INT. J. CONTROL, 1998, VOL. 71, NO. 4, 653–685

Design for general H_∞ almost disturbance decoupling problem with measurement feedback and internal stability—an eigenstructure assignment approach

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Feedback controllers, explicitly parameterized in a single parameter ε, are constructed to solve the well-known H_∞ almost disturbance decoupling problem with measurement feedback and with internal stability (H_∞-ADDPMS) for continuous-time linear systems. The construction of these feedback laws utilizes eigenstructure assignment algorithms and does not involve the solution of any parameterized algebraic Riccati equations. As a result, the coefficients of the feedback laws are explicitly given as polynomial matrices in ε. The results generalize the earlier work of Özcetin *et al.* by allowing the system to have invariant zeros on the imaginary axis. Such a generalization is motivated by the recent development of low-gain feedback design technique.

1. Introduction

We revisit the problem of H_∞ almost disturbance decoupling with measurement feedback and internal stability for continuous-time linear systems. The problem of almost disturbance decoupling has a vast history behind it, occupying a central part of classical as well as modern control theory. Several important problems, such as robust control, decentralized control, non-interactive control, model reference or tracking control, H_2 and H_∞ optimal control problems can all be recast into an almost disturbance decoupling problem. Roughly speaking, the basic almost disturbance decoupling problem is to find an output feedback control law such that in the closed-loop system the disturbances are quenched, say in an L_p sense, up to any prespecified degree of accuracy while maintaining internal stability. Such a problem was originally formulated by Willems (1981, 1982) and labelled ADDPMS (the almost disturbance decoupling problem with measurement feedback and internal stability). In a case where, instead of a measurement feedback, a state feedback is used, the above problem is termed an ADDPS (the almost disturbance decoupling problem with internal stability). The prefix H_∞ in the acronyms H_∞-ADDPMS and H_∞-ADDPS is used to specify that the degree of accuracy in disturbance quenching is measured in an L_2-sense.

There is extensive literature on the almost disturbance decoupling problem (see, for example, the recent work of Weiland and Willems 1989, and Özcetin *et al.* 1993 a, b, and references therein). In Weiland and Willems (1989), several variations of the

Received 15 December 1997. Revised 27 April 1998.
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0020-7179/98 $12.00

disturbance decoupling problems and their solvability conditions are summarized, and the necessary and sufficient conditions are given under which the H_∞-ADDPMS and H_∞-ADDPS for continuous-time linear systems are solvable. These conditions are given in terms of geometry subspaces and for strictly proper systems (i.e. without direct feedthrough terms from the control input to the output to be controlled and from the disturbance input to the measurement output). Under these conditions, Ozcetin *et al.* (1993 a) constructs feedback laws, parameterized explicitly in a single parameter ε, that solve the H_∞-ADDPMS and the H_∞-ADDPS. These results were later extended to proper systems (i.e. with direct feedthrough terms) in Ozcetin *et al.* (1993 b). The construction of these feedback laws utilizes explicit eigenstructure assignment algorithms and avoids the solution of any parameterized algebraic Riccati equations. As a result, the structure and design of the controllers do not require an explicit value of the parameter ε. As pointed out in Ozcetin *et al.* (1993 a, b), such a design approach has several distinct advantages, the prominent ones among which are that it is a 'one-shot' design and does not encounter arbitrarily small or large numbers, and hence it is not plagued by numerical 'stiffness'.

We note that in all the results mentioned above, the internal stability was always with respect to a closed set in the complex plane. Such a closeness restriction, while facilitating the development of the above results, excludes systems with disturbance affected purely imaginary invariant zero dynamics from consideration. Only recently was this 'final' restriction on the internal stability restriction removed by Scherer (1992), thus allowing purely imaginary invariant zero dynamics to be affected by the disturbance. More specifically, the work of Scherer (1992) gave a set of necessary and sufficient conditions under which the H_∞-ADDPMS and H_∞-ADDPS, with internal stability being with respect to the open left-half plane, are solvable for general proper linear systems. When the stability is with respect to the open left-half plane, the H_∞-ADDPMS and H_∞-ADDPS will be referred to as the general H_∞-ADDPMS and the general H_∞-ADDPS, respectively.

The objective of this paper is to generalize the direct eigenstructure assignment approach to design for H_∞-ADDPMS of Ozcetin *et al.* (1993 a, b) by allowing the system to have invariant zeros on an imaginary axis. We would like to note that a drastic level of complexity is added to the design in a general setting when one allows the system to have invariant zeros on the imaginary axis. Thus, due to the generality of the problem considered, there is inherently a certain degree of complexity in presenting our design algorithm. We, however, would like to emphasize that although our design algorithm appears to be complex, our step-by-step presentation facilitates its numerical implementation. In fact, our experience with its implementation in Matlab indicates that it is straightforward to come up with a stabilizing controller which quenches the disturbance to any desired level of accuracy.

To state our problem more precisely, we consider the general H_∞-ADDPMS and the general H_∞-ADDPS for the following general continuous-time linear system, Σ:

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= C_1 x + D_1 w \\ z &= C_2 x + D_2 u + D_{22} w \end{aligned}\right\} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measurement, $w \in \mathbb{R}^q$ is the disturbance and $z \in \mathbb{R}^p$ is the output to be controlled. A, B, E, C_1, C_2,

D_1, D_2 and D_{22} are constant matrices of appropriate dimensions. For convenient references in the future development, throughout this paper, we define Σ_P to be the subsystem characterized by the matrix quadruple (A, B, C_2, D_2), and Σ_Q to be the subsystem characterized by the matrix quadruple (A, E, C_1, D_1). The following dynamic feedback control laws, Σ_c, are investigated:

$$\left.\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \end{aligned}\right\} \tag{2}$$

The controller Σ_c of equation (2) is said to be internally stabilizing when applied to the system Σ, if the following matrix is asymptotically stable:

$$A_{\mathrm{cl}} := \begin{bmatrix} A + BD_cC_1 & BC_c \\ B_cC_1 & A_c \end{bmatrix} \tag{3}$$

i.e. all its eigenvalues lie in the open left-half complex plane. Denote by T_{zw} the corresponding closed-loop transfer matrix from the disturbance w to the output to be controlled z, i.e.

$$T_{zw} = \begin{bmatrix} C_2 + D_2D_cC_1 & D_2C_c \end{bmatrix} \left(sI - \begin{bmatrix} A + BD_cC_1 & BC_c \\ B_cC_1 & A_c \end{bmatrix} \right)^{-1} \begin{bmatrix} E + BD_cD_1 \\ B_cD_1 \end{bmatrix} + D_2D_cD_1 + D_{22} \tag{4}$$

The H_∞ norm of the transfer matrix T_{zw} is given by

$$\| T_{zw} \|_\infty := \sup_{\omega \in [0,\infty)} \sigma_{\max}\left[T_{zw}(j\omega) \right] \tag{5}$$

where $\sigma_{\max}[\cdot]$ denotes the largest singular value. Then the general H_∞-ADDPMS and the general H_∞-ADDPS can be formally defined as follows.

Definition 1: The general H_∞ almost disturbance decoupling problem with measurement feedback and internal stability (the general H_∞-ADDPMS) for equation (1) is said to be solvable if, for any given positive scalar $\gamma > 0$, there exists at least one controller of the form of equation (2) such that

(1) in the absence of disturbance, the closed-loop system comprising the system equations (1) and the controller equations (2) is asymptotically stable, i.e. the matrix A_{cl} as given by equation (3) is asymptotically stable;
(2) the closed-loop system has an L_2-gain, from the disturbance w to the controlled output z, that is less than or equal to γ, i.e.

$$\|z\|_{L_2} \le \gamma \|w\|_{L_2}, \qquad \forall w \in L_2 \text{ and for } (x(0), x_c(0)) = (0, 0) \tag{6}$$

Equivalently, the H_∞-norm of the closed-loop transfer matrix from w to z, T_{zw}, is less than or equal to γ, i.e. $\| T_{zw} \|_\infty \le \gamma$.

In the case that $C_1 = I$ and $D_1 = 0$, the general H_∞-ADDPMS as defined above becomes the general H_∞-ADDPS, where only a static state feedback instead of the dynamic output feedback in equation (2) is necessary.

As stated earlier, the objective of this paper is to construct families of feedback control laws of the form in equations (2), parameterized in a single parameter, say ε, that, under the necessary and sufficient conditions of Scherer (1992), solve the above-defined general H_∞-ADDPMS and H_∞-ADDPS for general systems whose subsystems Σ_P and Σ_Q may have invariant zeros on the imaginary axis. The feedback control laws we are to construct are observer-based. A family of static state feedback control laws parameterized in a single parameter is first constructed to solve the general H_∞-ADDPS. A class of observers parameterized in the same parameter ε is then constructed to implement the state feedback control laws and thus obtain a family of dynamic measurement feedback control laws parameterized in a single parameter ε that solve the general H_∞-ADDPMS. The basic tools we use in the construction of such families of feedback control laws are: (1) the special coordinate basis, developed by Sannuti and Saberi (1987) and Saberi and Sannuti (1990), in which a linear system is decomposed into several subsystems corresponding to its finite and infinite zero structures as well as its invertibility structures; (2) a block diagonal controllability canonical form that puts the dynamics of imaginary invariant zeros into a special canonical form under which the low-gain design technique can be applied; (3) the H_∞ low-and-high gain design technique that is to be fully developed here. The development of such an H_∞ low-and-high gain design technique was originated in Lin (1998) and Lin *et al.* (1997) in the context of H_∞-ADDPMS for special classes of non-linear systems that specialize to a SISO (and hence square invertible) linear system having no invariant zeros in the open right-half plane.

The outline of this paper is as follows: Section 2 recalls the background materials on the solvability conditions for the general H_∞-ADDPMS and the special coordinate basis of linear systems, § 3 deals with control law design for the H_∞-ADDPS, § 4 deals with the construction of both full- and reduced order output feedback controllers that solve the general H_∞-ADDPMS, and finally, the concluding remarks are made in § 5.

Throughout this paper, the following notation will be used: X' denotes the transpose of matrix X; I denotes an identity matrix with appropriate dimensions; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$ is the set of all complex numbers; $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$ are the open left-half complex plane, the imaginary axis and the open right-half complex plane, respectively; $\mathrm{Ker}(X)$ is the kernel of X; $\mathrm{Im}(X)$ is the image of X; $\lambda(X)$ is the set of eigenvalues of a real square matrix X; and $\sigma_{\max}(X)$ denotes the maximum singular value of matrix X. The following definitions of geometric subspaces of linear systems will also be used intensively throughout the paper:

Definition 2: Consider a linear system Σ_* characterized by a matrix quadruple (A, B, C, D). The weakly unobservable subspaces of Σ_*, $\mathcal{V}^X$, and the strongly controllable subspaces of Σ_*, $\mathcal{S}^X$, are defined as follows:

(1) $\mathcal{V}^X(\Sigma_*)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A+BF)$-invariant and contained in $\mathrm{Ker}(C+DF)$ such that the eigenvalues of $(A+BF)|\mathcal{V}^X$ are contained in $\mathbb{C}^X \subseteq \mathbb{C}$ for some constant matrix F;

(2) $\mathcal{S}^X(\Sigma_*)$ is the minimal $(A+KC)$-invariant subspace of $\mathbb{R}^n$ containing $\mathrm{Im}(B+KD)$ such that the eigenvalues of the map which is induced by $(A+KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}^X$ are contained in $\mathbb{C}^X \subseteq \mathbb{C}$ for some constant matrix K.

Furthermore, we denote $\mathcal{V}^- = \mathcal{V}^\times$ and $\mathcal{S}^- = \mathcal{S}^\times$ if $\mathbb{C}^\times = \mathbb{C}^- \cup \mathbb{C}^0$, $\mathcal{V}^+ = \mathcal{V}^\times$ and $\mathcal{S}^+ = \mathcal{S}^\times$ if $\mathbb{C}^\times = \mathbb{C}^+$, and finally $\mathcal{V}^* = \mathcal{V}^\times$ and $\mathcal{S}^* = \mathcal{S}^\times$ if $\mathbb{C}^\times = \mathbb{C}$.

Definition 3: Consider a linear system Σ_* characterized by a matrix quadruple (A, B, C, D). For any $\lambda_0 \in \mathbb{C}$, we define

$$\mathcal{S}_{\lambda_0}(\Sigma_*) := \left\{ x \in \mathbb{C}^n \;\middle|\; \exists u \in \mathbb{C}^{n+m} : \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{bmatrix} A - \lambda_0 I & B \\ C & D \end{bmatrix} u \right\} \tag{7}$$

and

$$\mathcal{V}_{\lambda_0}(\Sigma_*) := \left\{ x \in \mathbb{C}^n \;\middle|\; \exists u \in \mathbb{C}^{m} : 0 = \begin{bmatrix} A - \lambda_0 I & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\} \tag{8}$$

$\mathcal{V}_{\lambda_0}(\Sigma_*)$ and $\mathcal{S}_{\lambda_0}(\Sigma_*)$ are associated with the so-called state zero directions of Σ_* if λ_0 is an invariant zero of Σ_*.

2. Background materials

We first recall the special coordinate basis of linear time-invariant systems introduced by Sannuti and Saberi (1987) and Saberi and Sannuti (1990). Such a special coordinate basis has the distinct feature of explicitly displaying the finite and infinite zero structures as well as the invertibility structure of a given system. Connections between the special coordinate basis and the various invariant subspaces of geometric theory, as needed for our development, are also given.

Let us consider a linear time-invariant (LTI) system Σ_* characterized by the quadruple (A, B, C, D) or in the state space form,

$$\left. \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \right\} \tag{9}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output of Σ_*, respectively. It is simple to verify that there exist non-singular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

where m_0 is the rank of matrix D. In fact, U can be chosen as an orthogonal matrix. Hence hereafter, without loss of generality, it is assumed that the matrix D has the form given on the right hand side of equation (10). One can now rewrite the system of equations (9) as

$$\left. \begin{aligned} \dot{x} &= Ax + \begin{bmatrix} B_0 & B_1 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{aligned} \right\} \tag{11}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. We now have the theorem given below.

Theorem 1 (SCB): *Given the linear system* Σ_* *of* (10), *there exist*

(1) *coordinate free non-negative integers* n_a^-, n_a^0, n_a^+, n_b, n_c, n_d, $m_d \le m - m_0$ *and* q_i, $i = 1, \dots, m_d$,

(2) *non-singular state, output and input transformations* Γ_s, Γ_o *and* Γ_i *which take the given* Σ_* *into a special coordinate basis that displays explicitly both the finite and infinite zero structures of* Σ_*.

The special coordinate basis referred to above is described by the following set of equations:

$$x = \Gamma_s \tilde{x}, \qquad y = \Gamma_o \tilde{y}, \qquad u = \Gamma_i \tilde{u}, \tag{12}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \qquad x_a = \begin{pmatrix} x_a^- \\ x_a^0 \\ x_a^+ \end{pmatrix}, \qquad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix} \tag{13}$$

$$\tilde{y} = \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix}, \qquad y_d = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_d} \end{pmatrix}, \qquad \tilde{u} = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \qquad u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix} \tag{14}$$

and

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{ad}^- y_d + L_{ab}^- y_b \tag{15}$$

$$\dot{x}_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 y_0 + L_{ad}^0 y_d + L_{ab}^0 y_b \tag{16}$$

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + B_{0a}^+ y_0 + L_{ad}^+ y_d + L_{ab}^+ y_b \tag{17}$$

$$\dot{x}_b = A_{bb} x_b + B_{0b} y_0 + L_{bd} y_d, \qquad y_b = C_b x_b \tag{18}$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} y_0 + L_{cb} y_b + L_{cd} y_d + B_c \left[E_{ca}^- x_a^- + E_{ca}^0 x_a^0 + E_{ca}^+ x_a^+ \right] + B_c u_c \tag{19}$$

$$y_0 = C_{0c} x_c + C_{0a}^- x_a^- + C_{0a}^+ x_a^0 + C_{0a}^+ x_a^+ + C_{0d} x_d + C_{0b} x_b + u_0 \tag{20}$$

and for each $i = 1, \dots, m_d$,

$$\dot{x}_i = A_{q_i} x_i + L_{i0} y_0 + L_{id} y_d + B_{q_i} \left[u_i + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} x_j \right] \tag{21}$$

$$y_i = C_{q_i} x_i, \qquad y_d = C_d x_d \tag{22}$$

Here the states x_a^-, x_a^0, x_a^+, x_b, x_c *and* x_d *are of dimensions* n_a^-, n_a^0, n_a^+, n_b, n_c *and* $n_d = \sum_{i=1}^{m_d} q_i$, *respectively while* x_i *is of dimension* q_i *for each* $i = 1, \dots, m_d$. *The control vectors* u_0, u_d *and* u_c *are respectively of dimensions* m_0, m_d *and* $m_c = m - m_0 - m_d$, *while the output vectors y_0, y_d and y_b are of dimensions $p_0 = m_0$, $p_d = m_d$ and $p_b = p - p_0 - p_d$, respectively. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:*

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \qquad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_{q_i} = [1, 0, \ldots, 0] \tag{23}$$

Assuming that x_i, $i = 1, 2, \ldots, m_d$, are arranged such that $q_i \le q_{i+1}$, the matrix L_{id} has the particular form

$$L_{id} = \begin{bmatrix} L_{i1} & L_{i2} & \cdots & L_{ii-1} & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

Also, the last row of each L_{id} is identically zero. Moreover, we have $\lambda(A_{aa}^-) \subset \mathbb{C}^-$, $\lambda(A_{aa}^0) \subset \mathbb{C}^0$ and $\lambda(A_{aa}^+) \subset \mathbb{C}^+$. Also, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable.

Proof: See Sannuti and Saberi (1987) and Saberi and Sannuti (1990). The software realizations of the decomposition can be found in LAS by Chen (1988) and in MATLAB by Lin (1989). □

We can rewrite the special coordinate basis of the quadruple (A, B, C, D) given by Theorem 1 in a more compact form:

$$\tilde{A} = \Gamma_s^{-1}(A - B_0 C_0)\Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix} \tag{25}$$

$$\tilde{B} = \Gamma_s^{-1}\begin{bmatrix} B_0 & B_1 \end{bmatrix}\Gamma_i = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \qquad \tilde{D} = \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

$$\tilde{C} = \Gamma_o^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix} \tag{27}$$

In what follows, we state some important properties of the above special coordinate basis which are pertinent to our present work and which will be used throughout this paper.

Property 1: The given system Σ_* is observable (detectable) if and only if the pair $(A_{\rm obs}, C_{\rm obs})$ is observable (detectable), where

$$A_{\rm obs} := \begin{bmatrix} A_{aa} & 0 \\ B_c E_{ca} & A_{cc} \end{bmatrix}, \qquad C_{\rm obs} := \begin{bmatrix} C_{0a} & C_{0c} \\ E_{da} & E_{dc} \end{bmatrix}, \tag{28}$$

and where

$$A_{aa} := \begin{bmatrix} A_{aa}^- & 0 & 0 \\ 0 & A_{aa}^0 & 0 \\ 0 & 0 & A_{aa}^+ \end{bmatrix} \tag{29}$$

$$C_{0a} := \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ \end{bmatrix}, \qquad E_{da} := \begin{bmatrix} E_{da}^- & E_{da}^0 & E_{da}^+ \end{bmatrix}, \qquad E_{ca} := \begin{bmatrix} E_{ca}^- & E_{ca}^0 & E_{ca}^+ \end{bmatrix} \tag{30}$$

Also, define

$$B_{0a} := \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \end{bmatrix}, \qquad L_{ab} := \begin{bmatrix} L_{ab}^- \\ L_{ab}^0 \\ L_{ab}^+ \end{bmatrix}, \qquad L_{ad} := \begin{bmatrix} L_{ad}^- \\ L_{ad}^0 \\ L_{ad}^+ \end{bmatrix} \tag{31}$$

and

$$A_{\rm con} := \begin{bmatrix} A_{aa} & L_{ab} C_b \\ 0 & A_{bb} \end{bmatrix}, \qquad B_{\rm con} := \begin{bmatrix} B_{0a} & L_{ad} \\ B_{0b} & L_{bd} \end{bmatrix} \tag{32}$$

Similarly, Σ_* is controllable (stabilizable) if and only if the pair $(A_{\rm con}, B_{\rm con})$ is controllable (stabilizable).

Property 2: Invariant zeros of Σ_* are the eigenvalues of A_{aa}, which are the unions of the eigenvalues of A_{aa}^-, A_{aa}^0 and A_{aa}^+.

Property 3: Σ_* has $m_0 = \text{rank}\,(D)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ_* is given by $S_\infty^\star(\Sigma_*) = \{q_1, q_2, \ldots, q_{m_d}\}$, i.e., each q_i corresponds to an infinite zero of Σ_* of order q_i.

Property 4: The given system Σ_* is right invertible if and only if x_b (and hence y_b) are non-existent, left invertible if and only if x_c (and hence u_c) are non-existent, and invertible if and only if both x_b and x_c are non-existent. Moreover, Σ_* is degenerate if and only if it is neither left nor right invertible, i.e. both x_b and x_c are present.

By now it is clear that the special coordinate basis decomposes the state-space into several distinct parts. In fact, the state-space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \oplus \mathcal{X}_c \oplus \mathcal{X}_d \tag{33}$$

The following property shows interconnections between the special coordinate basis and various invariant geometric subspaces.

Property 5: Various components of the state vector of the special coordinate basis have the following geometrical interpretations:

(1) $x_a^- \oplus x_a^0 \oplus x_c \oplus x_d$ spans $\mathcal{S}^+(\Sigma_*)$;
(2) $x_a^+ \oplus x_c \oplus x_d$ spans $\mathcal{S}^-(\Sigma_*)$;
(3) $x_c \oplus x_d$ spans $\mathcal{S}^*(\Sigma_*)$;
(4) $x_a^- \oplus x_a^0 \oplus x_c$ spans $\mathcal{V}^-(\Sigma_*)$;
(5) $x_a^+ \oplus x_c$ spans $\mathcal{V}^+(\Sigma_*)$;
(6) $x_a^- \oplus x_a^0 \oplus x_a^+ \oplus x_c$ spans $\mathcal{V}^*(\Sigma_*)$.

Finally, we are ready to recall the necessary and sufficient conditions of Scherer (1992) under which the general H_∞-ADDPMS and H_∞-ADDPS are solvable.

Theorem 2: *Consider the general measurement feedback system in equation* (1) *with* $D_{22} = 0$. *Then the general* H_∞ *almost disturbance decoupling problem for equation* (1) *with internal stability* (H_∞-*ADDPMS*) *is solvable if and only if the following conditions are satisfied:*

(1) (A, B) *is stabilizable;*
(2) (A, C_1) *is detectable;*
(3) $\mathrm{Im}(E) \subset \mathcal{S}^+(\Sigma_P) \cap \{\cap_{\lambda_0 \in \mathbb{C}^0} \mathcal{S}_{\lambda_0}(\Sigma_P)\}$;
(4) $\mathrm{Ker}(C_2) \supset \mathcal{V}^+(\Sigma_Q) \cup \{\cup_{\lambda_0 \in \mathbb{C}^0} \mathcal{V}_{\lambda_0}(\Sigma_Q)\}$;
(5) $\mathcal{V}^+(\Sigma_Q) \subset \mathcal{S}^+(\Sigma_P)$.

It is simple to verify that for the case that all states of the system in equation (1) are fully measurable, i.e. $C_1 = I$ and $D_1 = 0$, then the solvability conditions for the general H_∞-ADDPS reduce to the following: (1) (A, B) is stabilizable; (2) $D_{22} = 0$; (3) $\mathrm{Im}(E) \subset \mathcal{S}^+(\Sigma_P) \cap \{\cap_{\lambda_0 \in \mathbb{C}^0} \mathcal{S}_{\lambda_0}(\Sigma_P)\}$. Moreover, in this case, a static state feedback control law, i.e. $u = Fx$, exists that solves the general H_∞-ADDPS, where F is a constant matrix and might be parameterized by certain tuning parameters.

3. Solution to the H_∞-ADDPS

In this section, we consider feedback control law design for the general H_∞ almost disturbance decoupling problem with internal stability and with full state feedback, where internal stability is with respect to the open left-half plane, i.e. the general H_∞-ADDPS. More specifically, we present a design procedure that constructs a family of parameterized static state feedback control laws,

$$u = F(\varepsilon)x, \tag{34}$$

that solves the general H_∞-ADDPS for the following system,

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= x \\ z &= C_2 x + D_2 u + D_{22} w \end{aligned}\right\} \tag{35}$$

That is, under this family of state feedback control laws, the resulting closed-loop system is asymptotically stable for sufficiently small ε, and the H_∞-norm of the closed-loop transfer matrix from w to z, $T_{zw}(s, \varepsilon)$, tends to zero as ε tends to zero, where

$$T_{zw}(s, \varepsilon) = [C_2 + D_2 F(\varepsilon)][sI - A - BF(\varepsilon)]^{-1} E + D_{22} \tag{36}$$

Clearly, $D_{22}=0$ is a necessary condition for the solvability of the general H_∞-ADDPS. We present an algorithm for obtaining this $F(\varepsilon)$, following the asymptotic time-scale and eigenstructure assignment (ATEA) procedure. The ATEA design procedure was originally conceived by Saberi and Sannuti (1989) and was used to solve many control problems (see, for example, Chen 1991, Özcetin *et al.* 1993a, Lin 1994, Saberi *et al.* 1995, to name only a few). It uses the special coordinate basis of the given system (see Theorem 1) to decompose the system into several subsystems according to its finite and infinite zero structures as well as its invertibility structures. The new component here is the low-gain design for the part of the zero dynamics corresponding to all purely imaginary invariant zeros. As will be clear shortly, the low-gain component is critical in handling the case when the zero dynamics corresponding to purely imaginary invariant zeros is affected by disturbance. It is well known that the disturbance affected purely imaginary zero dynamics is difficult to handle and has always been excluded from consideration until recently.

Step S.1. (decomposition of Σ_P) Transform the subsystem Σ_P, i.e. the quadruple (A, B, C_2, D_2), into the special coordinate basis (SCB) as given by Theorem 1 in §2. Denote the state, output and input transformation matrices as Γ_{sP}, Γ_{oP} and Γ_{iP}, respectively.

Step S.2. (gain matrix for the subsystem associated with x_c). Let F_c be any arbitrary $m_c \times n_c$ matrix subject to the constraint that

$$A^c_{cc}=A_{cc}-B_cF_c \tag{37}$$

is a stable ε matrix. Note that the existence of such an F_c is guaranteed by the property of SCB, i.e. (A_{cc}, B_c) is controllable.

Step S.3. (gain matrix for the subsystems associated with x_a^+ and x_b). Let

$$F^+_{ab}:=\begin{bmatrix} F^+_{a0} & F_{b0} \\ F^+_{ad} & F_{bd} \end{bmatrix} \tag{38}$$

be any arbitrary $(m_0+m_d)\times(n_a^+ + n_b)$ matrix subject to the constraint that

$$A^{+c}_{ab}:=\begin{bmatrix} A^+_{aa} & L^+_{ab}C_b \\ 0 & A_{bb} \end{bmatrix}-\begin{bmatrix} B^+_{0a} & L^+_{ad} \\ B_{0b} & L_{bd} \end{bmatrix}F^+_{ab} \tag{39}$$

is a stable matrix. Again, note that the existence of such an F_{ab} is guaranteed by the stabilizability of (A, B) and Property 1 of the special coordinate basis. For future use, let us partition $\begin{bmatrix} F^+_{ad} & F_{bd} \end{bmatrix}$ as

$$\begin{bmatrix} F^+_{ad} & F_{bd} \end{bmatrix}=\begin{bmatrix} F^+_{ad1} & F_{bd1} \\ F^+_{ad2} & F_{bd2} \\ \vdots & \vdots \\ F^+_{adm_d} & F_{bdm_d} \end{bmatrix} \tag{40}$$

where F^+_{adi} and F_{bdi} are of dimensions $1\times n_a^+$ and $1\times n_b$, respectively.

Step S.4. (gain matrix for the subsystem associated with x_a^0). The construction of this gain matrix is carried out in the following sub-steps.

Step S.4.1. (preliminary coordinate transformation). Recalling the definition of $(A_{\rm con}, B_{\rm con})$, i.e. equation (32), we have

$$A_{\rm con} - B_{\rm con}\begin{bmatrix} 0 & 0 & F_{ab}^{+} \end{bmatrix} = \begin{bmatrix} A_{aa}^{-} & 0 & A_{aab}^{-} \\ 0 & A_{aa}^{0} & A_{aab}^{0} \\ 0 & 0 & A_{ab}^{+c} \end{bmatrix}$$

$$B_{\rm con} = \begin{bmatrix} B_{0a}^{-} & L_{ad}^{-} \\ B_{0a}^{0} & L_{ad}^{0} \\ B_{0ab}^{+} & L_{abd}^{+} \end{bmatrix} \tag{41}$$

where

$$B_{0ab}^{+} = \begin{bmatrix} B_{0a}^{+} \\ B_{0b} \end{bmatrix}, \qquad L_{abd}^{+} = \begin{bmatrix} L_{ad}^{+} \\ L_{bd} \end{bmatrix}$$

$$A_{aab}^{0} = \begin{bmatrix} 0 & L_{ab}^{0} C_b \end{bmatrix} - \begin{bmatrix} B_{0a}^{0} & L_{ad}^{0} \end{bmatrix} F_{ab}^{+} \tag{42}$$

and

$$A_{aab}^{-} = \begin{bmatrix} 0 & L_{ab}^{-} C_b \end{bmatrix} - \begin{bmatrix} B_{0a}^{-} & L_{ad}^{-} \end{bmatrix} F_{ab}^{+}$$

Clearly $(A_{\rm con} - B_{\rm con}F_{ab}^{+}, B_{\rm con})$ remains stabilizable. Construct the following nonsingular transformation matrix,

$$\Gamma_{ab} = \begin{bmatrix} I_{n_a^-} & 0 & 0 \\ 0 & 0 & I_{n_a^+ + n_b} \\ 0 & I_{n_a^0} & T_a^0 \end{bmatrix}^{-1} \tag{43}$$

where T_a^0 is the unique solution to the following Lyapunov equation:

$$A_{aa}^{0} T_a^0 - T_a^0 A_{ab}^{+c} = A_{aab}^{0} \tag{44}$$

We note here that such a unique solution to the above Lyapunov equation always exists since all the eigenvalues of A_{aa}^{0} are on the imaginary axis and all the eigenvalues of A_{ab}^{+c} are in the open left-half plane. It is now easy to verify that

$$\Gamma_{ab}^{-1}(A_{\rm con} - B_{\rm con}F_{ab}^{+})\Gamma_{ab} = \begin{bmatrix} A_{aa}^{-} & A_{aab}^{-} & 0 \\ 0 & A_{ab}^{+c} & 0 \\ 0 & 0 & A_{aa}^{0} \end{bmatrix} \tag{45}$$

$$\Gamma_{ab}^{-1} B_{\rm con} = \begin{bmatrix} B_{0a}^{-} & L_{ad}^{-} \\ B_{0ab}^{+} & L_{abd}^{+} \\ B_{0a}^{0} + T_a^0 B_{0ab}^{+} & L_{ad}^{0} + T_a^0 L_{abd}^{+} \end{bmatrix} \tag{46}$$

Hence, the matrix pair (A^0_{aa}, B^0_a) is controllable, where

$$B^0_a = \begin{bmatrix} B^0_{0a} + T^0_a B^+_{0ab} & L^0_{ad} + T^0_a L^+_{abd} \end{bmatrix}$$

Step S.4.2. (further coordinate transformation). Find non-singular transformation matrices Γ^0_{sa} and Γ^0_{ia} such that (A^0_{aa}, B^0_a) can be transformed into the block diagonal controllability canonical form,

$$(\Gamma^0_{sa})^{-1} A^0_{aa} \Gamma^0_{sa} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_l \end{bmatrix}$$

$$(\Gamma^0_{sa})^{-1} B^0_a \Gamma^0_{ia} = \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1l} & \star \\ 0 & B_2 & \cdots & B_{2l} & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_l & \star \end{bmatrix}$$

where l is an integer and for $i = 1, 2, \ldots, l$,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a^i_{n_i} & -a^i_{n_i-1} & -a^i_{n_i-2} & \cdots & -a^i_1 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

We note that all the eigenvalues of A_i are on the imaginary axis. Here the $\star$s represent submatrices of less interest. We note that the existence of the above canonical form was shown by Wonham (1979), while its software realization can be found in Chen (1997).

Step S.4.3. (subsystem design). For each (A_i, B_i), let $F_i(\varepsilon) \in \mathbb{R}^{1\times n_i}$ be the state feedback gain such that

$$\lambda\{A_i + B_i F_i(\varepsilon)\} = -\varepsilon + \lambda(A_i) \in \mathbb{C}^-$$

Note that $F_i(\varepsilon)$ is unique.

Step S.4.4. (composition of gain matrix for the subsystem associated with x^0_a). Let

$$F_a^0(\varepsilon) := \Gamma_{ia}^0 \begin{bmatrix} F_1(\varepsilon) & 0 & \cdots & 0 & 0 \\ 0 & F_2(\varepsilon) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & F_{l-1}(\varepsilon) & 0 \\ 0 & 0 & \cdots & 0 & F_l(\varepsilon) \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} (\Gamma_{sa}^0)^{-1} \tag{47}$$

where $\varepsilon \in (0, 1]$ is a design parameter whose value is to be specified later.

Clearly, we have

$$\|F_a^0(\varepsilon)\| \le f_a^0 \varepsilon, \qquad \varepsilon \in (0, 1], \tag{48}$$

for some positive constant f_a^0, independent of ε. For future use, we define and partition $F_{ab}(\varepsilon) \in \mathbb{R}^{(m_0+m_d)\times(n_a+n_b)}$ as

$$F_{ab}(\varepsilon) = \begin{bmatrix} F_{ab0}(\varepsilon) \\ F_{abd}(\varepsilon) \end{bmatrix} = \begin{bmatrix} 0_{m_0\times n_a^-} & 0_{m_0\times(n_a^+ + n_b)} & F_{a0}^0(\varepsilon) \\ 0_{m_d\times n_a^-} & 0_{m_d\times(n_a^+ + n_b)} & F_{ad}^0(\varepsilon) \end{bmatrix} \Gamma_{ab}^{-1} \tag{49}$$

and

$$F_{abd}(\varepsilon) = \begin{bmatrix} F_{abd1}(\varepsilon) \\ F_{abd2}(\varepsilon) \\ \vdots \\ F_{abdm_d}(\varepsilon) \end{bmatrix} \tag{50}$$

where $F_{a0}^0(\varepsilon)$ and $F_{ad}^0(\varepsilon)$ are defined as

$$F_a^0(\varepsilon) = \begin{bmatrix} F_{a0}^0(\varepsilon) \\ F_{ad}^0(\varepsilon) \end{bmatrix} \tag{51}$$

We also partition $F_{ad}^0(\varepsilon)$ as,

$$F_{ad}^0(\varepsilon) = \begin{bmatrix} F_{ad1}^0(\varepsilon) \\ F_{ad2}^0(\varepsilon) \\ \vdots \\ F_{adm_d}^0(\varepsilon) \end{bmatrix} \tag{52}$$

Step S.5. (gain matrix for the subsystem associated with x_d). This step makes use of subsystems, $i = 1$ to m_d, represented by equation (21) of §2. Let $\Lambda_i = \{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{iq_i}\}$, $i = 1$ to m_d, be the sets of q_i elements all in

$\mathbb{C}^-$, which are closed under complex conjugation, where q_i and m_d are as defined in Theorem 1 but associated with the special coordinate basis of Σ_P. Let $\Lambda_d := \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{m_d}$. For $i = 1$ to m_d, we define

$$p_i(s) := \prod_{j=1}^{q_i}(s - \lambda_{ij}) = s^{q_i} + F_{i1}s^{q_i-1} + \cdots + F_{iq_i-1}s + F_{iq_i}, \tag{53}$$

and

$$\widetilde{F}_i(\varepsilon) := \frac{1}{\varepsilon^{q_i}} F_i S_i(\varepsilon), \tag{54}$$

where

$$F_i = \begin{bmatrix} F_{iq_i} & F_{iq_i-1} & \cdots & F_{i1} \end{bmatrix}, \quad S_i(\varepsilon) = \operatorname{diag}\left\{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q_i-1}\right\} \tag{55}$$

Step S.6. (compositon of parameterized gain matrix $F(\varepsilon)$). In this step, various gains calculated in Steps S.2–S.5 are put together to form a composite state feedback gain matrix $F(\varepsilon)$. Let

$$\widetilde{F}_{abd}(\varepsilon) := \begin{bmatrix} F_{abd1}(\varepsilon)F_{1_{q1}}(\varepsilon)/\varepsilon^{q1} \\ F_{abd2}(\varepsilon)F_{2_{q2}}(\varepsilon)/\varepsilon^{q2} \\ \vdots \\ F_{abdm_d}(\varepsilon F_{m_d q_{m_d}}/\varepsilon^{q_{m_d}} \end{bmatrix} \tag{56}$$

$$\widetilde{F}^{+}_{ad}(\varepsilon) := \begin{bmatrix} F^{+}_{ad1}F_{1_{q1}}/\varepsilon^{q1} \\ F^{+}_{ad2}F_{2_{q2}}/\varepsilon^{q2} \\ \vdots \\ F^{+}_{adm_d}F_{m_d q_{m_d}}/\varepsilon^{q_{m_d}} \end{bmatrix} \tag{57}$$

and

$$\widetilde{F}_{bd}(\varepsilon) := \begin{bmatrix} F^{+}_{bd1}F_{1_{q1}}/\varepsilon^{q1} \\ F^{+}_{bd2}F_{2_{q2}}/\varepsilon^{q2} \\ \vdots \\ F^{+}_{bdm_d}F_{m_d q_{m_d}}/\varepsilon^{q_{m_d}} \end{bmatrix} \tag{58}$$

Then define the state feedback gain $F(\varepsilon)$ as

$$F(\varepsilon) := -\Gamma_{iP}\left(\widetilde{F}^{\star}_{abcd}(\varepsilon) + \widetilde{F}_{abcd}(\varepsilon)\right)\Gamma_{sP}^{-1} \tag{59}$$

where

$$\tilde{F}^{\star}_{abcd}(\varepsilon) = \begin{bmatrix} C^-_{0a} & C^0_{0a} & C^+_{0a} + F^+_{a0} & C_{0b} + F_{b0} & C_{0c} & C_{0d} \\ E^-_{da} & E^0_{da} & E^+_{da} + \tilde{F}^+_{ad}(\varepsilon) & E_{db} + \tilde{F}_{bd}(\varepsilon) & E_{dc} & \tilde{F}_d(\varepsilon) + E_d \\ E^-_{ca} & E^0_{ca} & E^+_{ca} & 0 & F_c & 0 \end{bmatrix} \tag{60}$$

$$\tilde{F}_{abcd}(\varepsilon) = \begin{bmatrix} F_{ab0}(\varepsilon) & 0 & 0 \\ \tilde{F}_{abd}(\varepsilon) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{61}$$

and where

$$E_d = \begin{bmatrix} E_{11} & \dots & E_{1m_d} \\ \vdots & \ddots & \vdots \\ E_{m_d1} & \dots & E_{m_dm_d} \end{bmatrix} \tag{62}$$

$$\tilde{F}_d(\varepsilon) = \mathrm{diag}\left\{\tilde{F}_1(\varepsilon), \quad \tilde{F}_2(\varepsilon), \quad \dots, \quad \tilde{F}_{m_d}(\varepsilon)\right\} \tag{63}$$

Note that in principle one might take the perturbation or the LMI approach to solving the general H_∞-ADDPS. However, these approaches, especially the perturbation one, often have numerical difficulties in dealing with systems that have invariant zeros on the imaginary axis and/or infinite zeros. Our approach does not have such a problem as it does not involve the solution of any parameterized algebraic Riccati equations. Furthermore, the resulting feedback laws from our method are explicitly given as polynomial matrices in ε.

We have the following theorem.

Theorem 3: *Consider the given system* (36) *satisfying the following conditions:* (1) (A, B) *is stabilizable;* (2) $D_{22} = 0$; (3) $\mathrm{Im}(E) \subset \mathcal{S}^+(\Sigma_P) \cap \left\{\cap_{\lambda_0 \in \mathbb{C}^0} \mathcal{S}_{\lambda_0}(\Sigma_P)\right\}$. *Then the closed-loop system comprising equation* (35) *and the static state feedback control law* $u = F(\varepsilon)x$, *with* $F(\varepsilon)$ *given by equation* (59), *has the following properties: for any given* $\gamma > 0$, *there exists a positive scalar* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon \le \varepsilon^*$.

(1) *The closed-loop system is asymptotically stable, i.e.* $\lambda\{A + BF(\varepsilon)\} \subset \mathbb{C}^-$.

(2) *The* H_∞*-norm of the closed-loop transfer matrix from the disturbance* w *to the controlled output* z *is less than* γ, *i.e.* $\|T_{zw}(s, \varepsilon)\|_\infty < \gamma$.

Hence, by Definition 1, *the control law* $u = F(\varepsilon)x$ *solves the general* H_∞*-ADDPS for equation* (36).

Proof: See Appendix A. □

4. Solution to the H_∞-ADDPMS

In this section we present the designs of both full order and reduced order output feedback controllers that solve the general H_∞-ADDPMS for the system given in equation (1). Here, by full-order controller, we mean that the order of the controller is exactly the same as the given system (1), i.e. is equal to n. A reduced order controller, on the other hand, refers to a controller whose dynamic order is less than n. We will assume that $D_{22}=0$ in the system given in equation (1) throughout this section. If $D_{22}\neq 0$, one can easily show that the solvability of the general H_∞-ADDPMS implies the existence of a Conant matrix S such that

$$D_{22}+D_2SD_1=0 \tag{64}$$

Then, by applying a pre-output feedback control law $u=Sy+v$ to the system in equation (1), the resulting system we obtain will have a zero direct feedthrough matrix from w to z. Hence, it is without loss of any generality to assume that the matrix $D_{22}=0$.

4.1. *Full order output feedback controller design*

The following is a step-by-step algorithm for constructing a parameterized full-order output feedback controller that solves the general H_∞-ADDPMS:

Step F.C.1. (construction of the gain matrix $F_P(\varepsilon)$). Define an auxiliary system

$$\left.\begin{aligned} \dot{x}&=Ax+Bu+Ew\\ y&=x\\ z&=C_2x+D_2u+D_{22}w \end{aligned}\right\} \tag{65}$$

and then perform Steps S.1–S.6 of the previous section to the above system to obtain a parameterized gain matrix $F(\varepsilon)$. We let $F_P(\varepsilon)=F(\varepsilon)$.

Step F.C.2. (construction of the gain matrix $K_Q(\varepsilon)$). Define another auxiliary system

$$\left.\begin{aligned} \dot{x}&=A'x+C_1'u+C_2'w\\ y&=x\\ z&=E'x+D_1'u+D_{22}'w \end{aligned}\right\} \tag{66}$$

and then perform Steps S.1–S.6 of the previous section to the above system to get the parameterized gain matrix $F(\varepsilon)$. We let $K_Q(\varepsilon)=F(\varepsilon)'$.

Step F.C.3. (construction of the full order controller $\Sigma_{FC}(\varepsilon)$). Finally, the parameterized full order output feedback controller, $\Sigma_{FC}(\varepsilon)$, is given by

$$\left.\begin{aligned} \dot{x}_c&=A_{FC}(\varepsilon)x_c+B_{FC}(\varepsilon)y,\\ u&=C_{FC}(\varepsilon)x_c+D_{FC}(\varepsilon)y \end{aligned}\right\} \tag{67}$$

where

$$\left.\begin{aligned} A_{FC}(\varepsilon)&:=A+BF_P(\varepsilon)+K_Q(\varepsilon)C_1\\ B_{FC}(\varepsilon)&:=-K_Q(\varepsilon)\\ C_{FC}(\varepsilon)&:=F_P(\varepsilon)\\ D_{FC}(\varepsilon)&:=0 \end{aligned}\right\} \tag{68}$$

We have the following theorem.

Theorem 4: *Consider the system given in equation* (1) *with* $D_{22}=0$ *satisfying all the conditions in Theorem* 2. *Then the closed-loop system comprising equation* (1) *and the full order output feedback controller* (60) *has the following properties: For any given* $\gamma > 0$, *there exists a positive scalar* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon \le \varepsilon^*$:

(1) *the resulting closed-loop system is asymptotically stable*;
(2) *the* H_∞*-norm of the resulting closed-loop transfer matrix from the disturbance* w *to the controlled output* z *is less than* γ, *i.e.* $\|T_{zw}(s,\varepsilon)\|_\infty < \gamma$.

By Definition 1, *the control law in equation* (67) *solves the general* H_∞*-ADDPMS for equation* (1).

Proof: See Appendix B. □

4.2. *Reduced-order output feedback controller design*

In this subsection, we follow the procedure of Chen *et al.* (1992) to design a reduced order output feedback controller. We will show that such a controller structure with appropriately chosen gain matrices also solves the general H_∞-ADDPMS for the system in equation (1). First, without loss of generality and for simplicity of presentation, we assume that the matrices C_1 and D_1 are already in the form

$$C_1 = \begin{bmatrix} 0 & C_{1,02} \\ I_k & 0 \end{bmatrix} \quad \text{and} \quad D_1 = \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix} \tag{69}$$

where $k = \ell - \operatorname{rank}(D_1)$ and $D_{1,0}$ is of full rank. Then the system given in equation (1) can be written as

$$\left.\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} w \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} 0 & C_{1,02} \\ I_k & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix} w \\ z &= \begin{bmatrix} C_{2,1} & C_{2,2} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_2 u + D_{22} w \end{aligned}\right\} \tag{70}$$

where the original state x is partitioned to two parts, x_1 and x_2, and y is partitioned to y_0 and y_1 with $y_1 \equiv x_1$. Thus, one needs to estimate only the state x_2 in the reduced-order controller design. Next, define an auxiliary subsystem Σ_{QR} characterized by a matrix quadruple (A_R, E_R, C_R, D_R), where

$$(A_R, E_R, C_R, D_R) = \left(A_{22}, E_2, \begin{bmatrix} C_{1,02} \\ A_{12} \end{bmatrix}, \begin{bmatrix} D_{1,0} \\ E_1 \end{bmatrix} \right) \tag{71}$$

The following is a step-by-step algorithm that constructs the reduced order output feedback controller for the general H_∞-ADDPMS.

Step R.C.1. (construction of the gain matrix $F_{\mathrm{P}}(\varepsilon)$). Define an auxiliary system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= x \\ z &= C_2 x + D_2 u + D_{22} w \end{aligned}\right\} \tag{72}$$

and then perform Steps S.1–S.6 of §3 to the above system to get the parameterized gain matrix $F(\varepsilon)$. We let $F_{\mathrm{P}}(\varepsilon) = F(\varepsilon)$.

Step R.C.2. (construction of the gain matrix $K_{\mathrm{R}}(\varepsilon)$). Define another auxiliary system

$$\left.\begin{aligned} \dot{x} &= A_{\mathrm{R}}' x + C_{\mathrm{R}}' u + C_{2,2}' w \\ y &= x \\ z &= E_{\mathrm{R}}' x + D_{\mathrm{R}}' u + D_{22}' w \end{aligned}\right\} \tag{73}$$

and then perform Steps S.1–S.6 of §3 to the above system to get the parameterized gain matrix $F(\varepsilon)$. We let $K_{\mathrm{R}}(\varepsilon) = F(\varepsilon)'$.

Step R.C.3. (construction of the reduced order controller $\Sigma_{\mathrm{RC}}(\varepsilon)$). Let us partition $F_{\mathrm{P}}(\varepsilon)$ and $K_{\mathrm{R}}(\varepsilon)$ as,

$$F_{\mathrm{P}}(\varepsilon) = \begin{bmatrix} F_{\mathrm{P}1}(\varepsilon) & F_{\mathrm{P}2}(\varepsilon) \end{bmatrix} \quad \text{and} \quad K_{\mathrm{R}}(\varepsilon) = \begin{bmatrix} K_{\mathrm{R}0}(\varepsilon) & K_{\mathrm{R}1}(\varepsilon) \end{bmatrix} \tag{74}$$

in conformity with the partition

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

respectively. Then define

$$G_{\mathrm{R}}(\varepsilon) = \begin{bmatrix} -K_{\mathrm{R}0}(\varepsilon), & A_{21} + K_{\mathrm{R}1}(\varepsilon) A_{11} - (A_{\mathrm{R}} + K_{\mathrm{R}}(\varepsilon) C_{\mathrm{R}}) K_{\mathrm{R}1}(\varepsilon) \end{bmatrix} \tag{75}$$

Finally, the parameterized reduced-order output feedback controller, $\Sigma_{\mathrm{RC}}(\varepsilon)$, is given by

$$\left.\begin{aligned} \dot{x}_c &= A_{\mathrm{RC}}(\varepsilon) x_c + B_{\mathrm{RC}}(\varepsilon) y \\ u &= C_{\mathrm{RC}}(\varepsilon) x_c + D_{\mathrm{RC}}(\varepsilon) y \end{aligned}\right\} \tag{76}$$

where

$$\left.\begin{aligned} A_{\mathrm{RC}}(\varepsilon) &:= A_{\mathrm{R}} + B_2 F_{\mathrm{P}2}(\varepsilon) + K_{\mathrm{R}}(\varepsilon) C_{\mathrm{R}} + K_{\mathrm{R}1}(\varepsilon) B_1 F_{\mathrm{P}2}(\varepsilon) \\ B_{\mathrm{RC}}(\varepsilon) &:= G_{\mathrm{R}}(\varepsilon) + \begin{bmatrix} B_2 + K_{\mathrm{R}1}(\varepsilon) B_1 \end{bmatrix} \begin{bmatrix} 0, F_{\mathrm{P}1}(\varepsilon) - F_{\mathrm{P}2}(\varepsilon) K_{\mathrm{R}1}(\varepsilon) \end{bmatrix} \\ C_{\mathrm{RC}}(\varepsilon) &:= F_{\mathrm{P}2}(\varepsilon) \\ D_{\mathrm{RC}}(\varepsilon) &:= \begin{bmatrix} 0, F_{\mathrm{P}1}(\varepsilon) - F_{\mathrm{P}2}(\varepsilon) K_{\mathrm{R}1}(\varepsilon) \end{bmatrix} \end{aligned}\right\} \tag{77}$$

We have the following theorem.

Theorem 5: *Consider the system given in equation* (1) *with* $D_{22} = 0$ *satisfying all the conditions in Theorem* 2. *Then the closed-loop system comprising equation* (1) *and the reduced order output feedback controller in equation* (76) *have the following properties: for any given* $\gamma > 0$, *there exists a positive scalar* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon^*$:

(1) *the resulting closed-loop system is asymptotically stable;*

(2) *the* H_∞*-norm of the resulting closed-loop transfer matrix from the disturbance* w *to the controlled output* z *is less than* γ, *i.e.* $\|T_{zw}(s,\varepsilon)\|_\infty < \gamma$.

By Definition 1, *the control law in equation* (76) *solves the general* H_∞*-ADDPMS for equation* (1).

Proof: See Appendix C. □

5. Conclusions

In this paper we have presented several explicit procedures for constructing solutions for the general H_∞-ADDPS and H_∞-ADDPMS, in which the given systems are allowed to have invariant zeros on the imaginary axis of the complex plane. Our approach is decentralized in nature. We used a low-gain design technique to handle the subsystem associated with the zero dynamics on the imaginary axis and a high-gain technique to deal with the subsystem associated with the infinite zero structure of the given system. Hence, our approach can be termed as a low-and-high gain method.

6. Appendix A — Proof of Theorem 3

Under the feedback control law $u = F(\varepsilon)x$, the closed-loop system on the special coordinate basis can be written as follows:

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- z_0 + L_{ad}^- z_d + L_{ab}^- z_b + E_a^- w \tag{78}$$

$$\dot{x}_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 z_0 + L_{ad}^0 z_d + L_{ab}^0 z_b + E_a^0 w \tag{79}$$

$$\dot{x}_{ab}^+ = A_{ab}^{+c} x_{ab}^+ - B_{0ab}^+ F_{a0}^0(\varepsilon)\left[x_a^0 + T_a^0 x_{ab}^+\right] + L_{abd}^+\left[F_{ad}^+, F_{bd}\right] x_{ab}^+ + L_{abd}^+ z_d + E_{ab}^+ w \tag{80}$$

$$z_b = \left[0_{m_b \times n_a^+}, C_b\right] x_{ab}^+ \tag{81}$$

$$\dot{x}_c = A_{cc}^c + L_{c0} z_0 + L_{cb} z_b + L_{cd} z_d + E_c w \tag{82}$$

$$z_0 = -\left[F_{a0}^+, F_{b0}\right] x_{ab}^+ - F_{a0}^0(\varepsilon)(x_a^0 + T_a^0 x_{ab}^+) \tag{83}$$

$$\begin{aligned} \dot{x}_i = {} & A_{q_i} x_i + L_{i0} z_0 + L_{id} z_d - \frac{1}{\varepsilon^{q_i}} B_{q_i}\Big[F_{adi}^+ F_{iq_i} x_a^+ + F_{bdi} F_{iq_i} x_b \\ & + F_{adi}^0(\varepsilon) F_{iq_i}\left[x_a^0 + T_a^0 x_{ab}^+\right] + F_i S_i(\varepsilon) x_i\Big] + E_i w, \end{aligned} \tag{84}$$

$$z_i = C_{q_i} x_i, \qquad i = 1, 2, \ldots, m_d \tag{85}$$

where $x_{ab}^+ = \left[(x_a^+)', x_b'\right]'$ and B_{0ab}^+ and L_{abd}^+ are as defined in Step S.4.1 of the state feedback design algorithm. We have also used Condition 2 of the theorem, i.e. $D_{22}=0$, and E_a^-, E_a^0, E_{ab}^+, E_b, E_c and E_i, $i=1,2,\ldots,m_d$, are defined as follows,

$$\Gamma_{sP}^{-1}E=\left[(E_a^-)' \quad (E_a^0)' \quad (E_{ab}^+)' \quad E_c' \quad E_1' \quad E_2' \quad \cdots \quad E_{m_d}'\right]' \tag{86}$$

Condition 3 of the theorem then implies that

$$E_{ab}^+=0, \tag{87}$$

and

$$\mathrm{Im}\,(E_a^0)\subset \mathcal{S}\,(A_{aa}^0):=\cap_{w\in\lambda(A_{aa}^0)}\mathrm{Im}\{wI-A_{aa}^0\} \tag{88}$$

To complete the proof, we will make two state transformations on the closed-loop system in equations (78)–(85). The first state transformation is given as follows:

$$\bar{x}_{ab}=\Gamma_{ab}^{-1}x_{ab}, \qquad \bar{x}_c=x_c, \tag{89}$$

$$\bar{x}_{i1}=x_{i1}+F_{adi}^+x_a^+ + F_{bdi}x_b+F_{adi}^0(\varepsilon)\left[x_a^0+T_a^0x_{ab}^+\right], \qquad i=1,2,\ldots,m_d, \tag{90}$$

$$\bar{x}_{ij}=x_{ij}, \qquad j=2,3,\ldots,q_i, \qquad i=1,2,\ldots,m_d \tag{91}$$

where $x_{ab}=\left[(x_a^-)',(x_a^0)',(x_{ab}^+)'\right]'$ and $\bar{x}_{ab}=\left[(\bar{x}_a^-)',(\bar{x}_{ab}^+)',(\bar{x}_a^0)'\right]'$. In the new state variables in equations (89)–(91), the closed-loop system becomes

$$\dot{\bar{x}}_a^- = A_{aa}^-\bar{x}_a^- + A_{aab}^-\bar{x}_{ab}^+ - \left[B_{0a}^-,\, L_{ad}^-\right]F_a^0(\varepsilon)\bar{x}_a^0 + L_{ad}^-\bar{z}_d + E_a^-w, \tag{92}$$

$$\dot{\bar{x}}_{ab}^+ = A_{ab}^{+c}\bar{x}_{ab}^+ - \left[B_{0ab}^+,\, L_{abd}^+\right]F_a^0(\varepsilon)\bar{x}_a^0 + L_{abd}^+\bar{z}_d \tag{93}$$

$$\dot{\bar{x}}_a^0 = (A_{aa}^0 - B_a^0F_a^0(\varepsilon))\bar{x}_a^0 + (L_{ad}^0 + T_a^0L_{abd}^+)\bar{z}_d + E_a^0w \tag{94}$$

$$\dot{\bar{x}}_c = A_{cc}^c\bar{x}_c + \left(L_{cb}\left[0,\, C_b\right] - \left[L_{c0},\, L_{cd}\right]F_{ab}^+\right)\bar{x}_{ab}^+ - \left[L_{c0},\, L_{cd}\right]F_a^0(\varepsilon)\bar{x}_a^0 + L_{cd}\bar{z}_d + E_cw \tag{95}$$

$$z_0 = -\left[F_{a0}^+,\, F_{b0}\right]\bar{x}_{ab}^+ - F_{a0}^0(\varepsilon)\bar{x}_a^0, \tag{96}$$

$$\begin{aligned}\dot{\bar{x}}_i = {} & A_{q_i}\bar{x}_i - \frac{1}{\varepsilon^{q_i}}B_{q_i}F_iS_i(\varepsilon)\bar{x}_i + L_{iab}^+(\varepsilon)\bar{x}_{ab}^+ + L_{ia}^{01}(\varepsilon)F_a^0(\varepsilon)\bar{x}_a^0 \\ & + L_{ia}^{02}(\varepsilon)F_a^0(\varepsilon)A_{aa}^0\bar{x}_a^0 + \bar{L}_{id}(\varepsilon)\bar{z}_d + \bar{E}_i(\varepsilon)w\end{aligned} \tag{97}$$

$$\bar{z}_i = z_i + \left[F_{adi}^+, F_{bdi}\right]\bar{x}_{ab}^+ + F_{adi}^0\bar{x}_a^0 = C_{q_i}\bar{x}_i, \;\; i=1,2,\ldots,m_d, \tag{98}$$

$$\bar{z}_d = \left[\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_{m_d}\right]', \tag{99}$$

where A_{aab}^-, A_{aab}^0, B_a^0 and L_{abd}^+ are as defined in Step S.4.1 of the state feedback control law design algorithm, and $L_{iab}^+(\varepsilon)$, $L_{ia}^{01}(\varepsilon)$, $L_{ia}^{02}(\varepsilon)$, $\bar{L}_{id}(\varepsilon)$ and $\bar{E}_i(\varepsilon)$ are defined in an obvious way and, by equation (48), satisfy

$$\left\|L_{iab}^+(\varepsilon)\right\|\le l_{iab}^+, \qquad \left\|L_{ia}^{01}(\varepsilon)\right\|\le l_{ia}^{01}, \qquad \left\|L_{ia}^{02}(\varepsilon)\right\|\le l_{ia}^{02}, \qquad \left\|\bar{L}_{id}(\varepsilon)\right\|\le \bar{l}_{id},$$
$$\left\|\bar{E}_i(\varepsilon)\right\|\le \bar{e}_i, \;\; \varepsilon\in(0,1], \tag{100}$$

for some non-negative constants l_{iab}^+, l_{ia}^{01}, l_{ia}^{02}, $\bar{l}_{id}$ and $\bar{e}_i$ independent of ε.

We now proceed to construct the second transformation. We need to recall the following preliminary results from Lin *et al.* (1997).

Lemma 1: *Let the triple $(A_i, B_i, F_i(\varepsilon))$ be as given in Steps S.4.2 and S.4.3 of the state feedback design algorithm. Then, there exists a nonsingular state transformation matrix $Q_i(\varepsilon) \in \mathbb{R}^{n_i \times n_i}$ such that:*

(1) *$Q_i(\varepsilon)$ transforms $A_i - B_i F_i(\varepsilon)$ into a real Jordan form, i.e.*

$$Q_i^{-1}(\varepsilon)(A_i - B_i F_i(\varepsilon))Q_i(\varepsilon) = J_i(\varepsilon) = \text{blkdiag}\{J_{i0}(\varepsilon), J_{i1}(\varepsilon), J_{i2}(\varepsilon), \dots, J_{ip_i}(\varepsilon)\}, \tag{101}$$

where

$$J_{i0}(\varepsilon) = \begin{bmatrix} -\varepsilon & 1 & & \\ & \ddots & \ddots & \\ & & -\varepsilon & 1 \\ & & & -\varepsilon \end{bmatrix}_{r_{i0} \times r_{i0}} \tag{102}$$

and for each $j = 1$ to p_i,

$$J_{ij}(\varepsilon) = \begin{bmatrix} J_{ij}^{\star}(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & J_{ij}^{\star}(\varepsilon) & I_2 \\ & & & J_{ij}^{\star}(\varepsilon) \end{bmatrix}_{2r_{ij} \times 2r_{ij}}, \qquad J_{ij}^{\star}(\varepsilon) = \begin{bmatrix} -\varepsilon & \beta_{ij} \\ -\beta_{ij} & -\varepsilon \end{bmatrix}, \tag{103}$$

with $\beta_{ij} > 0$ for all $j = 1$ to p_i and $\beta_{ij} \neq \beta_{ik}$ for $j \neq k$;

(2) *both $\|Q_i(\varepsilon)\|$ and $\|Q_i^{-1}(\varepsilon)\|$ are bounded, i.e.*

$$\|Q_i(\varepsilon)\| \le \theta_i, \qquad \|Q_i^{-1}(\varepsilon)\| \le \theta_i, \qquad \varepsilon \in (0, 1] \tag{104}$$

for some positive constant θ_i, independent of ε;

(3) *if $E_i \in \mathbb{R}^{n_i \times q}$ is such that*

$$\text{Im}(E_i) \subset \cap_{w \in \lambda(A_i)} \text{Im}(wI - A_i), \tag{105}$$

then there exists a $\delta_i \ge 0$, independent of ε, such that

$$\|Q_i^{-1}(\varepsilon)E_i\| \le \delta_i, \qquad \varepsilon \in (0, 1] \tag{106}$$

and, if we partition $Q_i^{-1}(\varepsilon)E_i$ according to that of $J_i(\varepsilon)$ as,

$$Q_i^{-1}(\varepsilon)E_i = \begin{bmatrix} E_{i0}(\varepsilon) \\ E_{i1}(\varepsilon) \\ \vdots \\ E_{ip_i}(\varepsilon) \end{bmatrix}, \qquad E_{i0}(\varepsilon) = \begin{bmatrix} E_{i01}(\varepsilon) \\ E_{i02}(\varepsilon) \\ \vdots \\ E_{i0r_{i0}}(\varepsilon) \end{bmatrix}_{r_{i0} \times 1},$$

$$E_{ij}(\varepsilon) = \begin{bmatrix} E_{ij1}(\varepsilon) \\ E_{ij2}(\varepsilon) \\ \vdots \\ E_{ijr_{ij}}(\varepsilon) \end{bmatrix}_{2r_{ij}\times 1}, \tag{107}$$

then there exists a $\beta_i \geq 0$, *independent of* ε, *such that for each* $j = 0$, *to* p_i,

$$\left\| E_{ijr_{ij}}(\varepsilon) \right\| \leq \beta_i \varepsilon \tag{108}$$

(4) *if we define a scaling matrix* $S_{ai}(\varepsilon)$ *as*

$$S_{ai}(\varepsilon) = \text{blkdiag}\left\{ S_{ai0}(\varepsilon), S_{ai1}(\varepsilon), S_{ai2}(\varepsilon), \ldots, S_{aip_i}(\varepsilon) \right\} \tag{109}$$

where

$$S_{ai0}(\varepsilon) = \text{diag}\left\{ \varepsilon^{r_{i0}-1}, \varepsilon^{r_{i0}-2}, \ldots, \varepsilon, 1 \right\} \tag{110}$$

and for $j = 1$ *to* p_i,

$$S_{aij}(\varepsilon) = \text{blkdiag}\left\{ \varepsilon^{r_{ij}-1} I_2, \varepsilon^{r_{ij}-2} I_2, \ldots, \varepsilon I_2, I_2 \right\} \tag{111}$$

then there exists a $\kappa_i \geq 0$ *independent of* ε *such that*

$$\left| F_i(\varepsilon) Q_i(\varepsilon) S_{ai}^{-1}(\varepsilon) \right| \leq \kappa_i \varepsilon, \qquad \left| F_i(\varepsilon) A_i Q_i(\varepsilon) S_{ai}^{-1}(\varepsilon) \right| \leq \kappa_i \varepsilon \tag{112}$$

Proof: This is a combination of Lemmas 1, 3 and 4 of Lin *et al.* (1997), and (2.2.13) of Lin (1994). □

Lemma 2: *Let*

$$\tilde{J}_i(\varepsilon) = \text{blkdiag}\{\tilde{J}_{i0}, \tilde{J}_{i1}(\varepsilon), \ldots, \tilde{J}_{ip_i}(\varepsilon)\} \tag{113}$$

where

$$\tilde{J}_{i0} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}_{r_{i0}\times r_{i0}} \tag{114}$$

and for each $j = 1$ *to* p_i,

$$\tilde{J}_{ij}(\varepsilon) = \begin{bmatrix} \tilde{J}_{ij}^{\star}(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & \tilde{J}_{ij}^{\star}(\varepsilon) & I_2 \\ & & & \tilde{J}_{ij}^{\star}(\varepsilon) \end{bmatrix}_{2r_{ij}\times 2r_{ij}}, \qquad \tilde{J}_{ij}^{\star}(\varepsilon) = \begin{bmatrix} -1 & \beta_{ij}/\varepsilon \\ -\beta_{ij}/\varepsilon & -1 \end{bmatrix} \tag{115}$$

with $\beta_{ij} > 0$ *for all* $j = 1$ *to* p_i *and* $\beta_j \neq \beta_k$ *for* $j \neq k$. *Then the unique positive definite solution* $\tilde{P}_i$ *to the Lyapunov equation*

$$\tilde{J}_i(\varepsilon)'\tilde{P}_i + \tilde{P}_i\tilde{J}_i(\varepsilon) = -I \tag{116}$$

is independent of ε.

Proof: This is Lemma 2 in Lin *et al.* (1997). □

We now define the following second state transformation on the closed-loop system:

$$\tilde{x}_a^- = \bar{x}_a^-, \qquad \tilde{x}_{ab}^+ = \bar{x}_{ab}^+ \tag{117}$$

$$\tilde{x}_a^0 = \left[(\tilde{x}_{a1}^0)', (\tilde{x}_{a2}^0)', \cdots (\tilde{x}_{al}^0)'\right]' = S_a(\varepsilon)Q^{-1}(\varepsilon)(\Gamma_{sa}^0)^{-1}\bar{x}_a^0 \tag{118}$$

$$S_a(\varepsilon) = \text{blkdiag}\{S_{a1}(\varepsilon), S_{a2}(\varepsilon), \ldots, S_{al}(\varepsilon)\}$$

$$Q(\varepsilon) = \text{blkdiag}\{Q_1(\varepsilon), Q_2(\varepsilon), \ldots, Q_l(\varepsilon)\}$$

$$\tilde{x}_c = \varepsilon\bar{x}_c \tag{119}$$

$$\tilde{x}_d = \left[\tilde{x}_1', \tilde{x}_2', \ldots, \tilde{x}_{m_d}'\right]', \qquad \tilde{x}_i = S_i(\varepsilon)\bar{x}_i, \qquad i = 1, 2, \ldots, m_d \tag{120}$$

under which the closed-loop system becomes

$$\dot{\tilde{x}}_a^- = A_{aa}^-\tilde{x}_a^- + A_{aab}^-(\varepsilon)\tilde{x}_{ab}^+ + A_{aa}^{-0}(\varepsilon)\tilde{x}_a^0 + L_{ad}^-\tilde{z}_d + E_a^- w \tag{121}$$

$$\dot{\tilde{x}}_{ab}^+ = A_{ab}^{+c}\tilde{x}_{ab}^+ + A_{aba}^{+0}(\varepsilon)\tilde{x}_a^0 + L_{abd}^+\tilde{z}_d \tag{122}$$

$$\dot{\tilde{x}}_a^0 = \tilde{J}(\varepsilon)\tilde{x}_a^0 + \tilde{B}(\varepsilon)\tilde{x}_a^0 + \tilde{L}_{ad}^0(\varepsilon)\tilde{z}_d + \tilde{E}_a^0(\varepsilon)w \tag{123}$$

$$\dot{\tilde{x}}_c = A_{cc}^c\tilde{x}_c + \varepsilon\left[A_{cab}^+\tilde{x}_{ab}^+ + A_{ca}^0(\varepsilon)\tilde{x}_a^0 + L_{cd}\tilde{z}_d + E_c w\right] \tag{124}$$

$$z_0 = -\left[F_{a0}^+, F_{b0}\right]\tilde{x}_{ab}^+ - \tilde{F}_{a0}^0(\varepsilon)\tilde{x}_a^0 \tag{125}$$

$$\varepsilon\dot{\tilde{x}}_i = (A_{q_i} - B_{q_i}F_i)\tilde{x}_i + \varepsilon\tilde{L}_{iab}^+(\varepsilon)\tilde{x}_{ab}^+ + \varepsilon\tilde{L}_{ia}^0(\varepsilon)\tilde{x}_a^0 + \varepsilon\tilde{L}_{id}(\varepsilon)\tilde{z}_d + \varepsilon\tilde{E}_i(\varepsilon)w \tag{126}$$

$$\tilde{z}_i = \bar{z}_i = z_i + \left[F_{adi}^+, F_{bdi}\right]\tilde{x}_{ab}^+ + \tilde{F}_{adi}^0(\varepsilon)\tilde{x}_a^0 = C_{q_i}\tilde{x}_i \tag{127}$$

$$\tilde{z}_d = \left[\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{m_d}\right]', \tag{128}$$

where

$$A_{aa}^{-0}(\varepsilon) = -\left[B_{0a}^+, L_{ad}^-\right]F_a^0(\varepsilon)\Gamma_{sa}^0 Q(\varepsilon)S_a^{-1}(\varepsilon) \tag{129}$$

$$A_{aba}^{+0}(\varepsilon) = -\left[B_{0ab}^+, L_{abd}^+\right]F_a^0(\varepsilon)\Gamma_{sa}^0 Q(\varepsilon)S_a^{-1}(\varepsilon) \tag{130}$$

$$\tilde{J}(\varepsilon) = \text{blkdiag}\left\{\varepsilon\tilde{J}_1(\varepsilon), \varepsilon\tilde{J}_2(\varepsilon), \ldots, \varepsilon\tilde{J}_l(\varepsilon)\right\} \tag{131}$$

$$\tilde{B}(\varepsilon) = \begin{bmatrix} 0 & \tilde{B}_{12}(\varepsilon) & \tilde{B}_{13}(\varepsilon) & \ldots & \tilde{B}_{1l}(\varepsilon) \\ 0 & 0 & \tilde{B}_{23}(\varepsilon) & \ldots & \tilde{B}_{2l}(\varepsilon) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{132}$$

$$\tilde{b}_{jk}(\varepsilon) = S_{aj}(\varepsilon)Q_j^{-1}(\varepsilon)B_{jk}F_k(\varepsilon)Q_k(\varepsilon)S_{ak}^{-1}(\varepsilon)$$

$$j = 1, 2, \ldots, l, \qquad k = j+1, j+2, \ldots, l$$

$$\tilde{L}^0_{ad}(\varepsilon) = S_a(\varepsilon)Q^{-1}(\varepsilon)(\Gamma^0_{sa})^{-1}(L^0_{ad} + T^0_a L^+_{abd}) \tag{133}$$

$$\tilde{E}^0_a(\varepsilon) = S_a(\varepsilon)Q^{-1}(\varepsilon)(\Gamma^0_{sa})^{-1}E^0_a, \qquad \tilde{E}^0_a(\varepsilon) = \left[(\tilde{E}^0_{a1}(\varepsilon))' \quad (\tilde{E}^0_{a2}(\varepsilon))' \quad \dots \quad (\tilde{E}^0_{al}(\varepsilon))'\right]' \tag{134}$$

$$A^+_{cab} = L_{cb}\left[0, C_b\right] - \left[L_{c0}, L_{cd}\right]F^+_{ab} \tag{135}$$

$$A^0_{ca}(\varepsilon) = -\left[L_{c0}, L_{cd}\right]F^0_a(\varepsilon)\Gamma^0_{sa}Q(\varepsilon)S^{-1}_a(\varepsilon) \tag{136}$$

$$\tilde{F}^0_{a0}(\varepsilon) = F^0_{a0}(\varepsilon)S^{-1}_a(\varepsilon)Q(\varepsilon)\Gamma^0_{sa} \tag{137}$$

$$\tilde{L}^+_{iab}(\varepsilon) = S_i(\varepsilon)L^+_{iab}(\varepsilon) \tag{138}$$

$$\tilde{L}^0_{ia}(\varepsilon) = S_i(\varepsilon)\left[L^{01}_{ia}(\varepsilon)F^0_a(\varepsilon) + L^{02}_{ia}(\varepsilon)F^0_a(\varepsilon)A^0_{aa}\right]\Gamma^0_{sa}Q(\varepsilon)S^{-1}_a(\varepsilon) \tag{139}$$

$$\tilde{L}_{id}(\varepsilon) = S_i(\varepsilon)\bar{L}_{id}(\varepsilon) \tag{140}$$

$$\tilde{E}_i(\varepsilon) = S_i(\varepsilon)\bar{E}_i(\varepsilon) \tag{141}$$

$$\tilde{F}^0_{adi}(\varepsilon) = F^0_{adi}(\varepsilon)\Gamma^0_{sa}Q(\varepsilon)S^{-1}_a(\varepsilon) \tag{142}$$

and where, for $i = 1$ to l, $\tilde{J}_i(\varepsilon)$ is as defined in Lemma 2.

By equations (48) and (100), and Lemma 1, we have that for all $\varepsilon \in (0, 1]$,

$$\|A^-_{aab}(\varepsilon)\| \le a^-_{aab}, \qquad \|\tilde{L}^0_{ad}(\varepsilon)\| \le \tilde{l}^0_{ad}, \qquad \|A^+_{cab}\| \le a^+_{cab} \tag{143}$$

$$\|A^{-0}_{aa}(\varepsilon)\| \le a^{-0}_{aa}\varepsilon, \qquad \|A^{+0}_{aba}(\varepsilon)\| \le a^{+0}_{aa}\varepsilon, \qquad \|A^0_{ca}(\varepsilon)\| \le a^0_{ca}\varepsilon, \qquad \|\tilde{F}^0_{a0}(\varepsilon)\| \le \tilde{f}^0_{a0}\varepsilon \tag{144}$$

for $i = 1$ to m_d,

$$\|\tilde{L}^+_{iab}(\varepsilon)\} \le \tilde{l}^+_{ab}, \qquad \|\tilde{L}^0_{ia}(\varepsilon)\| \le \tilde{l}^0_a(\varepsilon), \qquad \|\tilde{L}^0_{id}(\varepsilon)\| \le \tilde{l}_d, \qquad \|\tilde{F}^0_{adi}(\varepsilon)\| \le \tilde{f}^0_{ad}\varepsilon,$$
$$\|\tilde{E}_i(\varepsilon)\| \le \tilde{e} \tag{145}$$

for $i = 1$ to l,

$$\|\tilde{E}^0_{ai}(\varepsilon)\| \le \tilde{e}^0_a\varepsilon \tag{146}$$

and finally, for $j = 1$ to l, $k = j + 1$ to l,

$$\|\tilde{B}_{jk}(\varepsilon)\| \le \tilde{b}_{jk}\varepsilon \tag{147}$$

where a^-_{aab}, $\tilde{l}^0_{ad}$, a^+_{cab}, a^{-0}_{aa}, a^{+0}_{aa}, $\tilde{e}^0_a$, a^0_{ca}, $\tilde{f}^0_{a0}$, $\tilde{l}^+_{ab}$, $\tilde{l}^0_a$, $\tilde{l}_d$, $\tilde{f}^0_{ad}$, $\tilde{b}_{jk}$ and $\tilde{e}$ are some positive constants, independent of ε.

We next construct a Lyapunov function for the closed loop system in equations (121)–(128). We do this by composing Lyapunov functions for the subsystems. For the subsystem of $\tilde{x}^-_a$, we choose a Lyapunov function,

$$V^-_a(\tilde{x}^-_a) = (\tilde{x}^-_a)'P^-_a\tilde{x}^-_a \tag{148}$$

where $P^-_a > 0$ is the unique solution to the Lyapunov equation

$$(A^-_{aa})'P^-_a + P^-_a A^-_{aa} = -I \tag{149}$$

and for the subsystem of $\tilde{x}^+_{ab}$, we choose a Lyapunov function

$$V^+_{ab}(\tilde{x}^+_{ab}) = (\tilde{x}^+_{ab})'P^+_{ab}\tilde{x}^+_{ab} \tag{150}$$

where $P_{ab}^{+} > 0$ is the unique solution to the Lyapunov equation

$$(A_{ab}^{+c})' P_{ab}^{+} + P_{ab}^{+} A_{ab}^{+c} = - I \tag{151}$$

The existence of such P_{aa}^{-} and P_{ab}^{+} is guaranteed by the fact that both A_{aa}^{-} and A_{ab}^{+c} are asymptotically stable. For the subsystem of $\tilde{x}_a^0 = \left[(\tilde{x}_{a1}^0)', (\tilde{x}_{a2}^0)', \ldots, (\tilde{x}_{al}^0)'\right]'$, we choose a Lyapunov function

$$V_a^0(\tilde{x}_a^0) = \sum_{i=1}^{l} \frac{(\alpha_a^0)^{i-1}}{\varepsilon} (\tilde{x}_{ai}^0)' P_{ai}^0 \tilde{x}_{ai}^0 \tag{152}$$

where α_a^0 is a positive scalar, whose value is to be determined later, and each P_{ai}^0 is the unique solution to the Lyapunov equation

$$\tilde{J}_i(\varepsilon)' P_{ai}^0 + P_{ai}^0 \tilde{J}_i(\varepsilon) = - I \tag{153}$$

which, by Lemma 2, is independent of ε. Similarly, for the subsystem $\tilde{x}_c$, choose a Lyapunov function

$$V_c(\tilde{x}_c) = \tilde{x}_c' P_c \tilde{x}_c \tag{154}$$

where $P_c > 0$ is the unique solution to the Lyapunov equation

$$(A_{cc}^c)' P_c + P_c A_{cc}^c = - I \tag{155}$$

The existence of such a P_c is again guaranteed by the fact that A_{cc}^c is asymptotically stable. Finally, for the subsystem of $\tilde{x}_d$, choose a Lyapunov function

$$V_d(\tilde{x}_d) = \sum_{i=1}^{m_d} \tilde{x}_i' P_i \tilde{x}_c \tag{156}$$

where each P_i is the unique solution to the Lyapunov equation

$$(A_{q_i} - B_{q_i} F_i)' P_i + P_i (A_{q_i} - B_{q_i} F_i) = - I \tag{157}$$

Once again, the existence of such P_i is due to the fact that $A_{q_i} - B_{q_i} F_i$ is asymptotically stable.

We now construct a Lyapunov function for the closed-loop system in equations (121)–(128) as follows:

$$V(\tilde{x}_a^-, \tilde{x}_{ab}^+, \tilde{x}_a^0, \tilde{x}_c, \tilde{x}_d) = V_a^-(\tilde{x}_a^-) + \alpha_{ab}^+ V_{ab}^+(\tilde{x}_{ab}^+) + V_a^0(\tilde{x}_a^0) + V_c(\tilde{x}_c) + \alpha_d V_d(\tilde{x}_d) \tag{158}$$

where $\alpha_{ab}^+ = 2\|P_a^-\|^2 (a_{aab}^-)^2$ and the value of α_d is to be determined.

Let us first consider the derivative of $V_a^0(\tilde{x}_a^0)$ along the trajectories of the subsystem $\tilde{x}_a^0$, and obtain that

$$\begin{aligned}\dot{V}_a^0(\tilde{x}_a^0) = &\sum_{i=1}^{l}\left[- (\alpha_a^0)^{i-1} (\tilde{x}_{ai}^0)' \tilde{x}_{ai}^0 + 2 \sum_{j=i+1}^{l} \frac{(\alpha_a^0)^{i-1}}{\varepsilon} (\tilde{x}_{ai}^0)' P_{ai}^0 \tilde{B}_{ij}(\varepsilon) \tilde{x}_j^0\right] \\ &+ 2 \sum_{i=1}^{l} \frac{(\alpha_a^0)^{i-1}}{\varepsilon} \left[(\tilde{x}_{ai}^0)' P_{ai}^0 \tilde{L}_{ad}^0(\varepsilon) \tilde{z}_d + (\tilde{x}_{ai}^0)' P_{ai}^0 \tilde{E}_a^0(\varepsilon) w\right]\end{aligned} \tag{159}$$

Using equation (147) it is straightforward to show that there exists an $\alpha_a^0 > 0$ such that

$$\dot{V}_a^0(\tilde{x}_a^0) \le -\tfrac{3}{4}\|\tilde{x}_a^0\|^2 + \frac{\alpha_1}{\varepsilon}\|\tilde{x}_a^0\|\cdot\|\tilde{z}_d\| + \alpha_2\|w\|^2 \tag{160}$$

for some non-negative constants α_1 and α_2, independent of ε.

In view of equation (160), the derivative of V along the trajectory of the closed-loop system in equations (121)–(128) can be evaluated as follows:

$$\begin{aligned}\dot{V} = &-(\tilde{x}_a^-)'\tilde{x}_a^- + 2(\tilde{x}_a^-)'P_a^- A_{aab}^-(\varepsilon)\tilde{x}_{ab}^+ + 2(\tilde{x}_a^-)'P_a^- A_{aa}^{-0}(\varepsilon)\tilde{x}_a^0 + 2(\tilde{x}_a^-)'P_a^- L_{ad}^-\tilde{z}_d \\ &+ 2(\tilde{x}_a^0)'P_a^- E_a^- w - \alpha_{ab}^+(\tilde{x}_{ab}^+)'\tilde{x}_{ab}^+ + 2\alpha_{ab}^+(x_{ab}^+)'P_{ab}^+A_{aba}^{+0}(\varepsilon)\tilde{x}_a^0 + 2\alpha_{ab}^+(x_{ab}^+)'P_{ab}^+L_{abd}^+\tilde{z}_d \\ &-\tfrac{3}{4}\|\tilde{x}_a^0\|^2 + \frac{\alpha_1}{\varepsilon}\|\tilde{x}_a^0\|\cdot\|\tilde{z}_d\| + \alpha_2\|w\|^2 - \tilde{x}_c'\tilde{x}_c + 2\varepsilon\tilde{x}_c'P_c\Big[A_{cab}^+\tilde{x}_{ab}^+ + A_{ca}^0(\varepsilon)\tilde{x}_a^0 \\ &+ L_{cd}\tilde{z}_d + E_c w\Big] + \alpha_d\sum_{i=1}^{m_d}\Big[-\frac{1}{\varepsilon}\tilde{x}_i'\tilde{x}_i + 2\tilde{x}_i'P_i\tilde{L}_{iab}^+(\varepsilon)\tilde{x}_{ab}^+ + 2\tilde{x}_i'P_i\tilde{L}_{ia}^0(\varepsilon)\tilde{x}_a^0 \\ &+ 2\tilde{x}_i'P_i\tilde{L}_{id}(\varepsilon)\tilde{z}_d + 2\tilde{x}_i'P_i\tilde{E}_i(\varepsilon)w\Big]\end{aligned} \tag{161}$$

Using the majorizations in equations (143)–(146) and noting the definition of α_{ab}^+ in equation (158), we can easily verify that there exists an $\alpha_d > 0$ and an $\varepsilon_1^* \in (0,1]$ such that, for all $\varepsilon \in (0,\varepsilon_1^*]$,

$$\dot{V} \le -\tfrac{1}{2}\|\tilde{x}_a^-\|^2 - \tfrac{1}{2}\|\tilde{x}_{ab}^+\|^2 - \tfrac{1}{2}\|\tilde{x}_a^0\|^2 - \frac{1}{2\varepsilon}\|\tilde{x}_d\|^2 + \alpha_3\|w\|^2 \tag{162}$$

for some positive constant α_3, independent of ε.

From equation (162), it follows that the closed-loop system in the absence of disturbance w is asymptotically stable. It remains to show that for any given $\gamma > 0$, there exists an $\varepsilon^* \in (0,\varepsilon_1^*]$ such that, for all $\varepsilon \in (0,\varepsilon^*]$,

$$\|z\|_{L_2} \le \gamma\|w\|_{L_2} \tag{163}$$

To this end, we integrate both sides of equation (162) from 0 to ∞. Noting that $V \ge 0$ and $V(t) = 0$ at $t = 0$, we have

$$\|\tilde{z}_d\|_{L_2} \le (\sqrt{2\alpha_3\varepsilon})\|w\|_{L_2} \tag{164}$$

which, when used in equation (160), results in

$$\|\tilde{x}_a^0\|_{L_2} \le \left(\sqrt{\frac{2\alpha_1^2\alpha_3}{\varepsilon} + \alpha_2}\right)\|w\|_{L_2} \tag{165}$$

Viewing $\tilde{z}_d$ as disturbance to the dynamics to $\tilde{x}_{ab}^+$ also results in

$$\|\tilde{x}_{ab}^+\|_{L_2} \le (\alpha_4\sqrt{\varepsilon})\|w\|_{L_2} \tag{166}$$

for some positive constant α_4, independent of ε.

Finally, recalling that

$$z = \Gamma_{oP}\begin{bmatrix} z_0 \\ \tilde{z}_d - F_{ab}^+\tilde{x}_{ab}^+ - \tilde{F}_{ad}^0(\varepsilon)\tilde{x}_a^0 \\ z_b \end{bmatrix} \tag{167}$$

where

$$\tilde{F}^0_{ad}(\varepsilon) = \begin{bmatrix} \tilde{F}^0_{ad1}(\varepsilon) \\ \tilde{F}^0_{ad2}(\varepsilon) \\ \vdots \\ F^0_{adm_d}(\varepsilon) \end{bmatrix} \tag{168}$$

with each $\tilde{F}_{adi}(\varepsilon)$ satisfying equation (145), we have

$$\|z\|_{L_2} \le \|\Gamma_{o\mathrm{P}}\|(\sqrt{2\alpha_3\varepsilon} + \alpha_4\|F^+_{ab}\|\sqrt{\varepsilon} + \alpha_5\sqrt{2\alpha_1^2\alpha_3\varepsilon + \alpha_2\varepsilon^2})\|w\|_{L_2} \tag{169}$$

for some positive constant α_5 independent of ε.

To complete the proof, we choose $\varepsilon^* \in (0, \varepsilon_1^*]$ such that

$$\|\Gamma_{o\mathrm{P}}\|(\sqrt{2\alpha_3\varepsilon} + \alpha_4\|F^+_{ab}\|\sqrt{\varepsilon} + \alpha_5\sqrt{2\alpha_1^2\alpha_3\varepsilon + \alpha_2\varepsilon^2}) \le \gamma \tag{170}$$

For the use in the proof of measurement feedback results, it is straightforward to verify from the closed-loop equations (121)–(128) that the transfer function from $E^0_a w$ to z is given by

$$T^0_{ao}(s) = T_{ao}(s, \varepsilon)\left[sI - A^0_{aa} + B^0_a F^0_a(\varepsilon)\right]^{-1} \tag{171}$$

where $T_{ao}(s, \varepsilon) \to 0$ pointwise in s as $\varepsilon \to 0$. □

Appendix B. Proof of Theorem 4

It is trivial to show the stability of the closed-loop system comprising the plant given in equation (1) and the full-order output feedback controller in equation (67). The closed-loop poles are given by $\lambda\{A + BF_{\mathrm{P}}(\varepsilon)\}$, which are in $\mathbb{C}^-$ for sufficiently small ε, as shown in Theorem 3, and $\lambda\{A + K_{\mathrm{Q}}(\varepsilon)C_1\}$, which can be dually shown to be in $\mathbb{C}^-$ for sufficiently small ε as well. In what follows, we will show that the full-order output feedback controller achieves the H_∞-ADDPMS for equation (1), which satisfies all five conditions of Theorem 2. Without loss of any generality but for simplicity of presentation, hereafter we assume throughout the rest of the proof that the subsystem Σ_{P}, i.e. the quadruple (A, B, C_2, D_2), has already been transformed into the special coordinate basis as given in Theorem 1. To be more specific, we have

$$A = B_0C_{2,0} + \begin{bmatrix} A^-_{aa} & 0 & 0 & L^-_{ab}C_b & 0 & L^-_{ad}C_d \\ 0 & A^0_{aa} & 0 & L^0_{ab}C_b & 0 & L^0_{ad}C_d \\ 0 & 0 & A^+_{aa} & L^+_{ab}C_b & 0 & L^+_{ad}C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd}C_d \\ B_cE^-_{ca} & B_cE^0_{ca} & B_cE^+_{ca} & L_{cb}C_b & A_{cc} & L_{cd}C_d \\ B_dE^-_{da} & B_dE^0_{da} & B_dE^+_{da} & B_dE_{bd} & B_dE_{dc} & A_{dd} \end{bmatrix} := B_0C_{2,0} + \tilde{A} \tag{172}$$

$$B = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix} \tag{173}$$

and

$$C_2 = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix},$$

$$C_{2,0} = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \end{bmatrix} \tag{174}$$

$$D_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{S}^+(\Sigma_P) = \mathrm{Im}\left\{ \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \right\} \tag{175}$$

It is simple to note that Condition 3 of Theorem 2 implies that

$$E = \begin{bmatrix} E_a^- \\ E_a^0 \\ 0 \\ 0 \\ E_c \\ E_d \end{bmatrix} \tag{176}$$

Next, for any $\zeta \in \mathcal{V}_{\lambda_0}(\Sigma_Q)$ with $\lambda_0 \in \mathbb{C}^0$, we partition ζ as follows:

$$\zeta = \begin{pmatrix} \zeta_a^- \\ \zeta_a^0 \\ \zeta_a^+ \\ \zeta_b \\ \zeta_c \\ \zeta_d \end{pmatrix} \tag{177}$$

Then, Condition 4 of Theorem 2 implies that $C_2\zeta = 0$, or equivalently

$$C_{2,0}\zeta = 0, \qquad C_b\zeta_b = 0 \qquad \text{and} \qquad C_d\zeta_d = 0 \tag{178}$$

By Definition 3, we have

$$\begin{bmatrix} A-\lambda_0 I & E \\ C_1 & D_1 \end{bmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} = 0 \tag{179}$$

for some appropriate vector η. Clearly, equations (179) and (176) imply that

$$(A-\lambda_0 I)\zeta = -E\eta = \begin{pmatrix} \star \\ \star \\ 0 \\ 0 \\ \star \\ \star \end{pmatrix} \tag{180}$$

where the $\star$ are some vectors of little interest. Note that equation (178) implies

$$(A-\lambda_0 I)\zeta = (B_0 C_{2,0} + \tilde{A} - \lambda_0 I)\zeta = (\tilde{A} - \lambda_0 I)\zeta$$

$$= \begin{bmatrix} \star \\ \star \\ (A_{aa}^{+} - \lambda_0 I)\zeta_a^{+} + L_{ab}^{+} C_b \zeta_b + L_{ad}^{+} C_d \zeta_d \\ (A_{bb} - \lambda_0 I)\zeta_b + L_{bd} C_d \zeta_d \\ \star \\ \star \end{bmatrix} = \begin{bmatrix} \star \\ \star \\ (A_{aa}^{+} - \lambda_0 I)\zeta_a^{+} \\ (A_{bb} - \lambda_0 I)\zeta_b \\ \star \\ \star \end{bmatrix} \tag{181}$$

and equations (180) and (181) imply that

$$(A_{aa}^{+} - \lambda_0 I)\zeta_a^{+} = 0 \qquad \text{and} \qquad (A_{bb} - \lambda_0 I)\zeta_b = 0 \tag{182}$$

Since A_{aa}^{+} has all its eigenvalues in $\mathbb{C}^{+}$, $(A_{aa}^{+} - \lambda_0 I)\zeta_a^{+} = 0$ implies that $\zeta_a^{+} = 0$. Similarly, since (A_{bb}, C_b) is completely observable, $(A_{bb} - \lambda_0 I)\zeta_b = 0$ and $C_b \zeta_b = 0$ imply $\zeta_b = 0$. Thus, ζ has the following property:

$$\zeta = \begin{pmatrix} \zeta_a^{-} \\ \zeta_a^{0} \\ 0 \\ 0 \\ \zeta_c \\ \zeta_d \end{pmatrix} \in \mathcal{S}^{+}(\Sigma_{\mathrm{P}}) \tag{183}$$

Obviously, equation (183) together with Condition 5 of Theorem 2 imply

$$\mathcal{S}^{+}(\Sigma_{\mathrm{P}}) \supset \mathcal{V}^{+}(\Sigma_{\mathrm{Q}}) \cup \left\{ \cup_{\lambda_0 \in \mathbb{C}^0} \mathcal{V}_{\lambda_0}(\Sigma_{\mathrm{Q}}) \right\} \tag{184}$$

Next, it is straightforward to verify that $A - sI$ can be partitioned as

$$A - sI = X_1 + X_2C_2 + X_3 + X_4 \tag{185}$$

where

$$X_1 := \begin{bmatrix} A_{aa}^- - sI & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^0 & L_{cb} C_b & A_{cc} - sI & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} - sI \end{bmatrix}$$

$$X_2 = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & L_{ad}^0 & L_{ab}^0 \\ B_{0a}^+ & L_{ad}^+ & L_{ab}^+ \\ B_{0b} & L_{bd} & 0 \\ B_{0c} & 0 & 0 \\ B_{0d} & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{aa}^+ - sI & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{bb} - sI & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{187}$$

and

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{aa}^0 - sI & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{188}$$

It is simple to see that

$$\operatorname{Im}(X_1) \subset \mathcal{S}^+(\Sigma_{\mathrm{P}}) \cap \left\{ \cap_{\lambda_0 \in \mathbb{C}^0} \mathcal{S}_{\lambda_0}(\Sigma_{\mathrm{P}}) \right\} \tag{189}$$

and

$$\operatorname{Ker}(X_3) \supset \mathcal{S}^+(\Sigma_{\mathrm{P}}) \supset \mathcal{V}^+(\Sigma_{\mathrm{Q}}) \cup \left\{ \cup_{\lambda_0 \in \mathbb{C}^0} \mathcal{V}_{\lambda_0}(\Sigma_{\mathrm{Q}}) \right\} \tag{190}$$

It follows from the proof of Theorem 3 that as $\varepsilon \to 0$

$$\left\| \left[C_2 + D_2 F_{\mathrm{P}}(\varepsilon) \right] \left[sI - A - BF_{\mathrm{P}}(\varepsilon) \right]^{-1} \right\|_\infty < \kappa_{\mathrm{P}} \tag{191}$$

where κ_{P} is a finite positive constant and is independent of ε. Moreover, under Condition 3 of Theorem 2, we have

$$\left[C_2 + D_2 F_{\mathrm{P}}(\varepsilon) \right] \left[sI - A - VF_{\mathrm{P}}(\varepsilon) \right]^{-1} E \to 0 \tag{192}$$

and

$$[C_2 + D_2F_P(\varepsilon)][sI - A - BF_P(\varepsilon)]^{-1}X_1 \to 0 \tag{193}$$

pointwise in s as $\varepsilon \to 0$. By equation (171), we have

$$[C_2 + D_2F_P(\varepsilon)][sI - A - BF_P(\varepsilon)]^{-1}X_4 \to 0 \tag{194}$$

pointwise in s as $\varepsilon \to 0$. Dually, one can show that

$$\left\| [sI - A - K_Q(\varepsilon)C_1]^{-1}[E + K_Q(\varepsilon)D_1] \right\|_\infty < \kappa_Q \tag{195}$$

where κ_Q is a finite positive constant and is independent of ε. If Condition 4 of Theorem 2 is satisfied, the following results hold:

$$C_2[sI - A - K_Q(\varepsilon)C_1]^{-1}[E + K_Q(\varepsilon)D_1] \to 0 \tag{196}$$

and

$$X_3[sI - A - K_Q(\varepsilon)C_1]^{-1}[E + K_Q(\varepsilon)D_1] \to 0 \tag{197}$$

pointwise is s as $\varepsilon \to 0$.

Finally, it is simple to verify that the closed-loop transfer matrix from the disturbance w to the controlled output z under the full-order output feedback controller in equation (67) is given by

$$\begin{aligned} T_{zw}(s, \varepsilon) = {} & [C_2 + D_2F_P(\varepsilon)][sI - A - BF_P(\varepsilon)]^{-1}E \\ & + C_2[sI - A - K_Q(\varepsilon)C_1]^{-1}[E + K_Q(\varepsilon)D_1] + [C_2 + D_2F_P(\varepsilon)] \\ & \times [sI - A - BF_P(\varepsilon)]^{-1}(A - sI)[sI - A - K_Q(\varepsilon)C_1]^{-1}[E + K_Q(\varepsilon)D_1] \end{aligned}$$

Using equation (185), we can re-write $T_{zw}(s, \varepsilon)$ as

$$\begin{aligned} T_{zw}(s, \varepsilon) = {} & [C_2 + D_2F_P(\varepsilon)][sI - A - BF_P(\varepsilon)]^{-1}E \\ & + C_2[sI - A - K_Q(\varepsilon)C_1]^{-1}[E + K_Q(\varepsilon)D_1] \\ & + [C_2 + D_2F_P(\varepsilon)][sI - A - BF_P(\varepsilon)]^{-1}(X_1 + X_2C_2 + X_3 + X_4) \\ & \times [sI - A - K_Q(\varepsilon)C_1]^{-1}[E + K_Q(\varepsilon)D_1] \end{aligned}$$

Following equations (191)–(197), and with some simple manipulations, it is straightforward to show that as $\varepsilon \to 0$, $T_{zw}(s, \varepsilon) \to 0$, pointwise in s, which is equivalent to $\|T_{zw}\|_\infty \to 0$ as $\varepsilon \to 0$. Hence, the full-order output feedback controller in equation (67) solves the H_∞-ADDPMS for the plant given in equation (1), provided that all five conditions of Theorem 2 are satisfied. □

Appendix C. Proof of Theorem 5

Again, it is trivial to show the stability of the closed-loop system comprising the plant given in equation (1) and the reduced-order output feedback controller in equation (76) as the closed-loop poles are given by $\lambda\{A + BF_P(\varepsilon)\}$ and $\lambda\{A_R + K_R(\varepsilon)C_R\}$, which are asymptotically stable for sufficiently small ε. Next, it is easy to compute the closed-loop transfer matrix from the disturbance w to the controlled output z under the reduced-order output feedback controller

$$T_{sw}(s,\varepsilon) = C_2\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}\left[sI - A_R - K_R(\varepsilon)C_R\right]^{-1}\left[E_R + K_R(\varepsilon)D_R\right]$$
$$+\left[C_2 + D_2F_P(\varepsilon)\right]\left[sI - A - BF_P(\varepsilon)\right]^{-1}(A - sI)\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}$$
$$\times\left[sI - A_R - K_R(\varepsilon)C_R\right]^{-1}\left[E_R + K_R(\varepsilon)D_R\right]$$
$$+\left[C_2 + D_2F_P(\varepsilon)\right]\left[sI - A - BF_P(\varepsilon)\right]^{-1}E$$

It was shown in Chen (1991) that

$$\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}\mathcal{V}^+(\Sigma_{QR}) = \mathcal{V}^+(\Sigma_Q) \tag{198}$$

Following the same lines of reasoning as in Chen (1991), one can also show that

$$\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}\cup_{\lambda_0\in\mathbb{C}^0}\mathcal{V}_{\lambda_0}(\Sigma_{QR}) = \cup_{\lambda_0\in\mathbb{C}^0}\mathcal{V}_{\lambda_0}(\Sigma_Q) \tag{199}$$

Hence, we have

$$\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}\left(\mathcal{V}^+(\Sigma_{QR}) \cup \left\{\cup_{\lambda_0\in\mathbb{C}^0}\mathcal{V}_{\lambda_0}(\Sigma_{QR})\right\}\right) = \mathcal{V}^+(\Sigma_Q) \cup \left\{\cup_{\lambda_0\in\mathbb{C}^0}\mathcal{V}_{\lambda_0}(\Sigma_Q)\right\} \tag{200}$$

The rest of the proof follows on the same lines as in Theorem 4. □

References

Chen, B. M., 1988, *Software Manual for the Special Coordinate Basis of Multivariable Linear Systems*. Washington State University Technical Report Number ECE 0094, Pullman, WA.

Chen, B. M., 1991, Theory of Loop Transfer Recovery for Multivariable Linear Systems, PhD dissertation, Washington State University, Pullman, WA, USA.

Chen, B. M., 1997, *Linear Systems and Control Toolbox*, Department of Electrical Engineering, National University of Singapore, Republic of Singapore.

Chen, B. M., Saberi, A., and Ly, U., 1992, Closed loop transfer recovery with observer based controllers—Part 1. Analysis; Part 2. Design. *Control and Dynamic Systems: Advances in Theory and Applications*, **51**, 247–348.

Lin, Z., 1989, *The Implementation of Special Coordinate Basis for Linear Multivariable Systems in Matlab*. Washington State University Technical Report Number ECE 0100, Pullman, WA.

Lin, Z., 1994, Global and Semi-Global Control Problems for Linear Systems Subject to Input Saturation and Minimum-Phase Input-Output Linearizable Systems, PhD dissertation, Washington State University, Pullman, WA, USA.

Lin, Z., 1998, Almost disturbance decoupling with global asymptotic stability for nonlinear systems with disturbance affected unstable zero dynamics. *Systems & Control Letters*, **33**, 163–169.

Lin, Z., Bao, X., and Chen, B. M., 1997, Further results on almost disturbance decoupling with global asymptotic stability for nonlinear systems. *Proceedings of the 36th Conference on Decision and Control*, pp. 2847–2852.

Ozcetin, H. K., Saberi, A., and Sannuti, P., 1993a, Design for H_∞ almost disturbance decoupling problem with internal stability via state or measurement feedback—singular perturbation approach. *International Journal of Control*, **55**, 901–944.

Ozcetin, H. K., Saberi, A., and Shmash, Y., 1993b, H_∞-almost disturbance decoupling for non-strictly proper systems—A singular perturbation approach. *Control-Theory & Advanced Technology*, **9**, 203–245.

Saberi, A., and Sannuti, P., 1989, Time-scale structure assignment in linear multivariable systems using high-gain feedback. *International Journal of Control*, **49**, 2191–2213.

Saberi, A., and Sannuti, P., 1990, Squaring down of non-strictly proper systems. *International Journal of Control*, **51**, 621–629.

Saberi, A., Sannuti, P., and Chen, B. M., 1995, *H_2 Optimal Control* (London: Prentice Hall).

Sannuti, P., and Saberi, A., 1987, A special coordinate basis of multivariable linear systems—finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, **45**, 1655–1704.

Scherer, C., 1992, H_∞ optimization without assumptions on finite or infinite zeros. *SIAM Journal on Control and Optimization*, **30**, 143–166.

Weiland, S., and Willems, J. C., 1989, Almost disturbance decoupling with internal stability. *IEEE Transactions on Automatic Control*, **34**, 277–286.

Willems, J. C., 1981, Almost invariant subspaces: An approach to high gain feedback design—Part I. Almost controlled invariant subspaces. *IEEE Transactions on Automatic Control*, **26**, 235–252.

Willems, J. C., 1982, Almost invariant subspaces: An approach to high gain feedback design—Part II. Almost conditionally invariant subspaces. *IEEE Transactions on Automatic Control*, **27**, 1071–1085.

Wonham, W. M., 1979, *Linear Multivariable Control: A Geometric Approach*, 2nd edn (New York: Springer-Verlag).