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Solvability conditions and solutions to perfect regulation problem under measurement output feedback

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Abstract

The problem of perfect regulation is to design a family of control laws for a given plant such that the resulting overall closed-loop system is internally stable and its controlled output can be reduced to zero arbitrarily fast from any initial condition. Such a problem was heavily studied by many researchers in the 1970s and early 1980s. However, to the best of our knowledge, all of the earlier results deal only with the problem under full state feedback. In this paper, we solve the long-standing problem of perfect regulation via measurement output feedback for general linear time-invariant multivariable systems. In particular, we derive necessary and sufficient conditions under which the problem of perfect regulation via measurement output feedback is solvable for general systems, and, under these conditions, construct two families of feedback laws, one of full order and the other reduced order, that solve the problem.

1. Introduction to the problem

The problem of perfect regulation is to design a family of control laws for a given plant such that the resulting overall closed-loop system is internally stable and its controlled output can be reduced to zero arbitrarily fast from any initial condition. Such a problem and its related topics were heavily investigated by many researchers in the 1970s and early 1980s. Kwakernaak and Sivan [7] derived a set of necessary and sufficient conditions for the solvability of the problem for square-invertible systems. Kimura [6] did a complete study of this problem under a crucial assumption, i.e., the limits of the closed-loop system eigenvalues should remain in the open left-half complex plane, which is equivalent to assuming that the given plant does not have invariant zeros on the imaginary axis. The problem investigated by Francis [5] was somewhat different. His result was concerning the set of initial conditions that can be reduced to zero arbitrarily fast. The problem of perfect regulation for non-strictly proper systems was examined by Scherzinger and Davison [12]. The necessary and sufficient conditions of [12] was also given under the assumption that the plant is free of the imaginary invariant zeros. The result of Lin et al. [9] shows that the problem is still solvable when the given plant has invariant zeros on the imaginary axis. In our opinion, the work of Lin et al. [9] is the most complete one up-to-date. Nevertheless, all the above mentioned results only dealt with the case when

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all the state variables of a given plant are available for feedback, which is usually not the case in practical situations. The problem of perfect regulation via measurement output feedback for linear multivariable systems still remains unsolved. Our objective here is to tackle this long-standing open problem.

We consider a linear time-invariant multivariable system Σ characterized by the following state-space equations:

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= C_1 x + D_1 u, \\ h &= C_2 x + D_2 u, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^q$ is the measurement output, and $h \in \mathbb{R}^p$ is the output to be regulated. A, B, C_1, D_1, C_2 and D_2 are constant matrices of appropriate dimensions. The problem of perfect regulation via measurement output feedback for the given plant (1) is to design a family of parameterized dynamic measurement output feedback control laws Σ_{cmp},

$$\begin{aligned} \dot{v} &= A_{\mathrm{cmp}}(\varepsilon) v + B_{\mathrm{cmp}}(\varepsilon) y, \\ u &= C_{\mathrm{cmp}}(\varepsilon) v + D_{\mathrm{cmp}}(\varepsilon) y, \end{aligned} \tag{2}$$

under which the closed-loop system has the following properties:

1. There exists a positive scalar $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, the resulting closed-loop system comprising the given plant Σ and the controller Σ_{cmp} is internally stable.
2. For any given initial condition x_0, the output to be regulated, i.e., $h(t)$, in the closed-loop system, which is clearly a function of ε, satisfies

$$\|h\|_2 = \int_0^\infty h'(t) h(t) \, \mathrm{d}t \to 0, \quad \text{as } \varepsilon \to 0. \tag{3}$$

We note that the second property basically means that the family of control laws is capable of regulating $h(t)$ to zero arbitrarily fast. The paper aims (i) to derive a set of necessary and sufficient conditions under which the above proposed problem is solvable, and (ii) under these conditions, to construct families of feedback laws that solve the problem.

Next, we note that it is without loss of generality to assume that $D_1 = 0$. This can be justified using the following arguments. If $D_1 \neq 0$, then one can define a new measurement output

$$y_{\mathrm{new}} := y - D_1 u = C_1 x, \tag{4}$$

which has a zero direct feedthrough term from u. Suppose we design a control law using this new measurement output, say

$$u = \mathscr{K}(s) y_{\mathrm{new}}. \tag{5}$$

Then, it can be converted to the following:

$$u = \mathscr{K}(s)(y - D_1 u) \quad \text{or} \quad u = [I + \mathscr{K}(s) D_1]^{-1} \mathscr{K}(s) y. \tag{6}$$

This shows that the control problem using y and y_{new} are equivalent, because they can be converted from one to the other. Thus, we will assume throughout the rest of this paper that matrix $D_1 = 0$. For simplicity of presentation, we also assume that matrices C_1, $[C_2 \ D_2]$ and $[B' \ D_2']$ are of full rank.

The outline of this paper is as follows: Section 2 presents the necessary and sufficient conditions under which the proposed perfect regulation problem is solvable. Numerical algorithms based on an eigenstructure assignment technique are given in Section 3 to construct solutions to the problem under the obtained solvability conditions. Two controller structures are considered: one has full order dynamics, i.e., the order of the controller is equal to n, and the other has reduced order dynamics, i.e., the order of the controller is $< n$. Finally, the concluding remarks are made in Section 4, where we also give a set of necessary and sufficient conditions for the solvability of the perfect regulation problem for plants with external disturbances.

Throughout this paper, the following notation will be used: X' denotes the transpose of matrix X; I denotes an identity matrix while I_k denotes an identity matrix of dimension $k \times k$; $\|h\|_2$ denotes the l_2-norm of a time-domain signal vector $h(t)$, while $\|H\|_2$ denotes the H_2-norm of a transfer matrix $H(s)$; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$ is the set of all complex numbers; $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$ are respectively, the left-half complex plane, the imaginary axis and the right-half complex plane; $\mathrm{Ker}\,(X)$ is the kernel of X; $\mathrm{Im}\,(X)$ is the image of X; and finally $\lambda(X)$ is the set of eigenvalues of a real square matrix X. We also introduce the following geometric subspaces:

Definition 1.1 (*Geometric Subspaces $\mathscr{V}^*$ and $\mathscr{S}^*$*)**.** The weakly unobservable subspaces of Σ, $\mathscr{V}^*$, and the strongly controllable subspaces of Σ, $\mathscr{S}^*$, are defined as follows:

1. $\mathscr{V}^*(\Sigma)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A+BF)$-invariant and is contained in $\mathrm{Ker}(C+DF)$ such that the eigenvalues of $(A+BF)|\mathscr{V}^*$ are contained in $\mathbb{C}$ for some constant matrix F.
2. $\mathscr{S}^*(\Sigma)$ is the minimal $(A+KC)$-invariant subspace of $\mathbb{R}^n$ containing $\mathrm{Im}(B+KD)$ such that the eigenvalues of the map which is induced by $(A+KC)$ on the factor space $\mathbb{R}^n/\mathscr{S}^*$ are contained in $\mathbb{C}$ for some constant matrix K.

2. Solvability conditions for general perfect regulation problem

We are now ready to present the first result of this paper, i.e., the necessary and sufficient conditions under which the problem of perfect regulation via measurement output feedback for general linear multivariable systems is solvable. We have the following theorem.

Theorem 2.1. *Consider the given system Σ of* (1). *The problem of perfect regulation via measurement output feedback for Σ is solvable if and only if the following conditions are satisfied*:

1. (A,B) *is stabilizable and* (A,C_1) *is detectable*;
2. (A,B,C_2,D_2) *is right invertible and is free of invariant zeros in* $\mathbb{C}^+$; *and*
3. $\mathrm{Ker}(C_2) \supseteq \mathrm{Ker}(C_1)$.

Proof. Since the property (3) is required of all initial conditions of Σ, it is simple to verify that the problem of perfect regulation problem via measurement output feedback for Σ is equivalent to an H_2 optimal control problem for the following auxiliary system;

$$\begin{aligned} \dot{x} &= Ax + Bu + Iw,\ x(0) = 0, \\ y &= C_1 x, \\ h &= C_2 x + D_2 u, \end{aligned} \tag{7}$$

where $w = \delta(t)$, a vector of impulse functions, for which the best achievable value of the H_2-norm of the closed-loop transfer matrix from w to h under measurement output feedback, say γ^*, is equal to zero. Following the result Saberi et al. [10, Theorem 5.5.1], one can show that the infimum γ^* of the H_2 optimal control problem for the above auxiliary system can be expressed as

$$\gamma^* = \{\mathrm{trace}(P) + \mathrm{trace}[(A'P + PA + C_2'C_2)Q]\}^{1/2}, \tag{8}$$

where P and Q are positive-semi-definite matrices and are, respectively, the so-called semi-stabilizing solutions to the following linear matrix inequalities:

$$\begin{bmatrix} A'P + PA + C_2'C_2 & PB + C_2'D_2 \\ B'P + D_2'C_2 & D_2'D_2 \end{bmatrix} \geqslant 0, \tag{9}$$

and

$$\begin{bmatrix} AQ + QA' + I & QC_1' \\ C_1 Q & 0 \end{bmatrix} \geqslant 0. \tag{10}$$

It follows from (8) that $\gamma^* = 0$, or equivalently to say that the proposed perfect regulation problem for Σ is solvable, if and only if

$$P = 0 \quad \text{and} \quad (A'P + PA + C_2'C_2)Q = C_2'C_2 Q = 0. \tag{11}$$

It then follows from the result of Saberi et al. [10] that the following two statements are equivalent:

1. $P = 0$ is a semi-stabilizing solution to the linear matrix inequality (9).
2. The pair (A, B) is stabilizable and the quadruple (A, B, C_2, D_2) is right invertible and is free of invariant zeros in $\mathbb{C}^+$.

Thus, our remaining task is to show that $C_2'C_2 Q = 0$ is equivalent to that (A, C_1) is detectable and

$$\mathrm{Ker}(C_2) \supseteq \mathrm{Ker}(C_1). \tag{12}$$

We note that the detectability of (A, C_1) is a necessary condition for the linear-matrix inequality (10) to have a semi-stabilizing solution Q. $C_2'C_2Q=0$ implies and is implied by that $C_2Q=0$. Let Σ_Q denote the quadruple $(A, I, C_1, 0)$. Following the result of [6, see Remark 5.4.3] and using the fact that the control matrix of Σ_Q is an identity matrix, which implies that Σ_Q is right invertible and has no finite zero structure, we have

$$\mathrm{Im}(Q) = \mathscr{V}^*(\Sigma_Q) \cap \mathscr{S}^*(\Sigma_Q). \tag{13}$$

It is simple to verify that for such a system Σ_Q, $\mathscr{S}^*(\Sigma_Q) = \mathbb{R}^n$. Hence,

$$\mathrm{Im}(Q) = \mathscr{V}^*(\Sigma_Q). \tag{14}$$

Again, because the control matrix of Σ_Q is an identity matrix, it follows from the definition of $\mathscr{V}^*$ that $\mathscr{V}^*(\Sigma_Q) = \mathrm{Ker}(C_1)$, which can be easily verified by choosing $F = -A$ in Definition 1.1. Thus, condition $C_2 Q = 0$ is equivalent to

$$C_2 \cdot \mathrm{Ker}(C_1) = 0 \quad \text{or} \quad \mathrm{Ker}(C_2) \supseteq \mathrm{Ker}(C_1). \tag{15}$$

This concludes the proof of Theorem 2.1. □

The following remarks are in order.

Remark 2.1. It is interesting to note that neither the two subsystems associated with the measurement output y, i.e., the quadruples (A, B, C_1, D_1) and $(A, I, C_1, 0)$, is required to be left invertible and/or minimum phase for the solvability of the problem of perfect regulation via measurement output feedback, as one would expect from the separation principle arguments. In fact, (A, B, C_1, D_1) can be non-invertible and/or of non-minimum phase.

Remark 2.2. For the state feedback case, i.e, $y = x$, or $C_1 = I$ and $D_1 = 0$, then $\mathrm{Ker}(C_1) = \{0\}$ and the third condition of Theorem 2.1 is automatically satisfied. The set of conditions in Theorem 2.1 is reduced to the following:

1. (A, B) is stabilizable; and
2. (A, B, C_2, D_2) is right invertible and is free of invariant zeros in $\mathbb{C}^+$.

The above result coincides with that of Lin et al. [9].

Remark 2.3. For the case when $h = y$ or h is contained in y, i.e., $h = My$ for some constant matrix M of appropriate dimensions, then the third condition of Theorem 2.1 is also automatically satisfied. The solvability conditions for the problem of perfect regulation via measurement output feedback for this class of systems are identical to those of the state feedback case. However, the condition that h is contained in y, is not necessary for the solvability of the proposed problem. We will illustrate this in a numerical example in the next section.

3. Solutions to general perfect regulation problem

We present in this section two algorithms that construct solutions to the problem of perfect regulation via measurement output feedback for system Σ of (1) provided that the solvability conditions in Theorem 2.1 satisfied. The first algorithm is to construct full-order solutions whose dynamical order is the same as that of Σ, while the second one is to construct reduced-order solutions whose dynamical order is less than that of Σ. Both of these algorithms involve finding a parameterized state feedback gain matrix $F(\varepsilon)$ such that $A+BF(\varepsilon)$ is asymptotically stable for sufficiently small ε, and

$$\|[C_2 + D_2F(\varepsilon)][sI - A - BF(\varepsilon)]^{-1}\|_2 \to 0 \quad \text{as } \varepsilon \to 0. \tag{16}$$

The above gain matrix can be obtained using the result of Lin et al. [9] (see [8] for a simplified procedure that results in a simpler gain matrix). Interestingly, it turns out that the observer gain design for our general perfect regulation problem is not dual to that of the state feedback gain, because the subsystem Σ_Q, i.e., the quadruple $(A, I, C_1, 0)$, is always right invertible and hence its dual system, i.e., $(A', C_1', I, 0)$, is always left invertible. The results of Lin et al. [9,8], which are applicable only to right invertible systems, cannot be applied to such this dual system.

3.1. Design of full-order solutions

The following step-by-step algorithm deals with the full-order solutions to the problem of perfect regulation via measurement output feedback.

Step F.1. Follow the result of Lin [8] to obtain a parameterized gain matrix $F(\varepsilon)$ which has the property as in (16).

Step F.2. Transform Σ_Q in to the special coordinate basis of Sannuti and Saberi [11] (see also Chen [3] for the detailed proofs of its properties). Since the control matrix of Σ_Q is an identity matrix, it can be shown that there exist nonsingular transformations Γ_s, Γ_o, Γ_s such that

$$\Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{cc} & L_{cd} \\ E_c & A_{dd} \end{bmatrix}, \quad \Gamma_s^{-1}\Gamma_i = \begin{bmatrix} 0 & I_{n-q} \\ I_q & 0 \end{bmatrix}, \quad \Gamma_o^{-1} C_1 \Gamma_s = [0 \ I_q]. \tag{17}$$

It can be verified that (A, C_1) is detectable if and only if (A_{cc}, E_c) is detectable.

Step F.3. Let K_c be a constant matrix of dimension $(n-q)\times q$ such that the eigenvalues of the matrix $A_{cc} - K_cE_c$ are in $\mathbb{C}^-$. This can always be done because of the fact that (A_{cc}, E_c) is detectable provided that (A, C_1) is detectable.

Step F.4. Next, we form a parameterized observer gain matrix,

$$K(\varepsilon) = -\Gamma_s \begin{bmatrix} L_{cd} + K_c/\varepsilon \\ A_{dd} + I_q/\varepsilon \end{bmatrix} \Gamma_o^{-1}. \tag{18}$$

Step F.5. Finally, the family of full-order measurement feedback parameterized control laws is given by (2) with

$$\begin{aligned} A_{\text{cmp}}(\varepsilon) &= A + BF(\varepsilon) + K(\varepsilon)C_1, \\ B_{\text{cmp}}(\varepsilon) &= -K(\varepsilon), \\ C_{\text{cmp}}(\varepsilon) &= F(\varepsilon), \\ D_{\text{cmp}}(\varepsilon) &= 0. \end{aligned} \tag{19}$$

This concludes the constructive algorithm of the full-order solution to the general perfect regulation problem. We have the following theorem.

Theorem 3.1. *Consider the given plant Σ of* (1) *and assume that the solvability conditions in Theorem* 2.1 *are satisfied. Then, the family of full-order measurement feedback control laws of the form as in* (2) *with its gain matrices as given in* (19) *solves the problem of perfect regulation for Σ.*

Proof. Using the result of Chen [1], one can show that the observer gain matrix $K(\varepsilon)$ of (18) has the following properties:

1. $A+K(\varepsilon)C_1$ has q eigenvalues in the neighborhood of $-1/\varepsilon$ and $n-q$ eigenvalues in the neighborhood of $\lambda(A_{cc}-K_cE_c)$. Hence, $A+K(\varepsilon)C_1$ is asymptotically stable for sufficiently small ε.
2. As $\varepsilon \to 0$,

$$\mathrm{Im}\{[sI-A-K(\varepsilon)C_1)^{-1}\} \to \mathscr{V}^*(\Sigma_Q)=\mathrm{Ker}(C_1). \tag{20}$$

Hence, we have

$$\|C_2[sI-A-K(\varepsilon)C_1)^{-1}\|_2 \to 0 \quad \text{as } \varepsilon \to 0. \tag{21}$$

Noting that the eigenvalues of the closed-loop system comprising the given plant (1) and the full-order control law (2) and (19) are given by the eigenvalues of $A+BF(\varepsilon)$ and $A+K(\varepsilon)C_1$. Thus, it is clear that the closed-loop system is asymptotically stable for sufficiently small ε. Next, utilizing the results of Chen et al. [2] with some straightforward manipulations, we can show that the Laplace transform of $h(t)$ in the closed-loop system, $H(s)$, is given by

$$\begin{aligned} H(s)=&[C_2+D_2F(\varepsilon)][sI-A-BF(\varepsilon)]^{-1}x_0+C_2[sI-A-K(\varepsilon)C_1]^{-1}x_0\\ &+[C_2+D_2F(\varepsilon)][sI-A-BF(\varepsilon)]^{-1}(A-sI)[sI-A-K(\varepsilon)C_1]^{-1}x_0. \end{aligned} \tag{22}$$

In view of (16), (21) and (22), it is clear that as $\varepsilon \to 0$,

$$\|H\|_2 \to 0 \Leftrightarrow \|h\|_2 \to 0. \tag{23}$$

This completes the proof of Theorem 3.1. □

3.2. Design of reduced-order solutions

Next, we will develop an algorithm that constructs reduced order measurement output feedback solutions, which have a dynamical order of $n-q$, to the problem of perfect regulation problem via measurement output feedback. For simplicity of presentation, we assume that matrix C_1 is already in the following form;

$$C_1=[I_q \quad 0], \tag{24}$$

and without loss of any generality, we can still assume that the disturbance matrix in (7) is an identity matrix. Then, (7) can be partitioned as follows:

$$\begin{aligned} &\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u+\begin{bmatrix}I_q & 0\\ 0 & I_{n-q}\end{bmatrix}w, \quad x_0=\begin{pmatrix}x_{10}\\ x_{20}\end{pmatrix},\\ &y=[I_q\ 0]\begin{pmatrix}x_1\\ x_2\end{pmatrix},\\ &h=[C_{21}\ C_{22}]\begin{pmatrix}x_1\\ x_2\end{pmatrix}+D_2u. \end{aligned} \tag{25}$$

Next, define Σ_{QR} to be a subsystem characterized by

$$(A_{\mathrm{R}},E_{\mathrm{R}},C_{\mathrm{R}},D_{\mathrm{R}})=(A_{22},[0\ I_{n-q}],A_{12},[I_q\ 0]). \tag{26}$$

The following is a step-by-step constructive algorithm.

Step R.1. Follow the result of Lin [8] to obtain a parameterized gain matrix $F(\varepsilon)$ which has the property as in (16). We partition $F(\varepsilon)$ in conformity with the partition of the state vector x in (25) as

$$F(\varepsilon) = [F_1(\varepsilon)\ F_2(\varepsilon)]. \tag{27}$$

Step R.2. It is straightforward to see that Σ_{QR} is right invertible and is free of finite and infinite zeros. Moreover, it can be verified that $(A_{\mathrm{R}}, C_{\mathrm{R}})$ is detectable if and only if (A, C_1) is detectable. Let K_{R} be a constant matrix of dimension $(n-q)\times q$ such that the eigenvalues of the matrix $A_{\mathrm{R}} + K_{\mathrm{R}}C_{\mathrm{R}}$ are in $\mathbb{C}^-$. We note that this reduced-order observer gain matrix K_{R} is independent of ε.

Step R.3. Finally, the family of reduced-order measurement feedback parameterized control laws is given by (2) with

$$\begin{aligned}
A_{\mathrm{cmp}}(\varepsilon) &= A_{22} + B_2F_2(\varepsilon) + K_{\mathrm{R}}A_{12} + K_{\mathrm{R}}B_1F_2(\varepsilon),\\
B_{\mathrm{cmp}}(\varepsilon) &= A_{21} + K_{\mathrm{R}}A_{11} - (A_{22} + K_{\mathrm{R}}A_{12})K_{\mathrm{R}} + (B_2 + K_{\mathrm{R}}B_1)[F_1(\varepsilon) - F_2(\varepsilon)K_{\mathrm{R}}],\\
C_{\mathrm{cmp}}(\varepsilon) &= F_2(\varepsilon),\\
D_{\mathrm{cmp}}(\varepsilon) &= F_1(\varepsilon) - F_2(\varepsilon)K_{\mathrm{R}}.
\end{aligned} \tag{28}$$

This concludes the constructive algorithm of the reduced-order solution to the problem of perfect regulation.

Theorem 3.2. *Consider the given plant Σ of* (1) *and assume that the solvability conditions in Theorem* 2.1 *are satisfied. Then, the family of reduced-order measurement output feedback control laws* (2) *with its gain matrices as given in* (28) *solves the problem of perfect regulation for the given Σ.*

Proof. It is straightforward to verify that the closed-loop system comprising the given plant (1) and the reduced order control law (2) and (28) is asymptotically stable for sufficiently small ε, because the closed-loop poles are the eigenvalues of $A + BF(\varepsilon)$ and $A_{\mathrm{R}} + K_{\mathrm{R}}C_{\mathrm{R}}$. Next, following the results of Chen et al. [2], one can show that the Laplace transform of the closed-loop response $h(t)$, $H(s)$, is given by

$$\begin{aligned}
H(s) = {} & [C_2 + D_2F(\varepsilon)][sI - A - BF(\varepsilon)]^{-1}x_0 + C_2\begin{pmatrix} 0 \\ I_{n-q}\end{pmatrix}(sI - A_{\mathrm{R}} - K_{\mathrm{R}}C_{\mathrm{R}})^{-1}(x_{20} + K_{\mathrm{R}}x_{10})\\
& + [C_2 + D_2F(\varepsilon)][sI - A - BF(\varepsilon)]^{-1}(A - sI)\begin{pmatrix} 0 \\ I_{n-q}\end{pmatrix}(sI - A_{\mathrm{R}} - K_{\mathrm{R}}C_{\mathrm{R}})^{-1}(x_{20} + K_{\mathrm{R}}x_{10}).
\end{aligned} \tag{29}$$

We note that $\mathrm{Ker}(C_2) \supseteq \mathrm{Ker}(C_1)$ implies

$$C_2\begin{pmatrix} 0 \\ I_{n-q}\end{pmatrix} = 0. \tag{30}$$

Hence, (16) implies that as $\varepsilon \to 0$,

$$\|H\|_2 \to 0 \Leftrightarrow \|h\|_2 \to 0. \tag{31}$$

This concludes the proof of Theorem 3.2. □

3.3. A numerical example

We consider a given linear time-invariant system of the form (1) characterized by

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [1 \quad 0], \quad D_1 = 0, \quad C_2 = [1 \quad 0], \quad D_2 = 1, \; x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}. \tag{32}$$

It is straightforward to verify that (A, B) is stabilizable, (A, C_1) is detectable, and (A, B, C_2, D_2) is invertible and has two invariant zeros at -1 and 0, respectively. Also, $\mathrm{Ker}(C_2) = \mathrm{Ker}(C_1)$. Hence, the solvability conditions of Theorem 2.1 are satisfied. We note that h and y are linearly independent.

First, following the algorithm for the full-order solutions, we obtain a family of parameterized control laws,

$$\begin{aligned} \dot{v} &= \begin{bmatrix} -1/\varepsilon & 1 \\ -1-1/\varepsilon & -\varepsilon \end{bmatrix} v + \begin{bmatrix} 1/\varepsilon - 1 \\ 1/\varepsilon + 1 \end{bmatrix} y, \\ u &= [-1 \quad -\varepsilon] v. \end{aligned} \tag{33}$$

It is straightforward to verify that the eigenvalues of the closed-loop system are given by -1, $-\varepsilon$, $-1+\mathrm{O}(\varepsilon)$ and $-1/\varepsilon + \mathrm{O}(1)$, and the closed-loop response $h(t)$ for $\varepsilon \ll 1$, is given by

$$h(t) \approx (\varepsilon x_{10} - x_{20})\varepsilon \mathrm{e}^{-\varepsilon t} + (x_{10} - \varepsilon x_{20})\mathrm{e}^{-t/\varepsilon} + 2(x_{20} - x_{10})\varepsilon \mathrm{e}^{-t}. \tag{34}$$

Thus, for $\varepsilon \ll 1$,

$$\|h\|_2 = \int_0^\infty h^2(t)\,\mathrm{d}t = \tfrac{1}{2}[(\varepsilon x_{10} - x_{20})^2 + (x_{10} - \varepsilon x_{20})^2]\varepsilon + \mathrm{O}(\varepsilon^2). \tag{35}$$

Clearly, as $\varepsilon \to 0$, $\|h\|_2 \to 0$. Thus, the perfect regulation problem for the given plant is solved by the full-order measurement output feedback control laws (33).

Next, we follow the algorithm for the reduced-order solutions and obtain a family of reduced order parameterized control laws,

$$\begin{aligned} \dot{v} &= -(1+\varepsilon)v - \varepsilon y, \\ u &= -\varepsilon v - (1+\varepsilon) y. \end{aligned} \tag{36}$$

It is again simple to verify that the eigenvalues of the closed-loop system are given by -1, $-\varepsilon$ and -1, and the closed-loop response $h(t)$ is given by

$$h(t) = (x_{20} - x_{10}/1 - \varepsilon)\varepsilon \mathrm{e}^{-t} + (\varepsilon x_{10} - x_{20}/1 - \varepsilon)\varepsilon \mathrm{e}^{-\varepsilon t}. \tag{37}$$

Thus, for $\varepsilon \ll 1$,

$$\|h\|_2 = \int_0^\infty h^2(t)\,\mathrm{d}t = \frac{1}{2}(\varepsilon x_{10} - x_{20})^2 \varepsilon + \mathrm{O}(\varepsilon^2). \tag{38}$$

Obviously, as $\varepsilon \to 0$, $\|h\|_2 \to 0$. Hence, the problem of perfect regulation for the given plant is also solved by the family of reduced-order measurement output feedback laws (36).

4. Concluding remarks

We have completely solved the problem of perfect regulation via measurement output feedback for general linear multivariable systems. A set of solvability conditions are obtained, which are all simple to verify. Under these solvability conditions, two algorithms are presented for the construction of both full- and reduced-order measurement output feedback laws that solve the problem perfect regulation.

We note that our results can be easily adapted to handle the case when there are external disturbances in the plant, i.e., the plant is characterized by

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew, \quad x(0) = x_0 \\ y &= C_1 x + \tilde{D}_1 u + D_1 w, \\ h &= C_2 x + D_2 u + D_{22} w, \end{aligned} \tag{39}$$

where the external disturbances w is independent zero mean white noise. For simplicity, we assume that $\tilde{D}_1 = 0$. Then, the necessary and sufficient conditions under which the problem of perfect regulation for the above system (39) with external disturbances, are as follows:

1. (A, B) is stabilizable and (A, C_1) is detectable;
2. $D_{22}+D_2SD_1=0$, where $S=-(D_2'D_2)^\dagger D_2'D_{22}D_1'(D_1D_1')^\dagger$ and where $(\cdot)^\dagger$ denotes the Moore–Penrose (pseudo) inverse;
3. (A, B, C_2, D_2) is right invertible and is free of invariant zeros in $\mathbb{C}^+$; and
4. $\mathrm{Ker}(C_2 + D_2SC_1) \supseteq C_1^{-1}\{\mathrm{Im}(D_1)\} := \{v \mid C_1v \in \mathrm{Im}(D_1)\}$.

The proof of such an assertion follows similar lines of reasoning as those of Theorem 2.1. Finally, we note that the above conditions are very different from those for the perfect tracking problem considered in [4].

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