Exact computation of infimum for a class of continuous-time H_∞ optimal control problem with a nonzero direct feedthrough term from the disturbance input to the controlled output

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Abstract

A noniterative method for the computation of infimum for a class of continuous-time H_∞ optimal control problem is considered in this paper. The problem formulation is fairly general and does not place any restrictions on any direct feedthrough terms of the given systems. The method is applicable to systems where (i) the transfer function from the disturbance input to the measurement output is free of imaginary axis invariant zeros and left invertible, and (ii) the transfer function from the control input to the controlled output of the given system is free of imaginary axis invariant zeros and right invertible. The result presented in this paper is a continuation of the previous work of the author and his co-workers (Chen et al., 1992), in which the direct feedthrough term from the disturbance input to the measurement output of the given system are required to be zero. © 1997 Elsevier Science B.V.

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1. Introduction and problem statement

Since the original formulation of the H_∞ optimal control problem in [18], a great deal of work has been done on the solution of this problem (see, for example, [1, 6–8, 10, 11, 16]). The solution to the H_∞ optimal control problem can be obtained from purely time-domain methods based on the γ-dependent algebraic Riccati equations (AREs) or linear matrix inequalities (LMIs). Typically in ARE or LMI approaches to H_∞ optimal control problems, the achieved design solution is suboptimal in the sense that the H_∞-norm of the closed-loop system transfer function from the disturbance input to the controlled output is less than a prescribed positive scalar, say γ.

In this paper, we address the problem of computing infimum in continuous-time H_∞ optimal control. In principle, the ARE approach (mainly for the regular case) or LMI approach (mainly for the singular case) to this problem provides an iterative scheme of approximating the infimum (denoted here by γ^*) of the H_∞-norm of the closed-loop transfer function. For example, utilizing the results of [16], an iterative procedure for approximating γ^* for the general singular case would proceed as follows: one starts with a $\gamma > 0$ and determines whether $\gamma > \gamma^*$ by first performing a loop shifting transformation to get rid of the direct feedthrough term from

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the disturbance input to the controlled output, and then solves two linear matrix inequalities and checks the positive semi-definiteness and stabilizing properties of these solutions. In the case where such positive semi-definite solutions exist and satisfy a *coupling condition*, we have $\gamma > \gamma^*$ and one repeats the above steps using a smaller value of γ. Obviously, this search procedure is exhaustive and can be very costly. More significantly, as γ gets close to γ^*, numerical solutions for these loop shifting transformation and LMIs can become highly sensitive and ill-conditioned. Thus, it might be difficult to obtain a meaningful approximation of γ^* from the above searching procedure or other iterative methods especially for the case when the direct feedthrough term from the disturbance input to the controlled output is nonzero. So in general the above iterative procedure should not be used to determine γ^*.

We propose a noniterative method for computing this γ^* for a class of continuous-time H_∞ optimal control problem in which the direct feedthrough term from the disturbance input to the controlled output of the given system is not necessarily zero, but the transfer function from the disturbance to the measurement output is left invertible, and the transfer function from the control input to the controlled output is right invertible. The work of this paper can be regarded as a continuation of our earlier work [3, 4] in which the direct feedthrough term from the disturbance input to the controlled output is required to be zero.

We consider in this paper the following standard linear time-invariant discrete time system Σ characterized by:

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Ew, \\
y &= C_1 x + D_{12} w, \\
z &= C_2 x + D_{21} u + D_{22} w,
\end{aligned}
\tag{1.1}
$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measurement, $w \in \mathbb{R}^q$ is the unknown disturbance and $z \in \mathbb{R}^p$ is the output to be controlled. A, B, E, C_1, D_{12}, C_2, D_{21} and D_{22} are constant matrices of appropriate dimension. Without loss of generality but for simplicity of presentation, we assume throughout this paper that matrices $[\, C_1 \quad D_{12} \,]$ and $[\, B' \quad D'_{21} \,]$ are of maximal rank. This is because if these two matrices are not of maximal rank, one can simply drop the redundant control inputs and measurement outputs to make them maximal rank. The H_∞ optimal control problem is to find an internally stabilizing proper controller such that the H_∞-norm of the overall closed-loop system is minimized. To be more specific, we will investigate dynamic feedback laws of the form:

$$
\Sigma_{\mathrm{c}} : \begin{cases} \dot{x}_{\mathrm{c}} = Kx_{\mathrm{c}} + Ly, \\ u = Mx_{\mathrm{c}} + Ny. \end{cases}
\tag{1.2}
$$

We will say that the controller Σ_{c} of (1.2) is internally stabilizing when applied to the system Σ, if the following matrix is asymptotically stable:

$$
A_{\mathrm{cl}} := \begin{bmatrix} A + BNC_1 & BM \\ LC_1 & K \end{bmatrix}
\tag{1.3}
$$

i.e., all its eigenvalues lie in the open left-half complex plane. Denote by G_{cl} the corresponding closed-loop transfer matrix. Then the H_∞ norm of the transfer matrix G_{cl} is given by

$$
\|G_{\mathrm{cl}}\|_\infty := \sup_{\omega \in [0, \infty)} \sigma_{\max}[G_{\mathrm{cl}}(\mathrm{j}\omega)]
$$

where $\sigma_{\max}[\cdot]$ denotes the largest singular value. The infimum γ^* can now be formally defined as

$$
\gamma^* := \inf\{\|G_{\mathrm{cl}}\|_\infty \mid \Sigma_{\mathrm{c}} \text{ internally stabilizes } \Sigma\}.
\tag{1.4}
$$

Given a $\gamma > \gamma^*$, the H_∞ optimal (or more precisely suboptimal) control problem is to find an internally stabilizing controller Σ_{c} such that the resulting $\|G_{\mathrm{cl}}\|_\infty < \gamma$. Also, Σ_{c} is said to be a γ suboptimal controller

for Σ if the corresponding $\|G_{cl}\|_\infty < \gamma$. The main purpose of this paper is to present a noniterative method that computes exactly this γ^* for Σ under the following assumptions:

(A1) (A, B) is stabilizable;

(A2) (A, B, C_2, D_{21}) is free of imaginary axis invariant zeros;

(A3) (A, B, C_2, D_{21}) is right invertible;

(A4) (A, C_1) is detectable;

(A5) (A, E, C_1, D_{12}) is free of imaginary axis invariant zeros;

(A6) (A, E, C_1, D_{12}) is left invertible.

Here we should point out that Assumptions (A1) and (A4) are necessary for any control problems, while (A2), (A5), (A3) and (A6) are not essential and can be relaxed. For example, Assumptions (A2) and (A5) can be easily removed using the technique reported in [2], while Assumptions (A3) and (A6) can be replaced by certain weaker geometric conditions as in [5]. We would keep these assumptions here in this paper merely to make the presentation of our results simpler. Also, note that (A3) and (A6) also imply that matrices $[C_2 \ D_{21}]$ and $[E' \ D'_{12}]$ are of maximal rank.

The paper is organized as follows: In Section 2, we recall the special coordinate basis of linear systems, which is instrumental to the development and derivation of the results in this paper. Section 3 gives the main results, namely, the noniterative algorithms for computation of γ^* for three common cases, i.e., the full information, the output feedback and the state feedback cases. Finally, the concluding remarks are drawn in Section 4.

Throughout this paper, X' denotes the transpose of matrix A. I denotes an identity matrix with appropriate dimension. $\mathbb{R}$ is the set of real numbers. $\lambda(X)$ is the set of eigenvalues of a real square matrix X. $\lambda_{max}(X)$ denotes the maximum eigenvalue of X where $\lambda(X) \subset \mathbb{R}$, and finally $\sigma_{max}(X)$ denotes the maximum singular value of matrix X.

2. Background materials

In this section, we should recall a theorem of the special coordinate basis of linear systems from [14, 15], which will be instrumental to the main results developed in the next sections. Consider the system described by

$$\dot{x} = Ax + Bu + Ew,$$
$$z = C_2 x + D_{21} u + D_{22} w. \tag{2.1}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a nonsingular matrix V that puts the direct feedthrough matrix D_{21} into the following form:

$$U D_{21} V = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.2}$$

where r is the rank of D_{21}. Without loss of generality, one can assume that the matrix D_{21} in Eq. (2.1) has the form as shown in Eq. (2.2). Thus the system in (2.1) can be rewritten as

$$\dot{x} = Ax + [B_0 \ B_1] \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + Ew,$$
$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix} x + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \begin{bmatrix} D_{22,0} \\ D_{22,1} \end{bmatrix} w \tag{2.3}$$

where B_0, B_1, $C_{2,0}$, $C_{2,1}$, $D_{22,0}$ and $D_{22,1}$ are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely,

$$u = V \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = Uz.$$

Also, note that the H_∞-norm of the system transfer function from w to z is unchanged when we apply an orthogonal transformation on the output z, and under any nonsingular transformations on the states and control inputs. We have the following theorem.

Theorem 2.1. *Consider the linear system as in* (2.1). *Assume that* (A, B, C_2, D_{21}) *is right invertible with no imaginary axis invariant zeros. Then, there exist nonsingular transformations* Γ_s, Γ_i *and* Γ_{or} *such that*

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \Gamma_i \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \qquad x = \Gamma_s \begin{pmatrix} x_a^+ \\ x_a^- \\ x_c \\ x_d \end{pmatrix}, \qquad \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{or} \end{bmatrix} \begin{pmatrix} z_0 \\ z_d \end{pmatrix}, \tag{2.4}$$

and

$$\Gamma_s^{-1}(A - B_0 C_{2,0})\Gamma_s = \begin{bmatrix} A_{aa}^+ & 0 & 0 & L_{ad}^+ C_d \\ 0 & A_{aa}^- & 0 & L_{ad}^- C_d \\ B_c E_{ca}^+ & B_c E_{ca}^- & A_{cc} & L_{cd} C_d \\ B_d E_{da}^+ & B_d E_{da}^- & B_d E_{dc} & A_{dd} \end{bmatrix}, \qquad \Gamma_s^{-1}E = \begin{bmatrix} E_a^+ \\ E_a^- \\ E_c \\ E_d \end{bmatrix}, \tag{2.5}$$

$$\Gamma_s^{-1}[B_0 \ B_1]\Gamma_i = \begin{bmatrix} B_{0a}^+ & 0 & 0 \\ B_{0a}^- & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \qquad D_{22} = \begin{bmatrix} D_{22,0} \\ D_{22,1} \end{bmatrix}, \tag{2.6}$$

$$\begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix} \Gamma_s = \begin{bmatrix} C_{0a}^+ & C_{0a}^- & C_{0c} & C_{0d} \\ 0 & 0 & 0 & \Gamma_{or} C_d \end{bmatrix}, \qquad D_{21}\Gamma_i = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

where the pair (A_{cc}, B_c) *is completely controllable and the subsystem* (A_{dd}, B_d, C_d) *is invertible and free of any invariant zeros. Sub-matrix* C_d *can be arranged as* $C_d = [0 \ I]$. *Also,* $\lambda(A_{aa}^+)$ *and* $\lambda(A_{aa}^-)$ *are, respectively, the sets of unstable and stable invariant zeros of* (A, B, C_2, D_{21}). *Moreover, the pair* (A, B) *is stabilizable if and only if the pair* $(A_{aa}^+, [B_{0a}^+, L_{ad}^+])$ *is controllable, where also,* (A, B, C_2, D_{21}) *is invertible if and only if* x_c *is nonexistent. For future use, we define an integer scalar* $n_a^+ := \dim(x_a^+)$.

Proof. The above theorem is a special case of the results reported in [15, 14]. The realization of this special coordinate basis can be found in the toolbox of [12]. □

3. Main results

Now, we are ready to present our main results, i.e., the noniterative algorithms for computing infimum, γ^*. This section is naturally divided into two subsections. The first subsection deals with the full information case, while the second subsection deals with the general output feedback case. The full state feedback problem is treated as a special case in a remark.

3.1. The full information case

We assume that $y = [x' \ w']'$ and the conditions (A1)–(A3) are satisfied. Without loss of generality but for simplicity of presentation of our results, we also assume that D_{21} is in the form of (2.2). In what follows, we state a step-by-step algorithm for the computation of infimum γ^* for the full information problem.

Step 1: Transform the following system:

$$\begin{aligned}
\dot{x} &= Ax + Bu + Ew, \\
z &= C_2 x + D_{21} u + D_{22} w,
\end{aligned} \tag{3.1}$$

into the special coordinate basis as in Section 2 and Theorem 2.1.

Step 2: Solve the following two continuous-time γ independent algebraic Lyapunov equations:

$$A_{aa}^+ S + S(A_{aa}^+)' = B_{0a}^+ (B_{0a}^+)' + L_{ad}^+ \Gamma_{or}^{-1} (L_{ad}^+ \Gamma_{or}^{-1})' \tag{3.2}$$

$$A_{aa}^+ T + T(A_{aa}^+)' = (E_a^+ - B_{0a}^+ D_{22,0} - L_{ad}^+ \Gamma_{or}^{-1} D_{22,1})(E_a^+ - B_{0a}^+ D_{22,0} - L_{ad}^+ \Gamma_{or}^{-1} D_{22,1})' \tag{3.3}$$

for positive definite solution S and positive semi-definite solution T. The existences of such solutions follows from [13] because of the fact that $(A_{aa}^+, [B_{0a}^+, L_{ad}^+])$ is completely controllable and A_{aa}^+ is an anti-stable matrix.

Step 3: Define a constant matrix,

$$M = \begin{bmatrix} D'_{22,1} D_{22,1} & 0 \\ 0 & TS^{-1} \end{bmatrix}. \tag{3.4}$$

The infimum, γ^*, is then given by

$$\gamma^* = \sqrt{\lambda_{\max}(M)}. \tag{3.5}$$

Proof of the algorithm. Without loss of generality, but for the simplicity of presentation, we will assume that the given system of (3.1) is already in the form of the special coordinate basis as in Theorem 2.1. Let us first apply a pre-state feedback law

$$\begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix} = - \begin{bmatrix} C_{0a}^+ & C_{0a}^- & C_{0c} & C_{0d} \\ E_{da}^+ & E_{da}^- & E_{dc} & 0 \\ E_{ca}^+ & E_{ca}^- & 0 & 0 \end{bmatrix} \begin{pmatrix} x_a^+ \\ x_a^- \\ x_c \\ x_d \end{pmatrix} + \begin{pmatrix} v_0 \\ v_d \\ v_c \end{pmatrix} \tag{3.6}$$

to the system in (3.1). Also, note that the sub-matrix C_d can be arranged as $C_d = [0 \ I]$. We further partition sub-matrices A_{dd}, B_d, B_{0d} and E_d, and x_d in conformity with C_d as follows:

$$A_{dd} = \begin{bmatrix} A_{dd00} & A_{dd01} \\ A_{dd10} & A_{dd11} \end{bmatrix}, \quad B_d = \begin{bmatrix} B_{d0} \\ B_{d1} \end{bmatrix}, \quad B_{0d} = \begin{bmatrix} B_{0d0} \\ B_{0d1} \end{bmatrix}, \quad E_d = \begin{bmatrix} E_{d0} \\ E_{d1} \end{bmatrix}, \quad x_d = \begin{pmatrix} x_{d0} \\ x_{d1} \end{pmatrix}. \tag{3.7}$$

Hence, the system of (3.1) with the pre-state feedback law of (3.6) can be re-written as

$$\dot{x} = \begin{bmatrix} A_{aa}^+ & 0 & 0 & 0 & L_{ad}^+ \\ 0 & A_{aa}^- & 0 & 0 & L_{ad}^- \\ 0 & 0 & A_{cc} & 0 & L_{cd} \\ 0 & 0 & 0 & A_{dd00} & A_{dd01} \\ 0 & 0 & 0 & A_{dd10} & A_{dd11} \end{bmatrix} x + \begin{bmatrix} B_{0a}^+ & 0 & 0 \\ B_{0a}^- & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d0} & B_{d0} & 0 \\ B_{0d1} & B_{d1} & 0 \end{bmatrix} v + \begin{bmatrix} E_a^+ \\ E_a^- \\ E_c \\ E_{d0} \\ E_{d1} \end{bmatrix} w, \tag{3.8}$$

$$z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Gamma_{or} \end{bmatrix} x + \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v + \begin{bmatrix} D_{22,0} \\ D_{22,1} \end{bmatrix} w.$$

Next, we partitioned the above system (3.8) as

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + [\, B_{0a}^+ \; L_{ad}^+ \,] \begin{pmatrix} v_0 \\ x_{d1} \end{pmatrix} + E_a^+ w, \tag{3.9}$$

$$\begin{pmatrix} \dot{x}_a^- \\ \dot{x}_c \\ \dot{x}_{d0} \\ \dot{x}_{d1} \end{pmatrix} = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ad}^- \\ 0 & A_{cc} & 0 & L_{cd} \\ 0 & 0 & A_{dd00} & A_{dd01} \\ 0 & 0 & A_{dd10} & A_{dd11} \end{bmatrix} \begin{pmatrix} x_a^- \\ x_c \\ x_{d0} \\ x_{d1} \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & B_c \\ B_{d0} & 0 \\ B_{d1} & 0 \end{bmatrix} \begin{pmatrix} v_d \\ v_c \end{pmatrix} + \begin{bmatrix} B_{0a}^- & E_a^- \\ B_{0c} & E_c \\ B_{0d0} & E_{d0} \\ B_{0d1} & E_{d1} \end{bmatrix} \begin{pmatrix} v_0 \\ w \end{pmatrix}, \tag{3.10}$$

$$\begin{pmatrix} z_0 \\ z_d \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} x_a^+ + \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{or} \end{bmatrix} \begin{pmatrix} v_0 \\ x_{d1} \end{pmatrix} + \begin{bmatrix} D_{22,0} \\ D_{22,1} \end{bmatrix} w. \tag{3.11}$$

It is then simple to verify using the properties of the special coordinate basis that the quadruple

$$\left(\begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ad}^- \\ 0 & A_{cc} & 0 & L_{cd} \\ 0 & 0 & A_{dd00} & A_{dd01} \\ 0 & 0 & A_{dd10} & A_{dd11} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & B_c \\ B_{d0} & 0 \\ B_{d1} & 0 \end{bmatrix}, [0 \; 0 \; 0 \; I], [0 \; 0] \right) \tag{3.12}$$

is right invertible and of minimum phase. Thus, there exists a feedback law

$$\begin{pmatrix} v_d \\ v_c \end{pmatrix} = \tilde{F} \begin{pmatrix} x_a^- \\ x_c \\ x_{d0} \\ x_{d1} \end{pmatrix} \tag{3.13}$$

that solves the almost disturbance decoupling problem for the system in (3.10) with an auxiliary controlled output x_{d1}. Following a similar procedure as in [17], where its author had proved the case when $D_{22} = 0$, one can show that the following two statements are equivalent:
1. There exists a static feedback law $u = F_1 x + F_2 w$ to the system (3.1) such that the resulting closed-loop system is internally stable and the H_∞-norm of the closed-loop transfer function from w to z is less than γ.
2. $D_{22,1}' D_{22,1} < \gamma^2 I$ and there exists a static feedback law $\tilde{u} = \tilde{F}_1 \tilde{x} + \tilde{F}_2 \tilde{w}$ to the following auxiliary system:

$$\dot{\tilde{x}} = A_{aa}^+ \tilde{x} + [\, B_{0a}^+ \; L_{ad}^+ \,] \tilde{u} + E_a^+ \tilde{w},$$

$$\tilde{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{or} \end{bmatrix} \tilde{u} + \begin{bmatrix} D_{22,0} \\ D_{22,1} \end{bmatrix} \tilde{w} \tag{3.14}$$

such that the resulting closed-loop system is internally stable and the H_∞-norm of the closed-loop transfer function from $\tilde{w}$ to $\tilde{z}$ is less than γ.
By applying a pre-disturbance feedback law

$$\tilde{u} = - \begin{bmatrix} D_{22,0} \\ \Gamma_{or}^{-1} D_{22,1} \end{bmatrix} \tilde{w} + \tilde{v} \tag{3.15}$$

to the auxiliary system (3.14), one can show that the second statement above is equivalent to the following two statements:

1. $D_{22,1}'D_{22,1} < \gamma^2 I$ and there exists a static feedback law $\tilde{v} = \hat{F}_1\tilde{x} + \hat{F}_2\tilde{w}$ to the following auxiliary system:

$$\dot{\tilde{x}} = A_{aa}^+\tilde{x} + [B_{0a}^+ \ L_{ad}^+]\tilde{v} + (E_a^+ - B_{0a}^+D_{22,0} - L_{ad}^+\Gamma_{or}^{-1}D_{22,1})\tilde{w},$$

$$\tilde{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\tilde{x} + \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{or} \end{bmatrix}\tilde{v} + 0\tilde{w} \tag{3.16}$$

such that the resulting closed-loop system is internally stable and the H_∞-norm of the closed-loop transfer function from $\tilde{w}$ to $\tilde{z}$ is less than γ.

2. $D_{22,1}'D_{22,1} < \gamma^2 I$ and there exists a positive definite solution P_{aa}^+ to the following Riccati equation:

$$0 = (A_{aa}^+)'P_{aa}^+ + P_{aa}^+A_{aa}^+ - P_{aa}^+[B_{0a}^+(B_{0a}^+)' + L_{ad}^+\Gamma_{or}^{-1}(L_{ad}^+\Gamma_{or}^{-1})']P_{aa}^+$$

$$+ P_{aa}^+(E_a^+ - B_{0a}^+D_{22,0} - L_{ad}^+\Gamma_{or}^{-1}D_{22,1})(E_a^+ - B_{0a}^+D_{22,0} - L_{ad}^+\Gamma_{or}^{-1}D_{22,1})'P_{aa}^+/\gamma^2.$$

Finally, following the result of [3], one can show that γ^* as given in (3.5) is indeed the infimum. This completes the proof of the algorithm. $\square$

3.2. The output feedback case

This subsection deals with the general measurement feedback problem. Again, we consider the given system of (1.1) and assume that (A1)–(A6) are satisfied. As in the previous subsection, we will first give a step-by-step noniterative algorithm that calculates the infimum, γ^*, and leave detailed justification in the proof of the algorithm.

Step A: Define an auxiliary full information problem for

$$\dot{x} = Ax + Bu + Ew,$$

$$y = \begin{pmatrix} 0 \\ I \end{pmatrix}x + \begin{pmatrix} I \\ 0 \end{pmatrix}w,$$

$$z = C_2x + D_{21}u + D_{22}w$$

and perform Steps 1 to 2 of the algorithm given in the previous subsection. For future use and in order to avoid any notation confusion, we rename the state transformation of the special coordinate basis for this subsystem as Γ_{sP} and matrix $D_{22,1}$ as $D_{22,1P}$. Also, rename S of (3.2) and T of (3.3) as S_P and T_P, respectively. Moreover, we denote n_{aP}^+ the number of the unstable invariant zeros of (A, B, C_2, D_{21}).

Step B: Define another auxiliary full information problem for

$$\dot{x} = A'x + C_1'u + C_2'w,$$

$$y = \begin{pmatrix} 0 \\ I \end{pmatrix}x + \begin{pmatrix} I \\ 0 \end{pmatrix}w,$$

$$z = E'x + D_{12}'u + D_{22}'w$$

and again perform Steps 1 to 2 of the algorithm given in Section 3.1 one more time but for this auxiliary. We also rename the state transformation of the special coordinate basis for this case as Γ_{sQ}, matrix $D_{22,1}$ as $D_{22,1Q}$, and S of (3.2) and T of (3.3) as S_Q and T_Q, respectively. Also, denote n_{aQ}^+ the number of the unstable invariant zeros of (A, E, C_1, D_{12}).

Step C: Partition

$$\Gamma_{sP}^{-1}(\Gamma_{sQ}^{-1})' = \begin{bmatrix} \Gamma & \star \\ \star & \star \end{bmatrix}, \tag{3.17}$$

where Γ is a $n_{aP}^+ \times n_{aQ}^+$ matrix, and define a constant matrix

$$M = \begin{bmatrix} D_{22,1P}'D_{22,1P} & 0 & 0 & 0 \\ 0 & T_P S_P^{-1} + \Gamma S_Q^{-1}\Gamma'S_P^{-1} & -\Gamma S_Q^{-1} & 0 \\ 0 & -T_Q S_Q^{-1}\Gamma'S_P^{-1} & T_Q S_Q^{-1} & 0 \\ 0 & 0 & 0 & D_{22,1Q}'D_{22,1Q} \end{bmatrix}. \tag{3.18}$$

Step D: The infimum γ^* is then given by

$$\gamma^* = \sqrt{\lambda_{\max}(M)}. \tag{3.19}$$

Proof of the algorithm. Once the result for the full information case is established, the proof of this algorithm follows along the similar lines of reasoning as in [4]. □

The following remark is regarding with the state feedback case.

Remark 3.1. It is simple to verify that for the state feedback case, the infimum γ^* is given by

$$\gamma^* = \sqrt{\lambda_{\max}(M)}, \tag{3.20}$$

where

$$M = \begin{bmatrix} D_{22}'D_{22} & 0 \\ 0 & T_P S_P^{-1} \end{bmatrix}. \tag{3.21}$$

Obviously, the infimum under the full state feedback is in general different from that under the full information feedback. They are identical, however, when the direct feedthrough term $D_{22} = 0$. This is a well-known fact in H_∞ literature. Of course, this can be easily seen from (3.4) and (3.21) as well.

We illustrate the above algorithm in the following example.

Example. Consider a system characterized by

$$A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

$$C_1 = [1 \quad -2 \quad 3 \quad -4], \qquad D_{12} = 0$$

and

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D_{22} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

It is simple to verify that the subsystem (A, B, C_2, D_{21}) is controllable and right invertible with one unstable invariant zero at 2 and one infinite zero of order 2, and the subsystem (A, E, C_1, D_{12}) is observable and

invertible with two unstable invariant zeros at $0.5 \pm j0.5916$ and one infinite zero of order 2. Thus, Assumptions (A1)–(A6) are satisfied. Following Step A of the algorithm for the output feedback case, we obtain

$$\Gamma_{sP} = I_4, \qquad \Gamma_{orP} = 1, \qquad A_{aaP}^+ = 2, \qquad B_{0aP}^+ = 1, \qquad L_{adP}^+ = 1,$$

$$E_{aP}^+ = 4, \qquad D_{22,0P} = 2, \qquad D_{22,1P} = 1$$

and

$$n_{aP}^+ = 1, \qquad S_P = 0.5, \qquad T_P = 0.25.$$

Here we append a subscript P to all submatrices associated with the corresponding special coordinate basis. Next, following Step B of the algorithm, we have

$$\Gamma_{orQ} = 1, \qquad \Gamma_{sQ} = \begin{bmatrix} 0.5723515 & 0 & 0.1 & 0.1 \\ -0.3012376 & -0.4866643 & -0.2 & 0.7 \\ -0.7530940 & 0.3244428 & 0.3 & -1.3 \\ 0.1204950 & 0.8111071 & -0.4 & 1.1 \end{bmatrix}, \qquad n_{aQ}^+ = 2,$$

$$A_{aaQ}^+ = \begin{bmatrix} 0.8842105 & -0.5101735 \\ 0.9753892 & 0.1157895 \end{bmatrix}, \qquad E_{aQ}^+ = \begin{bmatrix} -1.2230247 & -0.5241535 \\ 1.1679942 & 0.9408842 \end{bmatrix},$$

$$L_{adQ}^+ = \begin{bmatrix} -0.6289841 \\ 1.3756377 \end{bmatrix}, \qquad B_{0aQ}^+ = \emptyset, \qquad D_{22,0Q} = \emptyset, \qquad D_{22,1Q} = [2 \ 1]$$

and

$$S_Q = \begin{bmatrix} 0.5274947 & 0.5264991 \\ 0.5264991 & 3.7365053 \end{bmatrix}, \qquad T_Q = \begin{bmatrix} 0.5810175 & 0.9950273 \\ 0.9950273 & 3.2589825 \end{bmatrix}.$$

Again, we append here a subscript Q to all submatrices associated with the corresponding special coordinate basis. Finally, following Steps C and D, we obtain

$$\Gamma = [-1.2230247 \ 1.1679942],$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 9.7252904 & 3.0610640 & -0.7439148 & 0 \\ 0 & 2.0766328 & 0.9724337 & 0.1292764 & 0 \\ 0 & 1.2428740 & 1.1820112 & 0.7056473 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

and

$$\gamma^* = 3.2088448.$$

We would like to point out here that we have problems in obtaining this infimum using other approaches such as the perturbation approach of [19].

4. Concluding remarks

We have presented in this paper a noniterative method for the computation of infimum, γ^*, for a class of continuous-time H_∞ optimization problem in which the direct feedthrough term from the disturbance

input to the controlled output of the given system is nonzero. For the general output feedback problem that satisfies Assumptions (A1)–(A6), our method involves only transforming two subsystems into the special coordinate basis and solving two pairs of γ-independent Lyapunov equations. As we had mentioned in the introduction, some of the assumptions are not essential and can easily be removed or replaced by certain weaker conditions.

Finally, as suggested by one of the reviewers, it is interesting to compare our result with that of [9] in which the infimum was explicitly obtained for a special class of systems using the normalized coprime factorization approach. Following some simple manipulations, one can show that the class of systems considered in [9] can be re-expressed in the following state space form:

$$
\begin{aligned}
\dot{x} &= Ax + Bu - HR^{1/2}w, \\
y &= Cx + Du + R^{1/2}w, \\
z &= \begin{pmatrix} 0 \\ C \end{pmatrix} x + \begin{pmatrix} I \\ D \end{pmatrix} u + \begin{pmatrix} 0 \\ R^{1/2} \end{pmatrix} w,
\end{aligned}
\tag{4.1}
$$

where $R = I + DD'$, $H = -(ZC' + BD')R^{-1}$ and Z is the unique, positive definite solution of the following Riccati equation:

$$
(A - BD'R^{-1}C)Z + Z(A - BD'R^{-1}C)' - ZC'R^{-1}CZ + B(I - D'R^{-1}D)B' = 0.
\tag{4.2}
$$

It is simple to verify that the above system (4.1) does not satisfy Assumptions (A1)–(A6) of ours. Thus, the class of system considered in [9] is different from that considered in this paper. This means that there are rooms for improving or extending the results of [9] and ours.

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