INT. J. CONTROL, 1998, VOL. 71, NO. 6, 981–1003

# On properties of the special coordinate basis of linear systems

BEN M. CHEN†

In this paper, we provide, for the first time in the literature, rigorous and complete proofs to all the key properties of the special coordinate basis of linear time-invariant systems. The special coordinate basis decomposition or technique developed by Sannuti and Saberi in 1987 has a distinct feature of explicitly displaying the finite and infinite zero structures, the invertibility structures, as well as the invariant and almost invariant geometric subspaces of a given system. The technique has been extensively used in the literature to solve many system and control problems. We believe that the result of this paper is a complement of the seminal work of Sannuti and Saberi. It makes the theory of the special coordinate basis more complete.

## 1. Introduction

The special coordinate basis of linear time-invariant systems was first developed in the seminal work of Sannuti and Saberi (1987). Such a special coordinate basis decomposition or technique has a distinct feature of explicitly displaying the finite and infinite zero structures, the invertibility structures, as well as the invariant and almost invariant geometric subspaces of a given system. The technique has been extensively used in the literature to solve many system and control problems such as the squaring down and decoupling of linear systems (see, e.g. Sannuti and Saberi 1987, Saberi and Sannuti 1990), linear system factorizations (see, e.g. Chen *et al.* 1992a, Lin *et al.* 1996), model order reductions (see, e.g. Ozcetin *et al.* 1990), blocking zeros and strong stabilizability (see, e.g. Chen *et al.* 1992b), nested structural invariants (see, e.g. Saberi *et al.* 1992), zero placements (see, e.g. Chen and Zheng 1995), loop transfer recovery (see, e.g. Chen 1991, Saberi *et al.* 1993), $H_2$ optimal control (see, e.g. Chen *et al.* 1993, Saberi *et al.* 1995); $H_\infty$ optimal control (see, e.g. Chen 1998); disturbance decoupling (see, e.g. Ozcetin *et al.* 1993a, 1993b, Chen 1997), and control with saturations (see, e.g. Lin 1994), to name a few. The list is far from complete.

It is appropriate to trace a short history of the development of the technique of the special coordinate basis. The genesis of the concept of utilizing a special coordinate basis of a dynamic system first arose when dealing with high gain and cheap control problems (Sannuti 1983). At first, by separating the finite and infinite zero structures of what are now known as uniform rank systems, Sannuti (1983) showed the usefulness of utilizing the special coordinate basis in order to discuss the important features of high gain and cheap control problems. Then, Sannuti and Wason (1983) extended the concept of the special coordinate basis to general invertible

Received 10 February 1998. Revised 21 April 1998.
† Department of Electrical Engineering, National University of Singapore, Singapore 119260. e-mail: bmchen@nus.edu.sg.

systems and showed its significance in connection with multivariable root locus theory. A slight variation of the technique developed in Sannuti and Wason (1983) was used in Sannuti and Wason (1985) to bring into focus all the features of cheap control problems for general invertible systems. In their seminal paper, Sannuti and Saberi (1987) solidified the concept of the special coordinate basis of general linear strictly proper dynamic systems, and pointed out most of its important properties including those that are related to certain subspaces encountered in the geometric theory of linear systems. The required modifications in the development of the technique for general linear, not necessarily strictly proper dynamic, systems are given by Saberi and Sannuti (1990).

Unfortunately, all the properties of the special coordinate basis in the original work of Sannuti and Saberi (1987) were reported without detailed proofs. For some reason, their proofs are still missing in the literature. Although some of the properties of the special coordinate basis, e.g. controllability and observability, are quite obvious, some of them, e.g. the interconnections between the geometric subspaces and the subsystems of the special coordinate basis, are not transparent at all to general readers or even to researchers who are familiar with the technique. The goal of this paper is to give rigorous proofs to all the key properties of the special coordinate basis of linear systems once and for all. We will also take this opportunity to include some newly introduced properties. We believe that the results of this paper will make the theory of the special coordinate basis more complete.

The outline of this paper is as follows: In §2, we recall the special coordinate basis of linear systems and its properties. We should note that the original work of Sannuti and Saberi (1987) focused on continuous-time systems. We unify the theory for both continuous-time and discrete-time systems. Some fine tuning of the theory and new properties are also introduced. Section 3 presents the main results of this paper, i.e. the complete proofs of the key properties of the special coordinate basis. Finally, concluding remarks are drawn in §4.

Throughout this paper, the following notation will also be used: $X'$ denotes the transpose of matrix $X$; $I$ denotes an identity matrix with appropriate dimensions; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$ is the set of all complex numbers; $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$ are respectively the left-half complex plane, the imaginary axis and the right-half complex plane; $\mathbb{C}^{\odot}$, $\mathbb{C}^{\bigcirc}$ and $\mathbb{C}^{\otimes}$ are respectively the open unit disc, the unit circle and the set of complex numbers outside the unit circle; $\mathrm{Ker}\,(X)$ is the kernel of $X$; $\mathrm{Im}\,(X)$ is the image of $X$; and, finally, $\lambda(X)$ is the set of eigenvalues of a real square matrix $X$.

## 2. The special coordinate basis

In this section, we will make an attempt to unify the theory of the special coordinate basis of Sannuti and Saberi (1987) for both continuous-time and discrete-time systems. We will also recall all its key properties, which have been extensively used in the literature, and introduce some new ones. Let us consider a linear time-invariant (LTI) system $\Sigma_*$, which could be of either continuous-time or discrete-time, characterized by a matrix quadruple $(A_*, B_*, C_*, D_*)$ or in the state space form

$$\Sigma_* : \begin{cases} \delta(x) = A_* x + B_* u \\ y = C_* x + D_* u \end{cases} \tag{1}$$

where $\delta(x) = \dot{x}(t)$ if $\Sigma_*$ is a continuous-time system, or $\delta(x) = x(k+1)$ if $\Sigma_*$ is a discrete-time system. Similarly, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output of $\Sigma_*$. They represent $x(t)$, $u(t)$ and $y(t)$, respectively, if the given system is of continuous-time, or represent $x(k)$, $u(k)$ and $y(k)$, respectively, if $\Sigma_*$ is of discrete-time. Without loss of any generality, we assume that both $[B_*' \ \ D_*']$ and $[C_* \ \ D_*]$ are of full rank. The transfer function of $\Sigma_*$ is then given by

$$H_*(\varsigma) = C_*(\varsigma I - A_*)^{-1} B_* + D_* \tag{2}$$

where $\varsigma = s$, the Laplace transform operator, if $\Sigma_*$ is of continuous-time, or $\varsigma = z$, the $z$-transform operator, if $\Sigma_*$ is of discrete-time. It is simple to verify that there exist non-singular transformations $U$ and $V$ such that

$$UD_*V = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

where $m_0$ is the rank of matrix $D_*$. In fact, $U$ can be chosen as an orthogonal matrix. Hence hereafter, without loss of generality, it is assumed that the matrix $D_*$ has the form given on the right-hand side of (3). One can now rewrite system $\Sigma_*$ of (1) as

$$\left.\begin{aligned} \delta(x) &= A_* x + [B_{*,0} \ \ B_{*,1}] \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} C_{*,0} \\ C_{*,1} \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{aligned}\right\} \tag{4}$$

where the matrices $B_{*,0}$, $B_{*,1}$, $C_{*,0}$ and $C_{*,1}$ have appropriate dimensions. The following theorem unifies the result of Sannuti and Saberi (1987) for both continuous-time and discrete-time systems.

**Theorem 1 (SCB):** *Given the linear system $\Sigma_*$ of (1), there exist*

(1) *Coordinate free non-negative integers $n_a^-$, $n_a^0$, $n_a^+$, $n_b$, $n_c$, $n_d$, $m_d \le m - m_0$ and $q_i$, $i = 1, \ldots, m_d$, and*

(2) *Non-singular state, output and input transformations $\Gamma_s$, $\Gamma_o$ and $\Gamma_i$ which take the given $\Sigma_*$ into a special coordinate basis that displays explicitly both the finite and infinite zero structures of $\Sigma_*$.*

*The special coordinate basis is described by the following set of equations:*

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u} \tag{5}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad x_a = \begin{pmatrix} x_a^- \\ x_a^0 \\ x_a^+ \end{pmatrix}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix} \tag{6}$$

$$\tilde{y} = \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix}, \quad y_d = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_d} \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \quad u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix} \tag{7}$$

*and*

$$\delta(x_a^-) = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{ad}^- y_d + L_{ab}^- y_b \tag{8}$$

$$\delta(x_a^0) = A_{aa}^0 x_a^0 + B_{0a}^0 y_0 + L_{ad}^0 y_d + L_{ab}^0 y_b \tag{9}$$

$$\delta(x_a^+) = A_{aa}^+ x_a^+ + B_{0a}^+ y_0 + L_{ad}^+ y_d + L_{ab}^+ y_b \tag{10}$$

$$\delta(x_b) = A_{bb} x_b + B_{0b} y_0 + L_{bd} y_d, \quad y_b = C_b x_b \tag{11}$$

$$\delta(x_c) = A_{cc} x_c + B_{0c} y_0 + L_{cb} y_b + L_{cd} y_d + B_c \left[ E_{ca}^- x_a^- + E_{ca}^0 + E_{ca}^+ x_a^+ \right] + B_c u_c \tag{12}$$

$$y_0 = C_{0c} x_c + C_{0a}^- x_a^- + C_{0a}^+ x_a^0 + C_{0a}^+ x_a^+ + C_{0d} x_d + C_{0b} x_b + u_0 \tag{13}$$

*and, for each* $i = 1, \ldots, m_d$

$$\delta(x_i) = A_{q_i} x_i + L_{i0} y_0 + L_{id} y_d + B_{q_i} \left[ u_i + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} x_j \right] \tag{14}$$

$$y_i = C_{q_i} x_i, \quad y_d = C_d x_d \tag{15}$$

*Here the states* $x_a^-$, $x_a^0$, $x_a^+$, $x_b$, $x_c$ *and* $x_d$ *are respectively of dimensions* $n_a^-$, $n_a^0$, $n_a^+$, $n_b$, $n_c$ *and* $n_d = \sum_{i=1}^{m_d} q_i$, *while* $x_i$ *is of dimension* $q_i$ *for each* $i = 1, \cdots, m_d$. *The control vectors* $u_0$, $u_d$ *and* $u_c$ *are respectively of dimensions* $m_0$, $m_d$ *and* $m_c = m - m_0 - m_d$ *whereas the output vectors* $y_0$, $y_d$ *and* $y_b$ *are respectively of dimensions* $p_0 = m_0$, $p_d = m_d$ *and* $p_b = p - p_0 - p_d$. *The matrices* $A_{q_i}$, $B_{q_i}$ *and* $C_{q_i}$ *have the following form:*

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i - 1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = \begin{bmatrix} 1, & 0, & \ldots, & 0 \end{bmatrix} \tag{16}$$

*Assuming that* $x_i$, $i = 1, 2, \ldots, m_d$, *are arranged such that* $q_i \leq q_{i+1}$, *the matrix* $L_{id}$ *has the particular form*

$$L_{id} = \begin{bmatrix} L_{i1} & L_{i2} & \cdots & L_{ii-1} & 0 & \cdots & 0 \end{bmatrix} \tag{17}$$

*Also, the last row of each* $L_{id}$ *is identically zero. Moreover,*

(1) *If* $\Sigma_*$ *is a continuous-time system, then*

$$\lambda(A_{aa}^-) \subset \mathbb{C}^-, \quad \lambda(A_{aa}^0) \subset \mathbb{C}^0, \quad \lambda(A_{aa}^+) \subset \mathbb{C}^+ \tag{18}$$

(2) *If* $\Sigma_*$ *is a discrete-time system, then*

$$\lambda(A_{aa}^-) \subset \mathbb{C}^{\odot}, \quad \lambda(A_{aa}^0) \subset \mathbb{C}^{\circ}, \quad \lambda(A_{aa}^+) \subset \mathbb{C}^{\otimes} \tag{19}$$

*Also, the pair* $(A_{cc}, B_c)$ *is controllable and the pair* $(A_{bb}, C_b)$ *is observable.*

**Proof:** For strictly proper systems, using a modified structural algorithm of Silverman (1969), an explicit procedure for constructing the above special coordinate basis is given in Sannuti and Saberi (1987). The required modifications for non-strictly proper systems follow from Saberi and Sannuti (1990).

Here in Theorem 1, by another change of basis, the variable $x_a$ is further decomposed into $x_a^-$, $x_a^0$ and $x_a^+$. The software toolboxes that realize such a decomposition can be found in LAS by Chen (1988) or in MATLAB by Lin (1989). □

We can rewrite the special coordinate basis of the matrix quadruple $(A_*, B_*, C_*, D_*)$ given by Theorem 1 in a more compact form,

$$\tilde{A}_* = \Gamma_s^{-1}(A_* - B_{*,0}C_{*,0})\Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix} \tag{20}$$

$$\tilde{B}_* = \Gamma_s^{-1}[\,B_{*,0} \quad B_{*,1}\,]\Gamma_i = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix} \tag{21}$$

$$\tilde{C}_* = \Gamma_o^{-1}\begin{bmatrix} C_{*,0} \\ C_{*,1} \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix} \tag{22}$$

and

$$\tilde{D}_* = \Gamma_o^{-1}D_*\Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

In what follows, we state some key properties of the above special coordinate basis which are extensively used in the literature. The proofs of these properties will be given in the next section.

**Property 1:** The given system $\Sigma_*$ is observable (detectable) if and only if the pair $(A_{\rm obs}, C_{\rm obs})$ is observable (detectable), where

$$A_{\rm obs} := \begin{bmatrix} A_{aa} & 0 \\ B_c E_{ca} & A_{cc} \end{bmatrix}, \quad C_{\rm obs} := \begin{bmatrix} C_{0a} & C_{0c} \\ E_{da} & E_{dc} \end{bmatrix} \tag{24}$$

and where

$$A_{aa} := \begin{bmatrix} A_{aa}^- & 0 & 0 \\ 0 & A_{aa}^0 & 0 \\ 0 & 0 & A_{aa}^+ \end{bmatrix}, \quad C_{0a} := \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ \end{bmatrix} \tag{25}$$

$$E_{da} := [E_{da}^- \quad E_{da}^0 \quad E_{da}^+], \quad E_{ca} := [E_{ca}^- \quad E_{ca}^0 \quad E_{ca}^+] \tag{26}$$

Also, define

$$A_{\rm con} := \begin{bmatrix} A_{aa} & L_{ab}C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{\rm con} := \begin{bmatrix} B_{0a} & L_{ad} \\ B_{0b} & L_{bd} \end{bmatrix} \tag{27}$$

$$B_{0a} := \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \end{bmatrix}, \quad L_{ab} := \begin{bmatrix} L_{ab}^- \\ L_{ab}^0 \\ L_{ab}^+ \end{bmatrix}, \quad L_{ad} := \begin{bmatrix} L_{ad}^- \\ L_{ad}^0 \\ L_{ad}^+ \end{bmatrix} \tag{28}$$

Similarly, $\Sigma_*$ is controllable (stabilizable) if and only if the pair $(A_{\rm con}, B_{\rm con})$ is controllable (stabilizable).

The invariant zeros of a system $\Sigma_*$ characterized by $(A_*, B_*, C_*, D_*)$ can be defined via the Smith canonical form of the (Rosenbrock) system matrix (Rosenbrock 1970) of $\Sigma_*$ defined as the polynomial matrix $P_{\Sigma_*}(\varsigma)$,

$$P_{\Sigma_*}(\varsigma) := \begin{bmatrix} \varsigma I - A_* & -B_* \\ C_* & D_* \end{bmatrix} \tag{29}$$

We have the following definition for the invariant zeros (see also MacFarlane and Karcanias 1976).

**Definition 1** (invariant zeros): A complex scalar $\alpha \in \mathbb{C}$ is said to be an invariant zero of $\Sigma_*$ if

$$\text{rank}\,\{P_{\Sigma_*}(\alpha)\} < n + \text{normrank}\, H_*(\varsigma)\} \tag{30}$$

where normrank $\{H_*(\varsigma)\}$ denotes the normal rank of $H_*(\varsigma)$, which is defined as its rank over the field of rational functions of $\varsigma$ with real coefficients.

The special coordinate basis of Theorem 1 shows explicitly the invariant zeros and the normal rank of $\Sigma_*$. To be more specific, we have the following properties.

**Property 2:**

(1) The normal rank of $H_*(\varsigma)$ is equal to $m_0 + m_d$.

(2) Invariant zeros of $\Sigma_*$ are the eigenvalues of $A_{aa}$, which are the unions of the eigenvalues of $A_{aa}^-$, $A_{aa}^0$ and $A_{aa}^+$. Moreover, the given system $\Sigma_*$ is of minimum phase if and only if $A_{aa}$ has only stable eigenvalues, marginal minimum phase if and only if $A_{aa}$ has no unstable eigenvalue but has at least one marginally stable eigenvalue, and non-minimum phase if and only if $A_{aa}$ has at least one unstable eigenvalue.

In order to display various multiplicities of invariant zeros, let $X_a$ be a non-singular transformation matrix such that $A_{aa}$ can be transformed into a Jordan canonical form, i.e.

$$X_a^{-1} A_{aa} X_a = J = \text{blkdiag}\left\{J_1, J_2, \ldots, J_k\right\} \tag{31}$$

where $J_i$, $i = 1, 2, \ldots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_i = \text{diag}\left\{\alpha_i, \alpha_i, \ldots, \alpha_i\right\} + \begin{bmatrix} 0 & I_{n_i - 1} \\ 0 & 0 \end{bmatrix} \tag{32}$$

For any given $\alpha \in \lambda(A_{aa})$, let there be $\tau_\alpha$ Jordan blocks of $A_{aa}$ associated with $\alpha$. Let $n_{\alpha,1}$, $n_{\alpha,2}$, ..., $n_{\alpha,\tau_\alpha}$ be the dimensions of the corresponding Jordan blocks. Then we say $\alpha$ is an invariant zero of $\Sigma_*$ with multiplicity structure $S_\alpha^\star(\Sigma_*)$ (see also Saberi *et al.* 1991)

$$S_\alpha^\star(\Sigma_*) = \left\{n_{\alpha,1}, n_{\alpha,2}, \ldots, n_{\alpha,\tau_\alpha}\right\} \tag{33}$$

The geometric multiplicity of $\alpha$ is then simply given by $\tau_\alpha$, and the algebraic multiplicity of $\alpha$ is given by $\sum_{i=1}^{\tau_\alpha} n_{\alpha,i}$. Here we should note that the invariant zeros, together with their structures of $\Sigma_*$, are related to the structural invariant indices list $\mathcal{I}_1(\Sigma_*)$ of Morse (1973).

The special coordinate basis can also reveal the infinite zero structure of $\Sigma_*$. We note that the infinite zero structure of $\Sigma_*$ can be defined either in association with root-locus theory or as Smith–McMillan zeros of the transfer function at infinity. For the sake of simplicity, we consider the infinite zeros only from the point of view of Smith–McMillan theory here. To define the zero structure of $H_*(\varsigma)$ at infinity, one can use the familiar Smith–McMillan description of the zero structure at finite frequencies of a general, not necessarily square but strictly proper, transfer function matrix $H_*(\varsigma)$. Namely, a rational matrix $H_*(\varsigma)$ possesses an infinite zero of order $k$ when $H_*(1/z)$ has a finite zero of precisely that order at $z = 0$ (see Rosenbrock 1970, Verghese 1978, Pugh and Ratcliffe 1979, Commault and Dion 1982). The number of zeros at infinity together with their orders indeed defines an infinite zero structure. Owens (1978) related the orders of the infinite zeros of the root-loci of a square system with a non-singular transfer function matrix to $\mathcal{C}^*$ structural invariant indices; see list $\mathcal{I}_4$ of Morse (1973). This connection reveals that even for general, not necessarily strictly proper, systems, the *structure at infinity is in fact the topology of inherent integrations between the input and the output variables.* The special coordinate basis of Theorem 1 explicitly shows this topology of inherent integrations. The following property pinpoints this.

**Property 3:** $\Sigma_*$ has $m_0 = \text{rank}\,(D_*)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of $\Sigma_*$ is given by

$$S_\infty^\star(\Sigma_*) = \left\{q_1, q_2, \ldots, q_{m_d}\right\} \tag{34}$$

That is, each $q_i$ corresponds to an infinite zero of $\Sigma_*$ of order $q_i$. Note that for a single-input–single-output system $\Sigma_*$, we have $S_\infty^\star(\Sigma_*) = \left\{q_1\right\}$, where $q_1$ is the *relative degree* of $\Sigma_*$.

The special coordinate basis can also exhibit the invertibility structure of a given system $\Sigma_*$. The formal definitions of right invertibility and left invertibility of a linear system can be found in Moylan (1977). Basically, for the usual case when $\left[B_*' \quad D_*'\right]$

and $\left[ C_* \;\; D_* \right]$ are of maximal rank, the system $\Sigma_*$ or equivalently $H_*(\varsigma)$ is said to be left invertible if there exists a rational matrix function, say $L_*(\varsigma)$, such that

$$L_*(\varsigma) H_*(\varsigma) = I_m \tag{35}$$

$\Sigma_*$ or $H_*(\varsigma)$ is said to be right invertible if there exists a rational matrix function, say $R_*(\varsigma)$, such that

$$H_*(\varsigma) R_*(\varsigma) = I_p \tag{36}$$

$\Sigma_*$ is invertible if it is both left and right invertible, and $\Sigma_*$ is degenerate if it is neither left nor right invertible.

**Property 4:** The given system $\Sigma_*$ is right invertible if and only if $x_b$ (and hence $y_b$) are non-existent, left invertible if and only if $x_c$ (and hence $u_c$) are non-existent, and invertible if and only if both $x_b$ and $x_c$ are non-existent. Moreover, $\Sigma_*$ is degenerate if and only if both $x_b$ and $x_c$ are present.

The special coordinate basis can also be modified to obtain the structural invariant indices lists $\mathcal{I}_2$ and $\mathcal{I}_3$ of Morse (1973) for the given system $\Sigma_*$. In order to display $\mathcal{I}_2(\Sigma_*)$, we let $X_c$ and $X_i$ be non-singular matrices such that the controllable pair $(A_{cc}, B_c)$ is transformed into Brunovsky canonical form, i.e.

$$X_c^{-1} A_{cc} X_c = \begin{bmatrix} 0 & I_{\ell_1 - 1} & \cdots & 0 & 0 \\ \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{\ell_{m_c} - 1} \\ \star & \star & \cdots & \star & \star \end{bmatrix}, \quad X_c^{-1} B_c X_i = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix} \tag{37}$$

where $\star$ denote constant scalars or row vectors. Then we have

$$\mathcal{I}_2(\Sigma_*) = \left\{ \ell_1, \cdots, \ell_{m_c} \right\} \tag{38}$$

which is also called the controllability index of $(A_{cc}, B_c)$. Similarly, we have

$$\mathcal{I}_3(\Sigma_*) = \left\{ \mu_1, \cdots, \mu_{p_b} \right\} \tag{39}$$

where $\left\{ \mu_1, \cdots, \mu_{p_b} \right\}$ is the controllability index of the controllable pair $(A_{bb}', C_b')$.

By now it is clear that the special coordinate basis decomposes the state-space into several distinct parts. In fact, the state-space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \oplus \mathcal{X}_c \oplus \mathcal{X}_d \tag{40}$$

Here $\mathcal{X}_a^-$ is related to the stable invariant zeros, i.e. the eigenvalues of $A_{aa}^-$ are the stable invariant zeros of $\Sigma_*$. Similarly, $\mathcal{X}_a^0$ and $\mathcal{X}_a^+$ are respectively related to the invariant zeros of $\Sigma_*$ located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_b$ is related to the right invertibility, i.e. the system is right invertible if and only if $\mathcal{X}_b = \left\{0\right\}$, whereas $\mathcal{X}_c$ is related to left invertibility, i.e. the system is left invertible if and only if $\mathcal{X}_c = \left\{0\right\}$. Finally, $\mathcal{X}_d$ is related to zeros of $\Sigma_*$ at infinity.

There are interconnections between the special coordinate basis and various invariant geometric subspaces. To show these interconnections, we introduce the following geometric subspaces:

**Definition 2** (geometric subspaces $\mathcal{V}^X$ and $\mathcal{S}^X$): The weakly unobservable subspaces of $\Sigma_*$, $\mathcal{V}^X$, and the strongly controllable subspaces of $\Sigma_*$, $\mathcal{S}^X$, are defined as follows:

(1) $\mathcal{V}^X(\Sigma_*)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A_* + B_*F_*)$-invariant and contained in $\operatorname{Ker}(C_* + D_*F_*)$ such that the eigenvalues of $(A_* + B_*F_*)|\mathcal{V}^X$ are contained in $\mathbb{C}^X \subseteq \mathbb{C}$ for some constant matrix $F_*$.

(2) $\mathcal{S}^X(\Sigma_*)$ is the minimal $(A_* + K_*C_*)$-invariant subspace of $\mathbb{R}^n$ containing $\operatorname{Im}(B_* + K_*D_*)$ such that the eigenvalues of the map which is induced by $(A_* + K_*C_*)$ on the factor space $\mathbb{R}^n/\mathcal{S}^X$ are contained in $\mathbb{C}^X \subseteq \mathbb{C}$ for some constant matrix $K_*$.

Furthermore, we let $\mathcal{V}^- = \mathcal{V}^X$ and $\mathcal{S}^- = \mathcal{S}^X$, if $\mathbb{C}^X = \mathbb{C}^- \cup \mathbb{C}^0$; $\mathcal{V}^+ = \mathcal{V}^X$ and $\mathcal{S}^+ = \mathcal{S}^X$, if $\mathbb{C}^X = \mathbb{C}^+$; $\mathcal{V}^{\odot} = \mathcal{V}^X$ and $\mathcal{S}^{\odot} = \mathcal{S}^X$, if $\mathbb{C}^X = \mathbb{C}^{\odot} \cup \mathbb{C}^{\bigcirc}$; $\mathcal{V}^{\otimes} = \mathcal{V}^X$ and $\mathcal{S}^{\otimes} = \mathcal{S}^X$, if $\mathbb{C}^X = \mathbb{C}^{\otimes}$; and finally $\mathcal{V}^* = \mathcal{V}^X$ and $\mathcal{S}^* = \mathcal{S}^X$, if $\mathbb{C}^X = \mathbb{C}$.

Various components of the state vector of the special coordinate basis have the following geometrical interpretations.

**Property 5:**

(1) $x_a^- \oplus x_a^0 \oplus x_c$ spans $\begin{cases} \mathcal{V}^-(\Sigma_*) & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{V}^{\odot}(\Sigma_*) & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(2) $x_a^+ \oplus x_c$ spans $\begin{cases} \mathcal{V}^+(\Sigma_*) & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{V}^{\otimes}(\Sigma_*) & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(3) $x_a^- \oplus x_a^0 \oplus x_a^+ \oplus x_c$ spans $\mathcal{V}^*(\Sigma_*)$.

(4) $x_a^+ \oplus x_c \oplus x_d$ spans $\begin{cases} \mathcal{S}^-(\Sigma_*) & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{S}^{\odot}(\Sigma_*) & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(5) $x_a^- \oplus x_a^0 \oplus x_c \oplus x_d$ spans $\begin{cases} \mathcal{S}^+(\Sigma_*) & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{S}^{\otimes}(\Sigma_*) & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(6) $x_c \oplus x_d$ spans $\mathcal{S}^*(\Sigma_*)$.

Finally, we introduce two more subspaces of $\Sigma_*$. The original definitions of these subspaces were given by Scherer (1992).

**Definition 3** (Geometric Subspaces $\mathcal{V}_\lambda$ and $\mathcal{S}_\lambda$): For any $\lambda \in \mathbb{C}^-$, we define

$$\mathcal{V}_\lambda(\Sigma_*) := \left\{ \zeta \in \mathbb{C}^n \,\middle|\, \exists \omega \in \mathbb{C}^m : 0 = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} \right\} \tag{41}$$

and

$$\mathcal{S}_\lambda(\Sigma_*) := \left\{ \zeta \in \mathbb{C}^n \,\middle|\, \exists \omega \in \mathbb{C}^{n+m} : \begin{pmatrix} \zeta \\ 0 \end{pmatrix} = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} \omega \right\} \tag{42}$$

$\mathcal{V}_\lambda(\Sigma_*)$ and $\mathcal{S}_\lambda(\Sigma_*)$ are associated with the so-called state zero directions of $\Sigma_*$ if $\lambda$ is an invariant zero of $\Sigma_*$.

These subspaces $\mathcal{S}_\lambda(\Sigma_*)$ and $\mathcal{V}_\lambda(\Sigma_*)$ can also be easily obtained using the special coordinate basis. We have the following new property of the special coordinate basis, which has not been reported in any previous work.

**Property 6:**

$$\mathcal{S}_\lambda(\Sigma_*) = \operatorname{Im}\left\{\Gamma_s \begin{bmatrix} \lambda I - A_{aa} & 0 & 0 & 0 \\ 0 & Y_{b\lambda} & 0 & 0 \\ 0 & 0 & I_{n_c} & 0 \\ 0 & 0 & 0 & I_{n_d} \end{bmatrix}\right\} \tag{43}$$

where

$$\operatorname{Im}\{Y_{b\lambda}\} = \operatorname{Ker}\left[C_b(A_{bb} + K_b C_b - \lambda I)^{-1}\right] \tag{44}$$

and where $K_b$ is any appropriately dimensional matrix subject to the constraint that $A_{bb} + K_b C_b$ has no eigenvalue at $\lambda$. We note that such a $K_b$ always exists as $(A_{bb}, C_b)$ is completely observable.

$$\mathcal{V}_\lambda(\Sigma_*) = \operatorname{Im}\left\{\Gamma_s \begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \tag{45}$$

where $X_{a\lambda}$ is a matrix whose columns form a basis for the subspace

$$\{\zeta_a \in \mathbb{C}^{n_a} \mid (\lambda I - A_{aa})\zeta_a = 0\} \tag{46}$$

and

$$X_{c\lambda} := (A_{cc} + B_c F_c - \lambda I)^{-1} B_c \tag{47}$$

with $F_c$ being any appropriately dimensional matrix subject to the constraint that $A_{cc} + B_c F_c$ has no eigenvalue at $\lambda$. Again, we note that the existence of such an $F_c$ is guaranteed by the controllability of $(A_{cc}, B_c)$.

Clearly, if $\lambda \notin \lambda(A_{aa})$, then we have

$$\mathcal{V}_\lambda(\Sigma_*) \subseteq \mathcal{V}^{\mathrm{X}}(\Sigma_*) \tag{48}$$

and

$$\mathcal{S}_\lambda(\Sigma_*) \supseteq \mathcal{S}^{\mathrm{X}}(\Sigma_*) \tag{49}$$

Next, we would like to note that the subspaces $\mathcal{V}^{\mathrm{X}}(\Sigma_*)$ and $\mathcal{S}^{\mathrm{X}}(\Sigma_*)$ are dual in the sense that $\mathcal{V}^{\mathrm{X}}(\Sigma_*^\star) = \mathcal{S}^{\mathrm{X}}(\Sigma_*)^\perp$, where $\Sigma_*^\star$ is characterized by the quadruple $(A_*', C_*', B_*', D_*')$. Also, $\mathcal{S}_\lambda(\Sigma_*) = \mathcal{V}_{\bar{\lambda}}(\Sigma_*^\star)^\perp$.

## 3. Proofs of properties of the special coordinate basis

In this section, we provide detailed proofs for all the properties of the special coordinate basis listed in the previous section. We recall the following two lemmas whose results are quite well-known in the literature. The first lemma is about the effects of state feedback laws.

**Lemma 1:** *Consider a given system* $\Sigma_*$ *characterized by* $(A_*, B_*, C_*, D_*)$ *or in the state space form of* (1). *Also, consider a constant state feedback gain matrix* $F_* \in \mathbb{R}^{m\times n}$. *Then,* $\Sigma_{*F}$ *as characterized by the quadruple* $(A_* + B_*F_*, B_*, C_* + D_*F_*, D_*)$ *has the following properties:*

(1) $\Sigma_{*F}$ *is controllable (stabilizable) if and only if* $\Sigma_*$ *is a controllable (stabilizable);*
(2) *the normal rank of* $\Sigma_{*F}$ *is equal to that of* $\Sigma_*$*;*
(3) *the invariant zero structure of* $\Sigma_{*F}$ *is the same as that of* $\Sigma_*$*;*
(4) *the infinite zero structure of* $\Sigma_{*F}$ *is the same as that of* $\Sigma_*$*;*
(5) $\Sigma_{*F}$ *is (left- or right- or non-) invertible if and only if* $\Sigma_*$ *is (left- or right- or non-) invertible;*
(6) $\mathcal{V}^X(\Sigma_F) = \mathcal{V}^X(\Sigma_*)$ *and* $\mathcal{S}^X(\Sigma_{*F}) = \mathcal{S}^X(\Sigma_*)$*; and*
(7) $\mathcal{V}_\lambda(\Sigma_{*F}) = \mathcal{V}_\lambda(\Sigma_*)$ *and* $\mathcal{S}_\lambda(\Sigma_{*F}) = \mathcal{S}_\lambda(\Sigma_*)$.

**Proof:** Item 1 is obvious. Items 3, 4 and 5 are well known since all the lists of Morse, i.e. $\mathcal{I}_1$ to $\mathcal{I}_4$, are invariant under any state feedback laws. Furthermore, items 2 and 5 can be seen from the following simple manipulations:

$$\begin{aligned}
H_{*F}(\varsigma) &:= C_* + D_*F_*)(\varsigma I - A_* - B_*F_*)^{-1}B_* + D_* \\
&= (C_* + D_*F_*)(\varsigma I - A_*)^{-1}\left[I - B_*F_*(\varsigma I - A_*)^{-1}\right]^{-1}B_* + D_* \\
&= (C_* + D_*F_*)(\varsigma I - A_*)^{-1}B_*\left[I - F_*(\varsigma I - A_*)^{-1}B_*\right]^{-1} + D_* \\
&= \left[C_*(\varsigma I - A_*)^{-1}B_* + D_*\right]\left[I - F_*(\varsigma I - A_*)^{-1}B_*\right]^{-1} \\
&= H_*(\varsigma)\left[I - F_*(\varsigma I - A_*)^{-1}B_*\right]^{-1}
\end{aligned} \tag{50}$$

Since $\left[I - F_*(\varsigma I - A_*)^{-1}B_*\right]^{-1}$ is well defined almost everywhere on the complex plane, the results of items 2 and 5 follow.

For item 6, it is obvious from the definition of $\mathcal{V}^X$ that it is invariant under any state feedback laws. Next, for any subspace $\mathcal{S}$ that satisfies the following conditions:

$$(A_* + K_*C_*)\mathcal{S} \subseteq \mathcal{S} \tag{51}$$

$$\operatorname{Im}(B_* + K_*D_*) \subseteq \mathcal{S} \tag{52}$$

we have

$$(A_* + K_*C_* + B_*F_* + K_*D_*F_*)\mathcal{S} = (A_* + K_*C_*)\mathcal{S} + (B_* + K_*D_*)F_*\mathcal{S} \subseteq \mathcal{S}$$

Thus, $\mathcal{S}^X$ is also invariant under any state feedback laws.

Let us now prove item 7. Recalling the definition of $\mathcal{V}_\lambda$, we have

$$\mathcal{V}_\lambda(\Sigma_{*F}) = \left\{ \zeta \in \mathbb{C}^n \,\middle|\, \exists \omega \in \mathbb{C}^m : 0 = \begin{bmatrix} A_* + B_*F_* - \lambda I & B_* \\ C_* + D_*F_* & D_* \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} \right\}$$

Then, for any $\zeta \in \mathcal{V}_\lambda(\Sigma_{*F})$, there exists an $\omega \in \mathbb{C}^m$ such that

$$0 = \begin{bmatrix} A_* + B_* F_* - \lambda I & B_* \\ C_* + D_* F_* & D_* \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix} = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} \begin{bmatrix} I & 0 \\ F_* & I \end{bmatrix} \begin{pmatrix} \zeta \\ \omega \end{pmatrix}$$

or

$$0 = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} \begin{pmatrix} \zeta \\ \tilde{\omega} \end{pmatrix}$$

where $\tilde{\omega} = F_* \zeta + \omega$. Thus, $\zeta \in \mathcal{V}_\lambda(\Sigma_*)$ and hence $\mathcal{V}_\lambda(\Sigma_{*\mathrm{F}}) \subseteq \mathcal{V}_\lambda(\Sigma_*)$. Similarly, one can show that $\mathcal{V}_\lambda(\Sigma_*) \subseteq \mathcal{V}_\lambda(\Sigma_{*\mathrm{F}})$, and hence $\mathcal{V}_\lambda(\Sigma_*) = \mathcal{V}_\lambda(\Sigma_{*\mathrm{F}})$. The result that $\mathcal{S}_\lambda(\Sigma_{*\mathrm{F}}) = \mathcal{S}_\lambda(\Sigma_*)$ can be shown using similar arguments. □

The following lemma is about the effects of output injection laws.

**Lemma 2:** *Consider a given system* $\Sigma_*$ *characterized by* $(A_*, B_*, C_*, D_*)$ *or in the state space form of* (1). *Also, consider a constant output injection gain matrix* $K_* \in \mathbb{R}^{n \times p}$. *Then,* $\Sigma_{*\mathrm{K}}$ *as characterized by the quadruple* $(A_* + K_* C_*, B_* + K_* D_*, C_*, D_*)$ *has the following properties:*

(1) $\Sigma_{*\mathrm{K}}$ *is observable (detectable) if and only if* $\Sigma_*$ *is an observable (detectable);*
(2) *the normal rank of* $\Sigma_{*\mathrm{K}}$ *is equal to that of* $\Sigma_*$*;*
(3) *the invariant zero structure of* $\Sigma_{*\mathrm{K}}$ *is the same as that of* $\Sigma_*$*;*
(4) *the infinite zero structure of* $\Sigma_{*\mathrm{K}}$ *is the same as that of* $\Sigma_*$*;*
(5) $\Sigma_{*\mathrm{K}}$ *is (left- or right- or non-) invertible if and only if* $\Sigma_*$ *is (left- or right- or non-) invertible;*
(6) $\mathcal{V}^{\mathrm{X}}(\Sigma_{*\mathrm{K}}) = \mathcal{V}^{\mathrm{X}}(\Sigma_*)$ *and* $\mathcal{S}^{\mathrm{X}}(\Sigma_{*\mathrm{K}}) = \mathcal{S}^{\mathrm{X}}(\Sigma_*)$*; and*
(7) $\mathcal{V}_\lambda(\Sigma_{*\mathrm{K}}) = \mathcal{V}_\lambda(\Sigma_*)$ *and* $\mathcal{S}_\lambda(\Sigma_{*\mathrm{K}}) = \mathcal{S}_\lambda(\Sigma_*)$.

**Proof.** It is the dual version of Lemma 1. □

Now, we are ready to prove the properties of the special coordinate basis. Without loss of any generality but for simplicity of presentation, we assume throughout the rest of this section that the given system $\Sigma_*$ has already been transformed into the special coordinate basis of Theorem 1 or into the compact form of (20)–(23), i.e.

$$A_* = \begin{bmatrix} A_{aa} & L_{ab} C_b & 0 & L_{ad} C_d \\ 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca} & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da} & B_d E_{db} & B_d E_{dc} & A_{dd}^* + B_d E_{dd} + L_{dd} C_d \end{bmatrix} + B_{*,0} C_{*,0} \tag{53}$$

$$B_* = \begin{bmatrix} B_{*,0} & B_{*,1} \end{bmatrix} = \begin{bmatrix} B_{0a} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix} \tag{54}$$

and

$$C_* = \begin{bmatrix} C_{*,0} \\ C_{*,1} \end{bmatrix} = \begin{bmatrix} C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}, \quad D_* = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{55}$$

We further note that $A_{dd}^*$, $B_d$ and $C_d$ have the following forms:

$$A_{dd}^* = \text{blkdiag}\left\{A_{q_1}, \cdots, A_{q_{m_d}}\right\} \tag{56}$$

and

$$B_d = \text{blkdiag}\left\{B_{q_1}, \cdots, B_{q_{m_d}}\right\}, \quad C_d = \text{blkdiag}\left\{C_{q_1}, \cdots, C_{q_{m_d}}\right\} \tag{57}$$

where $A_{q_i}$, $B_{q_i}$ and $C_{q_i}$, $i = 1, 2, \cdots, m_d$, are defined as in (16).

**Proof of Property 1:** Let us define a state feedback gain matrix $F_*$ as follows:

$$F_* = -\begin{bmatrix} C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ E_{da} & E_{db} & E_{dc} & E_{dd} \\ E_{ca} & 0 & 0 & 0 \end{bmatrix} \tag{58}$$

Then, we have

$$A_* + B_* F_* = \begin{bmatrix} A_{aa} & L_{ab} C_b & 0 & L_{ad} C_d \\ 0 & A_{bb} & 0 & L_{bd} C_d \\ 0 & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ 0 & 0 & 0 & A_{dd}^* + L_{dd} C_d \end{bmatrix} \tag{59}$$

Noting that $(A_{cc}, B_c)$ is completely controllable, we have for any $\lambda \in \mathbb{C}$,

$$\text{rank}\begin{bmatrix} A_* + B_* F_* - \lambda I & B_* \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} A_{aa} - \lambda I & L_{ab} C_b & 0 & L_{ad} C_d & B_{0a} & 0 & 0 \\ 0 & A_{bb} - \lambda I & 0 & L_{bd} C_d & B_{0b} & 0 & 0 \\ 0 & L_{cb} C_b & A_{cc} - \lambda I & L_{cd} C_d & B_{0c} & 0 & B_c \\ 0 & 0 & 0 & A_{dd}^* + L_{dd} C_d - \lambda I & B_{0d} & B_d & 0 \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} A_{aa} - \lambda I & L_{ab} C_b & 0 & L_{ad} C_d & B_{0a} & 0 & 0 \\ 0 & A_{bb} - \lambda I & 0 & L_{bd} C_d & B_{0b} & 0 & 0 \\ 0 & 0 & A_{cc} - \lambda I & 0 & 0 & 0 & B_c \\ 0 & 0 & 0 & A_{dd}^* + L_{dd} C_d - \lambda I & B_{0d} & B_d & 0 \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} A_{\text{con}} - \lambda I & 0 & B_{\text{con}1} C_d & B_{\text{con}0} & 0 & 0 \\ 0 & A_{cc} - \lambda I & 0 & 0 & 0 & B_c \\ 0 & 0 & A_{dd}^* + L_{dd} C_d - \lambda I & B_{0d} & B_d & 0 \end{bmatrix} \tag{60}$$

where

$$A_{\rm con} = \begin{bmatrix} A_{aa} & L_{ab}C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{\rm con} = \begin{bmatrix} B_{\rm con0} & B_{\rm con1} \end{bmatrix} = \begin{bmatrix} B_{0a} & L_{ad} \\ B_{0b} & L_{bd} \end{bmatrix} \tag{61}$$

Also, noting the special structure of $(A^*_{dd}, B_d, C_d)$, it is simple to verify that $\begin{bmatrix} A_* + B_*F_* - \lambda I & B_* \end{bmatrix}$ is of maximal rank if and only if $\begin{bmatrix} A_{\rm con} - \lambda I & B_{\rm con} \end{bmatrix}$ is of maximal rank. By Lemma 1, we have that $(A, B)$ is controllable (stabilizable) if and only if $(A_{\rm con}, B_{\rm con})$ is controllable (stabilizable).

Similarly, one can show that $(A, C)$ is observable (detectable) if and only if $(A_{\rm obs}, C_{\rm obs})$ is observable (detectable). □

**Proof of Property 2:** Let us define a state feedback gain matrix $F_*$ as in (58) and an output injection gain matrix $K_*$ as follows:

$$K_* = -\begin{bmatrix} B_{0a} & L_{ad} & L_{ab} \\ B_{0b} & L_{bd} & 0 \\ B_{0c} & L_{cd} & L_{cb} \\ B_{0d} & L_{dd} & 0 \end{bmatrix} \tag{62}$$

We have

$$\breve{A}_* = A_* + B_*F_* + K_*C_* + K_*D_*F_* = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb} & 0 & 0 \\ 0 & 0 & A_{cc} & 0 \\ 0 & 0 & 0 & A^*_{dd} \end{bmatrix} \tag{63}$$

$$\breve{B}_* = B_* + K_*D_* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_d & 0 \end{bmatrix} \tag{64}$$

$$\breve{C}_* = C_* + D_*F_* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix} \tag{65}$$

and

$$\breve{D}_* = D_* = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{66}$$

Let $\breve{\Sigma}_*$ be characterized by the quadruple $(\breve{A}_*, \breve{B}_*, \breve{C}_*, \breve{D}_*)$. It is simple to verify that the transfer function of $\breve{\Sigma}_*$ is given by

$$\check{H}_*(\varsigma) = \check{C}_*(\varsigma I - \check{A}_*)^{-1}\check{B}_* + \check{D}_* = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & C_d(\varsigma I - A_{dd}^*)^{-1}B_d & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{67}$$

Furthermore, we can show that

$$C_d(\varsigma I - A_{dd}^*)^{-1}B_d = \begin{bmatrix} \frac{1}{\varsigma^{q_1}} & & \\ & \ddots & \\ & & \frac{1}{\varsigma^{q_{m_d}}} \end{bmatrix} \tag{68}$$

By Lemmas 1 and 2, we have

$$\text{normrank}\{H_*(\varsigma)\} = \text{normrank}\{\check{H}_*(\varsigma)\} = m_0 + m_d \tag{69}$$

Next, it follows from Lemmas 1 and 2 that the invariant zeros of $\Sigma_*$ and $\check{\Sigma}_*$ are equivalent. By the definition of the invariant zeros of a linear system, i.e. a complex scalar $\alpha$ is an invariant zero of $\check{\Sigma}_*$ if

$$\text{rank}\begin{bmatrix} \check{A}_* - \alpha I & \check{B}_* \\ \check{C}_* & \check{D}_* \end{bmatrix} < n + \text{normrank}\{\check{H}_*(\varsigma)\} = n + m_0 + m_d \tag{70}$$

and also noting the special structure of $(A_{dd}^*, B_d, C_d)$ and the facts that $(A_{bb}, C_b)$ is observable, and $(A_{cc}, B_c)$ is controllable, we have

$$\begin{aligned} \text{rank}\{P_{\check{\Sigma}_*}(\alpha)\} &= \text{rank}\begin{bmatrix} \check{A}_* - \alpha I & \check{B}_* \\ \check{C}_* & \check{D}_* \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} A_{aa} - \alpha I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{bb} - \alpha I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{cc} - \alpha I & 0 & 0 & 0 & B_c \\ 0 & 0 & 0 & A_{dd}^* - \alpha I & 0 & B_d & 0 \\ 0 & 0 & 0 & 0 & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & C_d & 0 & 0 & 0 \\ 0 & C_b & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= n_b + n_c + n_d + m_0 + m_d + \text{rank}\{A_{aa} - \alpha I\}. \end{aligned} \tag{71}$$

Clearly, the rank of $P_{\check{\Sigma}_*}(\alpha)$ drops below $n + m_0 + m_d$ if and only if $\alpha \in \lambda(A_{aa})$. Hence, the invariant zeros of $\check{\Sigma}_*$, or equivalently the invariant zeros of $\Sigma_*$, are given by the eigenvalues of $A_{aa}$, which are the union of $\lambda(A_{aa}^-)$, $\lambda(A_{aa}^0)$, and $\lambda(A_{aa}^+)$. This completes the proof of Property 2. □

**Proof of Property 3:** It follows from Lemmas 1 and 2 that the infinite zeros of $\Sigma_*$ and $\check{\Sigma}_*$ are equivalent. It is clear to see from (74) and (75) that the infinite zeros of $\check{\Sigma}_*$, or equivalently the infinite zeros of $\Sigma_*$, of order higher than 0, are given by

$$S_\infty^\star(\Sigma_*) = S_\infty^\star(\check{\Sigma}_*) = \{q_1, q_2, \cdots, q_{m_d}\} \tag{72}$$

Furthermore, $\check{\Sigma}_*$ or $\Sigma_*$ has $m_0$ infinite zeros of order 0. □

**Proof of Property 4:** Again, it follows from Lemmas 1 and 2 that $\Sigma_*$ or $H_*(\varsigma)$ is (left- or right- or non-) invertible if and only if $\check{\Sigma}_*$ or $\check{H}_*(\varsigma)$ is (left- or right- or non-) invertible. The results of Property 4 can be seen from the transfer function $\check{H}_*(\varsigma)$ in (67). □

**Proof of Property 5:** We will prove only the geometric subspace $\mathcal{V}^*(\Sigma_*)$, i.e.

$$\mathcal{V}^*(\Sigma_*) = \mathcal{X}_a \oplus \mathcal{X}_c = \mathrm{Im}\left\{ \Gamma_s \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} \right\} \tag{73}$$

Here $\Gamma_s = I_n$ since the given system $\Sigma_*$ is assumed to be already in the form of the special coordinate basis. It follows from Lemma 2 that $\mathcal{V}^*$ is invariant under any output injection laws. Let us choose an output injection gain matrix $K_*$ as in (69). Then, we have

$$\hat{A}_* = A_* + K_* C_* = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb} & 0 & 0 \\ B_c E_{ca} & 0 & A_{cc} & 0 \\ B_d E_{da} & B_d E_{db} & B_d E_{dc} & A_{dd}^* + B_d E_{dd} \end{bmatrix} \tag{74}$$

and

$$\hat{B}_* = B_* + K_* D_* = \check{B}_* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_d & 0 \end{bmatrix} \tag{75}$$

Let $\hat{\Sigma}_*$ be a system characterized by $(\hat{A}_*, \hat{B}_*, C_*, D_*)$. Then it is sufficient to show the property of $\mathcal{V}^*(\Sigma_*)$ by showing that

$$\mathcal{V}^*(\hat{\Sigma}_*) = \mathrm{Im}\left\{ \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} \right\} \tag{76}$$

First, let us choose a matrix $F_*$ as given in (58). Then, we have

$$\hat{A}_* + \hat{B}_* F_* = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb} & 0 & 0 \\ 0 & 0 & A_{cc} & 0 \\ 0 & 0 & 0 & A_{dd}^* \end{bmatrix} \tag{77}$$

and

$$C_* + D_* F_* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix} \tag{78}$$

It is now simple to see that for any

$$\zeta \in \mathcal{X}_a \oplus \mathcal{X}_c = \operatorname{Im}\left\{\begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix}\right\} \tag{79}$$

we have

$$\zeta = \begin{pmatrix} \zeta_a \\ 0 \\ \zeta_c \\ 0 \end{pmatrix} \tag{80}$$

and

$$(\hat{A}_* + \hat{B}_* F_*)\zeta = \begin{pmatrix} A_{aa}\zeta_a \\ 0 \\ A_{cc}\zeta_c \\ 0 \end{pmatrix} \in \operatorname{Im}\left\{\begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix}\right\} = \mathcal{X}_a \oplus \mathcal{X}_c \tag{81}$$

and

$$(C_* + D_* F_*)\zeta = 0 \tag{82}$$

Clearly, $\mathcal{X}_a \oplus \mathcal{X}_c$ is an $(\hat{A}_* + \hat{B}_* F_*)$-invariant subspace of $\mathbb{R}^n$ and is contained in Ker $(C_* + D_* F_*)$. By the definition of $\mathcal{V}^*$, we have

$$\mathcal{X}_a \oplus \mathcal{X}_c \subseteq \mathcal{V}^*(\hat{\Sigma}_*) \tag{83}$$

Conversely, for any $\zeta \in \mathcal{V}^*(\Sigma_*)$, by Definition 2, there exists a gain matrix $\hat{F}_* \in \mathbb{R}^{m\times n}$ such that

$$(\hat{A}_* + \hat{B}_* \hat{F}_*)\zeta \in \mathcal{V}^*(\hat{\Sigma}_*) \tag{84}$$

and

$$(C_* + D_* \hat{F}_*)\zeta = 0 \tag{85}$$

(84) and (85) imply that for any $\zeta \in \mathcal{V}^*(\Sigma_*)$

$$(C_* + D_*\hat{F}_*)(\hat{A}_* + \hat{B}_*\hat{F}_*)^k \zeta = 0, \quad k = 0, 1, \ldots, n-1 \tag{86}$$

Thus, (83) and (86) imply that

$$(C_* + D_*\hat{F}_*)(\hat{A}_* + \hat{B}_*\hat{F}_*)^k \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} = 0, \quad k = 0, 1, \ldots, n-1 \tag{87}$$

Next, let us partition this $\hat{F}_*$ as follows:

$$\hat{F}_* = \begin{bmatrix} F_{a0} - C_{0a} & F_{b0} - C_{0b} & F_{c0} - C_{0c} & F_{d0} - C_{0d} \\ F_{ad} - E_{da} & F_{bd} - E_{db} & F_{cd} - E_{dc} & F_{dd} - E_{dd} \\ F_{ac} - E_{ca} & F_{bc} & F_{cc} & F_{dc} \end{bmatrix} \tag{88}$$

We have

$$C_* + D_*\hat{F}_* = \begin{bmatrix} F_{a0} & F_{b0} & F_{c0} & F_{d0} \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix} \tag{89}$$

and

$$\hat{A}_* + \hat{B}_*\hat{F}_* = \begin{bmatrix} A_{aa} & 0 & 0 & 0 \\ 0 & A_{bb} & 0 & 0 \\ B_cF_{ac} & B_cF_{bc} & A_{cc} + B_cF_{cc} & B_cF_{dc} \\ B_dF_{ad} & B_dF_{bd} & B_dF_{cd} & A_{dd}^{**} \end{bmatrix} \tag{90}$$

where $A_{dd}^{**} = A_{dd}^{*} + B_dF_{dd}$. Then, using (87) with $k = 0$, we have

$$(C_* + D_*\hat{F}_*) \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} = 0 \tag{91}$$

which implies

$$F_{a0} = 0, \quad F_{c0} = 0 \tag{92}$$

and

$$C_* + D_*\hat{F}_* = \begin{bmatrix} 0 & \star & 0 & \star \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix} \tag{93}$$

where $\star$ are some matrices of not much interest. Using (87) with $k = 1$ together with (93), we have

$$C_dB_dF_{ad} = 0, \quad C_dB_dF_{cd} = 0 \tag{94}$$

and

$$(C_* + D_*\hat{F}_*)(\hat{A}_* + \hat{B}_*\hat{F}_*) = \begin{bmatrix} 0 & \bigstar & 0 & \bigstar \\ 0 & C_d B_d F_{bd} & 0 & C_d A_{dd}^{**} \\ 0 & C_b A_{bb} & 0 & 0 \end{bmatrix} \tag{95}$$

In general, one can show that for any positive integer $k$

$$C_d (A_{dd}^{**})^{k-1} B_d F_{ad} = 0, \quad C_d (A_{dd}^{**})^{k-1} B_d F_{cd} = 0 \tag{96}$$

and

$$(C_* + D_*\hat{F}_*)(\hat{A}_* + \hat{B}_*\hat{F}_*)^k = \begin{bmatrix} 0 & \bigstar & 0 & \bigstar \\ 0 & \bigstar & 0 & C_d (A_{dd}^{**})^k \\ 0 & C_b (A_{bb})^k & 0 & 0 \end{bmatrix} \tag{97}$$

As a by-product, we can easily show that $F_{ad} = 0$ and $F_{cd} = 0$, because of the fact that $(A_{dd}^{**}, B_d, C_d)$ is controllable, observable, invertible and is free of invariant zeros. Now, for any

$$\zeta = \begin{pmatrix} \zeta_a \\ \zeta_b \\ \zeta_c \\ \zeta_d \end{pmatrix} \in \mathcal{V}^*(\hat{\Sigma}_*) \tag{98}$$

it follows from (86) and (97) that

$$C_b (A_{bb})^k \zeta_b = 0, \quad k = 0, 1, \ldots, n-1 \tag{99}$$

which implies $\zeta_b = 0$ because $(A_{bb}, C_b)$ is completely observable, and

$$C_d (A_{dd}^{**})^k \zeta_d + \bigstar\, \zeta_b = C_d (A_{dd}^{**})^k \zeta_d = 0, \quad k = 0, 1, \ldots, n-1 \tag{100}$$

which implies $\zeta_d = 0$ because $(A_{dd}^{**}, C_d)$ is also completely observable. Hence,

$$\zeta = \begin{pmatrix} \zeta_a \\ 0 \\ \zeta_c \\ 0 \end{pmatrix} \in \mathrm{Im} \left\{ \begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \\ 0 & I_{n_c} \\ 0 & 0 \end{bmatrix} \right\} = \mathcal{X}_a \oplus \mathcal{X}_c \tag{100}$$

and

$$\mathcal{V}^*(\hat{\Sigma}_*) \subseteq \mathcal{X}_a \oplus \mathcal{X}_c \tag{102}$$

Obviously, (83) and (102) imply the result.

Similarly, one can follow the same procedure as in the above to show the properties of the other subspaces in Property 5. □

**Proof of Property 6:** Let us prove the property of $\mathcal{V}_\lambda(\Sigma_*)$. It follows from Lemmas 1 and 2 that $\mathcal{V}_\lambda$ is invariant under any state feedback and output injection laws. Thus, it is sufficient to prove the property of $\mathcal{V}_\lambda(\Sigma_*)$ by showing that

$$\mathcal{V}_\lambda(\check{\Sigma}_*) = \mathrm{Im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \tag{103}$$

where $\check{\Sigma}_*$ is as defined in the proof of Property 2, $X_{a\lambda}$ is a matrix whose columns form a basis for the subspace

$$\left\{\zeta_a \in \mathbb{C}^{n_a} \middle| (\lambda I - A_{aa})\zeta_a = 0\right\} \tag{104}$$

and

$$X_{c\lambda} = (A_{cc} + B_c F_c - \lambda I)^{-1} B_c \tag{105}$$

with $F_c$ being an appropriately dimensional matrix such that $A_{cc} + B_c F_c - \lambda I$ is invertible.

For any $\zeta \in \mathcal{V}_\lambda(\check{\Sigma}_*)$, by Definition 3, there exists a vector $\omega \in \mathbb{C}^m$ such that

$$\begin{bmatrix} \check{A}_* - \lambda I & \check{B}_* \\ \check{C}_* & \check{D}_* \end{bmatrix}\begin{pmatrix} \zeta \\ \omega \end{pmatrix} = 0 \tag{106}$$

or, equivalently,

$$\begin{bmatrix} A_{aa}-\lambda I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{bb}-\lambda I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{cc}-\lambda I & 0 & 0 & 0 & B_c \\ 0 & 0 & 0 & A_{dd}^*-\lambda I & 0 & B_d & 0 \\ 0 & 0 & 0 & 0 & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & C_d & 0 & 0 & 0 \\ 0 & C_b & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \zeta_a \\ \zeta_b \\ \zeta_c \\ \zeta_d \\ \omega_0 \\ \omega_d \\ \omega_c \end{pmatrix} = 0 \tag{107}$$

Hence, we have

$$(A_{aa} - \lambda I)\zeta_a = 0 \tag{108}$$

which implies that $\zeta_a \in \mathrm{Im}\{X_{a\lambda}\}$

$$\begin{bmatrix} A_{bb} - \lambda I \\ C_b \end{bmatrix}\zeta_b = 0 \tag{109}$$

which implies that $\zeta_b = 0$ since $(A_{bb}, C_b)$ is completely observable, and

$$\begin{bmatrix} A_{dd}^* - \lambda I & B_d \\ C_d & 0 \end{bmatrix}\begin{pmatrix} \zeta_d \\ \omega_d \end{pmatrix} = 0 \tag{110}$$

which implies that $\zeta_d = 0$ and $\omega_d = 0$ since $(A_{dd}^*, B_d, C_d)$ is square invertible and free of invariant zeros. We also have

$$(A_{cc} - \lambda I)\zeta_c + B_c\omega_c = 0 \tag{111}$$

which implies that

$$(A_{cc} + B_c F_c - \lambda I)\zeta_c + B_c(\omega_c - F_c\zeta_c) = 0 \tag{112}$$

or

$$\zeta_c = (A_{cc} + B_c F_c - \lambda I)^{-1} B_c(F_c\zeta_c - \omega_c) = X_{c\lambda}(F_c\zeta_c - \omega_c) \tag{113}$$

Hence $\zeta_c \in \operatorname{Im}\{X_{c\lambda}\}$. Clearly

$$\zeta \in \operatorname{Im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \Longrightarrow \mathcal{V}_\lambda(\check{\Sigma}_*) \subseteq \operatorname{Im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \tag{114}$$

Conversely, for any

$$\zeta = \begin{pmatrix} \zeta_a \\ \zeta_b \\ \zeta_c \\ \zeta_d \end{pmatrix} \in \operatorname{Im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \tag{115}$$

we have $\zeta_b = 0$, $\zeta_d = 0$, $\zeta_a \in \operatorname{Im}\{X_{a\lambda}\}$, which implies that $(\lambda I - A_{aa})\zeta_a = 0$, and $\zeta_c \in \operatorname{Im}\{X_{c\lambda}\}$, which implies that there exists a vector $\tilde{\omega}_c$ such that

$$\zeta_c = X_{c\lambda}\tilde{\omega}_c = (A_{cc} + B_c F_c - \lambda I)^{-1} B_c\tilde{\omega}_c \tag{116}$$

Thus, we have

$$(A_{cc} + B_c F_c - \lambda I)\zeta_c = B_c\tilde{\omega}_c \tag{117}$$

or

$$(A_{cc} - \lambda I)\zeta_c + B_c(F_c\zeta_c - \tilde{\omega}_c) = 0 \tag{118}$$

Let

$$\omega = \begin{pmatrix} \omega_0 \\ \omega_d \\ \omega_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F_c\zeta_c - \tilde{\omega}_c \end{pmatrix} \tag{119}$$

It is now straightforward to verify, using (107), that

$$\begin{bmatrix} \check{A}_* - \lambda I & \check{B}_* \\ \check{C}_* & \check{D}_* \end{bmatrix}\begin{pmatrix} \zeta \\ \omega \end{pmatrix} = 0 \tag{120}$$

By Definition 3, we have

$$\zeta \in \mathcal{V}_\lambda(\check{\Sigma}_*) \Longrightarrow \operatorname{Im}\left\{\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\} \subseteq \mathcal{V}_\lambda(\check{\Sigma}_*) \tag{121}$$

Finally, (114) and (121) imply the result.

The proof of $s_\lambda(\Sigma_*)$ follows from the same lines of reasoning. This concludes all the proofs to the properties of the special coordinate basis. □

## 4. Conclusion

We have presented in this paper rigorous and complete proofs to all the key properties of the special coordinate basis of linear time-invariant systems, developed by Sannuti and Saberi (1987). The results of this paper complement the work of Sannuti and Saberi and make the theory of Sannuti and Saberi more complete.

## References

Chen, B. M., 1988, *Software Manual for the Special Coordinate Basis of Multivariable Linear Systems*, Washington State University Technical Report No: ECE 0094 (Washington: Pullman).

Chen, B. M., 1991, *Theory of loop transfer recovery for multivariable linear systems*. PhD dissertation, Washington State University.

Chen, B. M., 1997, Solvability conditions for the disturbance decoupling problems with static measurement feedback. *International Journal of Control*, **68**, 51–60.

Chen, B. M., 1998, *$H_\infty$ Control and its Applications* (Berlin: Springer-Verlag).

Chen, B. M., and Zheng, D. Z., 1995, Simultaneous finite and infinite zero assignments of linear systems. *Automatica*, **31**, 643–648.

Chen, B. M., Saberi, A., and Sannuti, P., 1992a, Explicit expressions for cascade factorization of general non-minimum phase systems. *IEEE Transactions on Automatic Control*, **37**, 358–363.

Chen, B. M., Saberi, A., and Sannuti, P., 1992b, On blocking zeros and strong stabilizability of linear multivariable systems. *Automatica*, **28**, 1051–1055.

Chen, B. M., Saberi, A., Sannuti, P., and Shamash, Y., 1993, Construction and parameterization of all static and dynamic $H_2$-optimal state feedback solutions, optimal fixed modes and fixed decoupling zeros. *IEEE Transactions on Automatic Control*, **38**, 248–261.

Commault, C., and Dion, J. M., 1982, Structure at infinity of linear multivariable systems: A geometric approach. *IEEE Transactions on Automatic Control*, **27**, 693–696.

Lin, Z., 1989, *The Implementation of Special Coordinate Basis for Linear Multivariable Systems in Matlab*, Washington State University Technical Report No. ECE0100 (Washington: Pullman).

Lin, Z., 1994, Global and semi-global control problems for linear systems subject to input saturation and minimum-phase input–output linearizable systems,. PhD dissertation, Washington State University.

Lin, Z., Chen, B. M., Saberi, A., and Shamash, Y., 1996, Input–output factorization of discrete-time transfer matrices. *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, **43**, 941–945.

MacFarlane, A. G. J., and Karcanias, N., 1976, Poles and zeros of linear multivariable systems: A survey of the algebraic, geometric and complex variable theory. *International Journal of Control*, **24**, 33–74.

Morse, A. S., 1973, Structural invariants of linear multivariable systems. *SIAM Journal on Control*, **11**, 446–465.

Moylan, P., 1977, Stable inversion of linear systems. *IEEE Transactions on Automatic Control*, **22**, 74–78.

Owens, D. H., 1978, Invariant zeros of multivariable systems: A geometric analysis. *International Journal of Control*, **28**, 187–198.

Ozcetin, H. K., Saberi, A., and Sannuti, P., 1990, Special coordinate basis for order reduction of linear multivariable systems. *International Journal of Control*, **52**, 191–226.

Ozcetin, H. K., Saberi, A., and Shamash, Y., 1993a, Design for $H_\infty$ almost disturbance decoupling problem with internal stability via state or measurement feedback—Singular perturbation approach. *International Journal of Control*, **55**, 901–944.

Ozcetin, H. K., Saberi, A., and Shamash, Y., 1993b, $H_\infty$-almost disturbance decoupling for non-strictly proper systems—A singular perturbation approach. *Control—Theory & Advanced Technology*, **9**, 203–245.

Pugh, A. C., and Ratcliffe, P. A., 1979, On the zeros and poles of a rational matrix. *International Journal of Control*, **30**, 213–227.

Rosenbrock, H. H., 1970, *State-space and Multivariable Theory* (New York: John-Wiley).

Saberi, A., and Sannuti, P., 1990, Squaring down of non-strictly proper systems. *International Journal of Control*, **51**, 621–629.

Saberi, A., Chen, B. M., and Sannuti, P., 1991, Theory of LTR for non-minimum phase systems, recoverable target loops, recovery in a subspace—Part 1: Analysis. *International Journal of Control*, **53**, 1067–1160.

Saberi, A., Chen, B. M., and Sannuti, P., 1993, *Loop Transfer Recovery: Analysis and Design* (London: Springer-Verlag).

Saberi, A., Ozcetin, H. K., and Sannuti, P., 1992, New structural invariants of linear multivariable systems. *International Journal of Control*, **56**, 877–900.

Saberi, A., Sannuti, P., and Chen, B. M., 1995, *$H_2$ Optimal Control* (London: Prentice-Hall).

Sannuti, P., 1983, A direct singular perturbation analysis of high-gain and cheap control problems. *Automatica*, **19**, 41–51.

Sannuti, P., and Saberi, A., 1987, A special coordinate basis of multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, **45**, 1655–1704.

Sannuti, P., and Wason, H. S., 1983, A singular perturbation canonical form of invertible systems: Determination of multivariable root-loci. *International Journal of Control*, **37**, 1259–1286.

Sannuti, P., and Wason, H. S., 1985, Multiple time-scale decomposition in cheap control problems—Singulal control. *IEEE Transactions on Automatic Control*, **30**, 633–644.

Scherer, C., 1992, $H_\infty$-optimization without assumptions on finite or infinite zeros. *SIAM Journal on Control and Optimization*, **30**, 143–166.

Silverman, L., 1969, Inversion of multivariable linear systems. *IEEE Transactions on Automatic Control*, **14**, 270–276.

Verghese, G., 1978, *Infinite frequency behavior in generalized dynamical systems*. PhD dissertation, Stanford University.