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# Composite nonlinear control with state and measurement feedback for general multivariable systems with input saturation

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**Abstract**

In this paper, we present a design procedure of composite nonlinear feedback control for general multivariable systems with actuator saturation. We consider both the state feedback case and the measurement feedback case without imposing any restrictive assumption on the given systems. The composite nonlinear feedback control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with faster rise time, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to reduce overshoot and undershoot caused by the linear part. As such, a highly desired tracking performance with faster settling time and smaller overshoot can be obtained. The result is illustrated by a numerical example, which shows that the proposed design method yields a very satisfactory performance.


*Keywords:* Nonlinear control; Actuator saturation; Tracking control

## 1. Introduction and problem formulation

Every physical system in our real life has nonlinearities and very little can be done to overcome them. Many practical systems are sufficiently nonlinear so that important features of their performance may be completely overlooked if they are analyzed and designed through linear techniques (see e.g., [8]). When the actuator is saturated, the performance of the control system designed will seriously deteriorate. As such, the topic of nonlinear control for saturated linear systems has attracted considerable attentions in the past (see e.g. [6,7,10,12,13,16] to name a few).

Inspired by a work of Lin et al. [9], which was introduced to improve the tracking performance under state feedback laws for a class of second-order SISO systems subject to actuator saturation, Chen et al. [3] have recently extended the so-called composite nonlinear feedback (CNF) control technique to general SISO systems with measurement feedback. The work

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of Chen et al. [3] has been successfully applied to design an HDD servo system. It has also been demonstrated in [3] that the CNF design is capable of beating the time-optimal control in asymptotically tracking situations. The extension of the result of [9] to MIMO systems under state feedback was reported in [15]. However, the extension was made under a pretty odd assumption, which will be discussed later.

In this paper, we present a design procedure of the CNF control for improving tracking performance of general multivariable systems with actuator saturation. Generally, the CNF control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with faster rise time, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to reduce overshoot and undershoot caused by the linear part. More specifically, we consider a multivariable linear system $\Sigma$ with an amplitude-constrained actuator characterized by

$$\begin{aligned} \dot{x} &= Ax + B\,\mathrm{sat}(u), \quad x(0) = x_0, \\ y &= C_1 x, \\ h &= C_2 x + D_2\,\mathrm{sat}(u), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}^\ell$ are, respectively, the state, control input, measurement output and controlled output of the given system $\Sigma$. $A$, $B$, $C_1$ and $C_2$ are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \begin{pmatrix} \mathrm{sat}(u_1) \\ \mathrm{sat}(u_2) \\ \vdots \\ \mathrm{sat}(u_m) \end{pmatrix},$$

$$\mathrm{sat}(u_i) = \mathrm{sign}(u_i) \min(|u_i|,\ \bar{u}_i), \tag{2}$$

where $\bar{u}_i$ is the maximum amplitude of the $i$th control channel. The objective of this paper is to design an appropriate control law for (1) using the CNF approach such that the resulting controlled output will track some desired step references as fast and as smooth as possible. We will address the CNF control system design for the given system (1) for three different situations, namely, the state feedback case, the full order measurement feedback case and the reduced order measurement feedback case. For tracking purpose, the following assumptions on the given system are required: (i) $(A, B)$ is stabilizable; (ii) $(A, C_1)$ is detectable; and (iii) $(A, B, C_2, D_2)$ is right invertible and has no invariant zeros at $s = 0$. It is well understood in the literature that these assumptions are standard and necessary.

The paper is organized as follows: Section 2 deals with the theory of the CNF control for the state feedback case, whereas Section 3 deals with the detailed development of the CNF design with the full order measurement feedback and the reduced order measurement cases. We will address the issue on the selection of some key design parameters in Section 4. The proposed technique will then be illustrated by a numerical example in Section 5. Some concluding remarks will be drawn in Section 6.

## 2. The state feedback case

We first proceed to develop a composite nonlinear feedback control technique for the case when all the state variables of the plant $\Sigma$ of (1) are measurable, i.e., $y = x$. The design will be done in three steps, which is a natural extension of the results of Chen et al. [3]. We have the following step-by-step design procedure.

*Step* S1: Design a linear feedback law

$$u_{\mathrm{L}} = Fx + Gr, \tag{3}$$

where $r \in \mathbb{R}^l$ contains a set of step references. The state feedback gain matrix $F \in \mathbb{R}^{m \times n}$ is chosen such that the closed-loop system matrix $A + BF$ is asymptotically stable and the resulting closed-loop system transfer matrix, i.e., $D_2 + (C_2 + D_2 F)(sI - A - BF)^{-1}B$, has certain desired properties, e.g., having a small dominating damping ratio in each channel. We note that such an $F$ can be worked out using some well-studied methods such as the LQR, $H_\infty$ and $H_2$ optimization approaches (see, e.g., [1,2,11]). Furthermore, $G$ is an $m \times l$ constant matrix and is given by

$$G := G_0'(G_0 G_0')^{-1} \tag{4}$$

with $G_0 := D_2 - (C_2 + D_2 F)(A + BF)^{-1}B$. Here we note that both $G_0$ and $G$ are well defined because $A + BF$ is stable, and $(A, B, C_2, D_2)$ is right invertible and has no invariant zeros at $s = 0$, which implies

$(A+BF, B, C+D_2F, D_2)$ is right invertible and has no invariant zeros at $s=0$ (see e.g., Theorem 3.8.1 of [4]).

*Step* S2: Next, we compute

$$H := [I - F(A+BF)^{-1}B]G \tag{5}$$

and

$$x_{\mathrm{e}} := G_{\mathrm{e}}\, r := -(A+BF)^{-1}BG\, r. \tag{6}$$

Note that the definitions of $H$, $G_{\mathrm{e}}$ and $x_{\mathrm{e}}$ would become transparent later in our derivation. Given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$(A+BF)'P + P(A+BF) = -W, \tag{7}$$

for $P>0$. Such a $P$ exists since $A+BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_{\mathrm{N}}$ is given by

$$u_{\mathrm{N}} = \rho(r,y)B'P(x - x_{\mathrm{e}}), \tag{8}$$

where

$$\rho(r,y) = \mathrm{diag}\{\rho_1,\ldots,\rho_m\} = \begin{bmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_m \end{bmatrix}, \tag{9}$$

and $\rho_i = \rho_i(r,y)$, $i=1,2,\ldots,m$, are respectively some nonpositive functions, uniformly bounded and locally Lipschitz in $y$, which are used to change the closed-loop system damping ratios as the outputs approach the targets. The choice of these nonlinear functions and $W$ will be discussed in Section 4.

*Step* S3: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$u = u_{\mathrm{L}} + u_{\mathrm{N}} = Fx + Gr + \rho(r,y)B'P(x - x_{\mathrm{e}}). \tag{10}$$

This completes the design of the CNF controller for the state feedback case.

For further development, we partition $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{m\times n}$ and $H \in \mathbb{R}^{m\times l}$ as follows:

$$B = [\,B_1 \quad \cdots \quad B_m\,],$$
$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ \vdots \\ H_m \end{bmatrix}. \tag{11}$$

The following theorem shows that the closed-loop system comprising the given plant in (1) and the CNF control law of (10) is asymptotically stable. It also determines the magnitudes of the step functions in $r$ that can be tracked by such a control law without exceeding the control limit.

**Theorem 2.1.** *Consider the given system in* (1) *with* $y=x$, *which satisfies assumptions* (i) *and* (iii), *the linear control law of* (3) *and the composite nonlinear feedback control law of* (10). *For any* $\delta \in (0,1)$, *let* $c_\delta > 0$ *be the largest positive scalar such that for all* $x \in \boldsymbol{X}_\delta$, *where*

$$\boldsymbol{X}_\delta := \{x : x'Px \leqslant c_\delta\}, \tag{12}$$

*the following property holds*:

$$|F_i x| \leqslant (1-\delta)\bar{u}_i, \quad i=1,\ldots,m. \tag{13}$$

*Then, the linear control law of* (3) *is capable of driving the system controlled output* $h(t)$ *to track asymptotically a set of step references, i.e.,* $r$, *provided that the initial state* $x_0$ *and* $r$ *satisfy*

$$\tilde{x}_0 := (x_0 - x_{\mathrm{e}}) \in \boldsymbol{X}_\delta, \quad |H_i r| \leqslant \delta \bar{u}_i, \quad i=1,\ldots,m. \tag{14}$$

*Furthermore, for any nonpositive function* $\rho(r,y)$, *uniformly bounded and locally Lipschitz in* $y$, *the composite nonlinear feedback law in* (10) *is capable of driving the system controlled output* $h(t)$ *to track asymptotically the step command input of amplitude* $r$, *provided that the initial state* $x_0$ *and* $r$ *satisfy* (14).

**Proof.** Let us first define a new state variable $\tilde{x} = x - x_{\mathrm{e}}$. It is simple to verify that the linear feedback control law of (3) can be rewritten as

$$u_{\mathrm{L}}(t) = F\tilde{x}(t) + [I - F(A+BF)^{-1}B]Gr \tag{15}$$

$$= F\tilde{x}(t) + Hr, \tag{16}$$

and hence for all $\tilde{x} \in \boldsymbol{X}_\delta$ and, provided that $|H_i r| \leqslant \delta\bar{u}_i$, $i=1,\ldots,m$, the closed-loop system is linear and is given by

$$\dot{\tilde{x}} = (A+BF)\tilde{x} + Ax_{\mathrm{e}} + BHr. \tag{17}$$

Noting that

$$\begin{aligned}Ax_e + BHr &= \{B[I - F(A+BF)^{-1}B]G \\ &\quad - A(A+BF)^{-1}BG\}r \\ &= \{[I - BF(A+BF)^{-1}]BG \\ &\quad - A(A+BF)^{-1}BG\}r \\ &= \{I - BF(A+BF)^{-1} \\ &\quad - A(A+BF)^{-1}\}BGr = 0, \end{aligned} \tag{18}$$

the closed-loop system in (17) can then be simplified as

$$\dot{\tilde{x}} = (A+BF)\tilde{x}. \tag{19}$$

Similarly, the closed-loop system comprising the given plant in (1) and the CNF control law of (10) can be expressed as

$$\dot{\tilde{x}} = (A+BF)\tilde{x} + Bw, \tag{20}$$

where

$$w = \mathrm{sat}(F\tilde{x} + Hr + u_{\mathrm{N}}) - F\tilde{x} - Hr. \tag{21}$$

Clearly, for the given $x_0$ satisfying (14), we have $\tilde{x}_0 = (x_0 - x_e) \in \boldsymbol{X}_\delta$. We note that (20) is reduced to (19) if $\rho(r, y) = 0$.

Next, we define a Lyapunov function $V = \tilde{x}' P \tilde{x}$ and evaluate the derivative of $V$ along the trajectories of the closed-loop system in (20), i.e.,

$$\begin{aligned}\dot{V} &= \dot{\tilde{x}}' P\tilde{x} + \tilde{x}' P\dot{\tilde{x}} \\ &= \tilde{x}'(A+BF)'P\tilde{x} + \tilde{x}'P(A+BF)\tilde{x} + 2\tilde{x}'PBw \\ &= -\tilde{x}'W\tilde{x} + 2\tilde{x}'PBw. \end{aligned} \tag{22}$$

Note that for all

$$\begin{aligned}&\tilde{x} \in \boldsymbol{X}_\delta = \{\tilde{x} : \tilde{x}'P\tilde{x} \leqslant c_\delta\} \\ &\quad \Rightarrow |F_i\tilde{x}| \leqslant (1-\delta)\bar{u}_i, \quad i = 1, \ldots, m. \end{aligned} \tag{23}$$

In the remainder of this proof, we adopt similar lines of reasoning as those of Turner et al. [15] by considering the following different scenarios. For simplicity, we drop the dependent variables of the nonlinear function $\rho$ in the rest of this proof.

*Case* 1: All input channels are unsaturated. It is obvious that we have

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\tilde{x}'PB\rho B'P\tilde{x} \leqslant -\tilde{x}'W\tilde{x}. \tag{24}$$

*Case* 2: All input channels are exceeding their upper limits. In this case, we have

$$\begin{aligned}&F_i\tilde{x} + H_ir + \rho_i B_i'P\tilde{x} \geqslant \bar{u}_i, \\ &\quad i = 1, \ldots, m. \end{aligned} \tag{25}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (23) holds, and $r$ satisfying (14), we have

$$F_i\tilde{x} + H_ir \leqslant \bar{u}_i, \quad i = 1, \ldots, m, \tag{26}$$

and thus

$$\begin{aligned}w_i &= \mathrm{sat}(F_i\tilde{x} + H_ir + \rho_i B_i'P\tilde{x}) - F_i\tilde{x} - H_ir \\ &= \bar{u}_i - F_i\tilde{x} - H_ir \geqslant 0 \end{aligned} \tag{27}$$

and

$$\begin{aligned}&\rho_i B_i'P\tilde{x} \geqslant \bar{u}_i - (F_i\tilde{x} + H_ir) \geqslant 0 \\ &\qquad \Rightarrow B_i'P\tilde{x} = \tilde{x}'PB_i \leqslant 0. \end{aligned} \tag{28}$$

Hence,

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\sum_{i=1}^{m} \tilde{x}'PB_i\bar{w}_i \leqslant -\tilde{x}'W\tilde{x}. \tag{29}$$

*Case* 3: All input channels are exceeding their lower limits. For this case, we have

$$F_i\tilde{x} + H_ir + \rho_i B_i'P\tilde{x} \leqslant -\bar{u}_i, \quad i = 1, \ldots, m. \tag{30}$$

For all $\tilde{x} \in \boldsymbol{X}_\delta$, which implies (23) holds, and $r$ satisfying (14), we have

$$F_i\tilde{x} + H_ir \geqslant -\bar{u}_i, \quad i = 1, \ldots, m, \tag{31}$$

and thus

$$\begin{aligned}w_i &= \mathrm{sat}(F_i\tilde{x} + H_ir + \rho_i B_i'P\tilde{x}) - F_i\tilde{x} - H_ir \\ &= -u_i - F_i\tilde{x} - H_ir \leqslant 0 \end{aligned} \tag{32}$$

and

$$\begin{aligned}&\rho_i B_i'P\tilde{x} \leqslant -\bar{u}_i - (F_i\tilde{x} + H_ir) \leqslant 0 \\ &\qquad \Rightarrow B_i'P\tilde{x} = \tilde{x}'PB_i \geqslant 0. \end{aligned} \tag{33}$$

Hence,

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\sum_{i=1}^{m} \tilde{x}'PB_iw_i \leqslant -\tilde{x}'W\tilde{x}. \tag{34}$$

*Case* 4: Some control channels are saturated and some are unsaturated. In view of Cases 1–3, it is simple to note that for those unsaturated channels, we have

$$\tilde{x}'PB_iw_i = \rho_i\tilde{x}'PB_iB_i'P\tilde{x} \leqslant 0, \tag{35}$$

and those input channels whose signals exceed their upper limits, we have

$$w_i \geqslant 0, \quad \tilde{x}' P B_i \leqslant 0 \Rightarrow \tilde{x}' P B_i w_i \leqslant 0, \tag{36}$$

and finally for those channels whose signals exceed their lower limits,

$$w_i \leqslant 0, \quad \tilde{x}' P B_i \geqslant 0 \Rightarrow \tilde{x}' P B_i w_i \leqslant 0. \tag{37}$$

Thus, for this case, again we have

$$\dot{V} = -\tilde{x}' W \tilde{x} + 2 \sum_{i=1}^{m} \tilde{x}' P B_i w_i \leqslant -\tilde{x}' W \tilde{x}. \tag{38}$$

In conclusion, we have shown that

$$\dot{V} \leqslant -\tilde{x} W \tilde{x}, \quad \tilde{x} \in \boldsymbol{X}_\delta, \tag{39}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (20). Noting that $P > 0$, all trajectories of (20) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state $x_0$ and the step command input $r$ that satisfy (14), we have

$$\lim_{t\to\infty} x(t) = x_\mathrm{e}, \tag{40}$$

which implies

$$\begin{aligned} \lim_{t\to\infty} u(t) &= F \lim_{t\to\infty} x(t) + Gr \\ &\quad + \lim_{t\to\infty} \rho B' P [x(t) - x_\mathrm{e}] \\ &= F x_\mathrm{e} + Gr, \end{aligned} \tag{41}$$

since $\rho(r, y)$ is uniformly bounded. Hence,

$$\begin{aligned} \lim_{t\to\infty} h(t) &= C_2 \lim_{t\to\infty} x(t) + D_2 \,\mathrm{sat}\left[\lim_{t\to\infty} u(t)\right] \\ &= C_2 x_\mathrm{e} + D_2 (F x_\mathrm{e} + Gr) \\ &= (C_2 + D_2 F) x_\mathrm{e} + D_2 G r \\ &= -(C_2 + D_2 F)(A + BF)^{-1} \\ &\quad \times BGr + D_2 G r \\ &= [D_2 - (C_2 + D_2 F)(A + BF)^{-1} B] G r \\ &= G_0 G_0' (G_0 G_0')^{-1} r = r. \end{aligned} \tag{42}$$

This completes the proof of Theorem 2.1. □

Lastly, assuming that the dynamic equation of the given system is transformed into the following form:

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} \mathrm{sat}(u), \tag{43}$$

where $\bar{B}$ is nonsingular, Turner et al. [15] have solved the problem in a rather strange condition, i.e., $A_{11}$ is nonsingular. It was suggested in [15] to add some small perturbations to $A_{11}$ if it is singular. Recently, it has been pointed out by Turner and Postlethwaite [14] for the case when the system is stabilizable and $B$ is of full rank, there exists nonsingular state transformation that would convert the given system to the form of (43) with $A_{11}$ being nonsingular. Nonetheless, it is obvious from our development that such a transformation is totally unnecessary. We further note that our approach to the CNF design is much more elegant compared to that given in [15], and it carries over nicely to the measurement feedback cases in the following section.

## 3. The measurement feedback cases

The assumption that all the state variables of the given system $\Sigma$ are measurable is, in general, not practical. For example, in HDD servo systems (see [3]), the velocity of the actuator is usually hard to be measured. As such, in this section, we proceed to develop CNF design using only measurement information.

### 3.1. Full order measurement feedback case

We first deal with the full order measurement feedback case, in which the dynamical order of the controller is exactly the same as that of the given plant. The following is a step–by–step procedure for the CNF design using full order measurement feedback.

*Step* F1: We first construct a linear full order measurement feedback control law,

$$\begin{aligned} \dot{x}_\mathrm{v} &= (A + K C_1) x_\mathrm{v} - K y + B \,\mathrm{sat}(u_\mathrm{L}), \\ u_\mathrm{L} &= F(x_\mathrm{v} - x_\mathrm{e}) + H r, \end{aligned} \tag{44}$$

where $r$ is the set of step reference signals and $x_\mathrm{v}$ is the state of the controller. As usual, $K$, $F$ are gain matrices and are chosen such that $(A + KC_1)$ and $(A + BF)$ are asymptotically stable and the resulting closed-loop system has desired properties. Finally, $G$, $H$ and $x_\mathrm{e}$ are as defined in (4)–(6).

*Step* F2: Given a positive definite matrix $W_\mathrm{P} \in \mathbb{R}^{n\times n}$, solve the Lyapunov equation

$$(A + BF)' P + P(A + BF) = -W_\mathrm{P} \tag{45}$$

for $P>0$. As in the state feedback case, the linear control law of (44) obtained in the above step is to be combined with a nonlinear control law to form the following CNF controller

$$\dot{x}_{\mathrm{v}}=(A+KC_1)x_{\mathrm{v}}-Ky+B\,\mathrm{sat}(u),$$
$$u=F(x_{\mathrm{v}}-x_{\mathrm{e}})+Hr+\rho(r,y)B'P(x_{\mathrm{v}}-x_{\mathrm{e}}), \quad (46)$$

where $\rho(r,y)$ is as given in (9) with all its diagonal elements being respectively a nonpositive function, locally Lipschitz in $y$, which are to be chosen to improve the performance of the closed-loop system.

It turns out that, for the measurement feedback case, the choice of $\rho_i(r,y)$, $i=1,\ldots,m$, the nonpositive scalar functions, are not totally free. They are subject to certain constraints. We have the following result.

**Theorem 3.1.** *Consider the given system in* (1), *which satisfies the standard assumptions* (i)–(iii), *the full order linear measurement feedback control law of* (44) *and the composite nonlinear measurement feedback control law of* (46). *Given a positive define matrix* $W_{\mathrm{Q}}\in\mathbb{R}^{n\times n}$ *with*

$$W_{\mathrm{Q}}>F'B'PW_{\mathrm{P}}^{-1}PBF, \quad (47)$$

*let* $Q>0$ *be the solution to the Lyapunov equation,*

$$(A+KC_1)'Q+Q(A+KC_1)=-W_{\mathrm{Q}}. \quad (48)$$

*Note that such a* $Q$ *exists as* $A+KC_1$ *is asymptotically stable. For any* $\delta\in(0,1)$, *let* $c_\delta>0$ *be the largest positive scalar such that for all*

$$\begin{pmatrix} x\\ x_{\mathrm{v}}\end{pmatrix}\in \boldsymbol{X}_{\mathrm{F}\delta}:=\left\{\begin{pmatrix} x\\ x_{\mathrm{v}}\end{pmatrix}:\begin{pmatrix} x\\ x_{\mathrm{v}}\end{pmatrix}'\begin{bmatrix} P & 0\\ 0 & Q\end{bmatrix}\begin{pmatrix} x\\ x_{\mathrm{v}}\end{pmatrix}\leqslant c_\delta\right\}, \quad (49)$$

the following property holds:

$$\left|[\,F_i \quad F_i\,]\begin{pmatrix} x\\ x_{\mathrm{v}}\end{pmatrix}\right|\leqslant(1-\delta)\bar{u}_i,\quad i=1,\ldots,m. \quad (50)$$

*Then, the linear measurement feedback control law in* (46) *will drive the system's controlled output* $h(t)$ *to track asymptotically a set of step references, i.e.,* $r$, *from an initial state* $x_0$, *provided that* $x_0$, $x_{\mathrm{v}0}=x_{\mathrm{v}}(0)$ *and* $r$ satisfy

$$\begin{pmatrix} x_0-x_{\mathrm{e}}\\ x_{\mathrm{v}0}-x_0\end{pmatrix}\in \boldsymbol{X}_{\mathrm{F}\delta}\quad \textit{and}\quad |H_i r|\leqslant\delta\bar{u}_i,\quad i=1,\ldots,m. \quad (51)$$

*Furthermore, there exist positive scalars* $\rho_i^*>0$, $i=1,\ldots m$, *such that for any nonpositive functions* $\rho_i(r,y)$, $i=1,\ldots,m$, *locally Lipschitz in* $y$ *and* $|\rho_i(r,y)|\leqslant\rho_i^*$, $i=1,\ldots,m$, *the CNF control law of* (46) *will drive the system controlled output* $h(t)$ *to track asymptotically the reference* $r$ *from an initial* $x_0$, *provided that* $x_0$, $x_{\mathrm{v}0}$ *and* $r$ *satisfy* (51).

**Proof.** For simplicity, we again drop $r$ and $y$ in $\rho(r,y)$ throughout the proof of this theorem. Let $\tilde{x}=x-x_{\mathrm{e}}$ and $\tilde{x}_{\mathrm{v}}=x_{\mathrm{v}}-x$. The linear feedback control law of (44) can be written as

$$\dot{\tilde{x}}_{\mathrm{v}}=(A+KC_1)\tilde{x}_{\mathrm{v}},$$
$$u_{\mathrm{L}}=[\,F \quad F\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}+Hr. \quad (52)$$

Hence, for all

$$\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}\in \boldsymbol{X}_{\mathrm{F}\delta}\Rightarrow\left|[\,F_i \quad F_i\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}\right|\leqslant(1-\delta)\bar{u}_i,\quad i=1,\ldots,m, \quad (53)$$

and for any $r$ satisfying

$$|H_i r|\leqslant\delta\bar{u}_i,\quad i=1,\ldots,m, \quad (54)$$

each channel of $u_{\mathrm{L}}$, say $u_{\mathrm{L},i}$, has the following property

$$u_{\mathrm{L},i}=\left|[\,F_i \quad F_i\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}+H_i r\right|\leqslant\left|[\,F_i \quad F_i\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}\right|+|H_i r|\leqslant\bar{u}_i. \quad (55)$$

Thus, for all $\tilde{x}$ and $\tilde{x}_{\mathrm{v}}$ satisfying the condition as given in (53), the closed-loop system comprising the given plant and the linear control law of (44) can be rewritten as

$$\begin{pmatrix}\dot{\tilde{x}}\\ \dot{\tilde{x}}_{\mathrm{v}}\end{pmatrix}=\begin{bmatrix} A+BF & BF\\ 0 & A+KC_1\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}. \quad (56)$$

Similarly, the closed-loop system with the CNF control law of (46) can be expressed as

$$\begin{pmatrix}\dot{\tilde{x}}\\ \dot{\tilde{x}}_{\mathrm{v}}\end{pmatrix}=\begin{bmatrix} A+BF & BF\\ 0 & A+KC_1\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}+\begin{bmatrix} B\\ 0\end{bmatrix}w, \quad (57)$$

where

$$w = \mathrm{sat}\left[ [\,F \quad F\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} + Hr + \rho [\,B'P \quad B'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} \right] - [\,F \quad F\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} - Hr. \tag{58}$$

Clearly, for $x_0$ and $x_{\mathrm{v}0}$ satisfying (51), we have

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_{\mathrm{v}0} \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta}, \tag{59}$$

where $\tilde{x}_0 = \tilde{x}(0)$ and $\tilde{x}_{\mathrm{v}0} = \tilde{x}_{\mathrm{v}}(0)$. We note that (56) and (57) are identical when $\rho = 0$. Again, the results of Theorem 3.1 for both the linear and the nonlinear feedback cases can be proved in one shot.

Next, we define a Lyapunov function:

$$V = \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix}, \tag{60}$$

and evaluate the derivative of $V$ along the trajectories of the closed-loop system in (57), i.e.,

$$\dot{V} = \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}}' \end{pmatrix} \begin{bmatrix} -W_{\mathrm{P}} & PBF \\ F'B'P & -W_{\mathrm{Q}} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} + 2\tilde{x}'PBw. \tag{61}$$

Note that for all

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} \in \boldsymbol{X}_{\mathrm{F}\delta} \Rightarrow \left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} \right| \leqslant (1-\delta)\bar{u}_i, \quad i = 1, \ldots, m. \tag{62}$$

Again, as done in the full state feedback case, let us find the above derivative of $V$ for four different cases.

*Case* 1: All input channels are unsaturated. For this case, we have

$$\left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} + H_i r + \rho_i [\,B_i'P \quad B_i'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} \right| \leqslant \bar{u}_i, \quad i = 1, \ldots, m, \tag{63}$$

which implies

$$w_i = \rho_i [\,B_i'P \quad B_i'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} \tag{64}$$

and

$$\dot{V} = \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix}' \begin{bmatrix} -W_{\mathrm{P}} & PB(F+\rho B'P) \\ (F+\rho B'P)'B'P & -W_{\mathrm{Q}} \end{bmatrix} \times \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} + 2\tilde{x}'PB\rho B'P\tilde{x} \leqslant \begin{pmatrix} \hat{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix}' \begin{bmatrix} -W_{\mathrm{P}} & 0 \\ 0 & -\tilde{W}_{\mathrm{Q}} \end{bmatrix} \begin{pmatrix} \hat{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix}, \tag{65}$$

where

$$\hat{x} = \tilde{x} - W_{\mathrm{P}}^{-1}PB(F+\rho B'P)\tilde{x}_{\mathrm{v}} \tag{66}$$

and

$$\tilde{W}_{\mathrm{Q}} = W_{\mathrm{Q}} - (F+\rho B'P)'B'PW_{\mathrm{P}}^{-1} \times PB(F+\rho B'P). \tag{67}$$

Noting (47), i.e., $W_{\mathrm{Q}} > F'B'PW_{\mathrm{P}}^{-1}PBF$, and $\rho_i$ is locally Lipschitz, it is clear that there exist positive scalars $\rho_{i,1}^* > 0, i=1, \ldots, m$, such that for any nonpositive scalar function satisfying $|\rho_i| \leqslant \rho_{i,1}^*, i=1, \ldots, m$, we have $\tilde{W}_{\mathrm{Q}} > 0$ and hence $\dot{V} \leqslant 0$.

*Case* 2: All input channels are exceeding their upper limits. In such a situation, we have for all $i=1, \ldots, m$,

$$[\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} + H_i r + \rho_i [\,B_i'P \quad B_i'P\,] \times \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} \geqslant \bar{u}_i. \tag{68}$$

For all the trajectories inside $\boldsymbol{X}_{\mathrm{F}\delta}$,

$$\left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix} + H_i r \right| \leqslant \bar{u}_i, \tag{69}$$

we have for $i = 1, \ldots, m$,

$$0 \leqslant w_i \leqslant \rho_i [\,B_i'P \quad B_i'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix}. \tag{70}$$

Next, let us express

$$w_i = q_i \rho_i [\,B_i'P \quad B_i'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\mathrm{v}} \end{pmatrix}, \tag{71}$$

for some appropriate positive continuous function $q_i(t)$ bounded by 1 for all $t$. In this case, the derivative of $V$ becomes

$$\dot{V} = \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & PB(F+q\rho B'P) \\ (F+q\rho B'P)'B'P & -W_{\rm Q} \end{bmatrix} \times \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + 2\tilde{x}'PBq\rho B'P\tilde{x}$$
$$\leqslant \begin{pmatrix} \hat{x}_{+} \\ \tilde{x}_{\rm v} \end{pmatrix}' \begin{bmatrix} -W_{\rm P} & 0 \\ 0 & -\tilde{W}_{\rm Q+} \end{bmatrix} \begin{pmatrix} \hat{x}_{+} \\ \tilde{x}_{\rm v} \end{pmatrix}, \qquad (72)$$

where

$$q = \mathrm{diag}\{q_1, \ldots, q_m\}, \qquad (73)$$

$$\hat{x}_{+} = \tilde{x} - W_{\rm P}^{-1}PB(F+q\rho B'P)\tilde{x}_{\rm v} \qquad (74)$$

and

$$\tilde{W}_{\rm Q+} = W_{\rm Q} - (F+q\rho B'P)'B'PW_{\rm P}^{-1} \times PB(F+q\rho B'P). \qquad (75)$$

Again, noting (47), i.e., $W_{\rm Q} > F'B'PW_{\rm P}^{-1}PBF$, and $\rho_i$ is locally Lipschitz, it is clear that there exist positive scalars $\rho_{i,2}^* > 0$, $i = 1, \ldots, m$, such that for any scalar function satisfying $|\rho_i| \leqslant \rho_{i,2}^*$, $i = 1, \ldots, m$, we have $\tilde{W}_{\rm Q+} > 0$ and hence $\dot{V} \leqslant 0$.

*Case* 3: All input channels are exceeding their lower limits. In this case, we have for $i = 1, \ldots, m$,

$$[\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + H_i r + \rho_i [\,B_i'P \quad B_i'P\,] \times \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \leqslant -\bar{u}_i. \qquad (76)$$

For all the trajectories inside $\boldsymbol{X}_{{\rm F}\delta}$,

$$\left| [\,F_i \quad F_i\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} + H_i r \right| \leqslant \bar{u}_i, \qquad (77)$$

we have for $i = 1, \ldots, m$,

$$\rho_i [\,B_i'P \quad B_i'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \leqslant w_i \leqslant 0. \qquad (78)$$

Next, let us express

$$w_i = q_i \rho_i [\,B_i'P \quad B_i'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \qquad (79)$$

for some appropriate positive continuous function $q_i(t)$ bounded by 1 for all $t$. Following the similar arguments as in the previous case, we can show that there exist positive scalars $\rho_{i,3}^* > 0$, $i = 1, \ldots, m$, such that for any scalar function satisfying $|\rho_i| \leqslant \rho_{i,3}^*$, $i = 1, \ldots, m$, the corresponding $\dot{V} \leqslant 0$.

*Case* 4: Some control channels are saturated and some are unsaturated. Following the similar arguments as those in Cases 1–3, we can express that for $i = 1, \ldots, m$

$$w_i = q_i \rho_i [\,B_i'P \quad B_i'P\,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \qquad (80)$$

for some appropriate positive continuous function $q_i(t)$ bounded by 1 for all $t$, and show that there exist positive scalars $\rho_{i,4}^* > 0$, $i = 1, \ldots, m$, such that for any scalar function satisfying $|\rho_i| \leqslant \rho_{i,4}^*$, $i = 1, \ldots, m$, the corresponding $\dot{V} \leqslant 0$.

Finally, we let $\rho_i^* = \min\{\rho_{i,1}^*, \rho_{i,2}^*, \rho_{i,3}^*, \rho_{i,4}^*\}$. Then, we have for any nonpositive scalar function $\rho_i$ satisfying $|\rho_i| < \rho_i^*$, $i = 1, \ldots, m$

$$\dot{V} \leqslant 0, \quad \forall \begin{pmatrix} \tilde{x} \\ \tilde{x}_{\rm v} \end{pmatrix} \in \boldsymbol{X}_{{\rm F}\delta}. \qquad (81)$$

Thus, $\boldsymbol{X}_{{\rm F}\delta}$ is an invariant set of the closed-loop system in (57), and all trajectories starting from $\boldsymbol{X}_{{\rm F}\delta}$ will remain inside and asymptotically converge to the origin. This, in turn, indicates that, for the initial state of the given system $x_0$, the initial state of the controller $x_{\rm v0}$, and step command input $r$ that satisfy (51)

$$\lim_{t\to\infty} \tilde{x}_{\rm v}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x(t) = x_{\rm e}, \qquad (82)$$

and then it follows from (42) that the controlled output $h(t)$ converges asymptotically to the target reference $r$. This completes the proof of Theorem 3.1. □

### 3.2. Reduced order measurement feedback case

For the given system in (1), it is clear that there are $p$ state variables of the system, which are measurable if $C_1$ is of maximal rank. Thus, in general, it is not necessary to estimate these measurable state variables in measurement feedback laws. As such, we will proceed in this subsection to design a dynamic controller that has a dynamical order less than that of the given plant. For simplicity of presentation, we assume that $C_1$ is already in the form

$$C_1 = [\,I_p \quad 0\,]. \qquad (83)$$

Then, the system in (1) can be rewritten as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}\mathrm{sat}(u), \quad x_0 = \begin{pmatrix}x_{10}\\ x_{20}\end{pmatrix},$$

$$y = [\, I_p \quad 0\,]\begin{pmatrix}x_1\\ x_2\end{pmatrix},$$

$$h = C_2\begin{pmatrix}x_1\\ x_2\end{pmatrix} + D_2\,\mathrm{sat}(u), \tag{84}$$

where the original state $x$ is partitioned into two parts, $x_1$ and $x_2$ with $y \equiv x_1$. Thus, we will only need to estimate $x_2$ in the reduced order measurement feedback design. Next, we let $F$ be chosen such that (i) $A+BF$ is asymptotically stable, and (ii) $(C_2+D_2F)(sI-A-BF)^{-1}B + D_2$ has desired properties, and let $K_\mathrm{R}$ be chosen such that $A_{22} + K_\mathrm{R}A_{12}$ is asymptotically stable. Here we note that it can be shown that $(A_{22}, A_{12})$ is detectable if and only if $(A, C_1)$ is detectable. Thus, there exists a stabilizing $K_\mathrm{R}$. Again, such $F$ and $K_\mathrm{R}$ can be designed using an appropriate control technique. We then partition $F$ in conformity with $x_1$ and $x_2$:

$$F = [\, \bar{F}_1 \quad \bar{F}_2\,]. \tag{85}$$

We further partition $F_2$ as follows:

$$\bar{F}_2 = \begin{bmatrix}\bar{F}_{2,1}\\ \vdots\\ F_{2,m}\end{bmatrix}. \tag{86}$$

Also, let $G$, $H$ and $x_\mathrm{e}$ be as given in (4)–(6). The reduced order CNF controller is given by

$$\dot{x}_\mathrm{v} = (A_{22} + K_\mathrm{R}A_{12})x_\mathrm{v} + (B_2 + K_\mathrm{R}B_1)\,\mathrm{sat}(u) + [A_{21} + K_\mathrm{R}A_{11} - (A_{22} + K_\mathrm{R}A_{12})K_\mathrm{R}]y \tag{87}$$

and

$$u = F\left[\begin{pmatrix}y\\ x_\mathrm{v} - K_\mathrm{R}y\end{pmatrix} - x_\mathrm{e}\right] + Hr + \rho(r,y)B'P \times \left[\begin{pmatrix}y\\ x_\mathrm{v} - K_\mathrm{R}y\end{pmatrix} - x_\mathrm{e}\right], \tag{88}$$

where $\rho(r, y)$ is as given in (9).

Next, given a positive definite matrix $W_\mathrm{P} \in \mathbb{R}^{n\times n}$, let $P > 0$ be the solution to the Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W_\mathrm{P}. \tag{89}$$

Given another positive definite matrix $W_\mathrm{R} \in \mathbb{R}^{(n-p)\times(n-p)}$ with

$$W_\mathrm{R} > \bar{F}_2'B'PW_\mathrm{P}^{-1}PB\bar{F}_2, \tag{90}$$

let $Q_\mathrm{R} > 0$ be the solution to the Lyapunov equation

$$(A_{22} + K_\mathrm{R}A_{12})'Q_\mathrm{R} + Q_\mathrm{R}(A_{22} + K_\mathrm{R}A_{12}) = -W_\mathrm{R}. \tag{91}$$

Note that such $P$ and $Q_\mathrm{R}$ exist as $A + BF$ and $A_{22} + K_\mathrm{R}A_{12}$ are asymptotically stable. For any $\delta \in (0, 1)$, let $c_\delta$ be the largest positive scalar such that for all

$$\begin{pmatrix}x\\ x_\mathrm{v}\end{pmatrix} \in \boldsymbol{X}_{\mathrm{R}\delta} := \left\{\begin{pmatrix}x\\ x_\mathrm{v}\end{pmatrix} : \begin{pmatrix}x\\ x_\mathrm{v}\end{pmatrix}'\begin{bmatrix}P & 0\\ 0 & Q_\mathrm{R}\end{bmatrix}\begin{pmatrix}x\\ x_\mathrm{v}\end{pmatrix} \leqslant c_\delta\right\}, \tag{92}$$

the following property holds:

$$\left|[\, F_i \quad F_{2,i}\,]\begin{pmatrix}x\\ x_\mathrm{v}\end{pmatrix}\right| \leqslant \bar{u}_i(1-\delta), \quad i = 1,\ldots,m. \tag{93}$$

We have the following theorem.

**Theorem 3.2.** *Consider the given system in* (1), *which satisfies the usual assumptions* (i)–(iii). *Then, there exist positive scalars* $\rho_i^* > 0$, $i = 1,\ldots,m$, *such that for any nonpositive function* $\rho_i(r, y)$, $i = 1,\ldots,m$, *locally Lipschitz in* $y_i$ *and* $|\rho_i(r, y)| \leqslant \rho_i^*$, *the reduced order CNF law given by* (87) *and* (88) *will drive the system controlled output* $h(t)$ *to asymptotically track the reference* $r$ *from an initial state* $x_0$, *provided that* $x_0$, $x_{\mathrm{v}0}$ *and* $r$ *satisfy*

$$\begin{pmatrix}x_0 - x_\mathrm{e}\\ x_{\mathrm{v}0} - x_{20} - K_\mathrm{R}x_{10}\end{pmatrix} \in \boldsymbol{X}_{\mathrm{R}\delta},$$
$$|H_i r| \leqslant \delta\bar{u}_i, \quad i = 1,\ldots,m. \tag{94}$$

**Proof.** Let $\tilde{x} = x - x_\mathrm{e}$ and $\tilde{x}_\mathrm{v} = x_\mathrm{v} - x_2 - K_\mathrm{R}x_1$. Then, the closed-loop system comprising the given plant in (1) and the reduced order CNF control law of (87) and (88) can be expressed as

$$\begin{pmatrix}\dot{\tilde{x}}\\ \dot{\tilde{x}}_\mathrm{v}\end{pmatrix} = \begin{bmatrix}A + BF & B\bar{F}_2\\ 0 & A_{22} + K_\mathrm{R}A_{12}\end{bmatrix} \times \begin{pmatrix}\tilde{x}\\ \tilde{x}_\mathrm{v}\end{pmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}w, \tag{95}$$

where

$$w = \mathrm{sat}\left\{[\,F \quad \bar{F}_2\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix} + Hr + \rho(r,y)B'P \times \left[\tilde{x} + \begin{pmatrix}0\\ \tilde{x}_{\mathrm{v}}\end{pmatrix}\right]\right\} - [\,F \quad \bar{F}_2\,]\begin{pmatrix}\tilde{x}\\ \tilde{x}_{\mathrm{v}}\end{pmatrix} - Hr. \tag{96}$$

The rest of the proof follows along similar lines to the reasoning given in the full order measurement feedback case. □

## 4. Selecting design parameters $\rho(r, y)$ and $W$

The freedom to choose the function $\rho(r, y)$ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output $h$ approaches the set point. Since the main purpose of adding the nonlinear part to the CNF controllers is to speed up the settling time, or equivalently to contribute a significant value to the control input when the tracking error, $r - h$, is small. In general, we choose the nonlinear part to be in action when the control signal is far away from its saturation level, and thus it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the given plant in (1) and the three different types of control law can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + B\rho(r, y)B'P\tilde{x}. \tag{97}$$

We note that the additional term $\rho(r, y)$ does not affect the stability of the estimators. It is now clear that eigenvalues of the closed-loop system in (97) can be changed by the function $\rho(r, y)$. There are different types of nonlinear gains that have been suggested in the literature (see e.g., [3,9,15]). Assuming that $h$ is available, we follow the work of [3] to propose the following nonlinear gains:

$$\rho_i(r, h) = -\beta_i|\mathrm{e}^{-\alpha_i\|h(t)-r\|} - \mathrm{e}^{-\alpha_i\|h(0)-r\|}|, \quad i = 1, \ldots, m \tag{98}$$

which starts from 0 and gradually increases to a final gain of $-\beta_i|1 - \mathrm{e}^{-\alpha_i\|h(0)-r\|}|$ as $h$ approaches to the target reference $r$. $\alpha_i$ is used to determine the speed of change in $\rho_i$. Thus, one could properly select scalar gains $\alpha_i$ and $\beta_i$, $i = 1, \ldots, m$, to yield a desired performance.

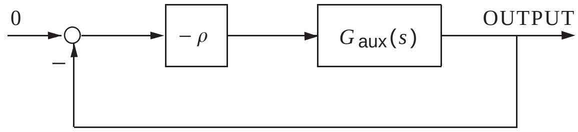


Fig. 1. Interpretation of the nonlinear function $\rho(r, y)$.

To examine the behavior of the closed-loop system (97) more explicitly, we define an auxiliary system $G_{\mathrm{aux}}(s)$ characterized by

$$\begin{aligned} G_{\mathrm{aux}}(s) &:= C_{\mathrm{aux}}(sI - A_{\mathrm{aux}})^{-1}B_{\mathrm{aux}} \\ &:= B'P(sI - A - BF)^{-1}B. \end{aligned} \tag{99}$$

Obviously, $G_{\mathrm{aux}}(s)$ is stable. The closed-loop system (97) can then be cast under the framework of the multivariable root locus theory as shown in Fig. 1 (we hereafter drop the dependent variables of $\rho$ for simplicity). We note that

$$C_{\mathrm{aux}}B_{\mathrm{aux}} = B'PB > 0 \tag{100}$$

which implies $G_{\mathrm{aux}}(s)$ is a square, invertible and uniform rank system with $m$ infinite zeros of order 1 and with $n - m$ invariant zeros. Noting that

$$\det(sI - A_{\mathrm{aux}} - B_{\mathrm{aux}}\cdot\rho\cdot C_{\mathrm{aux}}) = \det(\rho)\cdot\det\begin{bmatrix} sI - A_{\mathrm{aux}} & B_{\mathrm{aux}} \\ C_{\mathrm{aux}} & \rho^{-1}\end{bmatrix}, \tag{101}$$

it is clear that for any eigenvalue of the closed-loop system (97), i.e., $s \in \lambda(A + BF + B\rho B'P)$,

$$\det\begin{bmatrix} sI - A_{\mathrm{aux}} & B_{\mathrm{aux}} \\ C_{\mathrm{aux}} & \rho^{-1}\end{bmatrix} = 0. \tag{102}$$

Thus, when all diagonal elements of $\rho$, i.e., $\rho_i$, $i = 1, 2, \ldots, m$, approach to $-\infty$, the closed-loop eigenvalues of (97) approach to the zeros of $G_{\mathrm{aux}}(s)$ including the invariant zeros of $(A_{\mathrm{aux}}, B_{\mathrm{aux}}, C_{\mathrm{aux}})$ and those at infinity. Since it was shown that the closed-loop system remains stable for any $\rho$ whose diagonal elements are nonpositive, the invariant zeros of $G_{\mathrm{aux}}(s)$ has to be stable. Hence, $G_{\mathrm{aux}}(s)$ is of minimum phase.

It should be noted that there is freedom in preselecting the locations of these invariant zeros by selecting an appropriate $W$ in (7). In general, we should select the invariant zeros of $G_{\mathrm{aux}}(s)$, which are corresponding to the closed-loop poles of (97) for large $|\rho|$, such that the dominated ones have a large damping ratio, which in turn will generally yield a smaller overshoot. The following procedure for selecting an appropriate $W$ is adopted from that reported in [3]:

1. Given the pair $(A_{\rm aux}, B_{\rm aux})$ and the desired locations of the invariant zeros of $G_{\rm aux}(s)$, we follow the result reported in Chapter 9 of [4] on finite and infinite zero assignment to obtain an appropriate matrix $C_{\rm aux}$ such that the resulting $(A_{\rm aux}, B_{\rm aux}, C_{\rm aux})$ has the desired relative degree and invariant zeros.
2. Solve $C_{\rm aux} = B'P$ for a $P = P' > 0$. In general, the solution is nonunique as there are $n(n+1)/2$ elements in $P$ available for selection. However, if the solution does not exist, we go back to the previous step to re-select the invariant zeros.
3. Calculate $W$ using (7) and check if $W$ is positive definite. If $W$ is not positive definite, we go back to the previous step to choose another solution of $P$ or go to the first step to re-select the invariant zeros.

Another method for selecting $W$ is based on a trial and error approach by limiting the choice of $W$ to a diagonal matrix and adjusting its diagonal weights through simulation. The software package for realizing the CNF design reported in [5] was implemented based on such an approach. Generally, it will also yield a satisfactory result. We will illustrate such a design approach in the numerical example in the following section.

## 5. An illustrative example

We consider a two-input and two-output system characterized by (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -2 & -2 & -2 & -1 & -2 \\ 1 & 2 & 2 & 2 & 2 & 3 \\ -1 & -2 & -2 & -2 & -2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad x_0 = \begin{pmatrix} -0.6 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{pmatrix} \tag{103}$$

and

$$C_1 = C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{104}$$

The maximum amplitudes of both control channels are given by $\bar{u}_1 = \bar{u}_2 = 1$. The target references are

$$r = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{105}$$

Our aim is to design appropriate CNF controllers with full state feedback, full order measurement feedback and reduced order measurement feedback, which would control the controlled output of the system to track the command reference as fast as possible and as smoothly as possible. Following the procedures given in the previous sections and with appropriate selections of design parameters, we have obtained the following CNF control laws. We note that the state feedback gain $F$ is carried out by carefully examining the structural properties of the given system using the techniques reported in [4] whereas the full order and reduced order observer gain matrices are computed using the $H_2$ optimization technique given in [11].

1. Full state CNF controller:

$$u = Fx + Gr + \rho(r, y) F_{\rm n}(x - x_{\rm e}), \tag{106}$$

where

$$F = \begin{bmatrix} -1 & -1 & -3 & -2 & 2 & 2 \\ 1 & 2 & 2 & 0 & -1 & -3 \end{bmatrix},$$
$$G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The gain matrix $F_{\rm n}$ is given by

$$F_{\rm n} = B'P = \begin{bmatrix} 0.25 & 3.75 & 4.75 & 2.50 & 0.25 & -1.75 \\ -1.75 & -3.75 & -2.75 & 0.25 & 9.00 & 10.75 \end{bmatrix},$$

where $P$ is the solution of the Lyapunov equation (7) with $W = I$. Finally,

$$x_{\rm e} = [2 \quad -1 \quad 1 \quad -1 \quad 0 \quad 0]'$$

and

$$\rho(r, y) = \mathrm{diag}\{\rho_1(r, h), \;\; \rho_2(r, h)\}, \tag{107}$$

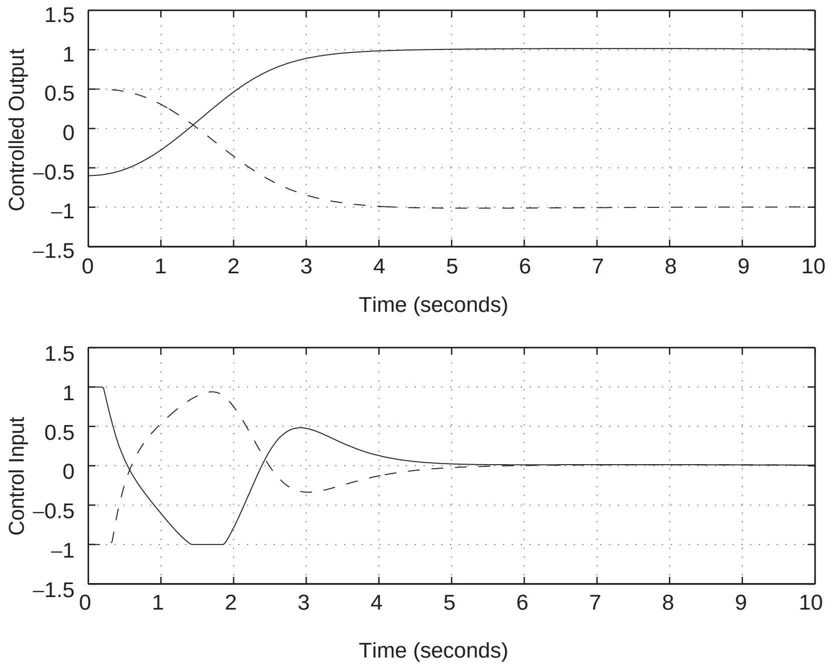


Fig. 2. Simulation result for the full state CNF case.

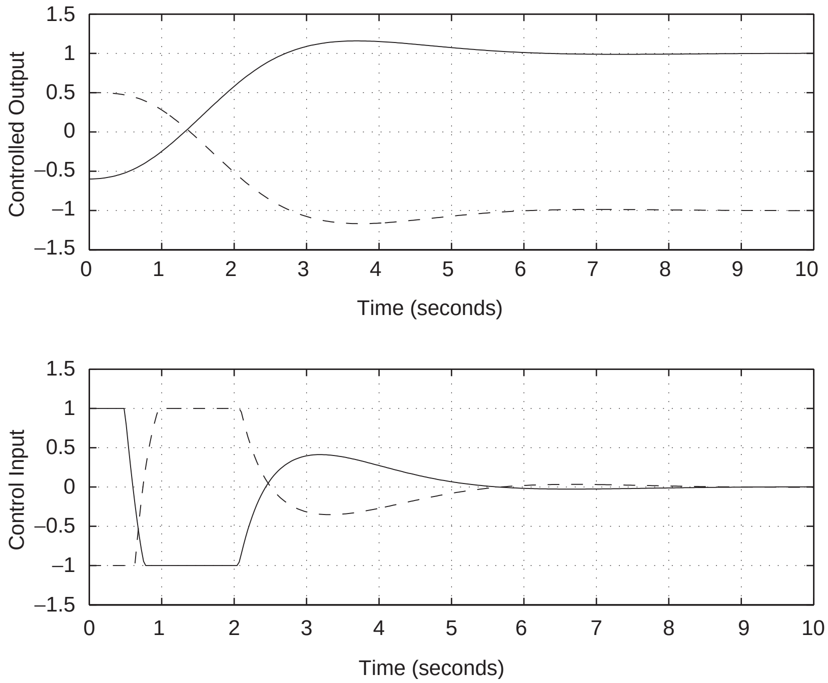


Fig. 3. Simulation result for the full state $H_2$ linear feedback case.

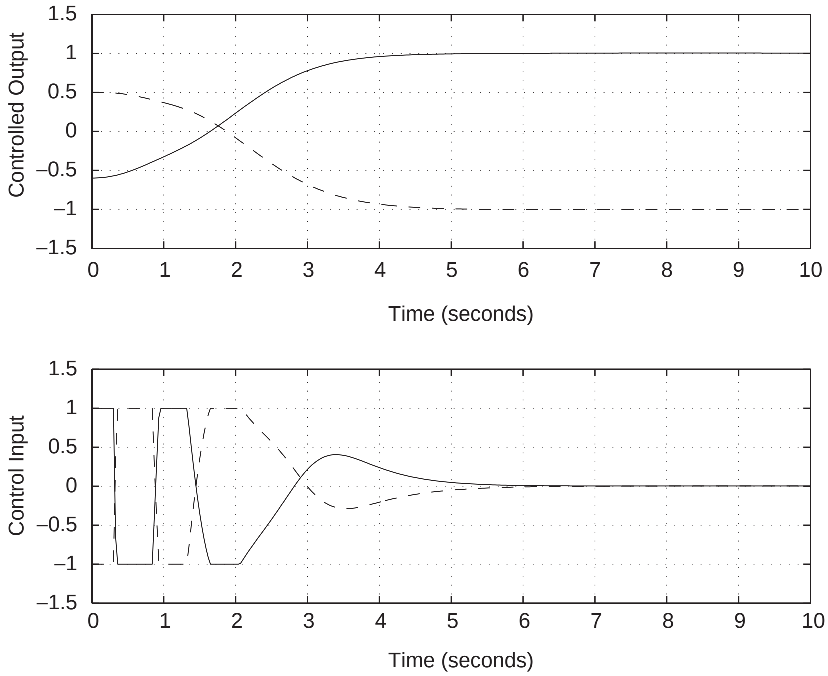


Fig. 4. Simulation result for the full order measurement CNF case.

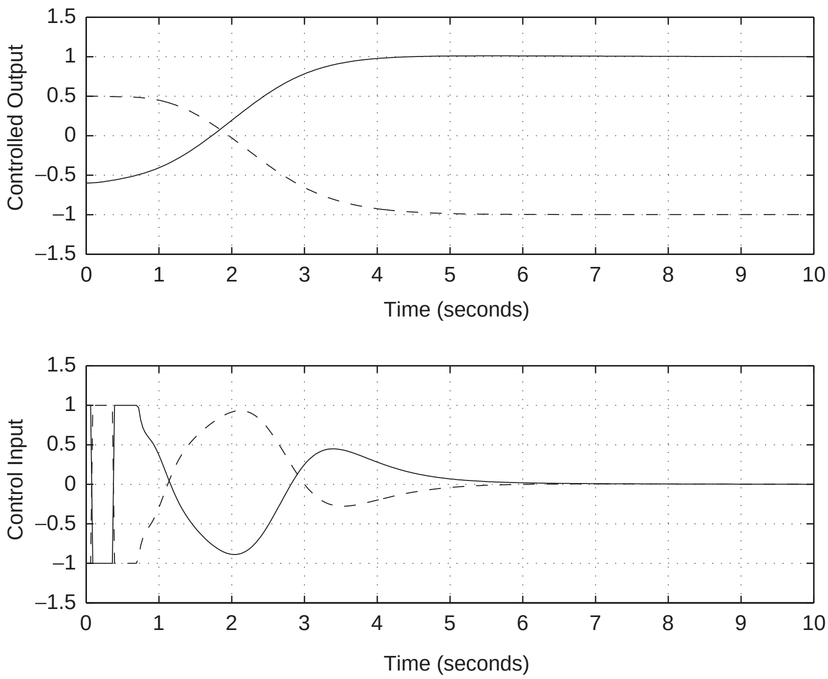


Fig. 5. Simulation result for the reduced order measurement CNF case.

where

$$\rho_1(r,h) = -2.8|\mathrm{e}^{-\|h(t)-r\|} - \mathrm{e}^{-\|h(0)-r\|}| \quad (108)$$

and

$$\rho_2(r,h) = -1.7|\mathrm{e}^{-\|h(t)-r\|} - \mathrm{e}^{-\|h(0)-r\|}|. \quad (109)$$

2. Full order measurement CNF controller:

$$\dot{x}_\mathrm{v} = (A + KC_1)x_\mathrm{v} - Ky + B\,\mathrm{sat}(u),$$
$$u = F(x_\mathrm{v} - x_\mathrm{e}) + Hr + \rho(r,y)F_\mathrm{n}(x_\mathrm{v} - x_\mathrm{e}), \quad (110)$$

where $F$, $F_\mathrm{n}$, $x_\mathrm{e}$ are as given in the state feedback case,

$$K = \begin{bmatrix} 65.9921 & -65.9537 \\ -57.5639 & 64.9515 \\ 92.5836 & -73.3967 \\ -27.4805 & 38.5006 \\ 26.4782 & -30.0729 \\ -34.9271 & 65.0887 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and $\rho(r,y)$ is slightly adjusted from that of (107) with $\rho_1(r,h)$ being modified as

$$\rho_1(r,h) = -2.5|\mathrm{e}^{-\|h(t)-r\|} - \mathrm{e}^{-\|h(0)-r\|}|. \quad (111)$$

3. Reduced order measurement CNF controller:

$$\dot{x}_\mathrm{v} = A_\mathrm{cmp}x_\mathrm{v} + K_\mathrm{cmp}y + B_\mathrm{cmp}\,\mathrm{sat}(u) \quad (112)$$

and

$$u = F\left[\begin{pmatrix} y \\ x_\mathrm{v} - K_\mathrm{R}y \end{pmatrix} - x_\mathrm{e}\right] + Hr + \rho(r,y)F_\mathrm{n}\left[\begin{pmatrix} y \\ x_\mathrm{v} - K_\mathrm{R}y \end{pmatrix} - x_\mathrm{e}\right], \quad (113)$$

where

$$A_\mathrm{cmp} = \begin{bmatrix} -99.0046 & 6.1086 & 92.8960 & 87.7874 \\ -76.6569 & 30.7676 & 43.8893 & 10.1217 \\ -96.9364 & -31.7442 & 128.6806 & 163.4248 \\ 11.1539 & 37.3526 & -49.5065 & -88.8591 \end{bmatrix},$$

$$B_\mathrm{cmp} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & -2 \\ 0 & 1 \end{bmatrix},$$

$$K_\mathrm{cmp} = 10^3 \times \begin{bmatrix} -1.4088 & 1.3335 \\ -1.1589 & 0.1610 \\ -1.4815 & 2.4787 \\ 0.1749 & -1.2475 \end{bmatrix},$$

$$K_\mathrm{R} = \begin{bmatrix} 99.0046 & -87.7874 \\ 74.1569 & -12.6217 \\ 98.4364 & -160.9248 \\ -13.1539 & 86.8591 \end{bmatrix}$$

and $F$, $H$, $x_\mathrm{e}$, $F_\mathrm{n}$ are the same as those given in the previous two cases whereas $\rho(r,y)$ is identical to that given in the full order measurement feedback case.

Using SIMULINK in MATLAB, we obtain a set of simulation results for the system with the CNF controllers in Figs. 2, 4 and 5. The initial conditions for the dynamics of both full and reduced order controllers are set to zero. The results are very satisfactory for all three cases. Note that the settling times for the full order and reduced order measurement feedback cases are slightly longer compared to those of the full state feedback case. For comparison, we include in Fig. 3 the simulation result of a carefully tuned state feedback linear control law using an $H_2$ optimization approach. Obviously, the CNF controller has a better performance compared to that of a best tuned linear controller.

## 6. Conclusions

We have extended the newly developed composite nonlinear feedback (CNF) tracking control technique to general multivariable linear systems with input saturations. The problem is solved for both the state feedback case and the measurement feedback case. The CNF control law consists of two parts, a linear component and a nonlinear component. The former is usually chosen to give fast rise time while the latter is added to smooth out the transient peaks or overshoots when the controlled output is approaching the target reference. We note that the hardest part in designing a CNF controller is perhaps the selection of the parameters, $\alpha_i$ and $\beta_i$, in the nonlinear function. However, with the software realization of the design method, such a problem can be easily overcome. Interested readers might contact us for a beta version of a toolkit implemented in MATLAB for the CNF design.

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