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Closed-form solutions to a class of H_∞-optimization problems

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A closed-form solution is presented to the H_∞ sub-optimal control problem via state or output feedback. The problem formulation is general and does not impose any restrictions on feedthrough terms between the control input and the controlled output or between the disturbance input and the measurement output variables. It is, however, required that both the transfer function from the disturbance input to the measurement output and the transfer function from the control input to the controlled output have no invariant zeros on the jω axis and that they satisfy certain geometric conditions. For the same class of systems, the conditions under which the H_∞ optimal control problem, via state feedback, has a solution are given. Moreover, explicit expressions for the optimal solutions are provided. Finally, the pole/zero cancellations in the closed-loop system resulting from the H_∞ optimal or sub-optimal state or output feedback are examined.

Nomenclature

A^{T}	transpose of A
A^{H}	complex conjugate transpose of A
I	an identity matrix of appropriate dimension
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	whole complex plane
$\mathbb{C}^-$	open left-half complex plane
$\mathbb{C}^+$	open right-half complex plane
$\mathbb{C}^0$	imaginary axis jω
$\sigma_{\max}(A)$	maximum singular value of A
$\lambda(A)$	the set of eigenvalues of A
$\lambda_{\max}(A)$	maximum eigenvalue of A where $\lambda(A) \subset \mathbb{R}$
$\rho(A)$	the spectral radius of A
$\mathrm{Ker}\,(V)$	kernel of V
$\mathrm{Im}\,(V)$	image of V

We say a square matrix A is stable if $\lambda(A) \in \mathbb{C}^-$ and A is anti-stable if $\lambda(A) \in \mathbb{C}^0 \cup \mathbb{C}^+$. We also define the following subspaces.

(i) $\mathcal{V}^g(A,\ B,\ C,\ D)$—the maximal subspace of $\mathbb{R}^n$ which is $(A + BF)$-invariant and contained in $\mathrm{Ker}\,(C + DF)$ such that the eigenvalues of $(A + BF)|\mathcal{V}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some F.

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(ii) $\mathcal{S}^g(A, B, C, D)$—the minimal $(A + KC)$-invariant subspace of $\mathbb{R}^n$ containing $\operatorname{Im}(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some K.

For the cases that $\mathbb{C}_g = \mathbb{C}$, $\mathbb{C}_g = \mathbb{C}^-$ and $\mathbb{C}_g = \mathbb{C}^0 \cup \mathbb{C}^+$, we replace the index g in $\mathcal{V}^g$ and $\mathcal{S}^g$ by '$*$', '$-$' and '$+$' respectively.

1. Introduction

Consider a linear time-invariant generalized plant P given in Fig. 1 with the control input u, the disturbance w, the measurement output y, and the controlled output z.

The H_∞ sub-optimal control problem is as follows: for a given number $\gamma > 0$ find a proper control law $u = K_{\text{sub}}(s)y$, if existent, such that the closed-loop system (as given in Fig. 1) is internally stable and that the H_∞ norm of the transfer function from w to z, denoted by T_{zw}, is less than γ, i.e. $\|T_{zw}\|_\infty < \gamma$. The H_∞-optimal control problem on the other hand is to find a proper control law $u = K_{\text{opt}}(s)y$, if existent, such that the closed-loop system is internally stable and $\|T_{zw}\|_\infty$ is minimized. The H_∞ sub-optimal control problem has attracted a lot of attention in the last decade ever since the work of Zames (1981). Several techniques to solve this problem are developed including the following:

(*a*) interpolation approach (e.g. Limbeer and Anderson 1988);

(*b*) frequency domain approach (e.g. Francis 1987);

(*c*) Polynomial approach (e.g. Kwakernaak 1986);

(*d*) J-spectral factorization approach (e.g. Kimura 1989);

(*e*) Time-domain approach (e.g. Doyle *et al*. 1988 and Tadmor 1990).

In our view, the time-domain approach yields the most intuitively appealing results. All the above techniques have been developed for the so-called regular case for which the plant Σ should satisfy the following assumptions.

(i) The subsystem from the control input to the controlled output should not have invariant zeros on the imaginary axis and the direct feedthrough matrix of this system should be injective.

(ii) The subsystem from the disturbance to the measurement output should not have invariant zeros on the imaginary axis and the direct feedthrough matrix of this system should be surjective.

The singular case refers to systems that do not satisfy at least one of the above conditions. Note that identical assumptions are assumed in the linear quadratic gaussian control problem. The restrictions on the feedthrough matrices

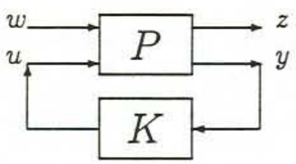

Figure 1. The standard H_∞-optimization problem.

of the above assumptions for the H_∞ sub-optimal control problem have recently been removed by Stoorvogel (1992).

Doyle *et al*. (1988) proposed, for the regular case, an H_∞ sub-optimal controller which is observer-based and can be obtained by solving two parameter-dependent and indefinite algebraic Riccati equations (hereafter referred to as H_∞-AREs). The solutions for these H_∞-AREs should also satisfy a certain coupling condition. This controller is known as a central controller and plays a crucial role in the characterization of all sub-optimal solutions for a given γ. At the present time, one must solve these H_∞-AREs to obtain any H_∞ sub-optimal controller. However, it is well known that these H_∞-AREs for some values of γ can become highly sensitive and ill-conditioned. This difficulty also arises in the verification of the coupling condition. These numerical difficulties are likely to be more severe for the singular case.

The goal of this paper is to provide closed-form solutions to the H_∞ sub-optimal control problem. Here, by closed-form solutions, we mean solutions which are explicitly parametrized in terms of γ and are obtained without explicitly requiring a value for γ. Hence, one can easily tune the parameter γ in order to obtain the desired level of disturbance attenuation. Such a design can be called a 'one-shot' design. We provide these closed-form solutions for a class of singular H_∞ sub-optimal control problems for which the subsystem from the control input u to the controlled output z and the subsystem from the disturbance w to the measurement output y satisfy certain geometric conditions. Moreover, for this class of systems we also provide conditions under which the H_∞ optimal control problem, via state feedback, has a solution and explicit expression for the solutions will also be given. Finally, the issue of pole/zero cancellations in the closed-loop system resulting from the H_∞ optimal or sub-optimal state or output feedback control laws is examined.

Some significant attributes of our method of generating the closed-form solutions in the H_∞ sub-optimal control problem are as follows.

(*a*) No H_∞-AREs are solved in generating the closed-form solutions. As such, all the numerical difficulties associated with the H_∞-AREs are alleviated.

(*b*) The value for γ can be adjusted on-line when the closed-form solution to the H_∞ sub-optimal control problem is implemented either by software or hardware. Since the effect of such a 'knob' on the performance and the robustness of the closed-loop system is straightforward, it should be very appealing from a practical point of view.

(*c*) Having closed-form solutions to the H_∞ sub-optimal control problem enables us to understand the behaviour of the controller (i.e. high-gain, bandwidth, etc) as the parameter γ approaches the infimal value of the H_∞ norm of T_{zw} over all stabilizing controllers.

The outline of this paper is as follows. In § 2 we introduce the problem statement. In § 3 we provide some preliminaries on the special coordinate basis (s.c.b.) and its properties. The s.c.b. transformation is instrumental in the derivations of the results given in this paper. Section 4 gives a closed-form solution to the H_∞ sub-optimal state feedback control problem, while § 5 provides a closed-form solution to the H_∞ sub-optimal output feedback control problem. Section 6 draws the conclusions of this paper.

2. Problem formulation

Consider a generalized system Σ with a state-space description

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1 x \qquad\quad + D_1 w \\ z = C_2 x + D_2 u \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^k$ is the external signal or disturbance, $z \in \mathbb{R}^l$ is the controlled output and $y \in \mathbb{R}^p$ is the measurement output. The transfer function $P(s)$ from $[w^{\mathrm{T}}, u^{\mathrm{T}}]^{\mathrm{T}}$ to $[z^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$ is

$$P(s) := \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & E & B \\ \hline C_2 & 0 & D_2 \\ C_1 & D_1 & 0 \end{array}\right]$$

The following assumptions are made.

Assumption A1: (A, B) *is stabilizable.*

Assumption A2: *The system* (A, B, C_2, D_2) *has no invariant zeros in* $\mathbb{C}^0$.

Assumption A3: $\mathrm{Im}(E) \subseteq \mathcal{V}^-(A, B, C_2, D_2) + \mathcal{S}^-(A, B, C_2, D_2)$.

Assumption A4: (A, C_1) *is detectable.*

Assumption A5: *The system* (A, E, C_1, D_1) *has no invariant zeros in* $\mathbb{C}^0$.

Assumption A6: $\mathrm{Ker}(C_2) \supseteq \mathcal{V}^-(A, E, C_1, D_1) \cap \mathcal{S}^-(A, E, C_1, D_1)$.

Remark 2.1: Assumptions (A1) and (A4) are, of course, necessary for the existence of any stabilizing controller. We also would like to mention that if (A, B, C_2, D_2) is right invertible, then A3 holds and if (A, E, C_1, D_1) is left invertible, (A6) holds. □

Remark 2.2: It might be helpful to interpret our conditions (A3) and (A6) in the context of 'block characterization' of the H_∞ optimal control problem, which stems from the frequency-domain approach in early 1980s. This 'block characterization' in the frequency-domain approach was considered to be an indicator of the degree of the 'complexity' of the problem, although in the opinion of these authors, such a 'block characterization' is proof technique dependent and cannot be used as a true measure of the 'complexity' of the problem. At any rate, let us first recall the definition of this 'block characterization'. We denote $P_1(s)$ and $P_2(s)$ as the Rosenbrock system matrices of the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) respectively, namely

$$P_1(s) := \begin{bmatrix} sI - A & B \\ C_2 & D_2 \end{bmatrix}, \quad P_2(s) := \begin{bmatrix} sI - A & E \\ C_1 & D_1 \end{bmatrix}$$

The H_∞ optimal control problem is said to be

(i) *general one block* if both $P_1(s)$ and $P_2^{\mathrm{T}}(s)$ have maximal row normal rank;

(ii) *general two block* if precisely one of the matrices $P_1(s)$ and $P_2^{\mathrm{T}}(s)$ has maximal row normal rank;

(iii) *general four block* if none of the matrices $P_1(s)$ and $P_2^{\mathrm{T}}(s)$ has maximal row normal rank.

Finally, the definition of the so-called one, two and four-block Nehari H_∞ control problem is the same as the above definitions with the exception that no zeros in $\mathbb{C}^0 \cup \{\infty\}$ in the systems (A, B, C_2, D_2) and $(A^{\mathrm{T}}, C_1^{\mathrm{T}}, E^{\mathrm{T}}, D_1^{\mathrm{T}})$ are allowed. Now it is easy to verify that the class of H_∞ optimal control problems considered here, namely the class of problems that satisfy conditions (A3) and (A6) are, in fact, a subset of the general four-block problem. Moreover, they subsume as special cases the one-block Nehari problem and the general one-block problem. □

We connect a feedback controller K from y to u as illustrated in Fig. 1. The closed-loop transfer function from the disturbance w to the controlled output z is then given by the linear fraction map

$$T_{zw} = \mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

The following definitions will be convenient in the following.

Definition 2.1—(P, γ)-admissible controller: Let P be a given generalized plant. A proper controller K is said to be (P, γ)-admissible if K stabilizes P and $\|\mathcal{F}(P, K)\|_\infty < \gamma$. □

Definition 2.2—The infimum of the H_∞ optimal control problem: For a given generalized plant P, the infimum of the H_∞ norm of the closed-loop transfer function $\mathcal{F}(P, K)$ over all stabilizing controllers K is denoted by γ_{o}^*, namely

$$\gamma_{\mathrm{o}}^* := \inf\{\gamma | \text{A } (P, \gamma)\text{-admissible controller exists}\}$$

In the case that the measurement output y is equal to the state (i.e. $C_1 = I$, $D_1 = 0$), we refer to γ_{o}^* as γ_{s}^*. It is well known (Zhou and Khargonekar 1988) that in this case γ_{s}^* is equal to the infimum of the H_∞ norm of $\mathcal{F}(P, F)$ over all stabilizing static feedback controllers $u = Fx$. □

Definition 2.3—The H_∞-optimal control problem: Given a generalized plant P find a proper controller $K_{\mathrm{opt}}(s)$ such that $\|\mathcal{F}(P, K_{\mathrm{opt}})\|_\infty = \gamma_{\mathrm{o}}^*$. □

The goal of this paper is to obtain a closed-form expression for a family of (P, γ)-admissible controllers for the class of generalized plants that satisfy Assumptions (A1) to (A6). We consider both state feedback and output feedback. Furthermore, for the same class of systems we obtain conditions under which the H_∞ optimal problem has a solution. The explicit expression for the optimal solutions will also be given.

3. A special coordinate basis

In the following we recapitulate the main results in a theorem and some properties of the special coordinate basis while leaving the detailed derivation and proofs to be found in Sannuti and Saberi (1987), Saberi and Sannuti (1990 a). Such an s.c.b. has a distinct feature of explicitly displaying the finite and infinite zero structure of a given system. Connections between the s.c.b. and the various geometric subspaces of the given system as needed for our development are also given.

Consider the system described by

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx + Du \end{aligned}\right\} \tag{3.1}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a non-singular matrix V that put the direct feedthrough matrix D into the following form

$$\bar{D} = UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

where m_0 is the rank of D. Thus, the system in (3.1) can be rewritten as

$$\left.\begin{aligned} \dot{x} &= Ax + [B_0 \quad B_1]\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \\ \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \end{aligned}\right\} \tag{3.3}$$

where B_0, B_1, C_0 and C_1 are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely

$$u = V\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = Uz$$

Note that the H_∞-norm of the system transfer function $T_{zw}(s)$ is unchanged when we apply an orthogonal transformation on the output z, and under any non-singular transformations on the states and control inputs. We have the following main theorem.

Theorem 3.1 (s.c.b.): *Consider the given system (A, B, C, D). There exist non-singular transformations Γ_s, Γ_o and Γ_i, an integer $m_f \leq m - m_o$, and integer indices q_i, $i = 1$ to m_f, such that*

$$x = \Gamma_s \bar{x}, \quad u = \Gamma_i \bar{u}, \quad z = \Gamma_o \bar{z}$$

$$\bar{x} = [(x_a^+)^T, x_b^T, (x_a^-)^T, x_c^T, x_f^T]^T$$

$$x_f = [x_1^T, x_2^T, \ldots, x_{m_f}^T]^T$$

$$\bar{z} = [z_0^T, z_f^T, z_b^T]^T, \quad z_f = [z_1, z_2, \ldots, z_{m_f}]^T$$

$$\bar{u} = [u_0^T, u_f^T, u_c^T]^T, \quad u_f = [u_1, u_2, \ldots, u_{m_f}]^T$$

we have the following system of equations

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + B_{0a}^+ z_0 + L_{af}^+ z_f + L_{ab}^+ z_b \tag{3.4}$$

$$\dot{x}_b = A_{bb} x_b + B_{0b} z_0 + L_{bf} z_f, \quad z_b = C_b x_b \tag{3.5}$$

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- z_0 + L_{af}^- z_f + L_{ab}^- z_b \tag{3.6}$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} z_0 + L_{cb} z_b + L_{cf} z_f + B_c[E_{ca}^- x_a^- + E_{ca}^+ x_a^+] + B_c u_c \tag{3.7}$$

and for each $i = 1$ to m_f,

$$\dot{x}_i = A_{qi}x_i + L_{i0}z_0 + L_{if}z_f + B_{qi}\left[u_i + E_{ia}x_a + E_{ib}x_b + E_{ic}x_c + \sum_{j=1}^{m_f} E_{ij}x_j\right] \tag{3.8}$$

$$z_i = C_{qi}x_i, \quad z_f = C_f x_f \tag{3.9}$$

$$z_0 = C_{0a}^- x_a^- + C_{0a}^+ x_a^+ + C_{0b}x_b + C_{0c}x_c + C_{0f}x_f + u_0 \tag{3.10}$$

Here, the states x_a^-, x_a^+, x_b, x_c and x_f are respectively of dimensions n_a^-, n_a^+, n_b, n_c and $n_f = \sum_{i=1}^{m_f} q_i$ while x_i is of dimension q_i for each $i = 1$ to m_f. The control vectors u_0, u_f and u_c are respectively of dimensions m_0, m_f and $m_c = m - m_0 - m_f$ while the output vectors z_0, z_f and z_b are respectively of dimensions $p_0 = m_0$, $p_f = m_f$ and $p_b = p - p_0 - p_f$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form

$$A_{qi} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1 \quad 0] \tag{3.11}$$

(Obviously for the case when $q_i = 1$, $A_{q_i} = 0$, $B_{q_i} = 1$ and $C_{q_i} = 1$.) Furthermore, we have $\lambda(A_{aa}^-) \in \mathbb{C}^-$, $\lambda(A_{aa}^+) \in \mathbb{C}^+$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable. Also, assuming that x_i are arranged such that $q_i \leq q_{i+1}$, the matrix L_{if} has the particular form,

$$L_{if} = [L_{i1}, L_{i2}, \ldots, L_{ii-1}, 0, 0, \ldots, 0]$$

Also, the last row of each L_{if} is identically zero.

Proof: The proof of this theorem can be found in Sannuti and Saberi (1987), Saberi and Sannuti (1990 a). □

Remark 3.1: We have utilized the structural algorithm in the construction of s.c.b. The numerical stability of the structural algorithm is well-known in the literature. We should also point out that we have implemented s.c.b. in a software package (Lin *et al*. 1991). This package has been commercially available for some time, and during this time we have had massive numerical experience with the implementation of s.c.b. which has shown us the stability of our algorithm. □

We shall note that the output transformation Γ_o is of form

$$\Gamma_o = U^{\mathrm{T}} \begin{bmatrix} I_{m_0} & 0 \\ 0 & \Gamma_{or} \end{bmatrix} \tag{3.12}$$

Moreover, we can rewrite the s.c.b. given by Theorem 3.1 in a more compact form

$$\bar{A} := \Gamma_s^{-1}(A - B_0C_0)\Gamma_s = \begin{bmatrix} A_{aa}^+ & L_{ab}^+C_b & 0 & 0 & L_{af}^+C_f \\ 0 & A_{bb} & 0 & 0 & L_{bf}C_f \\ 0 & L_{ab}^-C_b & A_{aa}^- & 0 & L_{af}^-C_f \\ B_cE_{ca}^+ & L_{cb}C_b & B_cE_{ca}^- & A_{cc} & L_{cf}C_f \\ B_fE_{fa}^+ & B_fE_{fb} & B_fE_{fa}^- & B_fE_{fc} & A_{ff} \end{bmatrix} \tag{3.13}$$

$$\bar{B} := \Gamma_s^{-1}[B_0 \ \ B_1]\Gamma_i = \begin{bmatrix} B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0a}^- & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix} \tag{3.14}$$

$$\bar{C} =: \Gamma_o^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^+ & C_{0b} & C_{0a}^- & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & C_b & 0 & 0 & 0 \end{bmatrix} \tag{3.15}$$

and

$$\bar{D} := \Gamma_o^{-1}D\Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.16}$$

In what follows, we state some important properties of the s.c.b. which are pertinent to our present work.

Property 3.1: The given system (A, B, C, D) is right-invertible if and only if x_b, and hence z_b, are non-existent; left-invertible if and only if x_c and hence u_c are non-existent; invertible if and only if both x_c and x_b are non-existent. □

Property 3.2: Invariant zeros of (A, B, C, D) are the eigenvalues of A_{aa}^- and A_{aa}^+. Moreover, the stable and unstable invariant zeros of (A, B, C, D) are the eigenvalues of A_{aa}^- and A_{aa}^+ respectively. □

Property 3.3: The pair (A, B) is stabilizable if and only if $(A_{\text{con}}, B_{\text{con}})$ is stabilizable where

$$A_{\text{con}} = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{\text{con}} = \begin{bmatrix} B_{0a}^+ & L_{af}^+ \\ B_{0b} & L_{bf} \end{bmatrix} \tag{3.17}$$

The pair (A, C) is detectable if and only if $(A_{\text{obs}}, B_{\text{obs}})$ is detectable where

$$A_{\text{obs}} = \begin{bmatrix} A_{aa}^+ & 0 \\ B_c E_{ca}^+ & A_{cc} \end{bmatrix}, \quad B_{\text{obs}} = \begin{bmatrix} C_{0a}^+ & C_{0c} \\ E_{fa}^+ & E_{fc} \end{bmatrix} \tag{3.18}$$

There are interconnections between the s.c.b. and various invariant and almost invariant geometric subspaces. The following property establishes such interconnections. □

Property 3.4:

(1) $x_a^- \oplus x_a^+ \oplus x_c$ spans $\mathcal{V}^*(A, B, C, D)$
(2) $x_a^- \oplus x_c$ spans $\mathcal{V}^-(A, B, C, D)$
(3) $x_a^+ \oplus x_c$ spans $\mathcal{V}^+(A, B, C, D)$
(4) $x_c \oplus x_f$ spans $\mathcal{S}^*(A, B, C, D)$
(5) $x_a^- \oplus x_c \oplus x_f$ spans $\mathcal{S}^+(A, B, C, D)$
(6) $x_a^+ \oplus x_c \oplus x_f$ spans $\mathcal{S}^-(A, B, C, D)$ □

4. State feedback design

In this section, we consider the case that $y = x$, i.e. consider a system

$$\Sigma_{\mathrm{P}}: \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = x \\ z = C_2 x + D_2 u \end{cases} \tag{4.1}$$

It is easy to verify that for this system, Assumptions (A4), (A5) and (A6) are automatically satisfied. Hence, we make only Assumptions (A1), (A2) and (A3) throughout this section. We introduce a procedure for obtaining the closed-form solutions for the H_∞ sub-optimal state feedback control problem utilizing an asymptotic time-scale and eigenstructure assignment (ATEA). The concept of the ATEA design procedure was proposed originally by Saberi and Sannuti (1989). It uses the special coordinate basis (s.c.b.) of the given system (See Theorem 3.1). We also give conditions under which the H_∞ optimal control problem has a solution. Furthermore, explicit expressions for these optimal solutions will be given. The following is a step-by-step algorithm to construct the closed-form of the sub-optimal state feedback laws.

Step 4.1.1

Transform the system (A, B, C_2, D_2) into s.c.b. as given by Theorem 3.1 in § 3. To all submatrices and transformations in the s.c.b. of (A, B, C_2, D_2), we append a subscript $_{\mathrm{P}}$ to signify their relation to the system Σ_{P}. Next, we compute

$$\bar{E} = \Gamma_{s\mathrm{P}}^{-1} E = [(E_{a\mathrm{P}}^{+})^{\mathrm{T}}, E_{b\mathrm{P}}^{\mathrm{T}}, (E_{a\mathrm{P}}^{-})^{\mathrm{T}}, E_{c\mathrm{P}}^{\mathrm{T}}, E_{f\mathrm{P}}^{\mathrm{T}}]^{\mathrm{T}} \tag{4.2}$$

It is simple to verify from the properties of s.c.b. that Assumption (A3) implies $E_{b\mathrm{P}} \equiv 0$. Also, for the economy of notation, we denote n_{P} the dimension of $\mathbb{R}^n/\mathcal{S}^+(A, B, C_2, D_2)$. We note that $n_{\mathrm{P}} = 0$ if and only if the system (A, B, C_2, D_2) is right invertible and is of minimum phase.

If the system (A, B, C_2, D_2) is of non-minimum phase and/or is not right invertible, we define

$$A_{11\mathrm{P}} = \begin{bmatrix} A_{aa\mathrm{P}}^{+} & L_{ab\mathrm{P}}^{+} C_{b\mathrm{P}} \\ 0 & A_{bb\mathrm{P}} \end{bmatrix}, \quad B_{11\mathrm{P}} = \begin{bmatrix} B_{0a\mathrm{P}}^{+} \\ B_{0b\mathrm{P}} \end{bmatrix}, \quad A_{13\mathrm{P}} = \begin{bmatrix} L_{af\mathrm{P}}^{+} \\ L_{bf\mathrm{P}} \end{bmatrix}$$

$$C_{21\mathrm{P}} = \Gamma_{or\mathrm{P}} \begin{bmatrix} 0 & 0 \\ 0 & C_{b\mathrm{P}} \end{bmatrix}, \quad C_{23\mathrm{P}} = \Gamma_{or\mathrm{P}} \begin{bmatrix} C_{f\mathrm{P}} C_{f\mathrm{P}}^{\mathrm{T}} \\ 0 \end{bmatrix}$$

and

$$A_{\mathrm{P}} = A_{11\mathrm{P}} - A_{13\mathrm{P}} (C_{23\mathrm{P}}^{\mathrm{T}} C_{23\mathrm{P}})^{-1} C_{23\mathrm{P}}^{\mathrm{T}} C_{21\mathrm{P}}$$

$$B_{\mathrm{P}} B_{\mathrm{P}}^{\mathrm{T}} = B_{11\mathrm{P}} B_{11\mathrm{P}}^{\mathrm{T}} + A_{13\mathrm{P}} (C_{23\mathrm{P}}^{\mathrm{T}} C_{23\mathrm{P}})^{-1} A_{13\mathrm{P}}^{\mathrm{T}}$$

$$C_{\mathrm{P}}^{\mathrm{T}} C_{\mathrm{P}} = C_{21\mathrm{P}}^{\mathrm{T}} C_{21\mathrm{P}} - C_{21\mathrm{P}}^{\mathrm{T}} C_{21\mathrm{P}} (C_{23\mathrm{P}}^{\mathrm{T}} C_{23\mathrm{P}})^{-1} C_{23\mathrm{P}}^{\mathrm{T}} C_{21\mathrm{P}}$$

Then, we solve for the unique positive definite solution S_{P} of the algebraic matrix Riccati equation

$$A_{\mathrm{P}} S_{\mathrm{P}} + S_{\mathrm{P}} A_{\mathrm{P}}^{\mathrm{T}} - B_{\mathrm{P}} B_{\mathrm{P}}^{\mathrm{T}} + S_{\mathrm{P}} C_{\mathrm{P}}^{\mathrm{T}} C_{\mathrm{P}} S_{\mathrm{P}} = 0 \tag{4.3}$$

together with the matrix T_{P} defined by

$$T_{\mathrm{P}} = \begin{bmatrix} T_{aa\mathrm{P}} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.4}$$

where T_{aaP} is the unique semi-positive solution of the algebraic matrix Lyapunov equation

$$A^{+}_{aaP}T_{aaP} + T_{aaP}(A^{+}_{aaP})^{\mathrm{T}} = E^{+}_{aP}(E^{+}_{aP})^{\mathrm{T}} \tag{4.5}$$

Note that the anti-stability of A^{+}_{aa} and the observability of (A_{bbP}, C_{bP}) imply that the pair $(-A_{\mathrm{P}}, C_{\mathrm{P}})$ is detectable, and Assumption (A1) implies that the pair $(A_{\mathrm{P}}, B_{\mathrm{P}})$ is stabilizable. Hence, the existence and uniqueness of the solutions for S_{P} and T_{aaP} follow from the results of Richardson and Kwong (1986). Then, it is shown by Chen *et al*. (1991, 1992) that the infimum of the H_∞ optimal control problem for Σ_{P}, i.e. γ^*_s, is given by

$$\gamma^*_s = \begin{cases} \sqrt{\lambda_{\max}(T_{\mathrm{P}}S_{\mathrm{P}}^{-1})} & \text{if } n_{\mathrm{P}} > 0 \\ 0 & \text{if } n_{\mathrm{P}} = 0 \end{cases} \tag{4.6}$$

Step 4.1.2

Let $\Delta_{c\mathrm{P}}$ be any arbitrary $m_{c\mathrm{P}} \times n_{c\mathrm{P}}$ matrix subject to the constraint that

$$A^{\mathrm{c}}_{cc\mathrm{P}} = A_{cc\mathrm{P}} - B_{c\mathrm{P}}\Delta_{c\mathrm{P}} \tag{4.7}$$

is a stable matrix. Note that the existence of such a $\Delta_{c\mathrm{P}}$ is guaranteed by the property of s.c.b., i.e. $(A_{cc\mathrm{P}}, B_{c\mathrm{P}})$ is controllable.

Step 4.1.3

Next, given any $\gamma \geqslant \bar{\gamma} > \gamma^*_s$, we define

$$F_{11}(\bar{\gamma}) := \begin{bmatrix} F^{+}_{a0}(\bar{\gamma}) & F_{b0}(\bar{\gamma}) \\ F^{+}_{a1}(\bar{\gamma}) & F_{b1}(\bar{\gamma}) \end{bmatrix} = \begin{bmatrix} B^{\mathrm{T}}_{11\mathrm{P}}P_0(\bar{\gamma}) \\ (C^{\mathrm{T}}_{23\mathrm{P}}C_{23\mathrm{P}})^{-1}[A^{\mathrm{T}}_{13\mathrm{P}}P_0(\bar{\gamma}) + C^{\mathrm{T}}_{23\mathrm{P}}C_{21\mathrm{P}}] \end{bmatrix} \tag{4.8}$$

where

$$P_0(\bar{\gamma}) := (S_{\mathrm{P}} - \bar{\gamma}^{-2}T_{\mathrm{P}})^{-1} \tag{4.9}$$

and define

$$A^{\mathrm{c}}_{11\mathrm{P}} := A_{11\mathrm{P}} - [B_{11\mathrm{P}}, A_{13\mathrm{P}}]F_{11}(\bar{\gamma})$$

We will show later that the eigenvalues of $A^{\mathrm{c}}_{11\mathrm{P}}$ are in $\mathbb{C}^-$. Let us partition $[F^{+}_{a1}(\bar{\gamma})\ \ F_{b1}(\bar{\gamma})]$ as

$$[F^{+}_{a1}(\bar{\gamma})\ \ F_{b1}(\bar{\gamma})] = \begin{bmatrix} F^{+}_{a11}(\bar{\gamma}) & F_{b11}(\bar{\gamma}) \\ F^{+}_{a12}(\bar{\gamma}) & F_{b12}(\bar{\gamma}) \\ \vdots & \vdots \\ F^{+}_{a1m_{f\mathrm{P}}}(\bar{\gamma}) & F_{b1m_{f\mathrm{P}}}(\bar{\gamma}) \end{bmatrix} \tag{4.10}$$

where $F^{+}_{a1i}(\bar{\gamma})$ and $F_{b1i}(\bar{\gamma})$ are of dimensions $1 \times n^{+}_{a\mathrm{P}}$ and $1 \times n_{b\mathrm{P}}$, respectively.

Step 4.1.4

This step makes use of subsystems, $i = 1$ to $m_{f_{\mathrm{P}}}$, represented by (3.8) of § 3. Let $\Lambda_i = \{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iq_i}\}$, $i = 1$ to $m_{f_{\mathrm{P}}}$, be the sets of q_i elements all in $\mathbb{C}^-$, which are closed under complex conjugation, where q_i and $m_{f_{\mathrm{P}}}$ are as defined in Theorem 3.1 but associated with the s.c.b. of (A, B, C_2, D_2). Let $\Lambda_{f_{\mathrm{P}}} := \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{m_{f\mathrm{P}}}$. For $i = 1$ to $m_{f_{\mathrm{P}}}$, we define

$$p_i(s) := \prod_{j=1}^{q_i}(s - \lambda_{ij}) = s^{q_i} + F_{i1}s^{q_i-1} + \cdots + F_{iq_i-1}s + F_{iq_i} \tag{4.11}$$

and

$$\widetilde{F}_i(\varepsilon, \Lambda_i) := \frac{1}{\varepsilon^{q_i}}[F_{iq_i}, \varepsilon F_{iq_i-1}, \ldots, \varepsilon^{q_i-1}F_{i1}] \tag{4.12}$$

Step 4.1.5

In this step, various gains calculated in Steps 4.1.2 to 4.1.4 are put together to form a composite state feedback gain for the given system Σ_{P}. Let

$$\widetilde{F}^+_{a1}(\bar{\gamma}, \varepsilon, \Lambda_{f_{\mathrm{P}}}) := \begin{bmatrix} F^+_{a11}(\bar{\gamma})F_{1q_1}/\varepsilon^{q_1} \\ F^+_{a12}(\bar{\gamma})F_{2q_2}/\varepsilon^{q_2} \\ \vdots \\ F^+_{a1m_{f\mathrm{P}}}(\bar{\gamma})F_{m_{f\mathrm{P}}q_{m_{f\mathrm{P}}}}/\varepsilon^{q_{m_{f\mathrm{P}}}} \end{bmatrix}$$

and

$$\widetilde{F}_{b1}(\bar{\gamma}, \varepsilon, \Lambda_{f_{\mathrm{P}}}) := \begin{bmatrix} F_{b11}(\bar{\gamma})F_{1q_1}/\varepsilon^{q_1} \\ F_{b12}(\bar{\gamma})F_{2q_2}/\varepsilon^{q_2} \\ \vdots \\ F_{b1m_{f\mathrm{P}}}(\bar{\gamma})F_{m_{f\mathrm{P}}q_{m_{f\mathrm{P}}}}/\varepsilon^{q_{m_{f\mathrm{P}}}} \end{bmatrix}$$

Then, define the state feedback gain $F(\bar{\gamma}, \varepsilon, \Lambda_{f_{\mathrm{P}}}, \Delta_{c\mathrm{P}})$ as

$$F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}}) = -\Gamma_{i\mathrm{P}}\widetilde{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}})\Gamma_{s\mathrm{P}}^{-1} \tag{4.13}$$

where

$$\widetilde{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}}) =$$

$$\begin{bmatrix} C^+_{0a\mathrm{P}} + F^+_{a0}(\bar{\gamma}) & C_{0b\mathrm{P}} + F_{b0}(\bar{\gamma}) & C^-_{0a\mathrm{P}} & C_{0c\mathrm{P}} & C_{0f\mathrm{P}} \\ E^+_{fa\mathrm{P}} + \widetilde{F}^+_{a1}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}) & E_{fb\mathrm{P}} + \widetilde{F}_{b1}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}) & E^-_{fa\mathrm{P}} & E_{fc\mathrm{P}} & \widetilde{F}_f(\varepsilon, \Lambda_{f\mathrm{P}}) + E_{f\mathrm{P}} \\ E^+_{ca\mathrm{P}} & E_{cb\mathrm{P}} & E^-_{ca\mathrm{P}} & \Delta_{c\mathrm{P}} & 0 \end{bmatrix}$$

and where

$$E_{f\mathrm{P}} = \begin{bmatrix} E_{11} & \cdots & E_{1m_{f\mathrm{P}}} \\ \vdots & \ddots & \vdots \\ E_{m_{f\mathrm{P}}1} & \cdots & E_{m_{f\mathrm{P}}m_{f\mathrm{P}}} \end{bmatrix}$$

and

$$\widetilde{F}_f(\varepsilon, \Lambda_{f\mathrm{P}}) = \mathrm{Diag}[\widetilde{F}_1(\varepsilon, \Lambda_1), \widetilde{F}_2(\varepsilon, \Lambda_2), \ldots, \widetilde{F}_{m_{f\mathrm{P}}}(\varepsilon, \Lambda_{m_{f\mathrm{P}}})]$$

We have the following theorem.

Theorem 4.1: *Consider a given generalized plant Σ_{P} satisfying Assumptions (A1) to (A3). Then, with state feedback gain given by* (4.13) *we have the following properties.*

(1) *For any $\gamma^*_s < \bar{\gamma} \leq \gamma$, for any $\Lambda_{f_{\mathrm{P}}} \subset \mathbb{C}^-$ which is closed under complex conjugation and for any $\Delta_{c\mathrm{P}}$ subject to the constraints that $A^{\mathrm{c}}_{cc\mathrm{P}}$ is stable (see* (4.7)*), there exists an $\varepsilon^* > 0$ such that for all $0 < \varepsilon \leq \varepsilon^*$, the state feedback control law, as given in* (4.13), *is (P, γ)-admissible. Namely, the closed-loop system comprising Σ_{P} and the state feedback law, $u = F(\bar{\gamma}, \varepsilon,$*

$\Lambda_{f\mathrm{P}}$, $\Delta_{c\mathrm{P}})x$, is internally stable and the H_∞-norm of the closed-loop transfer function from the disturbance w to the controlled output z is less than γ, i.e.

$$\|T_{zw}\|_\infty < \bar{\gamma} \leq \gamma$$

where $T_{zw} = \mathcal{F}(P, F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}}))$.

(2) *Moreover, as $\varepsilon \to 0$, the eigenvalues of the closed-loop system, i.e. the eigenvalues of $A + BF(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}})$, are given by*

$$\lambda(A^-_{aa\mathrm{P}}),\ \lambda(A^{\mathrm{c}}_{cc\mathrm{P}}),\ \lambda(A^{\mathrm{c}}_{11\mathrm{P}}) + 0(\varepsilon) \quad \text{and} \quad \frac{\Lambda_{f\mathrm{P}}}{\varepsilon} + 0(1)$$

Proof: For the proof see Appendix A. □

The following remarks are in order.

Remark 4.1—Interpretations of parameters ε, $\Lambda_{f\mathrm{P}}$ and $\Delta_{c\mathrm{P}}$: Theorem 4.1 shows that the closed-loop system under H_∞ sub-optimal state feedback laws, i.e. T_{zw}, has fast eigenvalues $\Lambda_{f\mathrm{p}}/\varepsilon$. So the set of parameter $\Lambda_{f\mathrm{P}}$ in the H_∞ sub-optimal gain $F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}})$ of (4.13) represents the asymptotes of these fast eigenvalues while ε represents their time-scale. The closed-loop system also has $\lambda(A^{\mathrm{c}}_{cc\mathrm{P}})$ as slow eigenvalues. These eigenvalues can be assigned to any desired locations in $\mathbb{C}^-$ by choosing an appropriate $\Delta_{c\mathrm{P}}$ (see (4.7)). Hence, the set of parameters $\Delta_{c\mathrm{P}}$ in the H_∞ sub-optimal state feedback gain prescribes the locations of these slow eigenvalues. □

Remark 4.2. Regular case: If D_2 is injective, it is obvious from our algorithm that $F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}}) = F(\bar{\gamma})$ does not depend on ε, $\Lambda_{f\mathrm{P}}$ and $\Delta_{c\mathrm{P}}$, and is given by

$$F(\bar{\gamma}) = -\Gamma_{i\mathrm{P}}[C^+_{0a\mathrm{P}} + F^+_{a0}(\bar{\gamma})\ C_{0b\mathrm{P}} + F_{b0}(\bar{\gamma})\ C^-_{0a\mathrm{P}}]\Gamma^{-1}_{s\mathrm{P}}$$

This corresponds to the regular case, and for $\bar{\gamma} = \gamma$, is the central controller given by Doyle *et al*. (1988). Moreover, if $\bar{\gamma} \to \infty$, the result reduces to the well-known LQG solution. □

The following theorem deals with pole/zero cancellations in the closed-loop system T_{zw} under the state feedback law $u = F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}})x$.

Theorem 4.2—Pole/zero cancellations: *$\lambda(A^-_{aa\mathrm{P}})$, the stable invariant zeros of the system (A, B, C_2, D_2), and $\lambda(A^{\mathrm{c}}_{cc\mathrm{P}})$ are the output decoupling zeros of T_{zw}. Hence, they cancel with the poles of T_{zw}.*

Proof: For the proof see Appendix B. □

We illustrate our algorithm with the following example.

Example: Consider a system (Chen *et al*. 1991) characterized by

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 1 \\ 0 & 0 \\ 0 & 0 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $C_1 = I$, $D_1 = 0$. It is easy to verify that (A, B) is stabilizable, and the system (A, B, C_2, D_2) is neither right nor left invertible and is of non-minimum phase with an invariant zero at $s = 1$. Moreover, it is already in the form of s.c.b. with $n_{aP}^{+} = 1$, $n_{aP}^{-} = 0$, $n_{bP} = 2$ and $n_{cP} = n_{fP} = 1$. Also, it is simple to see that $\mathrm{Im}(E) \subseteq \mathcal{V}^{-1}(A, B, C_2, D_2) + \mathcal{S}^{-}(A, B, C_2, D_2)$ since $E_{bP} = 0$. Then following our design procedure, we obtain

$$\gamma_s^* = 6{\cdot}4679044$$

and the closed-form of the sub-optimal state feedback gains,

$$F(\bar{\gamma}, \varepsilon, \lambda_{fP}, \Delta_{cP}) =$$

$$\begin{bmatrix} \dfrac{-0{\cdot}163673\bar{\gamma}^2}{0{\cdot}132909\bar{\gamma}^2 - 5{\cdot}560084} & -1 + \dfrac{0{\cdot}294790\bar{\gamma}^2\lambda_{fP}}{(0{\cdot}132909\bar{\gamma}^2 - 5{\cdot}560084)\varepsilon} & -1 \\ \dfrac{0{\cdot}185427\bar{\gamma}^2 - 3{\cdot}009097}{0{\cdot}132909\bar{\gamma}^2 - 5{\cdot}560084} & -1 + \dfrac{(0{\cdot}102145\bar{\gamma}^2 - 12{\cdot}824695)\lambda_{fP}}{(0{\cdot}132909\bar{\gamma}^2 - 5{\cdot}5560084)\varepsilon} & -1 \\ \dfrac{-0{\cdot}318336\bar{\gamma}^2 + 10{\cdot}696930}{0{\cdot}132909\bar{\gamma}^2 - 5{\cdot}560084} & -1 + \dfrac{(0{\cdot}163673\bar{\gamma}^2 - 2{\cdot}127749)\lambda_{fP}}{(0{\cdot}132909\bar{\gamma}^2 - 5{\cdot}560084)\varepsilon} & -1 \\ 0 & -1 & -\Delta_{cP} \\ 0 & \dfrac{\lambda_{fP}}{\varepsilon} & 0 \end{bmatrix}^{\mathrm{T}} \tag{4.14}$$

where the scalars $\lambda_{fP} < 0$ and $\Delta_{cP} > 1$ (note that Δ_{cP} must be greater than one in order to have stable A_{ccP}^{c}). We demonstrate our results in Fig. 2 through the plots of maximum singular values of the closed-loop transfer function matrix for several values of $\bar{\gamma}$ and ε. Note that in Fig. 2, we choose $\lambda_{fP} = -1$ and $\Delta_{cP} = 3$.

□

In the next theorem, we show the conditions under which the H_∞ optimal state feedback control problem has a solution. Furthermore, we examine the behaviour of the H_∞ sub-optimal solutions as $\gamma \to \gamma_s^*$.

Theorem 4.3: *Consider a generalized plant Σ_P satisfying Assumptions (A1) to (A3). Then*

(1) *the H_∞ optimal controller exists if and only if* $\mathrm{Im}(E) \subseteq \mathcal{V}^{-}(A, B, C_2, D_2)$. *Moreover, in this case $\gamma_s^* = 0$.*

(2) *if* $\mathrm{Im}(E) \not\subseteq \mathcal{V}^{-}(A, B, C_2, D_2)$, *any static or dynamic state feedback*

$$u = [C_{\mathrm{cmp}}(\gamma)(sI - A_{\mathrm{cmp}}(\gamma))^{-1}B_{\mathrm{cmp}}(\gamma) + D_{\mathrm{cmp}}(\gamma)]x$$

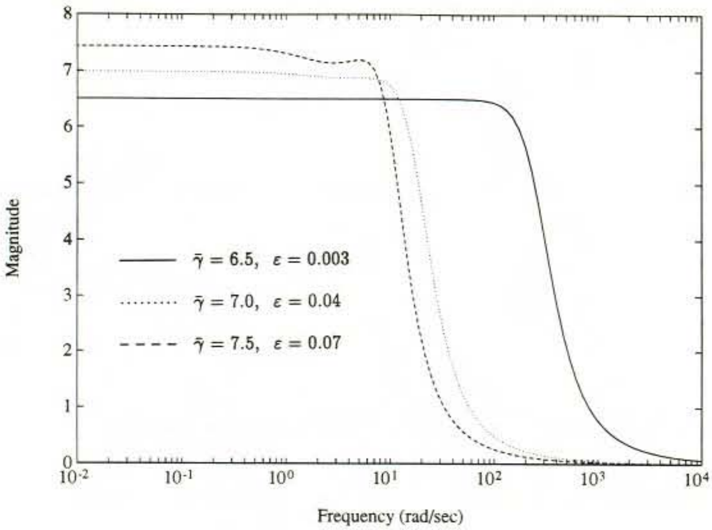

Figure 2. Maximum singular values of T_{zw} (state feedback case).

which is (P, γ)-admissible will exhibit high-gain, i.e.

$$\left\| \begin{bmatrix} A_{\mathrm{cmp}}(\gamma) & B_{\mathrm{cmp}}(\gamma) \\ C_{\mathrm{cmp}}(\gamma) & D_{\mathrm{cmp}}(\gamma) \end{bmatrix} \right\| \to \infty, \quad \text{as } \gamma \to \gamma_s^*$$

Proof: For the proof see Appendix C. □

In what follows we will provide an algorithm that produces a set of the H_∞ optimal state feedback laws.

Step 4.2.1. Transform the system (A, B, C_2, D_2) into s.c.b.

Step 4.2.2. Let F_{cc} be any arbitrary $m_{c\mathrm{P}} \times n_{c\mathrm{P}}$ matrix subject to the constraint that $A^{\mathrm{c}}_{cc\mathrm{P}} = A_{cc\mathrm{P}} - B_{c\mathrm{P}} F_{cc}$ is stable.

Step 4.2.3. Form matrices A_x and B_x as follows

$$A_x := \begin{bmatrix} A^+_{aa\mathrm{P}} & L^+_{ab\mathrm{P}} C_{b\mathrm{P}} & L^+_{af\mathrm{P}} C_{f\mathrm{P}} \\ 0 & A_{bb\mathrm{P}} & L_{bf\mathrm{P}} C_{f\mathrm{P}} \\ B_{f\mathrm{P}} E^+_{fa\mathrm{P}} & B_{f\mathrm{P}} E_{fb\mathrm{P}} & A_{ff\mathrm{P}} \end{bmatrix}, \quad B_x := \begin{bmatrix} B^+_{a0\mathrm{P}} & 0 \\ B_{b0\mathrm{P}} & 0 \\ B_{f0\mathrm{P}} & B_{f\mathrm{P}} \end{bmatrix}$$

Let F_x be any arbitrary $(m_{0\mathrm{P}} + m_{f\mathrm{P}}) \times (n^+_{a\mathrm{P}} + n_{b\mathrm{P}} + n_{f\mathrm{P}})$ matrix subject to the constraint that $A^{\mathrm{c}}_x = A_x - B_x F_x$ is stable. Here, we note that it is simple to verify from the properties of s.c.b. that the pair (A_x, B_x) is detectable if and only if (A, B) is detectable. Hence, such a F_x always exists. Next partition F_x as

$$F_x = \begin{bmatrix} F^+_{a0} & F_{b0} & F_{f0} \\ F^+_{a1} & F_{b1} & F_{f1} \end{bmatrix}$$

Step 4.2.4. Let

$$F = -\Gamma_{iP}\begin{bmatrix} C_{0aP}^{+} + F_{a0}^{+} & C_{0bP} + F_{b0} & C_{0aP}^{-} & C_{0cP} & C_{0fP} + F_{f0} \\ F_{a1}^{+} & F_{b1} & E_{faP}^{-} & E_{fcP} & F_{f1} \\ F_{ca}^{+} & F_{cb} & F_{ca}^{-} & F_{cc} & F_{cf} \end{bmatrix} \Gamma_{sP}^{-1} \quad (4.15)$$

where F_{ca}^{+}, F_{cb}, F_{ca}^{-} and F_{cf} are some arbitrary submatrices with appropriate dimensions.

We have the following result.

Theorem 4.4: *Given a generalized plant Σ_P satisfying Assumptions (A1) and (A2), and the condition* $\mathrm{Im}(E) \subseteq \mathcal{V}^{-}(A, B, C_2, D_2)$, *then for any state feedback law $u = Fx$ with F given by* (4.15), *the closed-loop is internally stable and the H_∞-norm of the closed-loop transfer function from w to z is equal to $\gamma_s^* = 0$. Moreover, eigenvalues of the closed-loop system are given by $\lambda(A_{aaP}^{-})$, $\lambda(A_{ccP}^{c})$ and $\lambda(A_x^c)$. In fact, if* $\mathrm{Im}(E) = \mathcal{V}^{-1}(A, B, C_2, D_2)$, *any stabilizing state feedback gain F which achieves the infimum is of the form* (4.15).

Proof: The proof follows from some simple algebra. □

5. Output feedback design

This section deals with H_∞ sub-optimal and optimal design using measurement output feedback. The output feedback controllers that we consider here are basically observer-based control laws and can be regarded as an extension of the central output feedback controller that was proposed by Doyle *et al*. (1988) for the regular case. We have modified the central output feedback controller of the regular case to deal with the singular case. This modification will be discussed later. The procedure for obtaining the closed-form of the H_∞ sub-optimal output feedback laws proceeds as follows.

Step 5.1.1.

1.1. Define an auxiliary system Σ_P as in (4.1) and proceed to perform Step 4.1.1 of § 4.

1.2. Define another auxiliary system Σ_Q as follows,

$$\Sigma_Q\colon \begin{cases} \dot{x} = A^{T}x + C_1^{T}u + C_2^{T}w \\ y = x \\ z = E^{T}x + D_1^{T}u \end{cases} \quad (5.1)$$

and proceed to perform Step 4.1.1 on Σ_Q. Note that under Assumptions (A4), (A5) and (A6), all Steps 4.1.1 through 4.1.5 in § 4 can be implemented on Σ_Q. Again, to all submatrices and transformations in the s.c.b. of (A^T, C_1^T, E^T, D_1^T), we append a subscript $_Q$ to signify their relation to the system Σ_Q. We also denote n_Q as the dimension of $\mathcal{V}^{+}(A, E, C_1, D_1)$. Moreover, we denote the solutions to (4.3) and (4.4) for Σ_Q as S_Q and T_Q, respectively.

1.3. Compute

$$\gamma_o^* = \sqrt{\lambda_{\max}(M)} \tag{5.2}$$

where

$$M := \begin{cases} \begin{bmatrix} T_P S_P^{-1} + \Gamma S_Q^{-1} \Gamma^T S_P^{-1} & -\Gamma S_Q^{-1} \\ -T_Q S_Q^{-1} \Gamma^T S_P^{-1} & T_Q S_Q^{-1} \end{bmatrix} & \text{if } n_P > 0 \text{ and } n_Q > 0 \\ T_P S_P^{-1} & \text{if } n_P > 0 \text{ and } n_Q = 0 \\ T_Q S_Q^{-1} & \text{if } n_P = 0 \text{ and } n_Q > 0 \\ 0 & \text{if } n_P = 0 \text{ and } n_Q = 0 \end{cases} \tag{5.3}$$

and where Γ is of dimension $n_P \times n_Q$ and satisfies the following

$$\Gamma_{sP}^{-1}(\Gamma_{sQ}^{-1})^T = \begin{bmatrix} \Gamma & \star \\ \star & \star \end{bmatrix}$$

As is shown by Chen *et al*. (1991), γ_o^* is indeed the infimum as defined in Definition 2.2.

Step 5.1.2.

2.1. For any $\gamma > \gamma_o^*$, apply the procedures of Steps 4.1.2 to 4.1.5 of § 4 to Σ_P to obtain the gain matrix $F(\bar\gamma, \varepsilon, \Lambda_{fP}, \Delta_{cP})$, where $\gamma_o^* < \bar\gamma \leq \gamma$.

2.2. For any $\gamma > \gamma_o^*$, apply the procedures of Steps 4.1.2 to 4.1.5 of § 4 to Σ_Q to obtain the gain matrix $F(\bar\gamma, \varepsilon, \Lambda_{fQ}, \Delta_{cQ})$. Let $L(\bar\gamma, \varepsilon, \Lambda_{fQ}, \Delta_{cQ}) := [F(\bar\gamma, \varepsilon, \Lambda_{fQ}, \Delta_{cQ})]^\dagger$, where $\gamma_o^* < \bar\gamma \leq \gamma$.

2.3. For any $\bar\gamma > \gamma_o^*$, also define

$$P(\bar\gamma) := (\Gamma_{sP}^{-1})^T \begin{bmatrix} (S_P - \bar\gamma^{-2} T_P)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{sP}^{-1} \tag{5.4}$$

and

$$Q(\bar\gamma) := (\Gamma_{sQ}^{-1})^T \begin{bmatrix} (S_Q - \bar\gamma^{-2} T_Q)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{sQ}^{-1} \tag{5.5}$$

It is shown by Chen *et al*. (1991, 1992) that both $P(\bar\gamma)$ and $Q(\bar\gamma)$ are positive semidefinite for all $\bar\gamma > \gamma_o^*$.

Step 5.1.3.

Construct the following full-order observer-based controller

$$\Sigma_{\text{cmp}}: \begin{cases} \dot v = A_{\text{cmp}} v + B_{\text{cmp}} y \\ u = C_{\text{cmp}} v \end{cases} \tag{5.6}$$

where

$$\begin{aligned} A_{\text{cmp}} = {} & A + \bar\gamma^{-2} E E^T P(\bar\gamma) + B F(\bar\gamma, \varepsilon, \Lambda_{fP}, \Delta_{cP}) + [I - \bar\gamma^{-2} Q(\bar\gamma) P(\bar\gamma)]^{-1} \\ & \cdot \{L(\bar\gamma, \varepsilon, \Lambda_{fQ}, \Delta_{cQ})[C_1 + \bar\gamma^{-2} D_1 E^T P(\bar\gamma)] \\ & + \bar\gamma^{-2} Q(\bar\gamma)[P(\bar\gamma) B + C_2^T D_2] F(\bar\gamma, \varepsilon, \Lambda_{fP}, \Delta_{cP}) \\ & + \bar\gamma^{-2} Q(\bar\gamma)[A^T P(\bar\gamma) + P(\bar\gamma) A + C_2^T C_2 + \bar\gamma^{-2} P(\bar\gamma) E E^T P(\bar\gamma)]\} \end{aligned} \tag{5.7}$$

and

$$B_{\mathrm{cmp}} = -[I - \bar{\gamma}^{-2}Q(\bar{\gamma})P(\bar{\gamma})]^{-1}L(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{Q}}, \Delta_{c\mathrm{Q}}), \quad C_{\mathrm{cmp}} = F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}}) \tag{5.8}$$

We have the following theorem.

Theorem 5.1: *Consider a generalized plant Σ satisfying Assumptions (A1) to (A6). Then for any $\gamma_o^* < \bar{\gamma} \leq \gamma$, for any $\Lambda_{f\mathrm{P}} \subset \mathbb{C}^-$ and $\Lambda_{f\mathrm{Q}} \subset \mathbb{C}^-$ which are closed under complex conjugation, and for any $\Delta_{c\mathrm{P}}$ and $\Delta_{c\mathrm{Q}}$ subject to the constraints that $A^{\mathrm{c}}_{cc\mathrm{P}}$ and $A^{\mathrm{c}}_{cc\mathrm{Q}}$ are stable matrices, there exists an $\varepsilon^* > 0$ such that for all $0 < \varepsilon \leq \varepsilon^*$, the Σ_{cmp} as given in* (5.6) *is $(P, \bar{\gamma})$-admissible; namely, the closed-loop system comprising of Σ and the output feedback controller Σ_{cmp}, is internally stable and the H_∞-norm of the closed-loop transfer function from the disturbance w to the controlled output z is less than γ, i.e.*

$$\|T_{zw}\|_\infty < \bar{\gamma} \leq \gamma$$

where $T_{zw} = \mathcal{F}(P, \Sigma_{\mathrm{cmp}})$.

Proof: For the proof see Appendix D. □

The following theorem deals with the issue of pole/zero cancellations and the closed-loop eigenvalues in the H_∞ sub-optimal output feedback control.

Theorem 5.2: *Consider a generalized plant satisfying Assumptions (A1) to (A6) with the H_∞ sub-optimal control Σ_{cmp} as given in* (5.6). *Then the following hold:*

(1) *$\lambda(A^-_{aa\mathrm{P}})$, the stable invariant zeros of the system (A, B, C_2, D_2), and $\lambda(A^{\mathrm{c}}_{cc\mathrm{P}})$ are the output decoupling zeros of the closed-loop system T_{zw}. Hence, they cancel with the poles of T_{zw}.*

(2) *$\lambda(A^-_{aa\mathrm{Q}})$, the stable invariant zeros of the system (A, E, C_1, D_1), and $\lambda(A^{\mathrm{c}}_{cc\mathrm{Q}})$ are the input decoupling zeros of the closed-loop system T_{zw}. Hence, they cancel with the poles of T_{zw}.*

(3) *As $\varepsilon \to 0$, the fast eigenvalues of the closed-loop system are asymptotically given by $\Lambda_{f\mathrm{P}}/\varepsilon + 0(1)$ and $\Lambda_{f\mathrm{Q}}/\varepsilon + 0(1)$.*

Proof: For the proof see Appendix E. □

The following remarks are in order.

Remark 5.1—Interpretations of parameters ε, $\Lambda_{f\mathrm{P}}$, $\Lambda_{f\mathrm{Q}}$, $\Delta_{c\mathrm{P}}$ and $\Delta_{c\mathrm{Q}}$: Again, as in Remark 4.1, the set of parameters $\Lambda_{f\mathrm{P}}$ and $\Lambda_{f\mathrm{Q}}$ represent the asymptotes of the fast eigenvalues of the closed-loop system while ε represents their time-scale. The set of parameters $\Delta_{c\mathrm{P}}$ and $\Delta_{c\mathrm{Q}}$ prescribe the locations of the slow eigenvalues of the closed-loop system corresponding to $\lambda(A^{\mathrm{c}}_{cc\mathrm{P}})$ and $\lambda(A^{\mathrm{c}}_{cc\mathrm{Q}})$. The eigenvalues can be assigned to any desired locations in $\mathbb{C}^-$ by choosing appropriate $\Delta_{c\mathrm{P}}$ and $\Delta_{c\mathrm{Q}}$. □

Remark 5.2—Regular case: If D_1 is surjective and D_2 is injective, it is simple to verify that $F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}}) = F(\bar{\gamma})$ and $L(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{Q}}, \Delta_{c\mathrm{Q}}) = L(\bar{\gamma})$ depend only on $\bar{\gamma}$. Moreover, we have

$$[P(\bar{\gamma})B + C_2^{\mathrm{T}}D_2]F(\bar{\gamma}) + [A^{\mathrm{T}}P(\bar{\gamma}) + P(\bar{\gamma})A + C_2^{\mathrm{T}}C_2 + \bar{\gamma}^{-2}P(\bar{\gamma})EE^{\mathrm{T}}P(\bar{\gamma})] = 0$$

Hence, Σ_{cmp} reduces to,

$$\Sigma_{\text{cmp}}: \begin{cases} \dot{v} = A_{\text{cmp}} v + B_{\text{cmp}} y \\ u = C_{\text{cmp}} v \end{cases}$$

where

$$\begin{aligned} A_{\text{cmp}} = A &+ \bar{\gamma}^{-2} E E^{\mathrm{T}} P(\bar{\gamma}) + B F(\bar{\gamma}) \\ &+ [I - \bar{\gamma}^{-2} Q(\bar{\gamma}) P(\bar{\gamma})]^{-1} L(\bar{\gamma}) [C_1 + \bar{\gamma}^{-2} D_1 E^{\mathrm{T}} P(\bar{\gamma})] \end{aligned}$$

and

$$B_{\text{cmp}} = -[I - \bar{\gamma}^{-2} Q(\bar{\gamma}) P(\bar{\gamma})]^{-1} L(\bar{\gamma}), \quad C_{\text{cmp}} = F(\bar{\gamma})$$

This corresponds to the regular case, and for $\bar{\gamma} = \gamma$, is the central controller given by Doyle *et al*. (1988). □

Remark 5.3: It is known that for a mixed sensitivity problem (Kwakernaak 1986, Postlewaite *et al*. 1990),

(*a*) the H_∞ design results in pole-zero cancellation between plant and controller at all of the stable poles of the uncompensated plant;

(*b*) moreover, the closed-loop poles include the mirror image positions of all unstable poles of the plant.

We would like to point out that none of these behaviours arise in the class of problem that we have considered in our paper. It is obvious to observe that the class of mixed sensitivity problems and our class of problems are disjoint since mixed sensitivity problems always involve a feedthrough term from the disturbance to the controlled output. □

Remark 5.4: We would also like to point out that both parametrized families of the state and the output feedback laws constructed in this paper, namely (4.13) and (5.6), are in fact minimizing sequences for the minimum entropy H_∞ control problem introduced by Mustafa and Glover (1990). More specifically, by letting $\varepsilon \to 0$ in (4.13) and (5.6) we are minimizing the entropy function. This can be shown by utilizing Theorems 7.9 and 7.11 of Stoorvogel (1992). As such, our results also show the asymptotic behaviour of the closed-loop poles when we have such minimizing sequences for the entropy function. □

Again, we illustrate our results in the following example.

Example (*continued*): Consider the example in the previous section with

$$C_1 = \begin{bmatrix} 0 & -2 & -3 & -2 & -1 \\ 1 & 2 & 3 & 2 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We first note that the pair (A, C_1) is detectable, and the system (A, E, C_1, D_1) is invertible (hence, Assumption A6 is automatically satisfied) and of non-minimum phase with invariant zeros at $\{-1{\cdot}630662, -3{\cdot}593415, 0{\cdot}521129 \pm \mathrm{j}0{\cdot}363042\}$. Then, following our design procedure, we obtain

$$\gamma_{\mathrm{o}}^* = 13{\cdot}6368725$$

and the closed-form to the output feedback sub-optimal controllers, as in (5.6) to (5.8), with $F(\bar{\gamma}, \varepsilon, \lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}})$ given by (4.14), $L(\bar{\gamma}, \varepsilon, \lambda_{f\mathrm{Q}}, \Delta_{c\mathrm{Q}})$ given by

$$\begin{bmatrix} \dfrac{-43\cdot91\bar{\gamma}^4 + 4257\cdot86\bar{\gamma}^2 - 97026\cdot13}{7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23} & -5 + 0\cdot090909\dfrac{\lambda_{fQ}}{\varepsilon} - \dfrac{(0\cdot24\bar{\gamma}^4 - 10\cdot14\bar{\gamma}^2)\lambda_{fQ}}{(7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23)\varepsilon} \\ \dfrac{-12\cdot45\bar{\gamma}^4 + 372\cdot65\bar{\gamma}^2 - 0\cdot02}{7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23} & -0\cdot363636 - \dfrac{(-2\cdot39\bar{\gamma}^4 + 190\cdot91\bar{\gamma}^2)\lambda_{fQ}}{(7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23)\varepsilon} \\ \dfrac{-48\cdot44\bar{\gamma}^4 + 1803\cdot08\bar{\gamma}^2 + 0\cdot02}{7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot12} & -0\cdot382726 - \dfrac{(2\cdot04\bar{\gamma}^4 - 108\cdot95\bar{\gamma}^2)\lambda_{fQ}}{(7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23)\varepsilon} \\ \dfrac{62\cdot57\bar{\gamma}^4 - 1212\cdot58\bar{\gamma}^2 - 38810\cdot46}{7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23} & -2\cdot545451 + 0\cdot272727\dfrac{\lambda_{fQ}}{\varepsilon} - \dfrac{(-1\cdot13\bar{\gamma}^4 + 14\cdot86\bar{\gamma}^2)\lambda_{fQ}}{(7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23)\varepsilon} \\ \dfrac{17\cdot80\bar{\gamma}^4 - 83\cdot04\bar{\gamma}^2 - 19405\cdot21}{7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^2 + 19405\cdot23} & -1\cdot272726 + 0\cdot363636\dfrac{\lambda_{fQ}}{\varepsilon} - \dfrac{(0\cdot69\bar{\gamma}^4 - 74\cdot56\bar{\gamma}^2)\lambda_{fQ}}{(7\cdot12\bar{\gamma}^4 - 790\cdot42\bar{\gamma}^4 + 19405\cdot23)\varepsilon} \end{bmatrix}$$

where $\lambda_{fQ} < 0$, and

$$P(\bar{\gamma}) = \frac{1}{0\cdot132909\bar{\gamma}^2 - 5\cdot560084} \times \begin{bmatrix} 0\cdot427699\bar{\gamma}^2 & -0\cdot296582\bar{\gamma}^2 & 0\cdot163673\bar{\gamma}^2 & 0 & 0 \\ -0\cdot296582\bar{\gamma}^2 & 0\cdot584154\bar{\gamma}^2 - 15\cdot833792 & -0\cdot185427\bar{\gamma}^2 + 3\cdot009097 & 0 & 0 \\ 0\cdot163673\bar{\gamma}^2 & -0\cdot185427\bar{\gamma}^2 + 3\cdot009097 & 0\cdot185427\bar{\gamma}^2 - 5\cdot136846 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q(\bar{\gamma}) = \frac{\bar{\gamma}^2}{0\cdot071193\bar{\gamma}^4 - 7\cdot904171\bar{\gamma}^2 + 194\cdot052288} \times \begin{bmatrix} 0\cdot083104\bar{\gamma}^2 - 3\cdot0576430 & 0\cdot124442\bar{\gamma}^2 - 3\cdot7265760 & 0\cdot484459\bar{\gamma}^2 - 18\cdot030782 & -0\cdot768087\bar{\gamma}^2 + 27\cdot934279 & -0\cdot249208\bar{\gamma}^2 + 8\cdot7345960 \\ 0\cdot124423\bar{\gamma}^2 - 3\cdot7265760 & 1\cdot778706\bar{\gamma}^2 - 122\cdot50790 & 0\cdot340500\bar{\gamma}^2 + 6\cdot5460280 & -1\cdot759522\bar{\gamma}^2 + 79\cdot188507 & -1\cdot184163\bar{\gamma}^2 + 70\cdot727376 \\ 0\cdot484459\bar{\gamma}^2 - 18\cdot030781 & 0\cdot340500\bar{\gamma}^2 + 6\cdot5460280 & 2\cdot917279\bar{\gamma}^2 - 113\cdot22255 & -4\cdot330299\bar{\gamma}^2 + 153\cdot81266 & -1\cdot256601\bar{\gamma}^2 + 36\cdot981101 \\ -0\cdot768087\bar{\gamma}^2 + 27\cdot934279 & -1\cdot759522\bar{\gamma}^2 + 79\cdot188509 & -4\cdot330299\bar{\gamma}^2 + 153\cdot81266 & 7\cdot332315\bar{\gamma}^2 - 272\cdot47959 & 2\cdot613520\bar{\gamma}^2 - 102\cdot79025 \\ -0\cdot249208\bar{\gamma}^2 + 8\cdot7345960 & -1\cdot184163\bar{\gamma}^2 + 70\cdot727376 & -1\cdot256601\bar{\gamma}^2 + 36\cdot981101 & 2\cdot613520\bar{\gamma}^2 - 102\cdot79025 & 1\cdot160281\bar{\gamma}^2 - 55\cdot552230 \end{bmatrix}$$

As in the previous example, we demonstrate our results (in Fig. 3) by the plots of maximum singular values of the closed-loop transfer function matrix for several values of $\bar{\gamma}$ and ε. Note that in Fig. 3, we choose $\lambda_{fP} = -1$, $\Delta_{cP} = 3$ and $\lambda_{fQ} = -1$. Note that since Σ_Q for this example is left invertible, the gain $L(\bar{\gamma}, \varepsilon, \lambda_{fQ}, \Delta_{cQ})$ depends only on $\bar{\gamma}$, ε and λ_{fQ}. □

6. Conclusions

In this paper we have presented a closed-form solution to the H_∞ suboptimal control problem. Our results are obtained under the assumptions that two subsystems $P_{12}(s)$ and $P_{21}(s)$ do not have invariant zeros on the jω axis and they satisfy some geometric conditions. We have made no restrictions on the direct feedthrough matrices from control input to the controlled output and from the disturbance input to the measurement output. For the same class of systems, we have also identified conditions under which the H_∞ optimal state feedback laws exist and these optimal solutions, when they exist, are given.

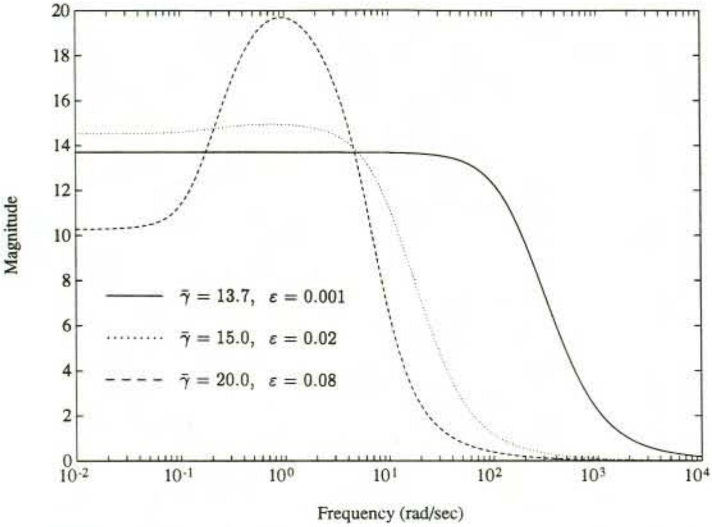

Figure 3. Maximum singular values of T_{zw} (output feedback case).

Our current effort is focused on weakening the assumptions posed on the transfer functions P_{12} and P_{21} and the issue of optimality for the case of output feedback.

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Appendix A

Proof of Theorem 4.1: We need to recall the following two lemmas in order to proceed with our proof of theorem 4.1.

Lemma A.1: *Let an auxiliary system Σ_{aux} be characterized by*

$$\Sigma_{\text{aux}}: \begin{cases} \dot{x}_x = A_x x_x + B_x u_x + E_x w_x \\ z_x = C_x x_x + D_x u_x \end{cases} \tag{A 1}$$

where

$$A_x = A_{11\text{P}}, \quad B_x = [B_{11\text{P}} \quad A_{13\text{P}}], \quad E_x = \begin{bmatrix} E_{a\text{P}}^{+} \\ 0 \end{bmatrix}$$

and

$$C_x = \Gamma_{o\text{P}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_{b\text{P}} \end{bmatrix}, \quad D_x = \Gamma_{o\text{P}} \begin{bmatrix} I & 0 \\ 0 & C_{f\text{P}} C_{f\text{P}}^{\text{T}} \\ 0 & 0 \end{bmatrix}$$

Then the state feedback law $u_x = -F_{11}(\bar{\gamma}) x_x$ applied to Σ_{aux} in (A 1) *is internally stable i.e.*

$$\lambda(A_{11\text{P}}^{\text{c}}) = \lambda\{A_{11\text{P}} - [B_{11\text{P}}, A_{13\text{P}}] F_{11}(\bar{\gamma})\} = \lambda\{A_x - B_x F_{11}(\bar{\gamma})\} \subset \mathbb{C}^{-} \tag{A 2}$$

and the resulting closed-loop transfer function from w_x to z_x has H_∞ norm less than $\bar{\gamma}$, i.e.

$$\|T_{z_x w_x}\|_\infty = \left\| \Gamma_{oP} \begin{bmatrix} -F_{11}(\bar{\gamma}) \\ [0 \;\; C_{bP}] \end{bmatrix} (sI - A_{11P}^{c})^{-1} \begin{bmatrix} E_{aP}^{+} \\ 0 \end{bmatrix} \right\|_\infty < \bar{\gamma} \tag{A 3}$$

Proof of Lemma A.1: We first note that Γ_{oP} is non-singular and $C_{fP}C_{fP}^{\mathrm{T}} = I$ which implies that D_x is injective. Furthermore, it is simple to verify that the invariant zeros of (A_x, B_x, C_x, D_x) are given by $\lambda(A_{aaP}^{+})$, and are not on the jω axis. Hence, Σ_{aux} satisfies the assumptions of a regular H_∞ control problem. Moreover, it is straightforward to verify that for any $\gamma_s^* < \bar{\gamma} \leq \gamma$, $P_0(\bar{\gamma}) = (S_P - \bar{\gamma}^{-2}T_P)^{-1} > 0$ is the solution of the following well-known H_∞-ARE

$$\begin{aligned} P_0(\bar{\gamma})A_x + A_x^{\mathrm{T}}P_0(\bar{\gamma}) + \bar{\gamma}^{-2}P_0(\bar{\gamma})E_xE_x^{\mathrm{T}}P_0(\bar{\gamma}) + C_x^{\mathrm{T}}C_x \\ - [P_0(\bar{\gamma})B_x + C_x^{\mathrm{T}}D_x](D_x^{\mathrm{T}}D_x)^{-1}[B_x^{\mathrm{T}}P_0(\bar{\gamma}) + D_x^{\mathrm{T}}C_x] = 0 \end{aligned} \tag{A 4}$$

with

$$\lambda(A_{xx}^{c}) := \lambda\{A_x + \bar{\gamma}^{-2}E_xE_x^{\mathrm{T}}P_0(\bar{\gamma}) - B_x(D_x^{\mathrm{T}}D_x)^{-1}(B_x^{\mathrm{T}}P_0(\bar{\gamma}) + D_xC_x)\} \in \mathbb{C}^-$$

Then the results of Lemma A.1 follow directly from Stoorvogel (1992). □

Lemma A.2: *Let (A, B, C), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, be right invertible and of minimum phase. Let $F(\varepsilon) \in \mathbb{R}^{m\times n}$ be parametrized in terms of ε and be of the form*

$$F(\varepsilon) = N(\varepsilon)\Gamma(\varepsilon)T(\varepsilon) + R(\varepsilon) \tag{A 5}$$

where $N(\varepsilon) \in \mathbb{R}^{m\times p}$, $\Gamma(\varepsilon) \in \mathbb{R}^{p\times p}$, $T(\varepsilon) \in \mathbb{R}^{p\times n}$ and $R(\varepsilon) \in \mathbb{R}^{m\times n}$. Also, $\Gamma(\varepsilon)$ is non-singular. Moreover, assume that the following conditions hold:

- (*a*) *$A + BF(\varepsilon)$ is asymptotically stable for all $0 < \varepsilon \leq \varepsilon^*$ where $\varepsilon^* > 0$;*
- (*b*) *$T(\varepsilon) \to WC$ as $\varepsilon \to 0$ where W is some $p \times p$ non-singular matrix;*
- (*c*) *as $\varepsilon \to 0$, $N(\varepsilon)$ tends to some finite matrix N such that $C(sI - A)^{-1}BN$ is invertible;*
- (*d*) *as $\varepsilon \to 0$, $R(\varepsilon)$ tends to some finite matrix R;*
- (*e*) *$\Gamma^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Then as $\varepsilon \to 0$, we have $\|C[sI - A - BF(\varepsilon)]^{-1}\|_\infty \to 0$.

Proof of Lemma A.2: This is a dual version of Lemma 2.2 given by Saberi and Sannuti (1990 b). The proof of this lemma follows from similar arguments as in Saberi and Sannuti (1990 b). □

Now we are ready to proceed with the proof of Theorem 4.1. Note that $F(\bar{\gamma}, \varepsilon, \Lambda_{fP}, \Delta_{fP})$ is constructed under the standard ATEA procedure. It can be shown using the techniques of the well-known singular perturbation theory as in Appendix B of Saberi *et al*. (1991) and Appendix A of Chen *et al*. (1992 b) that as $\varepsilon \to 0$, the eigenvalues of

$$A + BF(\bar{\gamma}, \varepsilon, \Lambda_{fP}, \Delta_{fP})$$

are given by $\lambda(A_{aaP}^{-}) \in \mathbb{C}^-$, $\lambda(A_{ccP}^{c}) \in \mathbb{C}^-$, $\Lambda_{fP}/\varepsilon \in \mathbb{C}^-$ and $\lambda(A_{11P}^{c}) \in \mathbb{C}^-$ (see Lemma A.1). Hence, the closed-loop is internally stable. Moreover, following the results of Saberi *et al*. 1991, Chen *et al*. 1992 b), it can be shown that for any $\lambda_f \in \Lambda_{f\mathrm{P}}/\varepsilon \in \mathbb{C}^-$, the corresponding right eigenvector, say $W(\varepsilon)$, satisfies

$$\lim_{\varepsilon\to 0} W(\varepsilon) = \bar{W} \in \mathcal{S}^+(A, B, C_2, D_2) \tag{A 6}$$

In fact, following the same arguments, one can show that as $\varepsilon \to 0$, the eigenvalues of

$$A + \bar{\gamma}^{-2} E E^{\mathrm{T}} P(\bar{\gamma}) + BF(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})$$

where $P(\bar{\gamma})$ is as defined in (5.4), are given by $\lambda(A^-_{aa\mathrm{P}}) \in \mathbb{C}^-$, $\lambda(A^{\mathrm{c}}_{cc\mathrm{P}}) \in \mathbb{C}^-$, $\Lambda_{f\mathrm{P}}/\varepsilon \in \mathbb{C}^-$ and $\lambda(A^{\mathrm{c}}_{xx}) \in \mathbb{C}^-$. We will use these properties later on in our proofs of other theorems. This proves the second part of Theorem 4.1.

In what follows, we will show that the state feedback law $u = F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})x$ yields

$$\|T_{zw}\|_\infty = \|[C_2 + D_2 F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})][sI - A - BF(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})]^{-1} E\|_\infty < \bar{\gamma} \leq \gamma$$

Without loss of generality, but for simplicity of presentation, we assume that the non-singular transformations $\Gamma_{s\mathrm{P}} = I$ and $\Gamma_{i\mathrm{P}} = I$, i.e. the system $(A, B, \Gamma^{-1}_{o\mathrm{P}} C_2, \Gamma^{-1}_{o\mathrm{P}} D_2)$ is in the form of s.c.b. In view of (4.13), let us partition $F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})$ as

$$F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}}) = \bar{F}_0(\bar{\gamma}) + \begin{bmatrix} 0 \\ \bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}}) \end{bmatrix}$$

where

$$\bar{F}_0(\bar{\gamma}) = -\begin{bmatrix} C^+_{0a\mathrm{P}} + F^+_{a0}(\bar{\gamma}) & C_{0b\mathrm{P}} + F_{b0}(\bar{\gamma}) & C^-_{0a\mathrm{P}} & C_{0c\mathrm{P}} & C_{0f\mathrm{P}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}}) = -\begin{bmatrix} E^+_{fa\mathrm{P}} + \tilde{F}^+_{a1}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}) & E_{fb\mathrm{P}} + \tilde{F}_{b1}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}) & E^-_{fa\mathrm{P}} & E_{fc\mathrm{P}} & \tilde{F}_f(\varepsilon, \Lambda_{f\mathrm{P}}) + E_{f\mathrm{P}} \\ E^+_{ca\mathrm{P}} & E_{cb\mathrm{P}} & E^-_{ca\mathrm{P}} & \Delta_{c\mathrm{P}} & 0 \end{bmatrix} \tag{A 7}$$

Then we have

$$\bar{C} = C_2 + D_2 F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}}) = \Gamma_{o\mathrm{P}} \begin{bmatrix} -F^+_{a0}(\bar{\gamma}) & -F_{b0}(\bar{\gamma}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{f\mathrm{P}} \\ 0 & C_{b\mathrm{P}} & 0 & 0 & 0 \end{bmatrix}$$

and

$$\bar{A} = A + B\bar{F}_0(\bar{\gamma}), \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_{c\mathrm{P}} \\ B_{f\mathrm{P}} & 0 \end{bmatrix} \tag{A 8}$$

With these definitions, we can write T_{zw} as

$$T_{zw} = \bar{C}[sI - \bar{A} - \bar{B}\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})]^{-1}E$$

Then, in view of (A 7), it can easily be seen that $\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})$ has the form

$$\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}}) = N\Gamma(\varepsilon)T(\varepsilon) + R$$

where

$$\Gamma(\varepsilon) = \mathrm{diag}\left[\frac{1}{\varepsilon^{q_1}}, \frac{1}{\varepsilon^{q_2}}, \ldots, \frac{1}{\varepsilon^{qm_{f\mathrm{P}}}}\right], \quad N = -\begin{bmatrix} I_{m_{f\mathrm{P}}} \\ 0 \end{bmatrix}$$

and

$$R = -\begin{bmatrix} E^{+}_{fa\mathrm{P}} & E_{fb\mathrm{P}} & E^{-}_{fa\mathrm{P}} & E_{fc\mathrm{P}} & E_{f\mathrm{P}} \\ E^{+}_{ca\mathrm{P}} & E_{cb\mathrm{P}} & E^{-}_{ca\mathrm{P}} & \Delta_{c\mathrm{P}} & 0 \end{bmatrix}$$

while $T(\varepsilon)$ satisfies

$$T(\varepsilon) \to TC_m$$

as $\varepsilon \to 0$, where

$$T = \mathrm{diag}[F_{1q_1}, F_{1q_2}, \ldots, F_{m_{f\mathrm{P}}qm_{f\mathrm{P}}}]$$

and

$$C_m = [F^{+}_{a1}(\bar{\gamma}) \quad F_{b1}(\bar{\gamma}) \quad 0 \quad 0 \quad C_{f\mathrm{P}}] \tag{A 9}$$

Using the same arguments as in Chen *et al*. (1992 a), it is straightforward to show that the triple $(\bar{A}, \bar{B}, C_m)$ is right invertible and of minimum phase. Thus, it follows from Lemma A.2 that

$$\|C_m[sI - \bar{A} - \bar{B}\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})]^{-1}\|_\infty \to 0$$

as $\varepsilon \to 0$. We should also note that following the same line of reasoning, one can show that the triple $(\bar{A} + \bar{\gamma}^{-2}EE^{\mathrm{T}}P(\bar{\gamma}), \bar{B}, C_m)$ is right invertible and of minimum phase, and moreover as $\varepsilon \to 0$,

$$\|C_m[sI - \bar{A} - \bar{\gamma}^{-2}EE^{\mathrm{T}}P(\bar{\gamma}) - \bar{B}\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})]^{-1}\|_\infty \to 0 \tag{A 10}$$

Next let

$$\bar{C} = \Gamma_{\mathrm{oP}}\begin{bmatrix} 0 \\ C_m \\ 0 \end{bmatrix} + C_e$$

where

$$C_e = \Gamma_{o\mathrm{P}}\begin{bmatrix} -F^{+}_{a0}(\bar{\gamma}) & -F_{b0}(\bar{\gamma}) & 0 & 0 & 0 \\ -F^{+}_{a1}(\bar{\gamma}) & -F_{b1}(\bar{\gamma}) & 0 & 0 & 0 \\ 0 & C_{b\mathrm{P}} & 0 & 0 & 0 \end{bmatrix}$$

We have

$$\|T_{zw}\|_\infty \to \|C_e[sI - \bar{A} - \bar{B}\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})]^{-1}E\|_\infty$$

as $\varepsilon \to 0$. Following the procedures of Saberi *et al*. (1991) and Chen *et al*. (1992 b), it can be shown that

$$C_e[sI - \bar{A} - \bar{B}\bar{F}(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{f\mathrm{P}})]^{-1}E \to \Gamma_{\mathrm{oP}}\begin{bmatrix} -F_{11}(\bar{\gamma}) \\ [0 \quad C_{b\mathrm{P}}] \end{bmatrix}(sI - A^{\mathrm{c}}_{11\mathrm{P}})^{-1}\begin{bmatrix} E^{+}_{a\mathrm{P}} \\ 0 \end{bmatrix}$$

pointwise in s as $\varepsilon \to 0$. Hence, the results of Theorem 4.1 follow from Lemma A.1. □

Appendix B

Proof of Theorem 4.2: Without loss of generality, but for simplicity of presentation, we assume that the non-singular state and input transformations $\Gamma_{sP} = I$ and $\Gamma_{iP} = I$, i.e. the system $(A, B, \Gamma_{oP}^{-1}C_2, \Gamma_{oP}^{-1}D_2)$ is in the form of s.c.b. Then, it is trivial to show that

$$A + BF(\bar{\gamma}, \varepsilon, \Lambda_{fP}, \Delta_{cP}) = \begin{bmatrix} \star & 0 & 0 & \star \\ \star & A_{aaP}^{-} & 0 & \star \\ \star & 0 & A_{ccP}^{c} & \star \\ \star & 0 & 0 & \star \end{bmatrix}$$

and

$$C_2 + D_2F(\bar{\gamma}, \varepsilon, \Lambda_{fP}, \Delta_{cP}) = \Gamma_{oP}\begin{bmatrix} \star & 0 & 0 & 0 \\ 0 & 0 & 0 & \star \\ \star & 0 & 0 & 0 \end{bmatrix}$$

where the $\star$ represents some submatrices which are of no interest to our proof. Hence, for any $\alpha \in \lambda(A_{aaP}^{-}) \cup \lambda(A_{ccP}^{c})$, the corresponding right eigenvector is in the kernel of $C_2 + D_2F(\bar{\gamma}, \varepsilon, \Lambda_{fP}, \Delta_{cP})$. This proves that α is an output decoupling zero of T_{zw}. □

Appendix C

Proof of Theorem 4.3: Following the formulation of Scherer (1989), it is straightforward to show that the infimum, γ_s^*, is attainable only when $P_0(\gamma_s^*) = [S_P - (\gamma_s^*)^{-2}T_P]^{-1} > 0$. In view of (4.6), we know that $P_0(\gamma_s^*) > 0$ if and only if $T_P \equiv 0$, which implies that $\gamma_s^* = 0$. Hence, the conditions under which the infimum, γ_s^*, is attainable, is equivalent to the solvability conditions of disturbance decoupling with internal stability to Σ_P. It is well-known (Stoorvogel and van der Woude 1991) that the problem of disturbance decoupling with internal stability to Σ_P is solvable if and only if $\mathrm{Im}(E) \subseteq \mathcal{V}^-(A, B, C_2, D_2)$. This proves the first part of Theorem 4.3.

In what follows, we will prove the second part of Theorem 4.3 by contradiction, which also follows the same line as in Scherer (1990). As was done by Scherer (1990), it is simple to convert the given plant

$$\Sigma_P: \begin{cases} \dot{x} = Ax + Bu + Ew \\ z = C_2x + D_2u \end{cases}$$

with the dynamic state feedback controller

$$\left.\begin{aligned} \dot{v} &= A_{\mathrm{cmp}}(\gamma)v + B_{\mathrm{cmp}}(\gamma)x \\ u &= C_{\mathrm{cmp}}(\gamma)v + D_{\mathrm{cmp}}(\gamma)x \end{aligned}\right\}$$

where $v \in \mathbb{R}^{d(\gamma)}$, into an auxiliary plant

$$\Sigma_d: \begin{cases} \dot{x}_d = A_dx_d + B_du_d + E_dw \\ z = C_dx_d + D_du_d \end{cases}$$

where

$$x_d = \begin{bmatrix} x \\ v \end{bmatrix}, \quad u_d = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A_d = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad E_d = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

and

$$C_d = [C_2 \ 0], \ D_d = [D_2 \ 0]$$

with a static state feedback law

$$u_d = F_{d(\gamma)}(\gamma)x_d = \begin{bmatrix} D_{\mathrm{cmp}}(\gamma) & C_{\mathrm{cmp}}(\gamma) \\ B_{\mathrm{cmp}}(\gamma) & A_{\mathrm{cmp}}(\gamma) \end{bmatrix} x_d$$

Now assume that the second part of Theorem 4.3 is false, then take any sequence of admissible $F_{d(\gamma_i)}(\gamma_i)$ with $\|\mathcal{F}(\Sigma_d, F_{d(\gamma_i)}(\gamma_i))\|_\infty \to \gamma_s^*$ as $\gamma_i \to \gamma_s^*$ and suppose that $d(\gamma_i)$ is bounded and the sequence is not of high-gain. Then, we can extract a subsequence (γ_{i_m}) such that $d(\gamma_{i_m}) \equiv d$, which is a constant, and $F_d(\gamma_{i_m})$ converges to some $F_d(\gamma_s^*)$ as $\gamma_{i_m} \to \gamma_s^*$. In the limit, we obtain $\lambda(A_d + B_dF_d(\gamma_s^*)) \subset \mathbb{C}^- \cup \mathbb{C}^0$ and for

$$T_{zw}(s) = [C_d + D_dF_d(\gamma_s^*)][sI - A_d - B_dF_d(\gamma_s^*)]^{-1}E_d$$

the equality

$$T_{zw}^{\mathrm{H}}(\mathrm{j}\omega)T_{zw}(\mathrm{j}\omega) = (\gamma_s^*)^2 I$$

holds for all $\omega \in \mathbb{R}$. Then, following the same procedure as Scherer (1989), it can be shown that $\lambda(A_d + B_dF_d(\gamma_s^*)) \in \mathbb{C}^-$ and hence $F_d(\gamma_s^*)$ is an admissible controller which achieves the infimum. This is a contradiction. Hence, the result follows. □

Appendix D

Proof of Theorem 5.1: For the sake of simplicity, in the following we drop the arguments of $F(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{P}}, \Delta_{c\mathrm{P}})$ and $L(\bar{\gamma}, \varepsilon, \Lambda_{f\mathrm{Q}}, \Delta_{c\mathrm{Q}})$. Also, we assume, without loss of generality, that $\bar{\gamma} = 1$. Thus, we will drop the dependency of $\bar{\gamma}$ in all the variables.

First, it is simple to verify that the positive semi-definite matrices P of (5.4) and Q of (5.5) satisfy

$$\boldsymbol{F}(P) := \begin{bmatrix} A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + PEE^{\mathrm{T}}P & PB + C_2^{\mathrm{T}}D_2 \\ B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2 & D_2^{\mathrm{T}}D_2 \end{bmatrix} \geqslant 0$$

and

$$\boldsymbol{G}(Q) := \begin{bmatrix} AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + QC_2^{\mathrm{T}}C_2Q & QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_1Q + D_1E^{\mathrm{T}} & D_1D_1^{\mathrm{T}} \end{bmatrix} \geqslant 0$$

respectively, i.e. P and Q are the solutions of the quadratic matrix inequalities $\boldsymbol{F}(P) \geqslant 0$ and $\boldsymbol{G}(Q) \geqslant 0$. Moreover, the following auxiliary system

$$\Sigma_{\mathrm{PQ}}: \begin{cases} \dot{x}_{\mathrm{PQ}} = A_{\mathrm{PQ}}x_{\mathrm{PQ}} + B_{\mathrm{PQ}}u_{\mathrm{PQ}} + E_{\mathrm{PQ}}w_{\mathrm{PQ}} \\ y_{\mathrm{PQ}} = C_{1\mathrm{P}}x_{\mathrm{PQ}} + D_{\mathrm{PQ}}w_{\mathrm{PQ}} \\ z_{\mathrm{PQ}} = c_{2\mathrm{P}}x_{\mathrm{PQ}} + D_{\mathrm{P}}u_{\mathrm{PQ}} \end{cases} \qquad \text{(D 1)}$$

where

$$F(P) = \begin{bmatrix} C_{2P}^{T} \\ D_{P}^{T} \end{bmatrix} [C_{2P} \;\; D_{P}], \quad G(Q) = \begin{bmatrix} E_{Q} \\ D_{PQ} \end{bmatrix} [E_{Q}^{T} \;\; D_{PQ}^{T}]$$

and

$$\begin{aligned} A_{PQ} &:= A + EE^{T}P + (I - QP)^{-1}QC_{2P}^{T}C_{2P} \\ B_{PQ} &:= B + (I - QP)^{-1}QC_{2P}^{T}D_{P} \\ E_{PQ} &:= (I - QP)^{-1}E_{Q} \\ C_{1P} &:= C_{1} + D_{1}E^{T}P \end{aligned}$$

has the following properties:

(1) $(A_{PQ}, B_{PQ}, C_{2P}, D_{P})$ is right invertible and of minimum phase.

(2) $(A_{PQ}, E_{PQ}, C_{1P}, D_{PQ})$ is left invertible and of minimum phase.

The following lemma is due to Stoorvogel (1992).

Lemma D.1: *For any given compensator Σ_{F} of the form*

$$\Sigma_{F}: \begin{cases} \dot{v} = A_{cmp}v + B_{cmp}y \\ u = C_{cmp}v + D_{cmp}y \end{cases}$$

The following two statements are equivalent:

(i) *Σ_{F} applied to the system Σ defined by* (2.1) *is internally stabilizing and the resulting closed-loop transfer function from w to z has an H_{∞} norm less than 1, i.e. $\|\mathcal{F}(\Sigma, \Sigma_{F})\|_{\infty} < 1$.*

(ii) *Σ_{F} applied to the new system Σ_{PQ} defined by* (D 1) *is internally stabilizing and the resulting closed loop transfer function from w_{PQ} to z_{PQ} has an H_{∞} norm less than 1, i.e. $\|\mathcal{F}(\Sigma_{PQ}, \Sigma_{F})\|_{\infty} < 1$.*

Hence, it is sufficient to show Theorem 5.1 by showing that Σ_{cmp} of (5.6) to (5.8) applied to Σ_{PQ} achieves almost disturbance decoupling with internally stability. Observing that

$$C_{2P}^{T}C_{2P} = A^{T}P + PA + C_{2}^{T}C_{2} + PEE^{T}P \quad \text{and} \quad C_{2P}^{T}D_{P} = PB + C_{2}^{T}D_{2}$$

it is simple to rewrite A_{cmp} of (5.7) as

$$A_{cmp} = A_{PQ} + B_{PQ}F + (I - QP)^{-1}LC_{1P}$$

Now it is trivial to see that Σ_{cmp} of (5.6) is simply the well-known full-order observer-based controller for the system Σ_{PQ} with state feedback gain F and observer gain $(I - QP)^{-1}L$. Hence, the well-known separation principle holds. Also, noting the facts that $(A_{PQ}, B_{PQ}, C_{2P}, D_{P})$ and $(A_{PQ}, E_{PQ}, C_{1P}, D_{PQ})$ are of minimum phase, and are right invertible and left invertible, respectively, it is sufficient to prove Theorem 5.1 by showing that as $\varepsilon \to 0$

(1) $A_{PQ} + B_{PQ}F$ is asymptotically stable;

(2) $\|[C_{2P} + D_{P}F][sI - A_{PQ} - B_{PQ}F]^{-1}\|_{\infty} \to 0$;

(3) $A_{PQ} + (I - QP)^{-1}LC_{1P}$ is asymptotically stable;

(4) $\|[sI - A_{PQ} - (I - QP)^{-1}LC_{1P}]^{-1}[E_{PQ} + (I - QP)^{-1}LD_{PQ}]\|_{\infty} \to 0$.

We shall introduce the following lemma for further development.

Lemma D.2: *As $\varepsilon \to 0$, we have*

(1) *$A + EE^{\mathrm{T}}P + BF$ is asymptotically stable and*

$$\|[C_{2\mathrm{P}} + D_{\mathrm{P}}F][sI - A - EE^{\mathrm{T}}P - BF]^{-1}\|_\infty \to 0 \tag{D 2}$$

(2) *$A + QC_2^{\mathrm{T}}C_2 + LC_1$ is asymptotically stable and*

$$\|[sI - A - QC_2^{\mathrm{T}}C_2 - LC_1]^{-1}[E_{\mathrm{Q}} + LD_{\mathrm{PQ}}]\|_\infty \to 0 \tag{D 3}$$

Proof of Lemma D.2: It is shown in Appendix A that for $\varepsilon \to 0$, $A + EE^{\mathrm{T}}P + BF$ is asymptotically stable. In what follows, we will show (D 2). By some elementary algebra, it can be shown that

$$C_{2\mathrm{P}} = \Gamma_{o\mathrm{P}}\begin{bmatrix} C_{0a\mathrm{P}}^{+} + F_{a0}^{+} & C_{0b\mathrm{P}} + F_{b0} & C_{0a\mathrm{P}}^{-} & C_{0c\mathrm{P}} & C_{0f\mathrm{P}} \\ F_{a1}^{+} & F_{b1} & 0 & 0 & C_{f\mathrm{P}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\Gamma_{s\mathrm{P}}^{-1}$$

and

$$D_{\mathrm{P}} = D_2 = \Gamma_{o\mathrm{P}}\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\Gamma_{i\mathrm{P}}^{-1}$$

Moreover

$$[C_{2\mathrm{P}} + D_{\mathrm{P}}F][sI - A - EE^{\mathrm{T}}P - BF]^{-1} = \begin{bmatrix} 0 \\ C_m \\ 0 \end{bmatrix}[sI - \bar{A} - EE^{\mathrm{T}}P - \bar{B}\bar{F}]^{-1}$$

where $\bar{A}$ and $\bar{B}$ are as in (A 8), $\bar{F}$ is as in (A 7) and C_m is given by (A 9). In view of (A 10), we have the result.

Item 2 of Lemma D.2 is the dual version of item 1. Hence, the results follow. This completes the proof of Lemma D.2. □

Next, we will first show that $A_{\mathrm{PQ}} + B_{\mathrm{PQ}}F$ is asymptotically stable for some sufficiently small ε and

$$\|[C_{2\mathrm{P}} + D_{\mathrm{P}}F][sI - A_{\mathrm{PQ}} - B_{\mathrm{PQ}}F]^{-1}\|_\infty \to 0$$

as $\varepsilon \to 0$. In view of Lemma D.2, we have

$$\begin{aligned} sI - A_{\mathrm{PQ}} - B_{\mathrm{PQ}}F &= sI - A - EE^{\mathrm{T}}P - BF - (I - QP)^{-1}QC_{2\mathrm{P}}^{\mathrm{T}}[C_{2\mathrm{P}} + D_{\mathrm{P}}F] \\ &= \{I - (I - QP)^{-1}QC_{2\mathrm{P}}^{\mathrm{T}}[C_{2\mathrm{P}} + D_{\mathrm{P}}F][sI - A - EE^{\mathrm{T}}P - BF]^{-1}\} \\ &\quad [sI - A - EE^{\mathrm{T}}P - BF] \\ &\to sI - A - EE^{\mathrm{T}}P - BF \text{ pointwise in } s \text{ as } \varepsilon \to 0. \end{aligned}$$

This implies that $A_{\mathrm{PQ}} + B_{\mathrm{PQ}}F$ is asymptotically stable for sufficiently small ε, and

$$\begin{aligned} [C_{2\mathrm{P}} + D_{\mathrm{P}}F][sI - A_{\mathrm{PQ}} - B_{\mathrm{PQ}}F]^{-1} &= [C_{2\mathrm{P}} + D_{\mathrm{P}}F][sI - A - EE^{\mathrm{T}}P - BF]^{-1} \\ &\quad \{I - (I - QP)^{-1}QC_{2\mathrm{P}}^{\mathrm{T}}[C_{2\mathrm{P}} + D_{\mathrm{P}}F][sI - A - EE^{\mathrm{T}}P - BF]^{-1}\}^{-1} \\ &\to 0 \text{ pointwise in } s \text{ as } \varepsilon \to 0. \end{aligned} \tag{D 4}$$

Again, in view of Lemma D.2 and

$$C_{2P}^{T}C_{2P} = A^{T}P + PA + C_2^{T}C_2 + PEE^{T}P$$
$$E_{Q}E_{Q}^{T} = AQ + QA^{T} + EE^{T} + QC_2^{T}C_2Q$$

we have the following induction

$$\begin{aligned}
&(I - QP)[sI - A_{PQ} - (I - QP)^{-1}LC_{1P}] \\
&\quad = [(I - QP)(sI - A - EE^{T}P) - QC_{2P}^{T}C_{2P} - LC_1 - LD_1E^{T}P] \\
&\quad = [sI - A - EE^{T}P - QC_{2P}^{T}C_{2P} - LC_1 - LD_1E^{T}P - sQP \\
&\qquad + QPA + QPEE^{T}P] \\
&\quad = [sI - A - EE^{T}P - Q(A^{T}P + PA + C_2^{T}C_2 + PEE^{T}P) \\
&\qquad - LC_1 - LD_1E^{T}P - sQP + QPA + QPEE^{T}P] \\
&\quad = [sI - A - QC_2^{T}C_2 - LC_1 - EE^{T}P - LD_1E^{T}P - QA^{T}P - sQP] \\
&\quad = [sI - A - QC_2^{T}C_2 - LC_1 - (E_QE_Q^{T} - AQ - QA^{T} - QC_2^{T}C_2Q)P \\
&\qquad - LD_1E^{T}P - QA^{T}P - sQP] \\
&\quad = [sI - A - QC_2^{T}C_2 - LC_1 - sQP + AQP + QC_2^{T}C_2QP \\
&\qquad - E_QE_Q^{T}LD_1E^{T}P] \\
&\quad = [(sI - A - QC_2^{T}C_2 - LC_1)(I - QP) - (E_Q + LD_{PQ})E_Q^{T}P] \\
&\quad = [sI - A - QC_2^{T}C_2 - LC_1] \\
&\qquad [(I - QP) - (sI - A - QC_2^{T}C_2 - LC_1)^{-1}(E_Q + LD_{PQ})E_Q^{T}P] \\
&\quad \to [sI - A - QC_2^{T}C_2 - LC_1](I - QP) \text{ pointwise in } s \text{ as } \varepsilon \to 0. \qquad \text{(D 5)}
\end{aligned}$$

Hence, $A_{PQ} + (I - QP)^{-1}LC_{1P}$ is asymptotically stable for sufficiently small ε. Now it follows from (D 5) that

$$\begin{aligned}
&[sI - A_{PQ} - (I - QP)^{-1}LC_{1P}]^{-1}[E_{PQ} + (I - QP)^{-1}LD_{PQ}] \\
&\quad \to (I - QP)^{-1}[sI - A - QC_2^{T}C_2 - LC_1]^{-1}(I - QP) \\
&\qquad [E_{PQ} + (I - QP)^{-1}LD_{PQ}] \\
&\quad = (I - QP)^{-1}[sI - A - QC_2^{T}C_2 - LC_1]^{-1}[E_Q + LD_{PQ}] \\
&\quad \to 0 \text{ pointwise in } s \text{ as } \varepsilon \to 0.
\end{aligned}$$

This completes the proof of Theorem 5.1. □

Appendix E

Proof of Theorem 5.2: As usual, for the sake of simplicity, we will assume that $\bar{\gamma} = 1$ and let $F = F(\bar{\gamma}, \varepsilon, \Lambda_{fP}, \Delta_{cP})$ and $L = L(\bar{\gamma}, \varepsilon, \Lambda_{fQ}, \Delta_{cQ})$. Then the closed loop system $T_{zw}(s)$ is given by

$$T_{zw}(s) = [C_2 \;\; D_2F]\left(sI - \begin{bmatrix} A & BF \\ -(I - QP)^{-1}LC_1 & A_{\mathrm{cmp}} \end{bmatrix}\right)^{-1}\begin{bmatrix} E \\ -(I - QP)^{-1}LD_1 \end{bmatrix}$$

It follows from Appendix B that for any $\alpha \in \lambda(A^-_{aaP}) \cup \lambda(A^c_{ccP}) \subseteq \lambda(A + BF)$, the corresponding right eigenvector, say W, i.e. $(A + BF)W = \alpha W$, satisfies $(C_2 + D_2F)W = 0$. Moreover, it is simple to verify that $(C_{2P} + D_PF)W = 0$ and $PW = 0$. By duality, one can show that for any $\beta \in \lambda(A^-_{aaQ}) \cup \lambda(A^c_{ccQ})$, $\beta \in \lambda(A + LC_1)$ and the corresponding left eigenvector, say V, i.e. $V^H(A + LC_1) = \beta V^H$, satisfies $V^H(E + LD_1) = 0$ and $V^HQ = 0$. In view of (5.7), we have

$$\begin{aligned} A_{\text{cmp}}W &= [A + EE^TP + BF + (I - QP)^{-1}QC_{2P}^T(C_{2P} + D_PF) \\ &\quad + (I - QP)^{-1}LC_1 + (I - QP)^{-1}LD_1E^TP]W \\ &= (I - QP)^{-1}LC_1W + (A + BF)W \end{aligned}$$

and

$$\begin{aligned} V^HA_{\text{cmp}} &= V^H(I - QP)[A + EE^TP + BF + (I - QP)^{-1}QC_{2P}^T(C_{2P} + D_PF) \\ &\quad + (I - QP)^{-1}LC_1 + (I - QP)^{-1}LD_1E^TP] \\ &= V^HBF + V^H(A + LC_1). \end{aligned}$$

Therefore,

$$\begin{aligned} \begin{bmatrix} A & BF \\ -(I - QP)^{-1}LC_1 & A_{\text{cmp}} \end{bmatrix} \begin{bmatrix} W \\ W \end{bmatrix} &= \begin{bmatrix} (A + BF)W \\ A_{\text{cmp}}W - (I - QP)^{-1}LC_1 \end{bmatrix} \\ &= \alpha \begin{bmatrix} W \\ W \end{bmatrix} \end{aligned}$$

and

$$[C_2 \;\; D_2F] \begin{bmatrix} W \\ W \end{bmatrix} = (C_2 + D_2F)W = 0$$

This shows that α is an output decoupling zero of $T_{zw}(s)$. Similarly

$$\begin{aligned} [V^H \;\; -V^H] &\begin{bmatrix} A & BF \\ -(I - QP)^{-1}LC_1 & A_{\text{cmp}} \end{bmatrix} \\ &= [V^H(I - QP)[A + (I - QP)^{-1}LC_1] \;\; V^H(BF - A_{cmp})] \\ &= \beta[V^H \;\; -V^H] \end{aligned}$$

and

$$[V^H \;\; -V^H] \begin{bmatrix} E \\ -(I - QP)^{-1}LD_1 \end{bmatrix} = V^H(E + LD_1) = 0$$

This implies that β is an input decoupling zero of $T_{zw}(s)$.

The first part of item 3 in Theorem 5.2 can be verified easily by using (A 6) and the fact that

$$\operatorname{Im}(P) = [\mathcal{S}^+(A, B, C_2, D_2)]^\perp$$

The second part is the dual of the first case. This completes the proof of theorem 5.2. □

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