x E_x + D_x' D_{22}) > 0. \tag{29}$$

2. There exists a $\tilde{P}_x > 0$ to the following continuous-time Riccati equation:

$$0 = \tilde{P}_x \tilde{A}_x + \tilde{A}_x' \tilde{P}_x + \tilde{C}_x' \tilde{C}_x - \begin{bmatrix} \tilde{B}_x' \tilde{P}_x + \tilde{D}_x' \tilde{C}_x \\ \tilde{E}_x' \tilde{P}_x + \tilde{D}_{22}' \tilde{C}_x \end{bmatrix}' \tilde{G}_x^{-1} \begin{bmatrix} \tilde{B}_x' \tilde{P}_x + \tilde{D}_x' \tilde{C}_x \\ \tilde{E}_x' \tilde{P}_x + \tilde{D}_{22}' \tilde{C}_x \end{bmatrix} \tag{30}$$

with

$$\tilde{D}_{22}' [I - \tilde{D}_x (\tilde{D}_x' \tilde{D}_x)^{-1} \tilde{D}_x'] \tilde{D}_{22} < \gamma^2 I \tag{31}$$

and

$$\tilde{G}_x := \begin{bmatrix} \tilde{D}_x' \tilde{D}_x & \tilde{D}_x' \tilde{D}_{22} \\ \tilde{D}_{22}' \tilde{D}_x & \tilde{D}_{22}' \tilde{D}_{22} - \gamma^2 I \end{bmatrix}. \tag{32}$$

Furthermore, the solutions to the above Riccati equations are related, if exist, by

$$P_x = 2(A_x' + F_x' B_x' + I)^{-1} \tilde{P}_x (A_x + B_x F_x + I)^{-1}. \tag{33}$$

Thus, it is equivalent to show that γ^* given by Eq. (23) is the infimum for system Σ of Eq. (1) by showing that it is an infimum for the auxiliary system in Eq. (25). This can be done by first showing the properties of the auxiliary system of Eq. (25) and then applying the results of [2]. We note that the matrix F_x in Step 2 of the algorithm is a pre-state feedback gain, which is introduced merely to deal with the situation when A_x has eigenvalues at -1 and the inverse of $I + A_x$ does not exist. For the sake of simplicity but without loss of generality, we will hereafter assume that A_x has no eigenvalues at -1 and $F_x = 0$. We will first show in the following that the quadruple $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ has a total number of n_x, where n_x is the dimension of A_x, unstable invariant zeros with some located at $\lambda\{(A_{\rm aa}^+ + I)^{-1}(A_{\rm aa}^+ - I)\}$ and the rest at 1. By the definition of $\tilde{D}_x$, i.e.,

$$\tilde{D}_x = D_x - C_x (A_x + I)^{-1} B_x = C_x (-I - A_x)^{-1} B_x + D_x \tag{34}$$

and also noting that from the special coordinate basis, the quadruple (A_x, B_x, C_x, D_x) is square invertible and A_x is assumed to have no eigenvalues at -1, it follows $\tilde{D}_x$ is a non-singular matrix. Hence $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ has a total number of n_x invariant zeros, which are complex numbers, say s, such that

$$\operatorname{rank}\begin{bmatrix} sI-\tilde{A}_x & -\tilde{B}_x \\ \tilde{C}_x & \tilde{D}_x \end{bmatrix} < n_x + p, \tag{35}$$

where p is the dimension of z of the given system (1). Noting that

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} sI-\tilde{A}_x & -\tilde{B}_x \\ \tilde{C}_x & \tilde{D}_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} sI-(A_x+I)^{-1}(A_x-I) & -2(A_x+I)^{-2}B_x \\ C_x & D_x - C_x(A_x+I)^{-1}B_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} s(A_x+I)-(A_x-I) & -2(A_x+I)^{-1}B_x \\ C_x & D_x - C_x(A_x+I)^{-1}B_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} (1+s)I-(1-s)A_x & -(1-s)B_x \\ C_x & D_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} (1+s)I-(1-s)A_{\mathrm{aa}}^{+} & -(1-s)L_{\mathrm{ad}}^{+}C_{\mathrm{d}} & -(1-s)B_{0\mathrm{a}}^{+} & 0 \\ -(1-s)B_{\mathrm{d}}E_{\mathrm{da}}^{+} & (1+s)I-(1-s)A_{\mathrm{dd}} & -(1-s)B_{0\mathrm{d}} & -(1-s)B_{\mathrm{d}} \\ 0 & 0 & I_r & 0 \\ 0 & C_{\mathrm{d}} & 0 & 0 \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} (1+s)I-(1-s)A_{\mathrm{aa}}^{+} & 0 & 0 & 0 \\ 0 & (1+s)I-(1-s)A_{\mathrm{dd}} & 0 & -(1-s)B_{\mathrm{d}} \\ 0 & 0 & I_{\mathrm{r}} & 0 \\ 0 & C_{\mathrm{d}} & 0 & 0 \end{bmatrix}.
\end{aligned}$$

It is obvious from the above expression that for any $s \in \lambda\{(A_{\mathrm{aa}}^{+}+I)^{-1}(A_{\mathrm{aa}}^{+}-I)\}$, the rank of matrix

$$\begin{bmatrix} sI-\tilde{A}_x & -\tilde{B}_x \\ \tilde{C}_x & \tilde{D}_x \end{bmatrix} \tag{36}$$

will drop lower than $n_x + p$. The only other scalar that causes the matrix to drop rank is $s = 1$ because of the property of the subsystem $(A_{\mathrm{dd}}, B_{\mathrm{d}}, C_{\mathrm{d}})$, which is invertible and free of invariant zeros (see Theorem 2.1). This proves our claim on the locations of the invariant zeros of $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$. Let us apply a pre-output feedback law,

$$\tilde{u}_x = -\tilde{D}_x^{-1}\tilde{C}_x\tilde{x}_x - \tilde{D}_x^{-1}\tilde{D}_{22}\tilde{w}_x + \tilde{v}_x \tag{37}$$

to the auxiliary system (25). Again, this pre-feedback law will not effect solutions to the H_∞ problem for Eq. (25) or to the solution $\tilde{P}_x$ of Eqs. (30)–(32). After applying this pre-feedback law, we obtain the following new system:

$$\begin{aligned}\dot{\tilde{x}}_x &= (\tilde{A}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{C}_x)\ \tilde{x}_x + \tilde{B}_x\tilde{v}_x\ + (\tilde{E}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{D}_{22})\tilde{w}_x,\\ \tilde{y}_x &= \begin{pmatrix}0\\ I\end{pmatrix}\ \tilde{x}_x\ + \begin{pmatrix}I\\ 0\end{pmatrix}\ \tilde{w}_x,\\ \tilde{z}_x &= 0\ \tilde{x}_x + \tilde{D}_x\tilde{v}_x\ + 0\ \tilde{w}_x.\end{aligned} \tag{38}$$

We are lucky enough to get rid of the direct feedthrough term $\tilde{D}_{22}$. Then it follows from the well-known results in H_∞ control theory that the existence condition of a γ suboptimal controller for Eq. (38) is equivalent to the existence of a $\tilde{P}_x > 0$ such that

$$\begin{aligned}0 = {} & \tilde{P}_x(\tilde{A}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{C}_x) + (\tilde{A}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{C}_x)'\tilde{P}_x - \tilde{P}_x\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{B}_x'\tilde{P}_x\\ & + \tilde{P}_x(\tilde{E}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{D}_{22})(\tilde{E}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{D}_{22})'\tilde{P}_x/\gamma^2\end{aligned} \tag{39}$$

is satisfied. Note that the solution $\tilde{P}_x$ to the above Riccati equation is identical to the solution that satisfies Eqs. (30)–(31).

Since the quadruple $(\tilde{A}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{C}_x, \tilde{B}_x, 0, \tilde{D}_x)$ in the auxiliary system of Eq. (38) is invertible and has no stable invariant zeros and infinite zeros, it satisfies the conditions of [2]. In fact, following the results of [2], we can show that

$$\gamma^* = \sqrt{\lambda_{\max}(\tilde{T}_x\tilde{S}_x^{-1})} \tag{40}$$

and for any $\gamma > \gamma^*$, the positive-definite solution $\tilde{P}_x$ of Eqs. (30)–(32) is given by

$$\tilde{P}_x = (\tilde{S}_x - \tilde{T}_x/\gamma^2)^{-1}. \tag{41}$$

It then follows from Eq. (33) that for any $\gamma > \gamma^*$, the positive definite solution P_x of Eqs. (26)–(29) is given by

$$P_x = 2(A_x' + I)^{-1}(\tilde{S}_x - \tilde{T}_x/\gamma^2)^{-1}(A_x + I)^{-1} \tag{42}$$

and hence γ^* can also be obtained from the following expression:

$$\gamma^* = \sqrt{\lambda_{\max}(T_xS_x^{-1})}, \tag{43}$$

where S_x and T_x are as defined in Eqs. (21) and (22), respectively.

Finally, the existences of the positive-definite solution $\tilde{S}_x$ and the positive semi-definite solution $\tilde{T}_x$ can be justified as follows: It is simple to see from Eq. (38) that the eigenvalues of $\tilde{A}_x - \tilde{B}_x\tilde{D}_x^{-1}\tilde{C}_x$ are also the invariant zeros of $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$, which are shown to be in the right-half complex plane. It then follows from [13] that the solution $\tilde{S}_x$ to the Lyapunov equation (19) is positive definite because $(\tilde{A}_x, \tilde{B}_x)$ is controllable, and the solution $\tilde{T}_x$ to the Lyapunov equation (20) is positive semi-definite. In fact, both of them are unique. This completes the proof of our algorithm. □

3.2. The output feedback case

This subsection deals with the general measurement feedback problem. Again, we consider the given system of Eq. (1) and assume that (A1)–(A6) are satisfied. As in the previous subsection, we will first give a step-by-step non-iterative algorithm that calculates the infimum, γ^*, and leave detailed justification in the proof of the algorithm.

Step A: Define an auxiliary full information problem for

$$\begin{aligned} x(k+1) &= A\ x(k) + B\ u(k) \quad + E\ w(k),\\ y(k) &= \begin{pmatrix} 0 \\ I \end{pmatrix} x(k) \qquad\quad + \begin{pmatrix} I \\ 0 \end{pmatrix} w(k),\\ z(k) &= C_2\ x(k) + D_2\ u(k)\ + D_{22}\ w(k) \end{aligned} \tag{44}$$

and perform Steps 1–3 of the algorithm given in the previous subsection. For future use and in order to avoid any notation confusion, we rename the state transformation of the special coordinate basis for this subsystem as Γ_{sP} and the dimension of A_x as n_{xP}. Also, rename S_x of Eq. (21) and T_x of Eq. (22) as S_{xP} and T_{xP}, respectively.

Step B: Define another auxiliary full information problem for

$$\begin{aligned} x(k+1) &= A'\ x(k) + C_1' u(k) \quad + C_2' w(k),\\ y(k) &= \begin{pmatrix} 0 \\ I \end{pmatrix} x(k) \qquad\quad + \begin{pmatrix} I \\ 0 \end{pmatrix} w(k),\\ z(k) &= E'\ x(k) + D_1'\ u(k)\ + D_{22}'\ w(k) \end{aligned} \tag{45}$$

and again perform Steps 1–3 of the algorithm given in Section 3.1 one more time but for this auxiliary. We also rename the state transformation of the special coordinate basis for this case as Γ_{sQ} and the dimension of A_x as n_{xQ}, and S_x of Eq. (21) and T_x of Eq. (22) as S_{xQ} and T_{xQ}, respectively.

Step C: Partition

$$\Gamma_{sP}^{-1}(\Gamma_{sQ}^{-1})' = \begin{bmatrix} \star & \star \\ \star & \Gamma \end{bmatrix}, \tag{46}$$

where Γ is a $n_{xP} \times n_{xQ}$ matrix, and define a constant matrix

$$M = \begin{bmatrix} T_{xP}S_{xP}^{-1} + \Gamma S_{xQ}^{-1}\Gamma' S_{xP}^{-1} & -\Gamma S_{xQ}^{-1} \\ -T_{xQ}S_{xQ}^{-1}\Gamma' S_{xP}^{-1} & T_{xQ}S_{xQ}^{-1} \end{bmatrix}. \tag{47}$$

Step D: The infimum γ^* is then given by

$$\gamma^* = \sqrt{\lambda_{\max}(M)}. \tag{48}$$

Proof of the Algorithm. The dirty work was done in the proof of the previous subsection. Once the result for the full information case is established, the proof of this algorithm follows the lines of reasoning given in [2, 3]. □

As was pointed out in [16], for discrete-time H_∞ control, the infimum for the full information problem is in general different from that of the full state feedback problem. For the state feedback case, i.e., $C_1 = I$ and $D_1 = 0$, we note that the subsystem (A, E, C_1, D_1) is always free of invariant zeros (and hence free of unit circle invariant zeros) and left invertible. Thus, as long as the other subsystem of the given system (1), i.e., (A, B, C_2, D_2) is free of unit circle invariant zeros and right invertible, one can apply the above algorithm to get the infimum γ^*. It turns out that for this special case Γ_{sQ}, n_{xQ}, S_{xQ} and T_{xQ} in Step B of the above algorithm can be directly obtained using the following simple procedure: Compute a non-singular transformation Γ_{sQ} such that

$$\Gamma_{sQ}' E = \begin{bmatrix} 0 \\ \hat{E} \end{bmatrix}, \tag{49}$$

where $\hat{E}$ is a $n_{xQ} \times n_{xQ}$ non-singular matrix. Then S_{xQ} and T_{xQ}, are, respectively, given by

$$S_{xQ} = (\hat{E}^{-1})'\hat{E}^{-1} \quad \text{and} \quad T_{xQ} = (\hat{E}^{-1})'(D_{22}' D_{22})\hat{E}^{-1}. \tag{50}$$

4. Concluding remarks

We have presented in this paper a non-iterative method for the computation of the infimum, γ^*, for a class of discrete-time H_∞ optimization problem in which the given system satisfies certain conditions, i.e., (i) the transfer function from the control input to the controlled output of the given system is free of the unit circle invariant zeros and right invertible, and (ii) the transfer function from the disturbance input to the measurement output is free of unit circle invariant zeros and left invertible.

Finally, we note that in principle one could tackle the proposed problem of this paper directly from the discrete-time domain instead of using the technique of bilinear transformation to transfer the problem to the continuous-time domain. The disadvantages of such an approach are: (i) it will be mathematically much more complicated as it involves solving discrete-time algebraic Riccati equations (DARE's), and (ii) it will bury the interconnections of structural properties between the continuous-time and discrete-time problems. We would like to point out further that most of results in the literature for solving DAREs involve iterative procedures except the one in [5]. The result of [5] again utilizes the bilinear transformation technique to derive a non-iterative method for solving general DARE's. Clearly, this technique is an efficient mean for dealing with many discrete-time problems. The mapping of structural properties, such as finite and infinite zero structures as well as invertibility structures, of general multivariable linear systems under bilinear transformation has been fully studied in a recent work of [6]. Utilizing the result of [6], one could easily obtain the insight of many discrete-time problems.

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