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Null controllability of planar bimodal piecewise linear systems

Xiaomeng Liu, Hai Lin* and Ben M. Chen

Department of Electrical and Computer Engineering, National University of Singapore, Singapore

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This article investigates the null controllability of planar bimodal piecewise linear systems, which consist of two second order LTI systems separated by a line crossing through the origin. It is interesting to note that even when both subsystems are controllable in the classical sense, the whole piecewise linear system may be not null controllable. On the other hand, a piecewise linear system could be null controllable even when it has uncontrollable subsystems. First, the evolution directions from any non-origin state are studied from the geometric point of view, and it turns out that the directions usually span an open half space. Then, we derive an explicit and easily verifiable necessary and sufficient condition for a planar bimodal piecewise linear system to be null controllable. Finally, the article concludes with several numerical examples and discussions on the results and future work.

Keywords: piecewise linear systems; null controllability; geometric method; hybrid systems; continuous-time systems

1. Introduction

Piecewise linear systems refer to a subclass of hybrid systems that the whole state space is partitioned into polyhedral regions and a linear dynamics is active on each of these regions. A large class of nonlinear systems (Rantzer and Johansson 2000; Johansson 2003) and lots of practical systems can be modelled as piecewise linear systems (Sontag 1981; Khalil 2002). For example, in Rantzer and Johansson (2000), it was proven that piecewise linear systems can be used to analyse smooth nonlinear dynamics with arbitrary accuracy. In Khalil (2002), the tunnel diode circuit was teated using framework of piecewise linear systems. Besides, piecewise linear systems can serve as an alternative system for the study of a particular hybrid system as indicated in Heemels, Schutter, and Bemporad (2001), where equivalences among five classes of hybrid systems including piecewise linear systems were established. Due to their theoretical and practical importance, piecewise linear systems have drawn considerable attention these years (see Hu and Lin 2000; Imura and Schaft 2000; Hu and Lin 2001; Feng 2002; Ferrari-Trecate, Cuzzola, Migone, and Morari 2002; Imura 2003, 2004).

Bemporad, Ferrari-Trecate, and Morari (2000) pointed out that observability and controllability properties of piecewise linear systems cannot be easily deduced from those of the component linear subsystems. Even if every subsystem is controllable, the whole piecewise linear system cannot always be controllable. For example, consider the following bimodal piecewise linear system:

$$\dot{x}_1 = \begin{cases} x_2 & \text{if } x_2 \geq 0, \\ -x_2 & \text{if } x_2 \leq 0, \end{cases}$$
$$\dot{x}_2 = u.$$

Each subsystem is controllable in the classical sense. However, the overall system is uncontrollable as the derivative of x_1 is always non-negative. Conversely, even if some subsystem is uncontrollable, the whole piecewise linear system can still be controllable. For example, consider the following bimodal piecewise linear system:

$$\dot{x}_1 = \begin{cases} u_1 & \text{if } x_2 \geq 0, \\ 0 & \text{if } x_2 \leq 0, \end{cases}$$
$$\dot{x}_2 = u_2.$$

The subsystem in $x_2 \geq 0$ is controllable and the subsystem in $x_2 \leq 0$ is uncontrollable because the derivative of x_1 is always 0. After simple observation, we can see that the whole system is controllable. Actually, due to the hybrid nature of piecewise linear systems, the controllability issues are far from being trivial as was pointed out in Blondel and Tsitsklis (1999), where it was shown that even for simple classes

*Corresponding author. Email: elelh@nus.edu.sg

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of piecewise linear systems, the controllability problem turns out to be undecidable. Although it is difficult to obtain explicit conditions for controllability of general piecewise linear systems, it is still possible to get some explicit necessary and/or sufficient conditions for some special subclasses of piecewise linear systems. In Veliov and Krasranov (1986), the authors investigated the controllability property of bimodal systems and a small-time local controllability condition was given. Geometric control method was adopted in this article, which may face difficulties in deriving good global controllability results of general bimodal or multi-modal PWL systems. In Camlibel, Heemels, and Schumacher (2003), Camlibel, Heemels, and Schumacher (2004) and Heemels and Brogliato (2003), bimodal systems with continuous dynamics on the switching surface were considered. For example, in Camlibel et al. (2003), the authors proposed a necessary and sufficient condition for the controllability of planar bimodal linear complementarity systems, which can be treated as a special class of piecewise linear systems. The controllability problem of conewise linear systems with dynamics continuous on the switching surface was studied in Camlibel, Heemels, and Schumacher (2008). The continuity assumption in the above work guarantees the well-posedness of PWL systems and plays a key role in the derivation of results in these work. Equivalence between the controllability of a special class of bimodal systems and that of open-loop switching systems using non-negative control was established in Bokor, Szabo, and Balas (2006), for which the general controllability problem for latter system turns out to be challenging. References Xie, Wang, Xun, and Zhao (2003) and Xu and Xie (2005) discussed the null controllability of discrete-time bimodal piecewise linear systems, in which some results that need to be checked case by case, were proposed.

In this article, attention is paid to the continuous-time bimodal piecewise linear systems. In particular, the null controllability problem is investigated and discontinuous systems are treated here. A self-contained geometric analysis method is introduced in this article. Specifically, first, the evolution directions from any non-origin state are studied from the geometric point of view, and it turns out that the directions usually span an open half space. After that, the whole state space is segmented into several spacial regions using the switching surface together with several new proposed dividing lines. Furthermore, using the classification discussion method according to the geometric position relation of system matrices, switching surface and the new proposed dividing lines, an explicit and easily verifiable geometric necessary and sufficient condition for the null controllability of planar bimodal piecewise linear systems is proposed.

The rest of this article is organised as follows: in Section 2, we introduce the class of systems to be studied, followed by null controllability study in Section 3, where one geometric necessary and sufficient condition, together with some necessary or sufficient conditions, for the null controllability is given. In Section 4, some examples are presented to illustrate the theoretical results. Finally, some concluding remarks are drawn in this article and some proofs are put into the Appendix.

2. Problem formulation

Consider the planar bimodal piecewise linear system with the following mathematical model:

$$\begin{cases} \dot{x}(t) & = A_1x(t) + bu(t) \qquad c^Tx \geq 0, \\ \dot{x}(t) & = A_2x(t) + bu(t) \qquad c^Tx \leq 0, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the control, A_1, A_2 and b ($b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$) are constant matrices with appropriate dimensions. c is a vector in $\mathbb{R}^2$. The whole state space is divided into two parts: $S_1 = \{x \in \mathbb{R}^2;\ c^Tx \geq 0\}$ and $S_2 = \{x \in \mathbb{R}^2;\ c^Tx \leq 0\}$, with one system active in each spacial part. Besides, on the switching surface $c^Tx = 0$, each of the two subsystems is possible to be active.

In the sequel, we will adopt the following definition of a trajectory of system (1).

Definition 2.1: An absolutely continuous function $x(\cdot) : [0, T] \to R^2$ is called (admissible) trajectory of system (1) if there exist a finite number of points $0 = t_0 < t_1 < \cdots < t_p = T$ and integers $i_1, i_2, \ldots, i_p \in \{1, 2\}$ such that for every $k \in \{1, \ldots, p\}$,

(i) $x(t) \in S_{i_k}$ for all $t \in [t_{k-1}, t_k]$;
(ii) there exists a piecewise continuous function $u(\cdot)$ such that $\dot{x}(t) = A_{i_k}x(t) + bu(t)$ for almost everywhere $t \in [t_{k-1}, t_k]$.

What follows is the definition of null controllability of system (1).

Definition 2.2 (Null controllability)**:** *A* non-zero state x of system (1) is called controllable, if there exists a trajectory $x(\cdot)$ of (1) such that $x(0) = x$ and $x(t_f) = 0$ for some $t_f > 0$. System (1) is said to be null controllable if any non-zero state x is controllable.

Remark 1: Since the switching sequences at switching surface together with the control input are the control signals that we can design, for any specific control input u and initial state, there exists a unique solution of system (1). We only concern when there exist control signals (including switching control at the switching surfaces) such that all initial states can be driven to the origin. Hence, our controllability checking can be applied to PWL systems that are not well-posed.

Our aim here is to find out under which condition, it is possible to drive any non-zero state in (1) to the origin with suitable choice of control input, namely that the continuous-time planar bimodal piecewise linear system (1) is null controllable.

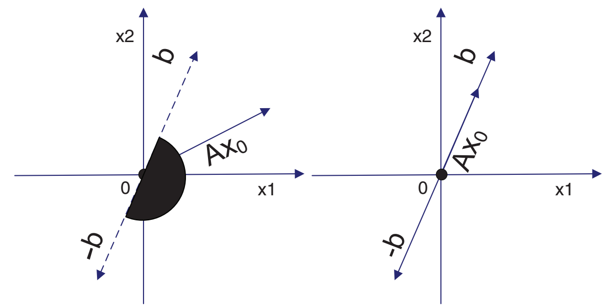

Figure 1. Graphical illustration of Lemma 3.1.

3. Null controllability

3.1 *Evolution directions*

A question arises naturally when people study the trajectory of some system dynamics: which directions can the state evolve at specific point x_0, i.e. what are the directions of tangent vectors or derivatives of state x_0?

Before answering this question, we need to introduce some notations first: in system (1), the line consisting of vectors b and $-b$ and crossing zero is defined as $d^T x = 0$, where $d = \left[\begin{smallmatrix}-b_2\\ b_1\end{smallmatrix}\right]$.

The line consisting of vectors b and $-b$ and crossing some point p is $d^T x = d^T p$. For notational convenience, this line will be represented by $d^T x(p)$ in the rest of this article. Let us use $\mathcal{X}_0$ to represent the set of evolution directions or derivative vectors at point x_0. It turns out that all the possible evolution directions of x_0 at a non-origin state x_0 usually span an open half space.

To answer the above question, what we need is to consider the coordinate centred at the point x_0 in a linear system. Then we can easily get the following lemma:

Lemma 3.1:

(i) $\mathcal{X}_0 = \{x | d^T x = 0\}$ *if* Ax_0 *is linearly dependent with* b;

(ii) $\mathcal{X}_0 = \{x | d^T x > 0\}$ *or* $\{x | d^T x < 0\}$ *if* Ax_0 *is linearly independent of* b.

Proof: For condition (i), since Ax_0 is linearly dependent with b, $Ax_0 + bu$, $u \in R$ can be any vector that belongs to the line consisting of b and $-b$ and crossing 0, i.e. $d^T x(0)$. For condition (ii), because Ax_0 is linearly independent of b, every vector f can be expressed as $f = \lambda_1 Ax_0 + \lambda_2 b$. If $d^T f > 0$, definitely, λ_1 is always positive (negative). Meanwhile, if another vector $f' = \lambda_1' Ax_0 + \lambda_2' b$ satisfies $d^T f' < 0$, definitely, λ_1' is always negative (positive). Now consider arbitrary vectors f and f' satisfying $d^T f >$ and $d^T f' < 0$, respectively. Suppose that $\lambda_1 > 0$ and $\lambda_1' < 0$. Then we have $f/\lambda_1 = Ax_0 + \lambda_2/\lambda_1 b$ and $-f'/|\lambda_1'| = Ax_0 + \lambda_2'/\lambda_1' b$. Consequently, $Ax_0 + bu$, $u \in R$ can and only be vector that satisfies $d^T(Ax_0 + bu) > 0$. Suppose that $\lambda_1 < 0$ and $\lambda_1' > 0$. $Ax_0 + bu, u \in R$ can and only be vector that satisfies $d^T(Ax_0 + bu) < 0$. Since Ax_0 is linearly independent of b, $Ax_0 \neq 0$. Therefore $Ax_0 + bu$, $u \in R$ cannot be any vector along the direction of b or $-b$. A graphical illustration is shown in Figure 1. □

Define another vector $E_i = [E_{i1}, E_{i2}] = d^T A_i = \left[\begin{smallmatrix}-b_2\\ b_1\end{smallmatrix}\right]^T A_i$, $i = \{1, 2\}$. The vector e_i which belongs to $E_i x = 0$ is $e_i = \left[\begin{smallmatrix}-E_{i2}\\ E_{i1}\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}E_{i2}\\ -E_{i1}\end{smallmatrix}\right]$. e_1 and e_2 need to be chosen in the way that $c^T e_1 \geq 0$, $c^T e_2 \leq 0$, because subsystem 1 is only active when $c^T x \geq 0$ and subsystem 2 is only active when $c^T x \leq 0$. Now for each of the two subsystems, in the whole state space, we have the following lemma.

Lemma 3.2:

(i) $\mathcal{X}_0 = \{x | d^T x = d^T x_0\}$ *if* $E_i x_0 = 0$;

(ii) $\mathcal{X}_0 = \{x | d^T x > d^T x_0\}$(*or* $\{x | d^T x < d^T x_0\}$) *if* $E_i x_0 > 0$. *Meanwhile,* $\mathcal{X}_0 = \{x | d^T x < d^T x_0\}$(*or* $\{x | d^T x > d^T x_0\}$) *if* $E_i x_0 < 0$.

Proof: This lemma is a direct corollary of Lemma 3.1. □

3.2 *Null controllability*

The following lemma presents a necessary condition for system (1) to be null controllable:

Lemma 3.3: *If both subsystems* (A_1, b) *and* $(A_2 b)$ *are uncontrollable in the classical sense, the piecewise linear system* (1) *is not null controllable.*

Proof: Suppose that both subsystems are uncontrollable. For any subsystem (A_i, b), $i = 1$ or 2, the controllability matrix is $[b, A_i b]$. Since (A_i, b) is not controllable, the controllability matrix now has rank 1 and is of the form $[b]$. For a linear system, the range space of controllability matrix, i.e. λb here, is actually the reachability and controllability spaces, i.e. the largest set of states that can be driven to zero. This implies that any state that does not belong to

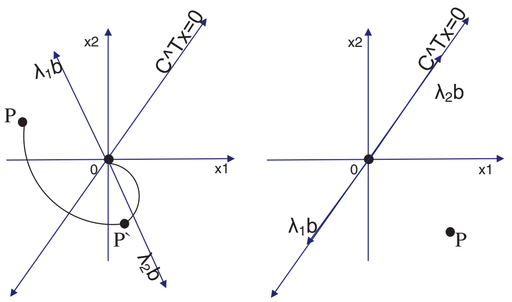

Figure 2. Graphical illustration of Lemma 3.3.

$\lambda_i b$ is not controllable and cannot be driven to zero under this linear dynamics. For the whole piecewise linear system (1), suppose that $c^T b \neq 0$, i.e. neither of the controllability spaces of two subsystems coincides with line $c^T x = 0$ as depicted in Figure 2. Consider an arbitrary state point p in $c^T x \geq 0$ but not in $\lambda_1 b$. Since (A_1, b) is not controllable, p cannot reach zero in $c^T x \geq 0$. If there exists a trajectory starting from p, crossing the line $c^T x = 0$ and reaching zero as depicted in the figure, there must exist another state p' of this trajectory included in $c^T x \leq 0$ but not in $\lambda_2 b$. Besides, the trajectory starting from p' and reaching zero stays entirely in $c^T x \leq 0$. This conflicts with the assumption that system (A_2, b) is not controllable and its controllable space is limited in $\lambda_2 b$. Therefore p cannot be driven to zero and the piecewise linear system (1) is not null controllable under this case. If $c^T b = 0$ as depicted in Figure 2(right), consider an arbitrary state p in $c^T x < 0$. The reachable set of subsystem (A_2, b) is now the line $c^T x = 0$ and there is no control input that can drive state p to zero or any point on $c^T x = 0$. Consequently, the piecewise linear system (1) is not null controllable. □

Remark 2: The necessary condition in this lemma can be applied to a more general model as follows:

$$\begin{cases} \dot{x}(t) = A_1 x(t) + B_1 u(t), & c^T x \geq 0, \\ \dot{x}(t) = A_2 x(t) + B_2 u(t), & c^T x \leq 0. \end{cases} \tag{2}$$

Compared with system (1), this planar bimodal piecewise linear system has different B matrices in the two subsystems and the control input u may not be scalar now. Besides, it is also a necessary condition for system (2) to be completely controllable.

Lemma 3.4: *The linear system $\dot{x} = A_i x + bu$, $i = 1$ or 2, is controllable if and only if $E_i b \neq 0$.*

Proof: If $b = 0$, the result follows directly. We assume that $b = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] \neq 0$. If the system (A_i, b) is not controllable, the controllability matrix $[b, A_i b]$ has rank 1, which means that column vector $A_i b$ is linearly dependent with b, i.e. $A_i b = \lambda b$. Furthermore, $E_i = \left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T A_i$ as defined. Therefore, $E_i b = \left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T A_i b = \lambda \left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] = 0$. On the other hand, if $E_i b = 0$, we have $\left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T [b, A_i b] = [\left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T b, \left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T A_i b] = [0, E_i b] = 0$. Since $b \neq 0$, $b_1 \neq 0$ or $b_2 \neq 0$. Therefore, the controllability matrix has rank less than 2 and the linear system (A_i, b) is uncontrollable. This completes the proof. □

Before proceeding further, we need to introduce the following definition for system (1).

Definition 3.5: Define the convex cone formed by e_1 and e_2 as $\mathcal{V}$: specifically, $\mathcal{V}$ is defined as an open convex cone if $e_1 \neq \lambda e_2$, $\lambda > 0$ and we say that a vector $v \in \mathcal{V}$ if there exist positive scalars λ_1 and λ_2 such that $v = \lambda_1 e_1 + \lambda_2 e_2$; When $e_1 = \lambda e_2$, $\lambda > 0$, we say that a vector $v \in \mathcal{V}$ if there exists positive scalar λ_i such that $v = \lambda_i * e_i$. Moreover, the condition that state x is outside $\mathcal{V}$ means that vector $x \notin \mathcal{V}$ and vector $x \neq \lambda_1 e_1$ and $x \neq \lambda_2 e_2$, $\lambda_1 > 0$, $\lambda_2 > 0$.

With the previous lemmas and definitions, we are in the position to present the main result of this article.

Theorem 3.6: *The bimodal piecewise linear system* (1) *is null controllable if and only if*:

(i) *there exist $i \in \{1, 2\}$ a scalar u, and a vector x outside $\mathcal{V}$ such that $A_i x + bu \in \mathcal{V}$;*
(ii) *the corresponding subsystem (A_i, b) is controllable.*

Remark 3: If $c^T e_i \neq 0$, definitely we have an unique e_i. Otherwise, when $c^T e_i = 0$, both e_i and $-e_i$ satisfy the requirement that $c^T e_i \geqq 0 (c^T e_i \leqq 0)$. Consequently, there are several convex cones formed by e_1 and e_2 (including $-e_i$). To satisfy conditions in Theorem 3.6, we should make sure for every cone, the two conditions should be satisfied.

If the matrix c has the form $c = \left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$, the following sufficient condition for system (1) to be null controllable can be given:

Corollary 3.7: *If the system matrices satisfy the following conditions*:

(i) $\left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T A_1 \left[\begin{smallmatrix} -c_2 \\ c_1 \end{smallmatrix}\right] \neq 0$ *and* $\left[\begin{smallmatrix} -b_2 \\ b_1 \end{smallmatrix}\right]^T A_2 \left[\begin{smallmatrix} -c_2 \\ c_1 \end{smallmatrix}\right] \neq 0$ *and*
(ii) $b = \lambda_3 e_1 + \lambda_4 e_2$, *for some* λ_3, λ_4 *that* $\lambda_3 \lambda_4 > 0$,

then the bimodal piecewise linear system (1) *is null controllable.*

Proof: This is actually a special case of the main theorem. For detailed proof, please refer to the Case A3 in the Appendix. □

Besides, we can get the following sufficient condition for system (1) to be not null controllable:

Corollary 3.8: *If the system matrices satisfy the following conditions*:

(i) $\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_1 \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} = 0$ *or* $\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_2 \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} = 0$ *and*

(ii) $\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_1 b = 0$ *or* $\begin{bmatrix} -b_2 \\ b_1 \end{bmatrix}^T A_2 b = 0$,

then the bimodal piecewise linear system (1) *is not null controllable*.

Proof: This is actually a combination of several cases of the main theorem. For detailed proof, please refer to the cases B1, B2 and C2 in the Appendix. □

4. Numerical examples

Example 4.1: Consider the system dynamics described in the following equations:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) & -x_1 + x_2 \geq 0, \\ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) & -x_1 + x_2 \leq 0. \end{cases} \tag{3}$$

The system matrices are as follows:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

After a simple calculation, it can be seen that: $d = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and furthermore, we can get the two dividing lines:

$$E_1 = d_1^T A_1 = [1 \quad -1] \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = [0 \quad 1],$$

$$E_1 x = 0 \Leftrightarrow x_2 = 0;$$

$$E_2 = d_2^T A_2 = [-1 \quad 1] \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} = [-1 \quad -2],$$

$$E_2 x = 0 \Leftrightarrow x_1 + 2x_2 = 0.$$

The refinement of the whole state space according to the dividing lines is shown in Figure 3. We can easily see that for the cone $\mathcal{V}$, there exists some vector $A_1 x + bu$, i.e. derivative vector of state $x, \in \mathcal{V}$, when x is in area ① outside $\mathcal{V}$ and also the subsystem (A_1, b) is controllable. According to Theorem 3.6, the system (3) is null controllable. Also, we can see that the conditions in Corollary 3.7 are also satisfied, with $e_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\lambda_3 = -3$ and $\lambda_4 = -1$. Next, some simulation results are presented to illustrate the null controllability. Here, we choose two typical states, design suitable control input and drive them to zero.

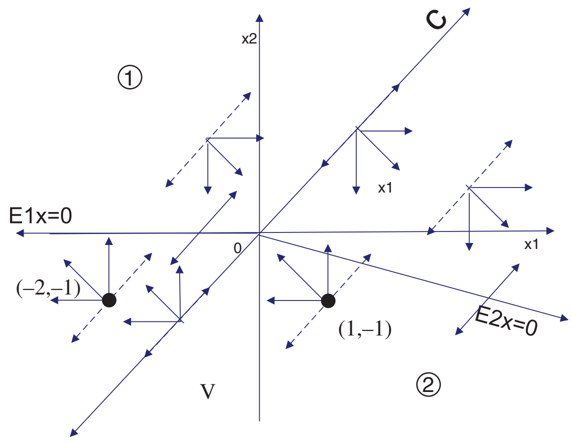

Figure 3. Refinement of state space of system (3).

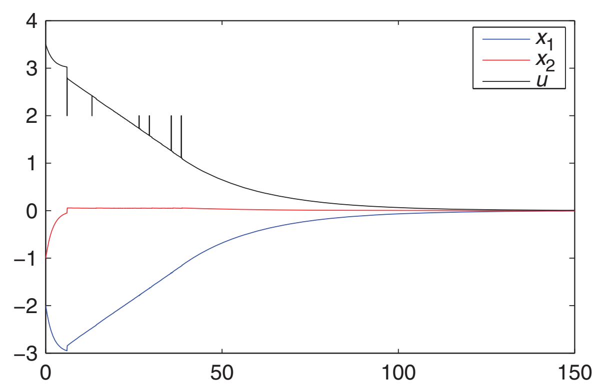

Figure 4. Trajectory and control input of driving $(-2, -1)$ to 0 in system (3).

Starting from state $(-2, -1)$, we can drive the system state towards $(-3, -1)$ and when the trajectory crosses $E_1 x = 0$, we will drive it along a line trajectory to zero. The simulation result is shown in Figure 4.

Starting from state $(1, -1)$, designing suitable u can ensure the system trajectory be driven along a line trajectory to zero. The simulation result is shown in Figure 5.

Example 4.2: Consider another system dynamics described in the following equations:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), & -x_1 + x_2 \geq 0, \\ \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), & -x_1 + x_2 \leq 0. \end{cases} \tag{4}$$

The system matrices are as follows:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

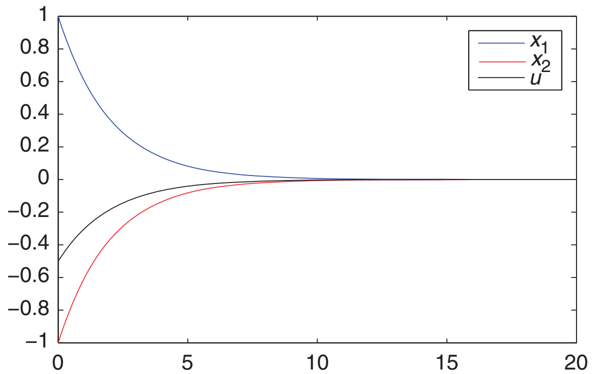

Figure 5. Trajectory and control input of driving $(1, -1)$ to 0 in system (3).

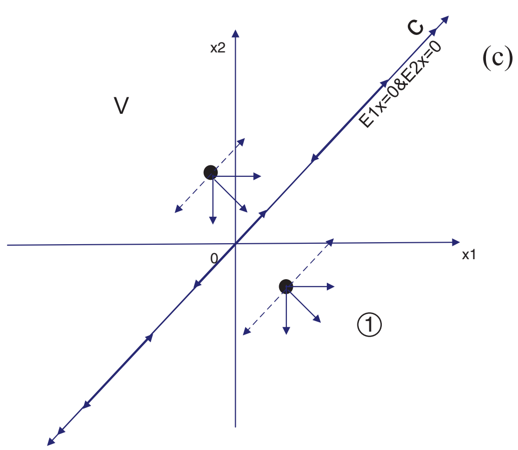

Figure 6. Refinement of state space of system (4).

Simple calculation yields that: $d = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and furthermore, we can get the two dividing lines:

$$E_1 = d_1^T A_1 = [1 \quad -1]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = [-1 \quad 1],$$
$$E_1 x = 0 \Leftrightarrow -x_1 + x_2 = 0;$$

$$E_2 = d_2^T A_2 = [-1 \quad 1]\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = [-1 \quad 1],$$
$$E_2 x = 0 \Leftrightarrow -x_1 + x_2 = 0.$$

The refinement of the whole state space according to the dividing lines is depicted in Figure 6. It can be easily seen that for the cone $\mathcal{V}$, there is no vector $A_i x + bu$, i.e. derivative vector of state $x, \in \mathcal{V}$, where x is outside $\mathcal{V}$ (here is area ①). According to Theorem 3.6 or Corollary 3.8, the system (4) is not null controllable. Actually, we can see that with such set of possible evolution directions as depicted in the figure for all states in area ①, arbitrary state in this area cannot be driven to zero.

5. Conclusions

In this article, we have investigated the null controllability of planar bimodal piecewise linear systems. An explicit and easily verifiable necessary and sufficient condition has been proposed in terms of the system parameters, followed by several necessary or sufficient conditions. The method of analysing the evolution directions of system states and the subsequent state space division brings us a deep insight of the relation between system trajectory and its controllability. We believe that using this kind of geometric analysis method, certain controllability of more general piecewise linear systems can also be considered, such as, high-order, multi-modal and complete controllability. Specifically, in general, the presented geometric analysis can be extended to multi-modal PWL systems. In multi-modal case, the analysis of evolution directions of systems states would be same as that in this article. The difference lies in the refinement of whole state space due to active area change of each mode. The procedure of state space division and the following trajectory or controllability study would be similar. However, the increase of system modes will make the classification more complex and would require a more complicated trajectory analysis under each case although it is doable. Considering extension to high-order PWL systems, our idea of evolution directions analysis of systems states and the following state space division can also be adopted. However, great difficulty results from that the trajectory analysis method used in this article would not work in high-order case because we use the planar aspect. What we need is to find a more efficient way to study the system trajectories within one cone or crossing the cones after space division. This is beyond our scope here and is the future research topic we will work on.

Acknowledgements

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Appendix A. Proof of Theorem 1

Proof: The basic idea here is to enumerate all the possible cases that there exists a vector $A_i x + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$ and (A_i, b) is controllable, and then prove that every non-zero state can be driven to zero. Conversely, all the possible cases that at least one of the two conditions stated in the theorem cannot be satisfied will be presented and proven that there exists some non-zero states that cannot be driven to zero. For simplicity, in all the following figures, let's use c, e_1 and e_2 to represent the lines $c^T x = 0$, $E_1 x = 0$ and $E_2 x = 0$, respectively. The derivative vectors or evolution directions of every state are depicted using the solid line with arrow. Besides, the dashed line with arrow represents the extreme derivative direction which cannot be achieved, which actually is the direction of vectors b and $-b$.

Case A: $c^T e_1 \neq 0$, $c^T e_2 \neq 0$.

As stated in Lemma 3.2, the evolving direction of one state p is actually along the line consisting of b and $-b$ and crossing p, i.e. $d^T x(p)$, or the right (left) open half plane divided by this line. Consequently, the geometric position relation of vector b and the cone $\mathcal{V}$ will clarify whether there exists vector $A_i x + bu$ that belongs to the cone $\mathcal{V}$:

Case A1: $b = \lambda_1 e_1$ or $b = \lambda_2 e_2$.

This is actually the case that the line consisting of b and $-b$ is parallel to or coincides with the boundary of $\mathcal{V}$. All the possible situations are shown in Figure A.1.

(a) Consider the case depicted in Figure A.1(a). It can be easily seen that there is no vector $A_i x + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$. Furthermore, consider one point p in area ①. We will show that starting from p, the system cannot reach any point in the right half space of line $d^T x(p)$. Suppose that there is a point q, who is reachable from p, in the right half space of line $d^T x(p)$ and outside the cone $\mathcal{V}$. If the system trajectory starting from p and reaching q crosses $\mathcal{V}$ as depicted using the dashed line, we can always find another point q' with the system trajectory from p to q' staying entirely outside the cone V. Thus, we can assume that the trajectory reaching q stays entirely outside the cone, which is represented by the solid line in the figure. Obviously, the trajectory must cross the line $d^T x(p)$ if it can reach the point q. We use p' to represent the crossing point. Consider another point that is infinitely close to p' in the right half space of the line $d^T x(p)$. The secant connecting these two points of this trajectory curve is the tangent which represents the derivative vector of p' when the two points are infinitely close. This implies that there is derivative vector of some point outside the cone $\mathcal{V}$ whose direction belongs to the right half plane of line $d^T x(p)$. However, this is impossible since there is no such derivative vector of any point outside the cone $\mathcal{V}$ as shown in the figure. Consequently p cannot be driven to zero which implies that the system under this case is not null controllable.

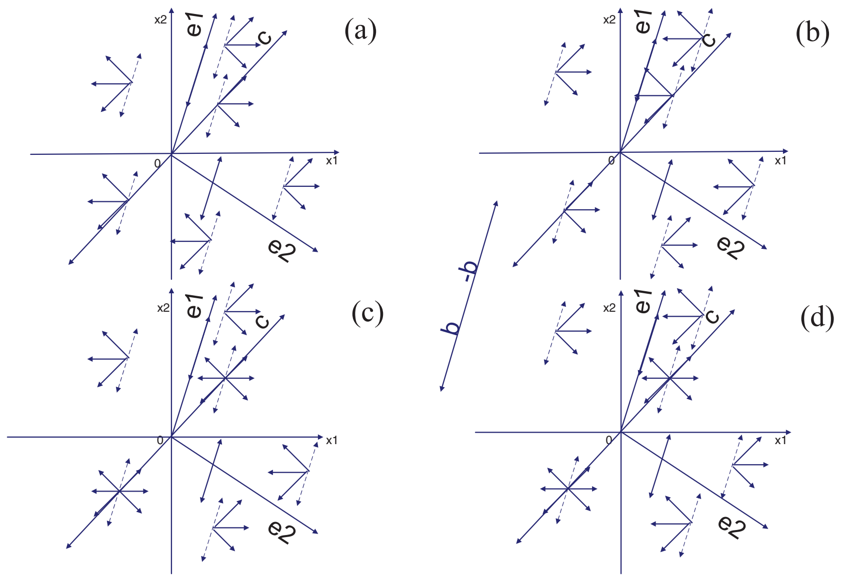

Figure A.1. Case A1 of $b = \lambda_1 e_1$ or $b = \lambda_2 e_2$.

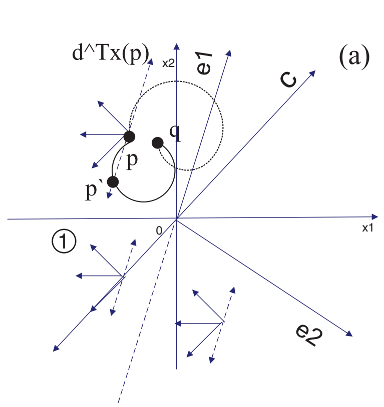

Figure A.1(a) Case (a) of $b = \lambda_1 e_1$ or $b = \lambda_2 e_2$.

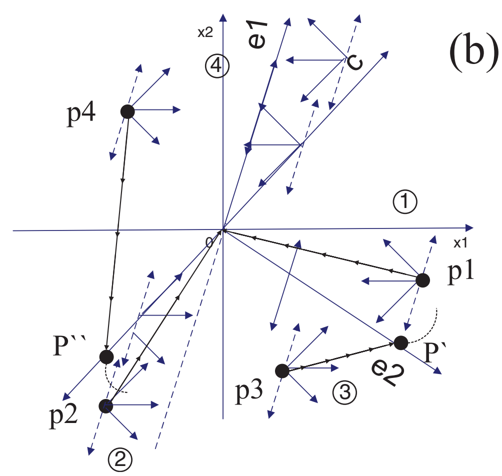

Figure A.1(b) Case (b) of $b = \lambda_1 e_1$ or $b = \lambda_2 e_2$.

(b) Consider the case depicted in Figure A.1(b). After simple observation, it can be seen that there are four types of points (states) according to the geometric position, indicated by ①②③④, respectively in Figure A.1(b) (the states belonging to the dividing lines and $c^T x = 0$ are not included. The analysis for these states is relatively easy, so we put it to the end of this case). For any point in area ② and ③, obviously, there exists some vector $A_2 x + bu \in \mathcal{V}$. Besides, subsystem (A_2, b) is controllable according to Lemma 3.4. For an arbitrary point p_1 in area ① p_1 is connected with zero using the solid line. Since the line is entirely contained in cone $\mathcal{V}$, each point on this line can have its derivative vector along the direction of $-p_1$ with suitable choice of u. Therefore, we can design the control input u and make the system dynamics as

$$\dot{x}(t) = A_2 x(t) + bu(t) = -\lambda(t) p_1, \ \lambda(t) > 0.$$

Solving this equation yields that the trajectory of this system is

$$x(t) = -p_1 \int_0^t \lambda(t) \mathrm{d}t + x_0.$$

The system (A_2, b) is controllable and vector b is not parallel with e_2, which implies that the derivative vector of system states with direction orthogonal to $E_2 x = 0$ can be chosen to be non-zero. Therefore, noting that $\lambda(t)$ is always positive scalar, suitable t can be chosen and the integral can be made equal to 1 since $\lambda(t)$ will not converge to zero now. Clearly x_0 equals to p_1 here. Hence, $x(T) = 0$ for some T, which implies that any state p_1 in area ① can be driven to zero. Using the similar analysis, it is easy to show that any point p_2 in area ② can be driven to zero and the possible trajectory is also depicted using solid line in the figure. Consider an arbitrary point p_3 in area ③. As shown above, there always exist some vector $A_i x + bu \in \mathcal{V}$ for any point in area ③. Choosing a derivative vector in the open cone $\mathcal{V}$, the corresponding line consisting of this derivative vector and p_3 surely intersects $E_2 x = 0$, which is the boundary of $\mathcal{V}$, at some point p'. As the part of line between p_3 and p' is entirely outside $\mathcal{V}$, all the points in this part have the same possible evolution directions. Consequently, we can design the control u and make every state in this part have derivative direction of vector $p' - p_3$. The system dynamics now become

$$\dot{x}(t) = A_2x(t) + bu(t) = \lambda(t)(p' - p_3),\ \lambda(t) > 0.$$

Solving this equation, we can get the trajectory of this system as

$$x(t) = (p' - p_3)\int_0^t \lambda(t)\mathrm{d}t + x_0.$$

With similar reason as above, some suitable t can be chosen and the integral can be made equal to 1. Besides x_0 equals to p_3 here. Thus, $x(T) = p'$ for some T. Furthermore, there are two reasons that the system dynamics cannot stay on $E_2x = 0$ or go back to the outside of $\mathcal{V}$. One is that on $E_2x = 0$, we can choose the u to let the derivative direction still point to the inside of cone $\mathcal{V}$ (b or $-b$ direction). The other is, actually, the dynamical equations of system do not have description about second derivative of the state. Therefore, sudden change, (here inverse change) of evolution trajectory of system state is not possible. Due to these two reasons, the system dynamics will not stay at the point p' or go back. The system trajectory will reach some point in area ① and using the former control design strategy, the system trajectory can be driven to zero. Finally, any state p_3 in area ③ has been proven that it can be driven to zero and is controllable. Area ④ is defined as the area $c^Tx > 0$ except the dividing line $E_1x = 0$. Consider an arbitrary point p_4 in area ④. According to Lemma 3.4, subsystem (A_1, b) is not controllable and its controllability space is limited in line $E_1x = 0$. Therefore p_4 cannot be driven to zero if its trajectory is only under linear dynamics $\dot{x} = A_1x + bu$ (even though it seems that we can drive p_4 directly to zero along a line trajectory, it is actually not possible because the derivative vector orthogonal to $E_1x = 0$ and the parameter $\lambda(t)$ will converge to zero due to the geometric relation of b and $E_1x = 0$, which means the integral of $\lambda(t)$ can reach 1 only when t towards infinity). Fortunately, similar to the discussion about the states in area ③, we can design control u and let p_4 be driven to some point p'' and then into area ② (① for the states in right half of area ④) and finally to zero. For the points on $E_1x = 0$ or $E_2x = 0$, using similar control trajectory as discussed above, as shown in the figure, designing suitable control u can ensure the points on $E_1x = 0$ be driven to zero along the boundary of $\mathcal{V}$ and the points on $E_2x = 0$ be driven into area ① and then driven to zero. The states on $c^Tx = 0$ can be treated as states in area ① or ② because we assume that any subsystem can be active on $c^Tx = 0$. All the states here can be driven to zero and therefore, the system is null controllable in this case.

(c) Consider the case depicted in Figure A.1(c). The analysis for this case is similar with the above case and the system is null controllable. The corresponding trajectory for every state driven to zero is shown in the figure (proof details omitted due to length limit of article).

(d) Consider the case depicted in Figure A.1(d). It can be easily seen that even though there exists some vector $A_1x + bu \in \mathcal{V}$ when x is in area ②, the subsystem (A_1, b) is not controllable. The conditions in the theorem are not satisfied. Furthermore, consider an arbitrary point p in area ① in Figure A.1(d) (note that the long dashed line is the line consisting of b and $-b$ and crossing 0. Points in area ① are the points in the left open half plane of this line and in $c^Tx < 0$). Using the same analysis as Case A1(a), it can be shown that starting from p, the system trajectory cannot reach any point in the right half space of line $d^Tx(p)$ under linear dynamics $\dot{x} = A_2x + bu$. The corresponding trajectory

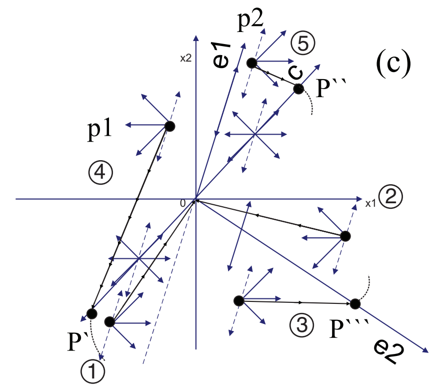

Figure A.1(c). Case (c) of $b = \lambda_1e_1$ or $b = \lambda_2e_2$.

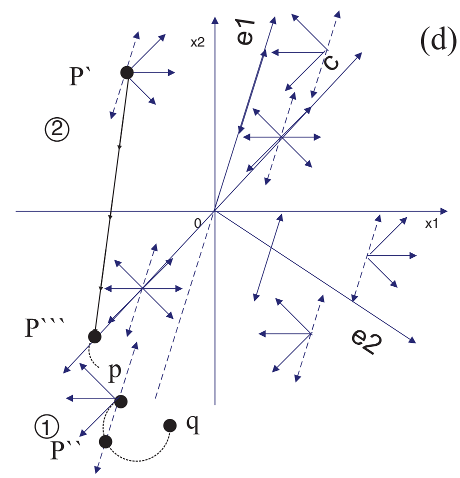

Figure A.1(d). Case (d) of $b = \lambda_1e_1$ or $b = \lambda_2e_2$.

starting from p may enter area ②. Consider an arbitrary point p' in area ②. Similar to the analysis about the states in area ④ of Case A1(b), it can be shown that starting from p', the system cannot reach any point on the line $E_1x = 0$ under linear dynamics $\dot{x} = A_1x + bu$. The corresponding trajectory may go into area ①. If the trajectory reaches the dividing line of area ① and ②, i.e. the left open segment of line $c^Tx = 0$, any one of the two subsystems maybe active. However, no matter which system is active, the corresponding trajectory still cannot reach zero according to the trajectory analysis for states in areas ① and ②. Consequently, any state p in area ① and any state p' in area ② cannot be driven to zero point which implies that the system is not null controllable.

Remark 4: All the proofs and graphical illustrations are based on $b = \lambda_1e_1$. For the case that $b = \lambda_2e_2$, the analysis method and result are similar. As a result, in this article, we will only give detailed analysis on $b = \lambda_1e_1$ to stand for the analysis of the case that $b = \lambda_1e_1$ or $b = \lambda_2e_2$ if without leading to confusion.

Case A2: b and $-b$ are outside $\mathcal{V}$.

All the possible situations are shown in Figure A.2.

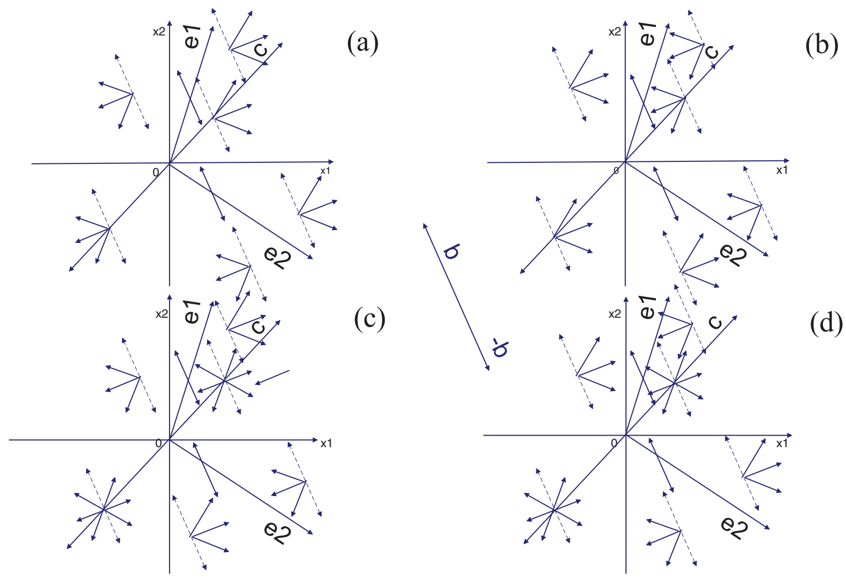

Figure A.2. Case A2. b and $-b$ are outside $\mathcal{V}$.

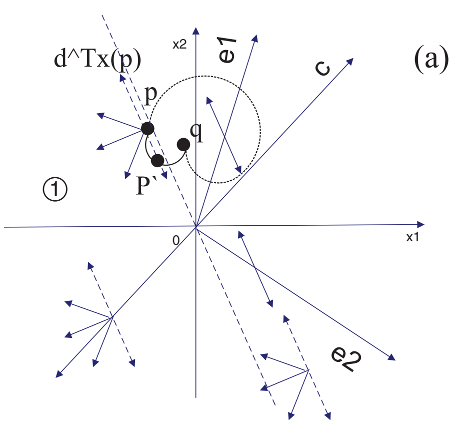

Figure A.2(a). Case (a) of b and $-b$ are outside $\mathcal{V}$.

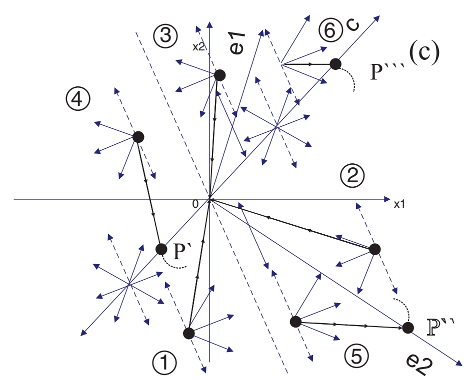

Figure A.2(c). Case (c) of b and $-b$ are outside $\mathcal{V}$.

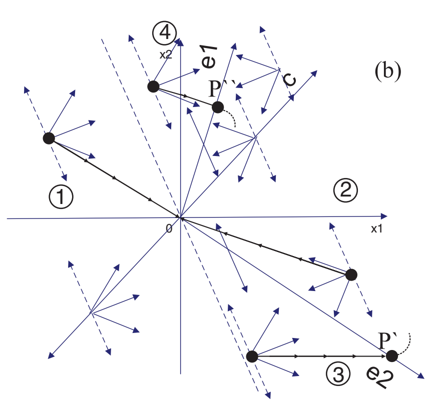

Figure A.2(b). Case (b) of b and $-b$ are outside $\mathcal{V}$.

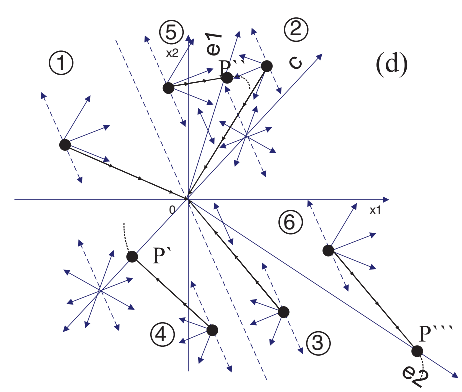

Figure A.2(d). Case (d) of b and $-b$ are outside $\mathcal{V}$.

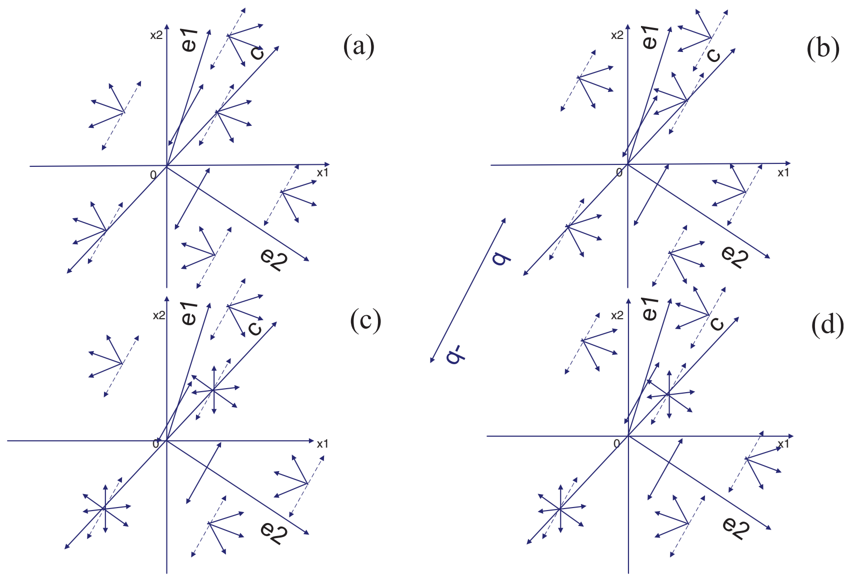

Figure A.3. Case A3. b or $-b$ is in $\mathcal{V}$.

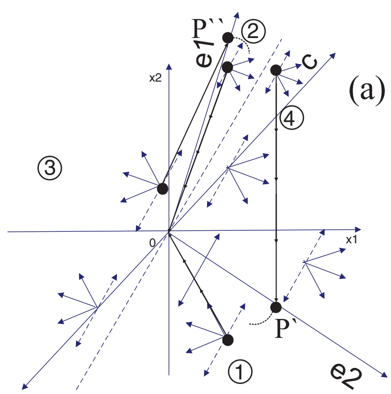

Figure A.3(a). Case (a) of b or $-b$ is in $\mathcal{V}$.

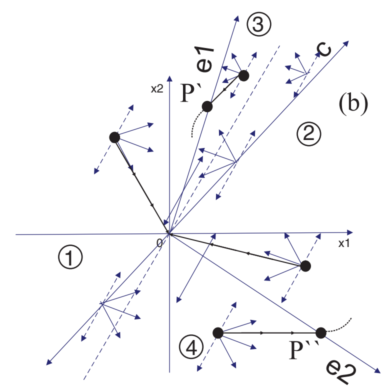

Figure A.3(b). Case (b) of b or $-b$ is in $\mathcal{V}$.

(a) Consider the case depicted in Figure A.2(a). It can be easily seen that there is no vector $A_i x + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$. Furthermore, consider one point p in area ① in Figure A.2(a) (note that the long dashed line is the line consisting of b and $-b$ and crossing 0. Points in area ① are the points in the left open half plane of this line). Using the same analysis as Case A1(a), it can be shown that starting from p, the system trajectory cannot reach any point in the right half space of line $d^T x(p)$. Consequently p cannot be driven to zero which implies that the system under this case is not null controllable.

(b) Consider the case depicted in Figure A.2(b). After simple observation, it is clear that there are four types of points according to the geometric position, indicated by ①②③④, respectively, in Figure A.2(b). For any point in area ①, ③ and ④, obviously, there exists some vector $A_i x + bu \in \mathcal{V}$ and (A_i, b) is controllable. Using the same analysis as Case A.1(b), it can be found that the points in ①② can be driven directly to zero along a line trajectory. Besides, the points in area ③(④) can be driven by a line trajectory to some point $p'(p'')$ and then into area ②(②) and finally driven to zero. The points on $E_1 x = 0$ $E_2\, x = 0$ or $c^T x = 0$ can easily be shown

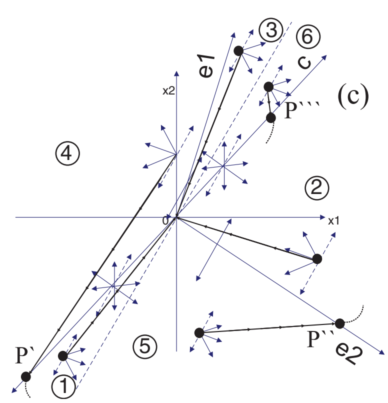

Figure A.3(c). Case (c) of b or $-b$ is in $\mathcal{V}$.

to be controllable too. All the states here can be driven to zero and therefore, the system is null controllable in this case.

(c) and (d) Consider the cases depicted in Figures A.2(c) and A.2(d). The analysis for these cases are similar as the

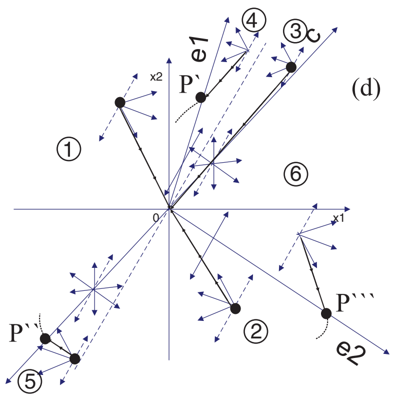

Figure A.3(d). Case (d) of b or $-b$ is in $\mathcal{V}$.

above case and the system is null controllable. The corresponding trajectory for every state driven to zero is shown in the figures (proof details omitted due to length limit of article).

Case A.3: b or $-b$ is in $\mathcal{V}$:

All the possible situations are shown in Figure A.3.

(a), **(b)**, **(c) and (d)** Consider the cases depicted in Figures A.3(a), A.3(b), A.3(c) and A.3(d) Figure A.3(a–d). The analysis for these cases are similar as Case A.2(b). There exists some vector $A_ix+bu \in \mathcal{V}$, when x is outside $\mathcal{V}$ and (A_i, b) is controllable. The corresponding trajectory for every state driven to zero is shown in the figures. All the states can be driven to zero and therefore the system is null controllable in these cases.

Remark 5: A special case contained in this case is that e_1 and e_2 are linearly dependent, which represents that line $E_1x=0$ coincides with $E_2x=0$. The proof for this special case is actually the same as the general case we presented above.

Appendix B. Case B: $c^Te_1 \neq 0\, c^Te_2=0$.

Remark 6: We can also assume that $c^Te_1=0$, $c^Te_2 \neq 0$. All the following proof would be the same, so we only prove this case with $c^Te_1 \neq 0$, $c^Te_2=0$.

Remark 7: As stated in Remark 3, from $c^Te_2=0$, we have e_2 and $-e_2$. It is necessary to consider simultaneously the convex cone formed by e_1 and e_2 and the convex cone e_1 and $-e_2$ when we verify the conditions stated in Theorem 3.6. For simplicity, we refer to the cone on the right side as cone $\mathcal{V}_1$ and the left one as $\mathcal{V}_2$.

Case B1: $b=\lambda_2e_2$.

This is actually the case that $d^Tx(p)$ is parallel to or coincides with $c^Tx=0$. All the possible situations are shown in Figure B.1.

(a) Consider the case depicted in Figure B.1(a). Considering cone $\mathcal{V}_1$, for any point in area ③, obviously, there exists some vector $A_1x+bu \in \mathcal{V}_1$ and (A_1, b) is controllable. However, for cone $\mathcal{V}_2$, even though for any point in area ①, there exists some vector $A_2x+bu \in \mathcal{V}_2$, (A_2, b) is uncontrollable. Therefore, the conditions in the theorem are not satisfied. For any state p in area ①, since subsystem (A_2, b) is uncontrollable, using the similar analysis about the states in area ④ of case A.1(b), it can be shown that starting from p, the system cannot reach any point on the line $E_2x=0$ (also $c^Tx=0$ here) under linear dynamics $\dot{x}=A_2x+bu$. Consequently, state p cannot be driven to zero and the system is not null controllable.

(b), (c), and (d) Consider the cases depicted in Figures B.1(b), B.2(c) and B.2(d). These cases are similar as the above case. In Case B.1(b), considering cone $\mathcal{V}_1$, for any point in area ①, there exists some vector $A_2x+bu \in \mathcal{V}_1$, but (A_2, b) is uncontrollable. In Case B.1(c), for cone $\mathcal{V}_2$, the area outside $\mathcal{V}_2$ is now consisting of ①, ③ and their dividing line. Obviously, there is no point with a derivative vector, i.e. a vector A_ix+bu, that is in the open cone $\mathcal{V}_2$. In Case B.1(d), for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now consisting of ①, ② and their dividing line. Clearly, there is no point with a derivative vector, i.e. a vector A_ix+bu, that is in the open cone $\mathcal{V}_1$. The conditions in the theorem are not satisfied in all these cases. For the same reason as in the above case or Case A.1(a), one state p in area ① cannot reach zero. Therefore, the piecewise linear systems in these cases are not null controllable.

Case B2: $b=\lambda_1e_1$.

This is actually the case that $d^Tx(p)$ is parallel to or coincides with $E_1x=0$. All the possible situations are shown in Figure B.2.

(a) Consider the case depicted in Figure B.1(a). Considering cone $\mathcal{V}_1$, for any point in area ①, obviously, there exists some vector $A_ix+bu \in \mathcal{V}_1$. However, for cone $\mathcal{V}_2$, the area outside $\mathcal{V}_2$ is now consisting of ①, ③ and their dividing line. Obviously, there is no point with a derivative vector, i.e. a vector A_ix+bu, that is in the open cone $\mathcal{V}_2$. Furthermore, similar to Case A1(a), any point p in area ③ cannot reach any point q in the left half space of line $d^Tx(p)$ in area ③ and ①. Therefore p cannot be driven to zero which implies that the system is not null controllable.

(b) Consider the case depicted in Figure B.2(b). This case is almost the same as the above case. The only difference is that under this case, it is that for cone $\mathcal{V}_1$ rather than cone $\mathcal{V}_2$, there is no desired vector $A_ix+bu \in \mathcal{V}_1$ (proof details omitted due to length limit of article).

(c) Consider the case depicted in Figure B.2(c). Considering cone $\mathcal{V}_1$, for any point in area ①, clearly, there exists some vector $A_2x+bu \in \mathcal{V}_1$ and (A_2, b) is controllable. However, considering cone $\mathcal{V}_2$, for any point in area ②, there exists some vector $A_1x+bu \in \mathcal{V}_1$, but (A_1, b) is uncontrollable. Therefore, the conditions in the theorem are not satisfied. Furthermore, similar to Case A.1(d), although any state p in area ① can reach some states in area ② or dividing line of areas ① and ② and any state p' in area ② can reach some states in area ① or dividing line of areas ① and ②, no state in area ①, area ② and their dividing line can be driven to zero. Thus, the system under this case is not null controllable.

(d) Consider the case depicted in Figure B.2(d). The analysis for this case is similar to the above case. Easily we can see for cone $\mathcal{V}_1$, even though for any point in area ②, there exists some vector $A_1x+bu \in \mathcal{V}_1$, (A_1, b) is uncontrollable. System (1) is not null controllable in this case (proof details omitted due to length limit of article).

Case B3: b or $-b$ is in $\mathcal{V}_2$.

Remark 8: We can also assume that b or $-b$ is in $\mathcal{V}_1$. All the following proof would be the same, so we only prove this case with b or $-b \in \mathcal{V}_2$.

All the possible situations are shown in Figure B.3.

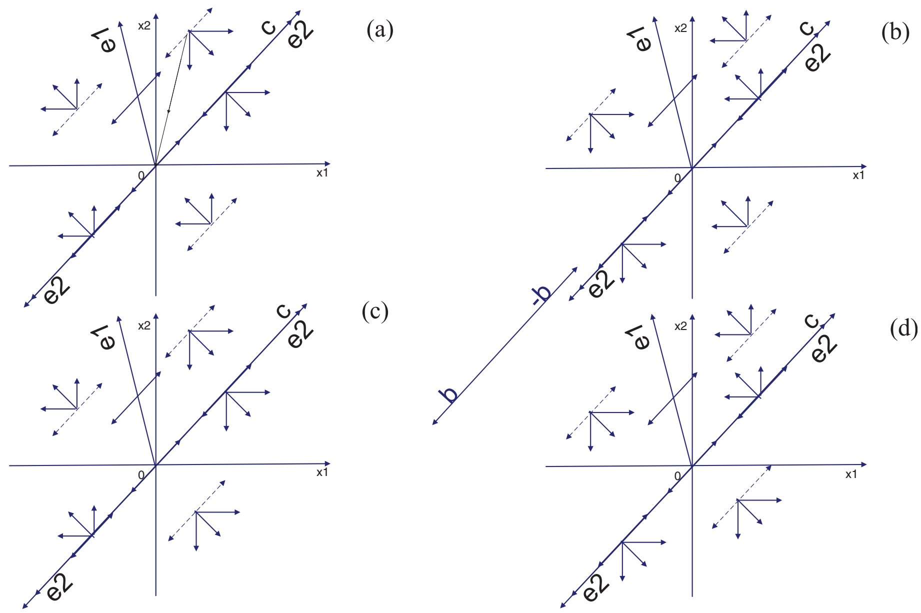

Figure B.1. Case B1. $b = \lambda_2 e_2$.

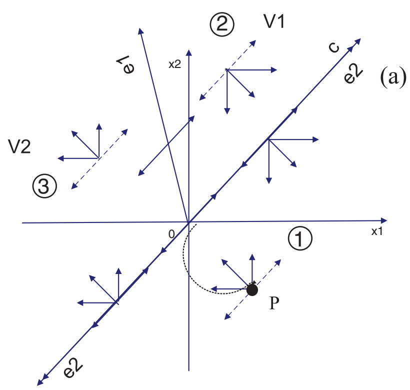

Figure B.1(a). Case (a) of $b = \lambda_2 e_2$.

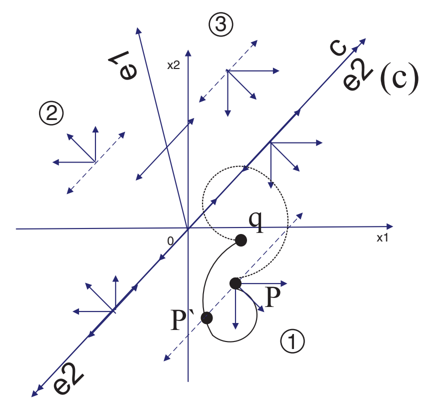

Figure B.1(c). Case (c) of $b = \lambda_2 e_2$.

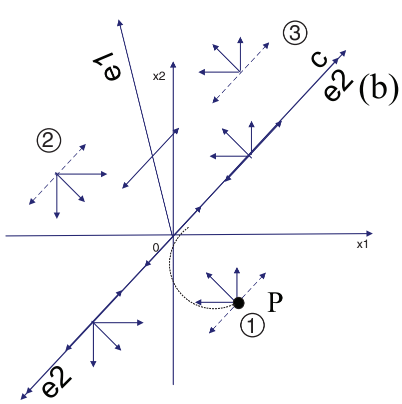

Figure B.1(b). Case (b) of $b = \lambda_2 e_2$.

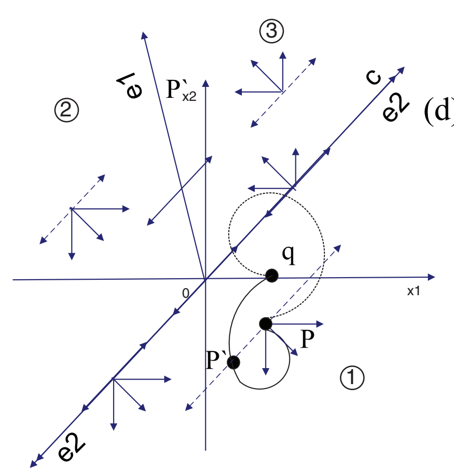

Figure B.1(d). Case (iv) of $b = \lambda_2 e_2$.

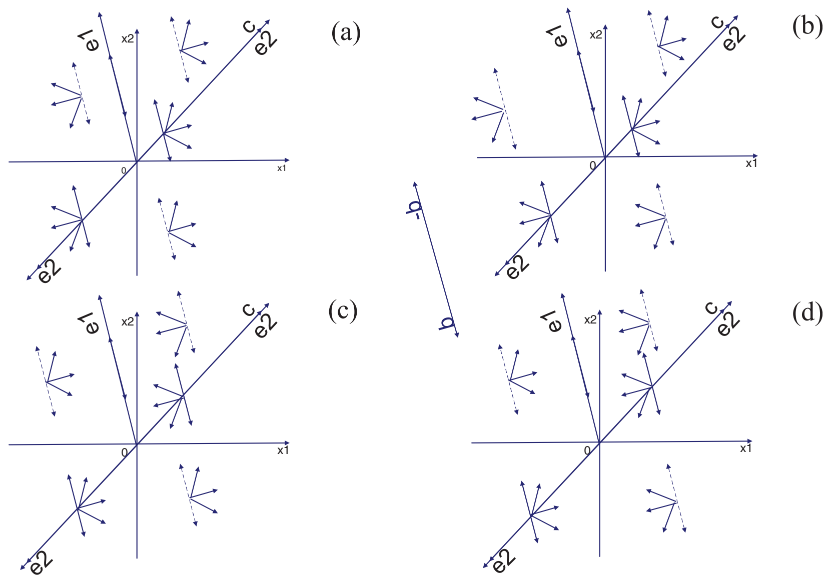

Figure B.2. Case B2. $b = \lambda_1 e_1$.

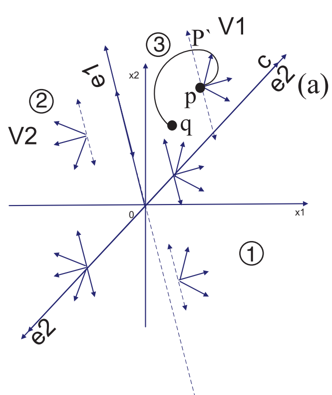

Figure B.2(a). Case (a) of $b = \lambda_1 e_1$.

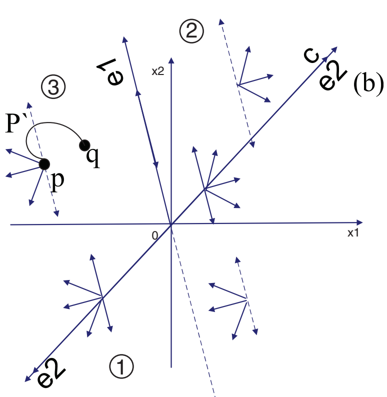

Figure B.2(b). Case (b) of $b = \lambda_1 e_1$.

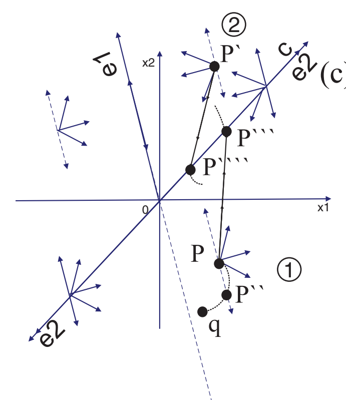

Figure B.2(c). Case (c) of $b = \lambda_1 e_1$.

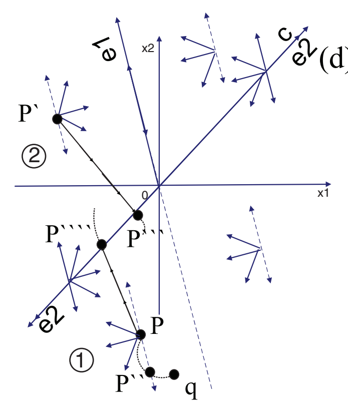

Figure B.2(d). Case (d) of $b = \lambda_1 e_1$.

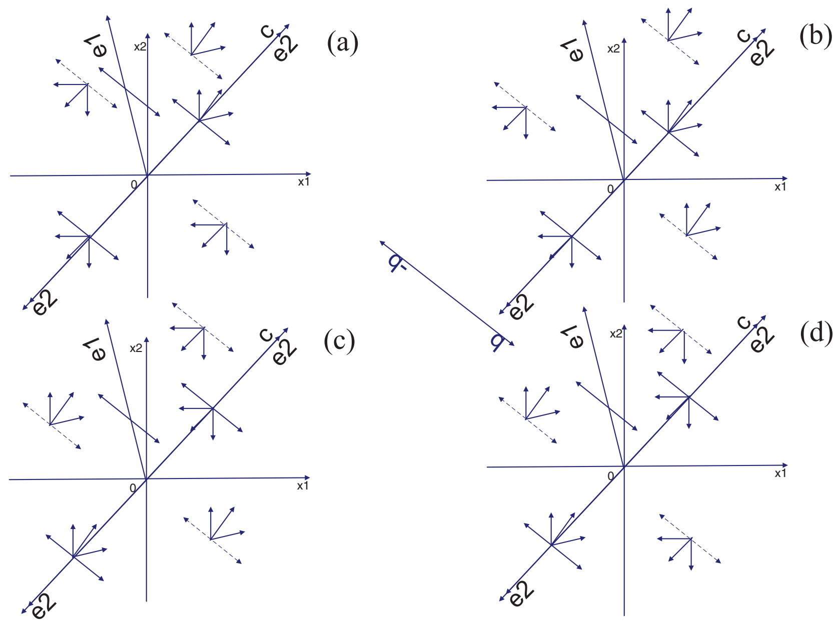

Figure B.3. Case B3. b or $-b$ is in $\mathcal{V}_2$.

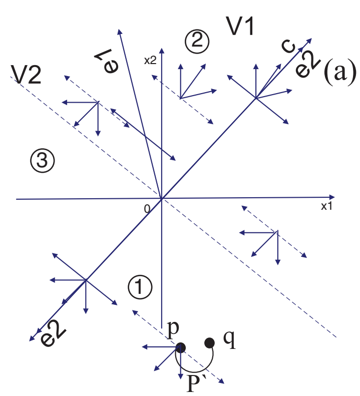

Figure B.3(a). Case (a) of b or $-b$ is in $\mathcal{V}_2$.

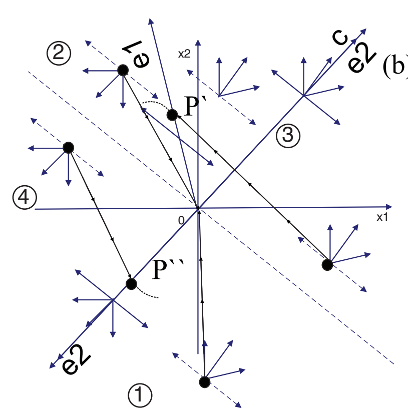

Figure B.3(b). Case (b) of b or $-b$ is in $\mathcal{V}_2$.

(a) Consider the case depicted in Figure B.3(a). Considering cone $\mathcal{V}_2$, for any point in area ①, obviously, there exists some vector $A_2x + bu \in \mathcal{V}_2$ and (A_2, b) is controllable. However, for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now consisting of ①, ③ and their dividing line. Clearly, there is no point with a derivation vector, i.e. a vector $A_ix + bu$, that is in the open cone $\mathcal{V}_1$. Furthermore, similar to Case A.1(a), some point p in area ① cannot reach any point q in the right half space of line $d^Tx(p)$ in area ① and ③. Consequently p cannot be driven to zero point which implies that the system under this case is not null controllable.

(b), (c), and (d) Consider the cases depicted in Figure B.3(b–d). Easily we can see for both cone $\mathcal{V}_1$ and cone $\mathcal{V}_2$, there exists the desired vector and the corresponding subsystem is controllable. The system (1) is null controllable under these cases. The corresponding trajectory for every state driven to zero is shown in the figures (proof details omitted due to length limit of article).

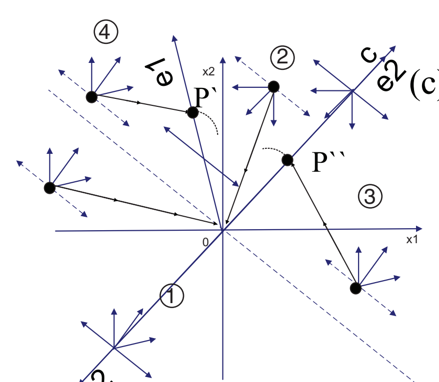

Figure B.3(c). Case (c) of b or $-b$ is in $\mathcal{V}_2$.

Appendix C. Case C: $c^Te_1 = c^Te_2 = 0$.

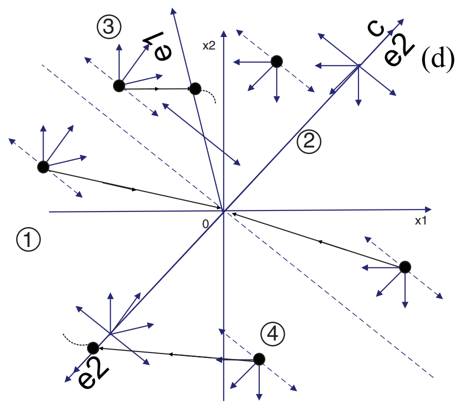

Figure B.3(d). Case (d) of b or $-b$ is in $\mathcal{V}_2$.

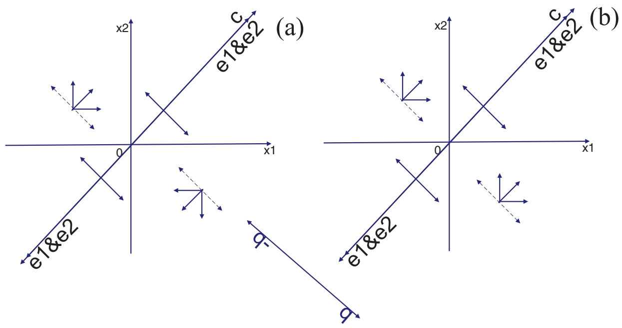

Figure C.1. Case C1: $b \neq \lambda_1 e_1$.

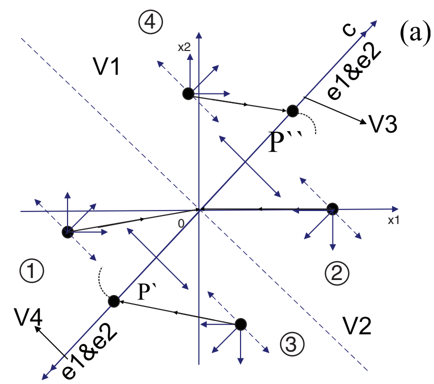

Figure C.1(a). Case (a) of $b \neq \lambda_1 e_1$.

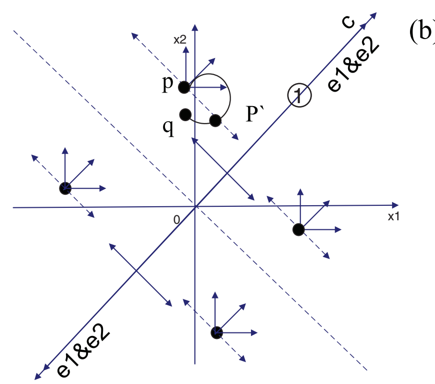

Figure C.1(b). Case (b) of $b \neq \lambda_1 e_1$.

Remark 9: As stated in Remark 3, from $c^T e_1 = c^T e_2 = 0$, it follows that e_1, $-e_1$, e_2 and $-e_2$ all satisfy the equation $c^T e_1 \geq 0$ and $c^T e_2 \leq 0$. Then we should consider simultaneously the convex cone formed by e_1 and e_2, the convex cone formed by$-e_1$ and e_2, convex cone formed by e_1 and $-e_2$ and convex cone formed by $-e_1$ and $-e_2$ when we verify the conditions stated in Theorem 3.6. According to Definition 3.5, $\mathcal{V}$ is defined as the open convex cone if $e_1 \neq \lambda e_2$, $\lambda > 0$ and we say a vector $v \in \mathcal{V}$ if $v = \lambda_1 e_1 + \lambda_2 e_2$, $\lambda_1 > 0$, $\lambda_2 > 0$. When $e_1 = \lambda e_2$, $\lambda > 0$, we say a vector $v \in \mathcal{V}$ if $v = \lambda_i * e_i$, $\lambda_i > 0$. For simplicity, in the following proof, we denote the open cone opening up as cone $\mathcal{V}_1$ and the open cone opening down as $\mathcal{V}_2$. For $e_1 = \lambda e_2$, $\lambda > 0$, we refer to the right side cone as $\mathcal{V}_3$ and the left side cone as $\mathcal{V}_4$.

Case C1: $b \neq \lambda_1 e_1$.

This is actually the case that $d^T x(p)$ is not parallel to or coincides with $c^T x = 0$. All the possible situations are shown in Figure C.1.

(a) Consider the case depicted in Figure C.1(a). There are four types of points according to the geometric position, indicated by ①②③④, respectively, in Figure C.1(a). First, considering cone $\mathcal{V}_1$ (the left open half plane of $c^T x = 0$ here), for any point in area ② and ③, clearly, there exists some vector $A_2 x + bu \in \mathcal{V}_1$ and (A_2, b) is controllable. Second, considering cone $\mathcal{V}_2$ (the right open half plane of $c^T x = 0$ here), for any point in area ① and ④, clearly, there exists some vector $A_1 x + bu \in \mathcal{V}_2$ and (A_1, b) is controllable. Third, considering cone $\mathcal{V}_3$ (the right half segment of line $c^T x = 0$), for any point in area ① and ④, clearly, there exists some vector $A_1 x + bu \in \mathcal{V}_3$ and (A_1, b) is controllable. Finally, considering cone $\mathcal{V}_4$ (the left half segment of line $c^T x = 0$), for any point in area ② and ③, clearly, there exists some vector $A_2 x + bu \in \mathcal{V}_4$ and (A_2, b) is controllable. The conditions in Theorem 3.6 are satisfied. Similarly, it can be shown that the points in ①② can be driven directly to zero along a line trajectory. Besides, the points in area ③ and ④ can be driven by a line trajectory to some point p' and p'' and then into area ① and ②, respectively, and finally driven to zero. The points on $c^T x = 0$ can be shown that they can be driven to area ① or ② or ③ or ④. Consequently, all the states here can be driven to zero and therefore the system is null controllable in this case.

(b) Consider the case depicted in Figure C.1(b). For cone $\mathcal{V}_4$, the area outside $\mathcal{V}_4$ is now consisting of all the areas except the left half of line $c^T x = 0$. Obviously, there is no state with a derivative vector, i.e. a vector $A_i x + bu$, that is in the cone $\mathcal{V}_4$. Furthermore, similar to Case A.1(a), any point p at area ① cannot reach any point q on the left half space of line $d^T x(p)$. Therefore p cannot be driven to zero which implies that the system under this case is not null controllable. All the possible situations are shown in Figure C.2.

Case C2: $b = \lambda_1 e_1$.

This is actually the case that $d^T x(p)$ is parallel to or coincides with $c^T x = 0$. All the possible situations are shown in Figure C.2.

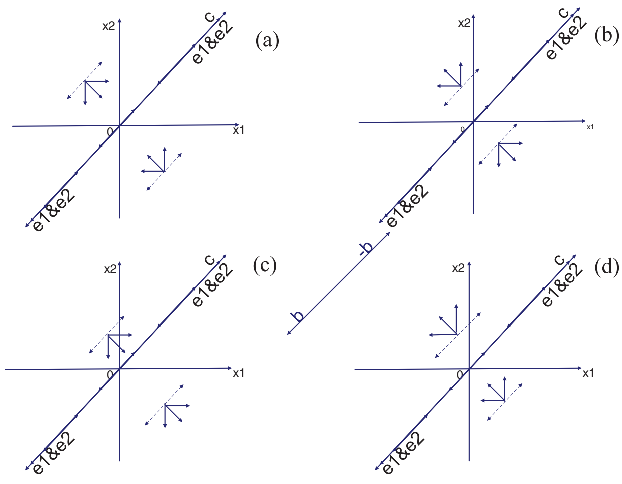

Figure C.2. Case C2. $b = \lambda_1 e_1$.

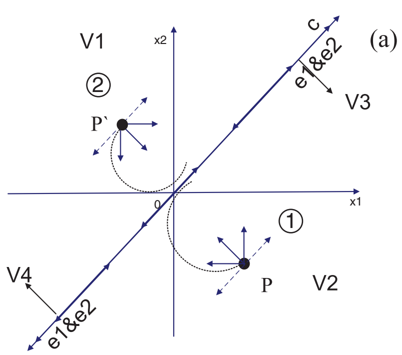

Figure C.2(a). Case (a) of $b = \lambda_1 e_1$.

(a) Consider the case depicted in Figure C.2(a). For cone $\mathcal{V}_1$, although for any point in area ①, there exists some vector $A_2 x + bu \in \mathcal{V}_1$, the subsystem (A_2, b) is uncontrollable. Therefore, the conditions in the theorem are not satisfied. Furthermore, similar to the discussion analysis about the states in area ④ of Case A.1(b), it can be shown that starting from arbitrary state p in area ①, the system cannot reach any point on the line $E_2 x = 0$ (also $c^T x = 0$ here) under linear dynamics $\dot{x} = A_2 x + bu$. Hence p cannot be driven to zero which implies that the system under this case is not null controllable.

(b), (c), and (d) Consider the cases depicted in Figures C.2(b–d). These cases are similar with the above case. In Case C.2(b), for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now area ①. Clearly, there is no state with a derivative vector, i.e. a vector $A_2 x + bu$, that is in the open cone $\mathcal{V}_1$. In Case C.2(c), for cone $\mathcal{V}_1$, the area outside $\mathcal{V}_1$ is now area ①. It is easy to see that there is no state with a derivative vector, i.e. a vector $A_2 x + bu$, that is in the open cone $\mathcal{V}_1$. In Case C.2(d), for cone $\mathcal{V}_2$, the area outside $\mathcal{V}_2$ is now area ①. It is clear that there is no state with a derivative vector, i.e. a vector $A_1 x + bu$, that is in the open cone $\mathcal{V}_2$. The conditions in the theorem are not satisfied in all these cases. For the same reason as in the above case or Case A.1(a), one state p in area ① cannot reach zero. Consequently, the piecewise linear systems in these cases are not null controllable.

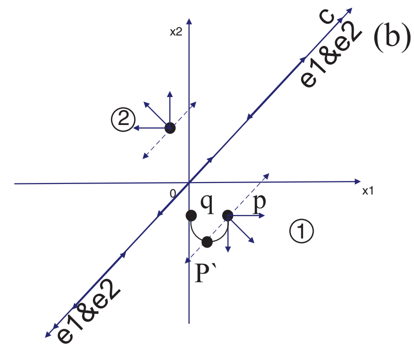

Figure C.2(b). Case (b) of $b = \lambda_1 e_1$.

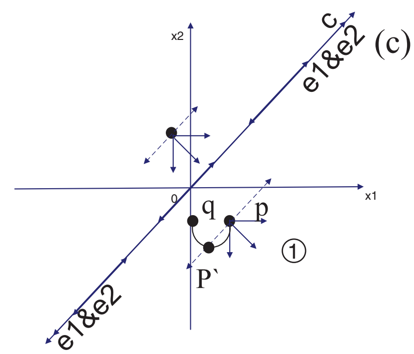

Figure C.2(c). Case (c) of $b = \lambda_1 e_1$.

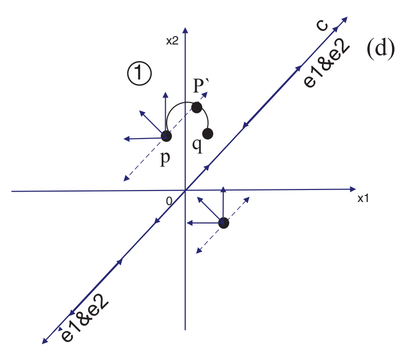

Figure C.2(d). Case (d) of $b = \lambda_1 e_1$.

In conclusion, all the cases that there exists a vector $A_i x + bu \in \mathcal{V}$ when x is outside $\mathcal{V}$ and the corresponding subsystem (A_i, b) is controllable, are proven to be null controllable. Besides, all the possible cases that at least one of the two conditions cannot be satisfied are proven that there always exists some non-zero state that cannot be driven to zero and the system (1) is not null controllable.