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Non-iterative computation of infimum in discrete-time H_∞-optimization and solvability conditions for the discrete-time disturbance decoupling problem

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A non-iterative method for the computation of the infimum for a class of discrete-time H_∞ optimal control problems, and the solvability conditions for the general discrete-time disturbance decoupling problem are given in this paper. The method for the computation of the infimum is applicable to systems where the transfer functions from the disturbance input to the measurement output and from the control input to the controlled output are free of unit circle invariant zeros and satisfy certain geometric conditions. The solvability conditions we obtained for the general discrete-time disturbance decoupling problem are also necessary and sufficient conditions.

1. Introduction and problem statement

A great deal of work has been done on the study of the H_∞ optimal control problem in both continuous-time setting (see for example, Doyle *et al.* 1989, Francis 1987, Glover 1984, Kimura 1989, Khargonekar *et al.* 1988, Stoorvogel 1992), and discrete-time setting (see for example, Basar and Bernard 1989, Stoorvogel 1992, Stoorvogel *et al.* 1994), since the original formulation of the problem by Zames (1981). On the other hand, the disturbance decoupling problem was first introduced by Willems in the early 1970s (see Weiland and Willems 1989, and Stoorvogel and van der Woude 1991, for recent results and related references). Recently, Stoorvogel (1992) has obtained a very interesting interconnection between the H_∞ optimal control problem and the disturbance decoupling problem. By performing certain system transformations, he was able to transform the solution of an H_∞ optimal control problem to the solution of an auxiliary disturbance decoupling problem.

In this paper, we first address the problem of the computation of the infimum in discrete-time H_∞ optimization. The algebraic Riccati equation, or ARE-based approach to this problem (see for example Stoorvogel *et al.* 1994) provides an iterative scheme of approximating the infimum (denoted here by γ^*) of the H_∞-norm of the closed-loop transfer function. As is well-known, this kind of search procedure is exhaustive and can be very costly. More seriously, as γ gets close to γ^*, numerical solutions for these AREs can become highly sensitive and ill-conditioned. So, in general, the iterative procedure for the computation of γ^* based on AREs is not reliable and thus should not be used to determine the infimum γ^*. Recently, Chen (1995b) proposed a non-iterative method for computing this γ^* for a class of discrete-time H_∞-optimization problems in which the transfer function from the disturbance to

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the measurement output is left invertible, and the transfer function from the control input to the output to be controlled is right invertible. In this paper, we extend his result by replacing the above conditions by certain weaker geometric conditions. The second result of the paper deals with the problem of discrete-time disturbance decoupling. We would like to point out that most of the results on disturbance decoupling in literature are in continuous-time setting. To the best of our knowledge, there has been no report in the literature that deal with the necessary and sufficient solvability conditions for the general discrete-time disturbance decoupling problem. We will derive for the first time a set of solvability conditions for this problem.

We consider in this paper the following standard linear time-invariant discrete time system Σ characterized by

$$\left.\begin{aligned} x(k+1) &= Ax(k)+Bu(k) &&+Ew(k) \\ y(k) &= C_1 x(k) &&+D_{12} w(k) \\ z(k) &= C_2 x(k)+D_{21} u(k) &&+D_{22} w(k) \end{aligned}\right\} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^l$ is the measurement, $w \in \mathbb{R}^q$ is the disturbance and $z \in \mathbb{R}^p$ is the output to be controlled. A, B, E, C_1, D_{12}, C_2, D_{21} and D_{22} are constant matrices of appropriate dimension. Throughout this paper, we assume that (A, B) is stabilizable and (A, C_1) is detectable and, as in most of the H_∞ control literature, we also assume that both the subsystems (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) are free of unit circle invariant zeros. Without loss of generality but for simplicity of presentation, we further assume that matrices $[C_1 \;\; D_{12}]$ and $[B' \;\; D_{21}']$ are of maximal rank. The H_∞ optimal control problem is to find an internally stabilizing causal controller such that the H_∞-norm of the overall closed-loop system is minimized. To be more specific, we will investigate dynamic feedback laws of the form

$$\Sigma_c : \begin{cases} x_c(k+1) = Kx_c(k) + Ly(k) \\ \qquad u(k) = Mx_c(k) + Ny(k) \end{cases} \tag{1.2}$$

We will say that the controller Σ_c of (1.2) is internally stabilizing when applied to the system Σ, if the following matrix is asymptotically stable

$$A_{cl} := \begin{bmatrix} A+BNC_1 & BM \\ LC_1 & K \end{bmatrix} \tag{1.3}$$

i.e. all its eigenvalues lie inside the open unit disc of the complex plane. Denote by G_{cl} the corresponding closed-loop transfer matrix. Then the H_∞ norm of the transfer matrix G_{cl} is given by

$$\|G_{cl}\|_\infty := \sup_{\omega \in [0, 2\pi]} \sigma_{\max}[G_{cl}(e^{j\omega})]$$

where $\sigma_{\max}[\cdot]$ denotes the largest singular value. The infimum γ^* can now be formally defined as

$$\gamma^* := \inf\{\|G_{cl}\|_\infty \mid \Sigma_c \text{ internally stabilizes } \Sigma\} \tag{1.4}$$

Given a $\gamma > \gamma^*$, the H_∞ optimal (or more precisely suboptimal) control problem is to find an internally stabilizing controller Σ_c such that the resulting $\|G_{cl}\|_\infty < \gamma$. Also, Σ_c is said to be a γ suboptimal controller for Σ if the corresponding $\|G_{cl}\|_\infty < \gamma$. The discrete-time disturbance decoupling problem for Σ of (1.1) is rather easy to define at this stage. It is simply to find an internally stabilizing controller Σ_c such that the resulting $G_{\mathrm{cl}} = 0$ and hence γ^* is equal to zero. The goals of this paper are to present a non-iterative method that computes exactly this γ^* for Σ under assumptions (A1) and (A2) given in §3, and to derive a set of necessary and sufficient conditions under which the disturbance decoupling problem for Σ is solvable.

The remainder of this paper is organized as follows. In §2, we recall some background material, i.e. the special coordinate basis of linear systems, which is instrumental to the derivation of the main results of the paper. Section 3 gives non-iterative algorithms for computation of γ^* for three common cases, i.e. the full information, the output feedback and the state feedback cases, while §4 derives a set of necessary and sufficient conditions under which the disturbance decoupling problem for Σ is solvable. Finally, concluding remarks are made in §5.

For a system characterized by a matrix quadruple (A, B, C, D), we define the following two geometric subspaces.

(1) $\mathscr{V}^-(A, B, C, D)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A+BF)$-invariant and contained in $\mathrm{Ker}\,(C+DF)$ such that the eigenvalues of $(A+BF)|\mathscr{V}^-$ are inside the open unit disc of the complex plane for some F.

(2) $\mathscr{S}^-(A, B, C, D)$ is the minimal $(A+KC)$-invariant subspace of $\mathbb{R}^n$ containing $\mathrm{Im}\,(B+KD)$ such that the eigenvalues of the map which is induced by $(A+KC)$ on the factor space $\mathbb{R}^n/\mathscr{S}^-$ are contained inside the open unit disc of the complex plane for some K.

Obviously, $\mathscr{V}^-(A, B, C, D) = \mathbb{R}^n/\mathscr{S}^-(A', C', B', D')$. Throughout this paper, the following notation will also be used:

$$
\begin{aligned}
X' &:= \text{transpose of matrix } X \\
X^\dagger &:= \text{generalized inverse of matrix } X \\
I &:= \text{identity matrix with appropriate dimension} \\
\mathrm{Ker}\,(X) &:= \text{kernel of } X \\
\mathrm{Im}\,(X) &:= \text{image of } X \\
\lambda(X) &:= \text{set of eigenvalues of a real square matrix } X \\
\lambda_{\max}(X) &:= \text{maximum eigenvalue of } X \text{ where } \lambda(X) \subset \mathbb{R} \\
\sigma_{\max}(X) &:= \text{maximum singular value of matrix } X \\
C^{-1}\{\mathscr{X}\} &:= \{x \mid Cx \in \mathscr{X}\}, \text{ where } \mathscr{X} \text{ is a subspace}
\end{aligned}
$$

2. Background material

In this section, we recall from Sannuti and Saberi (1987) and Saberi and Sannuti (1990) the special coordinate basis for linear systems. Consider the system described by

$$
\left.
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) + Ew(k) \\
z(k) &= C_2 x(k) + D_{21} u(k)
\end{aligned}
\right\} \tag{2.1}
$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a non-singular matrix V that put the direct feedthrough matrix D_{21} into the following form

$$
UD_{21}V = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{2.2}
$$

where r is in the rank of D_{21}. Without loss of generality one can assume that the matrix

D_{21} in (2.1) has the form as shown in (2.2). Thus, the system in (2.1) can be rewritten as

$$\left.\begin{aligned} x(k+1) &= Ax(k)+[B_0 \quad B_1]\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix}+Ew(k) \\ \begin{pmatrix} z_0(k) \\ z_1(k) \end{pmatrix} &= \begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix}x(k)+\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix} \end{aligned}\right\} \tag{2.3}$$

where B_0, B_1, $C_{2,0}$ and $C_{2,1}$ are matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely

$$u = V\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = Uz$$

Also, note that the H_∞-norm of the system transfer function from w to z remains unchanged when we apply an orthogonal transformation on the output z, and under any non-singular transformations on the states and control inputs. We have the following theorem.

Theorem 2.1: *Consider the linear system as given in* (2.1). *Assume that* (A, B, C_2, D_{21}) *has no invariant zeros on the unit circle. Then, there exist non-singular transformations* Γ_s, Γ_i *and* Γ_o *such that*

$$x = \Gamma_s\begin{pmatrix} x_c \\ x_a^- \\ x_a^+ \\ x_d \\ x_b \end{pmatrix}, \quad \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \Gamma_i\begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \quad z = \Gamma_o\begin{pmatrix} z_0 \\ z_d \\ z_b \end{pmatrix} \tag{2.4}$$

and

$$\Gamma_s^{-1}(A-B_0C_{2,0})\Gamma_s = \begin{bmatrix} A_{cc} & B_cE_{ca}^- & B_cE_{ca}^+ & L_{cd}C_d & L_{cb}C_b \\ 0 & A_{aa}^- & 0 & L_{ad}^-C_d & L_{ab}^-C_b \\ 0 & 0 & A_{aa}^+ & L_{ad}^+C_d & L_{ab}^+C_b \\ B_dE_{dc} & B_dE_{da}^- & B_dE_{da}^+ & A_{dd} & B_dE_{db} \\ 0 & 0 & 0 & L_{bd}C_d & A_{bb} \end{bmatrix},$$

$$\Gamma_s^{-1}E = \begin{bmatrix} E_c \\ E_a^- \\ E_a^+ \\ E_d \\ E_b \end{bmatrix} \tag{2.5}$$

$$\Gamma_s^{-1}[B_0 \quad B_1]\Gamma_i = \begin{bmatrix} B_{0c} & 0 & B_c \\ B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0d} & B_d & 0 \\ B_{0b} & 0 & 0 \end{bmatrix} \tag{2.6}$$

$$\Gamma_o^{-1}\begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0c} & C_{0a}^- & C_{0a}^+ & C_{0d} & C_{0b} \\ 0 & 0 & 0 & C_d & 0 \\ 0 & 0 & 0 & 0 & C_b \end{bmatrix}, \quad \Gamma_o^{-1}D_{21}\Gamma_i = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.7}$$

where the pair (A_{cc}, B_c) is completely controllable, the pair (A_{bb}, C_b) is completely observable, while the subsystem (A_{dd}, B_d, C_d) is invertible and free of invariant zeros. Also, $\lambda(A^+_{aa})$ and $\lambda(A^-_{aa})$ are respectively the sets of unstable and stable invariant zeros of (A, B, C_2, D_{21}). Moreover, the pair (A, B) is stabilizable if and only if the pair (A_{con}, B_{con}) is controllable, where

$$A_{con} := \begin{bmatrix} A^+_{aa} & L^+_{ad} C_d & L^+_{ab} C_b \\ B_d E^+_{da} & A_{dd} & B_d E_{db} \\ 0 & L_{bd} C_d & A_{bb} \end{bmatrix} \quad \text{and} \quad B_{con} := \begin{bmatrix} B^+_{0a} & 0 \\ B_{0d} & B_d \\ B_{0b} & 0 \end{bmatrix} \tag{2.8}$$

Also, (A, B, C_2, D_{21}) is left invertible if and only if x_c is non-existent; it is right invertible if and only if x_b is non-existent; it is invertible if and only if both x_b and x_c are non-existent. Furthermore, $x^-_a \oplus x_c$ spans the subspace $\mathscr{V}^-(A, B, C_2, D_{21})$ and $x^+_a \oplus x_c \oplus x_d$ spans the subspace $\mathscr{S}^-(A, B, C_2, D_{21})$. For further use, we define $n_x := \dim(x^+_a) + \dim(x_d) + \dim(x_b)$.

Proof: For the proof see Sannuti and Saberi (1987), and Saberi and Sannuti (1990), where the continuous-time counterpart was proven and the state variables x^-_a and x^+_a were not separated. The separation of x^-_a and x^+_a for discrete-time systems can be easily done using the algorithm given by Chen (1995a). One can slightly modify the m-file, scb.m, in the toolbox of Lin *et al.* (1991) to yield a Matlab function that realizes the above special coordinate basis. □

3. Non-iterative procedures for computing infimum

We present in this section our first result, i.e. the non-iterative algorithms for computing the infimum γ^* of discrete-time H_∞ optimization for Σ of (1.1). For the sake of simplicity, we will first assume that $D_{22} = 0$. The case when $D_{22} \neq 0$ will be discussed later. It is then without loss of generality to assume that $[C_2 \;\; D_{21}]$ and $[E' \;\; D'_{12}]$ are of maximal rank. Throughout this section, we also assume that the following two conditions are satisfied

(A1): $\operatorname{Im}(E) \subseteq \mathscr{V}^-(A, B, C_2, D_{21}) + \mathscr{S}^-(A, B, C_2, D_{21})$, and

(A2): $\operatorname{Ker}(C_2) \supseteq \mathscr{V}^-(A, E, C_1, D_{12}) \cap \mathscr{S}^-(A, E, C_1, D_{12})$

Note that these assumptions are not essential and might be further relaxed. Moreover, they are automatically satisfied if (A, B, C_2, D_{21}) is right invertible and (A, E, C_1, D_{12}) is left invertible. The general interpretations of the above conditions are rather simple under the special coordinate basis and will be given later.

This section is divided into three subsections. The first subsection deals with the full information case, while the second subsection deals with the general output feedback case. The full state feedback problem is then treated as a special case in the second subsection. A numerical example that illustrates our algorithms for the computation of the infimum γ^* is given in §3.3.

3.1. *The full information case*

We assume that $y = [x'w']'$, which implies that the condition (A2) is automatically satisfied. Without of loss generality but for simplicity of presentation of our results, we also assume that D_{21} is in the form of (2.2). In what follows, we state a step-by-step algorithm for the computation of the infimum γ^*.

Step 1. Transform the following system

$$\left.\begin{aligned} x(k+1) &= Ax(k)+Bu(k)+Ew(k) \\ z(k) &= C_2\, x(k)+D_{21}\, u(k) \end{aligned}\right\} \tag{3.1}$$

into the special coordinate basis as given by Theorem 2.1 and define A_x, B_x, B_{x0}, B_{x1}, E_x, C_x and D_x as follows

$$A_x := \begin{bmatrix} A_{\mathrm{aa}}^+ & L_{\mathrm{ad}}^+ C_{\mathrm{d}} & L_{\mathrm{ab}}^+ C_{\mathrm{b}} \\ B_{\mathrm{d}} E_{\mathrm{da}}^+ & A_{\mathrm{dd}} & B_{\mathrm{d}} E_{\mathrm{db}} \\ 0 & L_{\mathrm{bd}} C_{\mathrm{d}} & A_{\mathrm{bb}} \end{bmatrix}, \quad B_x := [B_{x0} \quad B_{x1}] := \begin{bmatrix} B_{0\mathrm{a}}^+ & 0 \\ B_{0\mathrm{d}} & B_{\mathrm{d}} \\ B_{0\mathrm{b}} & 0 \end{bmatrix}, \quad E_x := \begin{bmatrix} E_{\mathrm{a}}^+ \\ E_{\mathrm{d}} \\ E_{\mathrm{b}} \end{bmatrix} \tag{3.2}$$

and

$$C_x := \Gamma_{\mathrm{o}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{\mathrm{d}} & 0 \\ 0 & 0 & C_{\mathrm{b}} \end{bmatrix}, \quad D_x = \Gamma_{\mathrm{o}} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

Note that assumption (A1) implies and is implied by the fact that E_{b} in (3.2) is always equal to zero. Also, it follows from the property of the special coordinate basis that the pair (A_x, B_x) is completely controllable.

Step 2. Find a matrix F_x such that $A_x+B_x F_x$ has no eigenvalues at -1. Then define $\tilde{A}_x$, $\tilde{B}_x$, $\tilde{E}_x$, $\tilde{C}_x$, $\tilde{D}_x$ and $\tilde{D}_{22}$ as

$$\left.\begin{aligned} \tilde{A}_x &:= (A_x+B_x F_x+I)^{-1}(A_x+B_x F_x-I) \\ \tilde{B}_x &:= 2(A_x+B_x F_x+I)^{-2} B_x \\ \tilde{E}_x &:= 2(A_x+B_x F_x+I)^{-2} E_x \\ \tilde{C}_x &:= C_x+D_x F_x \\ \tilde{D}_x &:= D_x-(C_x+D_x F_x)(A_x+B_x F_x+I)^{-1} B_x \\ \tilde{D}_{22} &:= -(C_x+D_x F_x)(A_x+B_x F_x+I)^{-1} E_x \end{aligned}\right\} \tag{3.4}$$

Step 3. Solve the following continuous-time algebraic Riccati equation and algebraic Lyapunov equation, both independent of γ

$$\begin{aligned} 0 = [\tilde{A}_x-\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{C}_x]\tilde{S}_x+\tilde{S}_x[\tilde{A}_x-\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{C}_x]' \\ -\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{B}_x'+\tilde{S}_x[\tilde{C}_x'\tilde{C}_x-\tilde{C}_x'\tilde{D}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{C}_x]\tilde{S}_x \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 = [\tilde{A}_x-\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{C}_x]\tilde{T}_x+\tilde{T}_x[\tilde{A}_x-\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{C}_x]' \\ -[\tilde{E}_x-\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{D}_{22}][\tilde{E}_x-\tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{D}_{22}]' \end{aligned} \tag{3.6}$$

for positive definite solution $\tilde{S}_x$ and positive semi-definite solution $\tilde{T}_x$. For future use, we define

$$S_x := (A_x+B_x F_x+I)\,\tilde{S}_x(A_x'+F_x' B_x'+I)/2 \tag{3.7}$$

and

$$T_x := (A_x+B_x F_x+I)\,\tilde{T}_x(A_x'+F_x' B_x'+I)/2 \tag{3.8}$$

Step 4. The infimum, γ^*, is given by

$$\gamma^* = (\lambda_{\max}(\tilde{T}_x\tilde{S}_x^{-1}))^{1/2} = (\lambda_{\max}(T_x S_x^{-1}))^{1/2} \tag{3.9}$$

Proof of the algorithm: Following the results of Chen *et al.* (1994) and Stoorvogel *et al.* (1994), it is straightforward to show that the following three statements are equivalent.

(1) There exists a γ suboptimal controller for Σ of (1.1) with $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$ and $D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$.

(2) There exists a γ suboptimal controller for the following auxiliary system

$$\left.\begin{aligned} x_x(k+1) &= A_x x_x(k) + B_x u_x(k) + E_x w_x(k) \\ y_x(k) &= \begin{pmatrix} 0 \\ I \end{pmatrix} x_x(k) \qquad\qquad + \begin{pmatrix} I \\ 0 \end{pmatrix} w_x(k) \\ z_x(k) &= C_x x_x(k) + D_x u_x(k) \end{aligned}\right\} \tag{3.10}$$

where A_x, B_x, E_x, C_x and D_x are defined as in (3.2) and (3.3).

(3) There exists a γ suboptimal controller for the following auxiliary system

$$\left.\begin{aligned} \dot{\tilde{x}}_x &= \tilde{A}_x \tilde{x}_x + \tilde{B}_x \tilde{u}_x + \tilde{E}_x \tilde{w}_x \\ \tilde{y}_x &= \begin{pmatrix} 0 \\ I \end{pmatrix} \tilde{x}_x \qquad + \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{w}_x \\ \tilde{z}_x &= \tilde{C}_x \tilde{x}_x + \tilde{D}_x \tilde{u}_x + \tilde{D}_{22} \tilde{w}_x \end{aligned}\right\} \tag{3.11}$$

where $\tilde{A}_x$, $\tilde{B}_x$, $\tilde{E}_x$, $\tilde{C}_x$, $\tilde{D}_x$ and $\tilde{D}_{22}$ are as defined in (3.4).

We would like to note that items (2) and (3) above are also equivalent to the following.

(1) There exists a solution $P_x > 0$ to the following discrete-time algebraic Riccati equation

$$P_x = A_x' P_x A_x + C_x' C_x - \begin{bmatrix} B_x' P_x A_x + D_x' C_x \\ E_x' P_x A_x \end{bmatrix}' G_x(P_x)^{-1} \begin{bmatrix} B_x' P_x A_x + D_x' C_x \\ E_x' P_x A_x \end{bmatrix} \tag{3.12}$$

where

$$G_x(P_x) := \begin{bmatrix} D_x' D_x & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} B_x' \\ E_x' \end{bmatrix} P_x [B_x \quad E_x] \tag{3.13}$$

such that the following conditions are satisfied

$$V_x := B_x' P_x B_x + D_x' D_x > 0 \tag{3.14}$$

$$R_x := \gamma^2 I - E_x' P_x E_x + E_x' P_x B_x V_x^{-1} B_x' P_x E_x > 0 \tag{3.15}$$

(2) There exists a solution $\tilde{P}_x > 0$ to the following continuous-time algebraic Riccati equation

$$0 = \tilde{P}_x \tilde{A}_x + \tilde{A}_x' \tilde{P}_x + \tilde{C}_x' \tilde{C}_x - \begin{bmatrix} \tilde{B}_x' \tilde{P}_x + \tilde{D}_x' \tilde{C}_x \\ \tilde{E}_x' \tilde{P}_x + \tilde{D}_{22}' \tilde{C}_x \end{bmatrix}' \tilde{G}_x^{-1} \begin{bmatrix} \tilde{B}_x' \tilde{P}_x + \tilde{D}_x' \tilde{C}_x \\ \tilde{E}_x' \tilde{P}_x + D_{22}' \tilde{C}_x \end{bmatrix} \tag{3.16}$$

with

$$\tilde{D}_{22}'[I - \tilde{D}_x(\tilde{D}_x' \tilde{D}_x)^{-1} \tilde{D}_x'] \tilde{D}_{22} < \gamma^2 I \tag{3.17}$$

and

$$\tilde{G}_x := \begin{bmatrix} \tilde{D}_x' \tilde{D}_x & \tilde{D}_x' \tilde{D}_{22} \\ \tilde{D}_{22}' \tilde{D}_x & \tilde{D}_{22}' \tilde{D}_{22} - \gamma^2 I \end{bmatrix} \tag{3.18}$$

Furthermore, the solutions to the above Riccati equations, if they exist, are related by

$$P_x = 2(A_x' + F_x' B_x' + I)^{-1} \tilde{P}_x (A_x + B_x F_x + I)^{-1} \tag{3.19}$$

Thus, it is equivalent to show that γ^* given by (3.9) is the infimum for system Σ of (1.1) by showing that it is an infimum for the auxiliary system in (3.11). This can be done by first showing the properties of the auxiliary system of (3.11) and then applying the results of Chen *et al.* (1992b). We note that the matrix F_x in Step 2 of the algorithm is a pre-state feedback gain, which is introduced merely to deal with the situation when A_x has eigenvalues at -1 and the inverse of $I + A_x$ does not exist. For the sake of simplicity but without loss of generality, we will hereafter assume that A_x has no eigenvalues at -1 and $F_x = 0$. We will first show the following three facts associated with the auxiliary system (3.11): there exists a pre-disturbance feedback to the system in (3.11) in the form of

$$\tilde{u}_x = \tilde{F}_w \tilde{w}_x + \tilde{v}_x \tag{3.20}$$

such that

(1) $\tilde{D}_{22} + \tilde{D}_x \tilde{F}_w = 0$
(2) $\mathrm{Im}\,(\tilde{E}_x + \tilde{B}_x \tilde{F}_w) \subseteq \mathscr{V}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x) + \mathscr{S}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$, and
(3) $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ is left invertible, and is free of infinite zeros and stable invariant zeros as well as invariant zeros on the unit circle.

In fact, we will show that such an $\tilde{F}_w$ is given by

$$\tilde{F}_w = -(\tilde{D}_x' \tilde{D}_x)^{-1} \tilde{D}_x' \tilde{D}_{22} \tag{3.21}$$

In order to make our proof simpler, we first apply a pre-state feedback law

$$u_x = F_x x_x + v_x = -\begin{bmatrix} 0 & 0 & 0 \\ E_{\mathrm{da}}^+ & 0 & E_{\mathrm{db}} \end{bmatrix} x_x + v_x \tag{3.22}$$

to the system in (3.10) such that the resulting dynamic matrix $A_x + B_x F_x$ has the following format

$$\begin{bmatrix} A_{\mathrm{aa}}^+ & L_{\mathrm{ad}}^+ C_{\mathrm{d}} & L_{\mathrm{ab}}^+ C_{\mathrm{b}} \\ 0 & A_{\mathrm{dd}} & 0 \\ 0 & L_{\mathrm{bd}} C_{\mathrm{d}} & A_{\mathrm{bb}} \end{bmatrix} \tag{3.23}$$

while the rest of system matrices in (3.10) remain unchanged. Hence, it is without loss of generality to assume that A_x is already in the form of (3.23). Also, we assume that both A_{dd} and A_{bb} have no eigenvalues at -1. Then it is simple to verify that

$$(A_x + I)^{-1} = \begin{bmatrix} (A_{\mathrm{aa}}^+ + I)^{-1} & X_1 & X_2 \\ 0 & (A_{\mathrm{dd}} + I)^{-1} & 0 \\ 0 & -(A_{\mathrm{bb}} + I)^{-1} L_{\mathrm{bd}} C_{\mathrm{d}} (A_{\mathrm{dd}} + I)^{-1} & (A_{\mathrm{bb}} + I)^{-1} \end{bmatrix} \tag{3.24}$$

where

$$X_1 = -(A_{\mathrm{aa}}^+ + I)^{-1} [L_{\mathrm{ad}}^+ - L_{\mathrm{ab}}^+ C_{\mathrm{b}} (A_{\mathrm{bb}} + I)^{-1} L_{\mathrm{bd}}] C_{\mathrm{d}} (A_{\mathrm{dd}} + I)^{-1} \tag{3.25}$$

$$X_2 = -(A_{\mathrm{aa}}^+ + I)^{-1} L_{\mathrm{ab}}^+ C_{\mathrm{b}} (A_{\mathrm{bb}} + I)^{-1} \tag{3.26}$$

and

$$\tilde{D}_x = D_x - C_x(A_x+I)^{-1} B_x$$

$$= \Gamma_o \begin{bmatrix} I & 0 \\ -C_d(A_{dd}+I)^{-1} B_{0d} & -C_d(A_{dd}+I)^{-1} B_d \\ X_3 & C_b(A_{bb}+I)^{-1} L_{bd} C_d(A_{dd}+I)^{-1} B_d \end{bmatrix}$$

where

$$X_3 = C_b(A_{bb}+I)^{-1} L_{bd} C_d(A_{dd}+I)^{-1} B_{0d} - C_b(A_{bb}+I)^{-1} B_{0b} \tag{3.27}$$

Define

$$\tilde{\Gamma}_o = \Gamma_o \begin{bmatrix} I & 0 & 0 \\ -C_d(A_{dd}+I)^{-1} B_{0d} & -C_d(A_{dd}+I)^{-1} B_d & 0 \\ X_3 & C_b(A_{bb}+I)^{-1} L_{bd} C_d(A_{dd}+I)^{-1} B_d & I \end{bmatrix} \tag{3.28}$$

We note that $\tilde{\Gamma}_o$ is non-singular. This follows from the property of the special coordinate basis that the triple (A_{dd}, B_d, C_d) is square and invertible with no invariant zeros, and hence $C_d(A_{dd}+I)^{-1} B_d$ is non-singular. Then we have

$$\tilde{D}_x = \tilde{\Gamma}_o \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \tag{3.29}$$

and

$$\tilde{D}_{22} = -C_x(A_x+I)^{-1} E_x = \begin{bmatrix} 0 \\ -C_d(A_{dd}+I)^{-1} E_d \\ C_b(A_{bb}+I)^{-1} L_{bd} C_d(A_{dd}+I)^{-1} E_d \end{bmatrix} = \tilde{\Gamma}_o \begin{bmatrix} 0 \\ X_4 \\ 0 \end{bmatrix} \tag{3.30}$$

where

$$X_4 = [C_d(A_{dd}+I)^{-1} B_d]^{-1} C_d(A_{dd}+I)^{-1} E_d \tag{3.31}$$

It is now obvious to see that the following pre-disturbance feedback law to (3.11)

$$\tilde{u}_x = \tilde{F}_w \tilde{w}_x + \tilde{v}_x = -\begin{bmatrix} 0 \\ X_4 \end{bmatrix} \tilde{w}_x + \tilde{v}_x \tag{3.32}$$

is such that $\tilde{D}_{22} + \tilde{D}_x \tilde{F}_w = 0$. We also have

$$\tilde{E}_x + \tilde{B}\tilde{F}_w = 2(A_x+I)^{-2}(E_x + B_x \tilde{F}_w) = 2(A_x+I)^{-2} \begin{bmatrix} E_a^+ \\ E_d^* \\ 0 \end{bmatrix} \tag{3.33}$$

where

$$E_d^* = E_d - B_d[C_d(A_{dd}+I)^{-1} B_d]^{-1} C_d(A_{dd}+I)^{-1} E_d \tag{3.34}$$

This shows the first fact. Since $\tilde{D}_x$ is of maximal column rank, it follows that the above $\tilde{F}_w$ is also equivalent to $-(\tilde{D}_x' \tilde{D}_x)^{-1} \tilde{D}_x' \tilde{D}_{22}$. Next, let us proceed to prove the second fact, i.e.

$$\operatorname{Im}(\tilde{E}_x + \tilde{B}_x \tilde{F}_w) \subseteq \mathscr{V}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x) + \mathscr{S}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$$

We will have to apply several non-singular state transformations to the system

$$\left.\begin{aligned}\dot{\tilde{x}}_x &= \tilde{A}_x \tilde{x}_x + \tilde{B}_x \tilde{v}_x + (\tilde{E}_x + \tilde{B}_x \tilde{F}_w)\tilde{w}_x \\ \tilde{z}_x &= \tilde{C}_x \tilde{x}_x + \tilde{D}_x \tilde{v}_x\end{aligned}\right\} \tag{3.35}$$

and transform it into the form of the special coordinate basis as given in Theorem 2.1. First let us define a state transformation

$$\tilde{T}_x = (A_x + I)^{-2} \tag{3.36}$$

In view of (3.24), it is straightforward, although tedious, to verify that

$$\tilde{T}_x = \begin{bmatrix} (A_{\mathrm{aa}}^+ + I)^{-2} & \star & \star \\ 0 & (A_{\mathrm{dd}} + I)^{-2} & 0 \\ 0 & X_5 & (A_{\mathrm{bb}} + I)^{-2} \end{bmatrix} \tag{3.37}$$

where $\star$ represents matrices of not much interest and

$$X_5 = -(A_{\mathrm{bb}} + I)^{-1}[L_{\mathrm{bd}} C_{\mathrm{d}}(A_{\mathrm{dd}} + I)^{-1} + (A_{\mathrm{bb}} + I)^{-1} L_{\mathrm{bd}} C_{\mathrm{d}}](A_{\mathrm{dd}} + I)^{-1} \tag{3.38}$$

and

$$\begin{aligned}\bar{A}_x &:= \tilde{T}_x^{-1} \tilde{A}_x \tilde{T}_x = (A_x - I)(A_x + I)^{-1} \\ &= \begin{bmatrix} (A_{\mathrm{aa}}^+ - I)(A_{\mathrm{aa}}^+ + I)^{-1} & \star & 2(A_{\mathrm{aa}}^+ + I)^{-1} L_{\mathrm{ab}}^+ C_{\mathrm{b}}(A_{\mathrm{bb}} + I)^{-1} \\ 0 & (A_{\mathrm{dd}} - I)(A_{\mathrm{dd}} + I)^{-1} & 0 \\ 0 & 2(A_{\mathrm{bb}} + I)^{-1} L_{\mathrm{bd}} C_{\mathrm{d}}(A_{\mathrm{dd}} + I)^{-1} & (A_{\mathrm{bb}} - I)(A_{\mathrm{bb}} + I)^{-1} \end{bmatrix}\end{aligned} \tag{3.39}$$

$$\bar{B}_x := \tilde{T}_x^{-1} \tilde{B}_x = 2B_x = 2\begin{bmatrix} B_{0\mathrm{a}}^+ & 0 \\ B_{0\mathrm{d}} & B_{\mathrm{d}} \\ B_{0\mathrm{b}} & 0 \end{bmatrix} \tag{3.40}$$

$$\bar{E}_x := \tilde{T}_x^{-1}(\tilde{E}_x + \tilde{B}_x \tilde{F}_w) = 2\begin{bmatrix} E_{\mathrm{a}}^+ \\ E_{\mathrm{d}}^* \\ E_{\mathrm{b}} \end{bmatrix}, \quad \text{where } E_{\mathrm{b}} = 0 \tag{3.41}$$

$$\bar{C}_x := \tilde{C}_x \tilde{T}_x = \tilde{\Gamma}_{\mathrm{o}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -[C_{\mathrm{d}}(A_{\mathrm{dd}} + I)^{-1} B_{\mathrm{d}}]^{-1} C_{\mathrm{d}}(A_{\mathrm{dd}} + I)^{-2} & 0 \\ 0 & -C_{\mathrm{b}}(A_{\mathrm{bb}} + I)^{-2} L_{\mathrm{bd}} C_{\mathrm{d}}(A_{\mathrm{dd}} + I)^{-1} & C_{\mathrm{b}}(A_{\mathrm{bb}} + I)^{-2} \end{bmatrix} \tag{3.42}$$

$$\bar{D}_x := \tilde{D}_x = \tilde{\Gamma}_{\mathrm{o}} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \tag{3.43}$$

In order to bring the system of (3.35) into the standard form of the special coordinate basis, we will have to perform another state transformation that will cause the (3, 2) block of $\bar{C}_x$ in the right-hand side of (3.42) to vanish. The following transformation $\bar{T}_x$ will do the job,

$$\bar{T}_x = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & L_{\mathrm{bd}} C_{\mathrm{d}}(A_{\mathrm{dd}} + I)^{-1} & (A_{\mathrm{bb}} + I)^2 \end{bmatrix} \tag{3.44}$$

It is quite easy to verify this time that

$$\hat{A}_x := \bar{T}_x^{-1} \bar{A}_x \bar{T}_x$$
$$= \begin{bmatrix} (A_{aa}^+ - I)(A_{aa}^+ + I)^{-1} & \star & 2(A_{aa}^+ + I)^{-1} L_{ab}^+ C_b (A_{bb} + I) \\ 0 & (A_{dd} - I)(A_{dd} + I)^{-1} & 0 \\ 0 & 2(A_{bb} + I)^{-2} L_{bd} C_d (A_{dd} + I)^{-2} & (A_{bb} + I)^{-1}(A_{bb} - I) \end{bmatrix} \tag{3.45}$$

$$\hat{B}_x := \hat{B}_{x0} := \bar{T}_x^{-1} \bar{B}_x = 2 \begin{bmatrix} B_{0a}^+ & 0 \\ B_{0d} & B_d \\ \star & -(A_{bb} + I)^{-2} L_{bd} C_d (A_{dd} + I)^{-1} B_d \end{bmatrix} \tag{3.46}$$

$$\hat{E}_x := \bar{T}_x^{-1} \bar{E}_x = 2 \begin{bmatrix} E_a^+ \\ E_d^* \\ (A_{bb} + I)^{-2} [E_b - L_{bd} C_d (A_{dd}^+ + I)^{-1} E_d^*] \end{bmatrix} = 2 \begin{bmatrix} E_a^+ \\ E_d^* \\ 0 \end{bmatrix} \tag{3.47}$$

$$\hat{C}_x := \begin{bmatrix} \hat{C}_{x0} \\ \hat{C}_{x1} \end{bmatrix} := \bar{C}_x \bar{T}_x = \tilde{\Gamma}_o \begin{bmatrix} 0 & 0 & 0 \\ 0 & -[C_d (A_{dd} + I)^{-1} B_d]^{-1} C_d (A_{dd} + I)^{-2} & 0 \\ \hline 0 & 0 & C_b \end{bmatrix} \tag{3.48}$$

$$\hat{D}_x := \bar{D}_x = \tilde{D}_x \tag{3.49}$$

Then we have

$$\hat{A}_x - \hat{B}_{x0} \hat{C}_{x0} = \begin{bmatrix} (A_{aa}^+ - I)(A_{aa}^+ + I)^{-1} & \star & 2(A_{aa}^+ + I)^{-1} L_{ab}^+ C_b (A_{bb} + I) \\ 0 & A_{aa}^* & 0 \\ 0 & 0 & (A_{bb} + I)^{-1}(A_{bb} - I) \end{bmatrix} \tag{3.50}$$

where

$$A_{aa}^* = (A_{dd} - I)(A_{dd} + I)^{-1} + 2B_d [C_d (A_{dd} + I)^{-1} B_d]^{-1} C_d (A_{dd} + I)^{-2} \tag{3.51}$$

Define another non-singular state transformation

$$\hat{T}_x = \begin{bmatrix} I & 0 & \hat{T}_* \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.52}$$

with $\hat{T}_*$ being a solution to the following general Lyapunov equation

$$(I - A_{aa}^+)(I + A_{aa}^+)^{-1} \hat{T}_* + \hat{T}_* (A_{bb} + I)^{-1} (A_{bb} - I) = 2(A_{aa}^+ - I)^{-1} L_{ab}^+ C_b (A_{bb} + I) \tag{3.53}$$

It follows from Kailath (1980) that such a solution always exists and is unique if A_{aa}^+ and A_{bb} have no common eigenvalues. Then it is straightforward to verify that it would transform the $(1,3)$ block of $\hat{A}_x - \hat{B}_{x0} \hat{C}_{x0}$ in (3.50) to 0 while not changing the structures of other blocks. Hence, $\hat{T}_x$ would also transform the system $(\hat{A}_x, \hat{B}_x, \hat{C}_x, \hat{D}_x)$ and $\hat{E}_x$ into the standard form of the special coordinate basis as given in Theorem 2.1

since the pair $\{(A_{\rm bb}+I)^{-1}(A_{\rm bb}-I), C_{\rm b}\}$ is completely observable due to the complete observability of $(A_{\rm bb}, C_{\rm b})$. It is now clear from the properties of the special coordinate basis that

$$\operatorname{Im}(\hat{E}_x) \subseteq \mathscr{V}^-(\hat{A}_x, \hat{B}_x, \hat{C}_x, \hat{D}_x) + \mathscr{S}^-(\hat{A}_x, \hat{B}_x, \hat{C}_x, \hat{D}_x)$$

which is equivalent to

$$\operatorname{Im}(\tilde{E}_x + \tilde{B}_x \tilde{F}_w) \subseteq \mathscr{V}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x) + \mathscr{S}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$$

This proves the second fact. Moreover, it is also obvious from the properties of the special coordinate basis that $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ is left invertible with no infinite zeros and has some invariant zeros at

$$\lambda\{(A_{\rm aa}^+ - I)(A_{\rm aa}^+ + I)^{-1}\} \tag{3.54}$$

which are unstable, i.e. in the open right-half complex plane, due to the fact that $\lambda(A_{\rm aa}^+)$ are outside the unit disc, and the rest of the invariant zeros at $\lambda(A_{\rm aa}^*)$. In what follows, we will show that all the eigenvalues of $A_{\rm aa}^*$ of (3.51) are at 1. As $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ is left invertible, it is well-known that a complex scalar s is an invariant zero if and only if

$$\operatorname{rank}\begin{bmatrix} sI - \tilde{A}_x & -\tilde{B}_x \\ \tilde{C}_x & \tilde{D}_x \end{bmatrix} < n_x + p$$

where p is the dimension of z of the given system (1.1). Noting that

$$\begin{aligned}
r_x(s) &= \operatorname{rank}\begin{bmatrix} sI - \tilde{A}_x & -\tilde{B}_x \\ \tilde{C}_x & \tilde{D}_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} sI - (A_x+I)^{-1}(A_x - I) & -2(A_x+I)^{-2} B_x \\ C_x & D_x - C_x(A_x+I)^{-1} B_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} s(A_x+I) - (A_x - I) & -2(A_x+I)^{-1} B_x \\ C_x & D_x - C_x(A_x+I)^{-1} B_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} (1+s)I - (1-s)A_x & -(1-s)B_x \\ C_x & D_x \end{bmatrix} \\
&= \operatorname{rank}\begin{bmatrix} (1+s)I-(1-s)A_{\rm aa}^+ & (s-1)L_{\rm ad}^+ C_{\rm d} & (s-1)L_{\rm ab}^+ C_{\rm b} & (s-1)B_{\rm 0a}^+ & 0 \\ 0 & (1+s)I-(1-s)A_{\rm dd} & 0 & (s-1)B_{\rm 0d} & (s-1)B_{\rm d} \\ 0 & (s-1)L_{\rm bd} C_{\rm d} & (1+s)I-(1-s)A_{\rm bb} & (s-1)B_{\rm 0b} & 0 \\ 0 & 0 & 0 & I_{\rm r} & 0 \\ 0 & C_{\rm d} & 0 & 0 & 0 \\ 0 & 0 & C_{\rm b} & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$= \operatorname{rank} \begin{bmatrix} (1+s)I-(1-s)A_{aa}^{+} & 0 & 0 & 0 & 0 \\ 0 & (1+s)I-(1-s)A_{dd} & 0 & 0 & (s-1)B_d \\ 0 & 0 & (1+s)I-(1-s)A_{bb} & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 \\ 0 & C_d & 0 & 0 & 0 \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}$$

It is obvious to see that for any $s \in \lambda\{(A_{aa}^{+}+I)^{-1}(A_{aa}^{+}-I)\}, r_x(s) < n_x+p$, which verifies that some invariant zeros of $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ are given by (3.54). The only other scalar that causes $r_x(s)$ to drop below n_x+p is $s=1$ because the subsystem (A_{dd}, B_d, C_d) is invertible and free of invariant zeros, and the pair (A_{bb}, C_b) is completely observable. Thus, we can conclude that the remaining invariant zeros of $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ are at 1 and hence all eigenvalues of A_{aa}^{*} are at 1. This shows the third fact that we have claimed.

Next, let us apply a pre-disturbance feedback law

$$\tilde{u}_x = \tilde{F}_w \tilde{w}_x + \tilde{v}_x = -(\tilde{D}_x' \tilde{D}_x)^{-1} \tilde{D}_x' \tilde{D}_{22} \tilde{w}_x + \tilde{v}_x \tag{3.55}$$

to the auxiliary system (3.11). Again, this pre-feedback law will not affect solutions to the H_∞ problem for (3.11) or to the solution $\tilde{P}_x$ of (3.16)–(3.18). After applying this pre-feedback law, we obtain the following new system

$$\left.\begin{aligned} \dot{\tilde{x}}_x &= \tilde{A}_x \tilde{x}_x + \tilde{B}_x \tilde{v}_x + [\tilde{E}_x - \tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{D}_{22}]\,\tilde{w}_x \\ \tilde{y}_x &= \begin{pmatrix} 0 \\ I \end{pmatrix} \tilde{x}_x + \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{w}_x \\ \tilde{z}_x &= \tilde{C}_x \tilde{x}_x + \tilde{D}_x \tilde{v}_x + 0\, \tilde{w}_x \end{aligned}\right\} \tag{3.56}$$

Then it follows from the well-known results in H_∞ control theory (see for example, Stoorvogel 1992) that the existence condition of a γ suboptimal controller for (3.56) is equivalent to the existence of a $\tilde{P}_x > 0$ such that

$$\begin{aligned} 0 = \tilde{P}_x \tilde{A}_x + \tilde{A}_x' \tilde{P}_x + \tilde{C}_x' \tilde{C}_x - (\tilde{P}_x \tilde{B}_x + \tilde{C}_x' \tilde{D}_x)(\tilde{D}_x' \tilde{D}_x)^{-1}(\tilde{P}_x \tilde{B}_x + \tilde{C}_x' \tilde{D}_x)' \\ + \tilde{P}_x[\tilde{E}_x - \tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{D}_{22}][\tilde{E}_x - \tilde{B}_x(\tilde{D}_x'\tilde{D}_x)^{-1}\tilde{D}_x'\tilde{D}_{22}]'\,\tilde{P}_x/\gamma^2 \end{aligned} \tag{3.57}$$

is satisfied. Note that the solution $\tilde{P}_x$ to the above Riccati equation is identical to the solution that satisfies (3.16)–(3.17).

Now, in view of the properties of the auxiliary system of (3.56), i.e. the second and third facts that we have proved earlier, it satisfies the conditions of Chen *et al.* (1992b). In fact, following the results of Chen *et al.* (1992b), we can show that

$$\gamma^* = (\lambda_{\max}(\tilde{T}_x \tilde{S}_x^{-1}))^{1/2} \tag{3.58}$$

and for any $\gamma > \gamma^*$, the positive definite solution $\tilde{P}_x$ of (3.16)–(3.18) is given by

$$\tilde{P}_x = (\tilde{S}_x - \tilde{T}_x/\gamma^2)^{-1} \tag{3.59}$$

It then follows from (3.19) that for any $\gamma > \gamma^*$, the positive definite solution P_x of (3.12)–(3.15) is given by

$$P_x = 2(A'_x + I)^{-1}(\tilde{S}_x - \tilde{T}_x/\gamma^2)^{-1}(A_x + I)^{-1} \tag{3.60}$$

and hence γ^* can also be obtained from the following expression,

$$\gamma^* = (\lambda_{\max}(T_x S_x^{-1}))^{\frac{1}{2}} \tag{3.61}$$

where S_x and T_x are as defined in (3.7) and (3.8), respectively.

Finally, note that $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ is left invertible, and is free of infinite zeros and stable invariant zeros as well as invariant zeros on the unit circle. It follows from Richardson and Kwong (1986) that the solution $\tilde{S}_x$ to the Riccati equation (3.5) is positive definite because (A_x, B_x) is controllable, and the solution $\tilde{T}_x$ to the Lyapunov equation (3.6) is positive semi-definite. In fact, both of them are unique. This completes the prrof of our algorithm. □

The following remark deals with the case when the direct feedthrough term from the disturbance to the controlled output of (1.1) is non-zero, i.e. $D_{22} \neq 0$.

Remark 3.1: For the case when $D_{22} \neq 0$, the assumption (A1) should be replaced by the following conditions:

(1) $\tilde{D}_{22} := D_{22} - C_x(A_x + I)^{-1} E_x$ is in the range space of $\tilde{D}_x = D_x - C_x(A_x + I)^{-1} B_x$, and

(2) $\operatorname{Im}[\tilde{E}_x - \tilde{B}_x(\tilde{D}'_x \tilde{D}_x)^{-1} \tilde{D}'_x \tilde{D}_{22}] \subseteq \mathscr{V}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x) + \mathscr{S}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$.

Then our algorithm would carry through without any problems. We would also like to note that if (A, B, C_2, D_{21}) is right invertible, then $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ is invertible and $\tilde{D}_x$ is square and non-singular, and $\mathscr{V}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x) + \mathscr{S}^-(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x) = \mathbb{R}^{n_x}$. Hence, the above two conditions will be automatically satisfied. Also, in this case, our result will be reduced to that reported by Chen (1995b). □

3.2. *The output feedback case*

This subsection deals with the general measurement feedback problem. Again, we will first consider the given system of (1.1) with $D_{22} = 0$ and assume that (A1) and (A2) are satisfied. As in the previous subsection, we will give a step-by-step non-iterative algorithm for the computation of γ^*.

Step A. Define an auxiliary full information problem for

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ew(k) \\ y(k) &= \begin{pmatrix} 0 \\ I \end{pmatrix} x(k) \qquad + \begin{pmatrix} I \\ 0 \end{pmatrix} w(k) \\ z(k) &= C_2 x(k) + D_{21} u(k) \end{aligned}\right\} \tag{3.62}$$

and perform Steps 1 to 3 of the algorithm given in the previous subsection. For future use and in order to avoid notational confusion, we rename the state transformation of the special coordinate basis for this subsystem as Γ_{sP} and the dimension of A_x as n_{xP}. Also, rename S_x of (3.7) and T_x of (3.8) as S_{xP} and T_{xP}, respectively. For later use in §4, we rename E_x and B_{x1} respectively as E_{xP} and B_{x1P}.

Step B. Define another auxiliary full information problem for

$$\left.\begin{aligned} x(k+1) &= A'x(k)+C_1'u(k)+C_2'w(k) \\ y(k) &= \begin{pmatrix}0\\I\end{pmatrix}x(k) \qquad\quad +\begin{pmatrix}I\\0\end{pmatrix}w(k) \\ z(k) &= E'x(k)+D_{12}'u(k) \end{aligned}\right\} \tag{3.63}$$

and again perform Steps 1 to 3 of the algorithm given in §3.1 one more time but for this auxiliary system. We also rename the state transformation of the special coordinate basis for this case as Γ_{sQ} and the dimension of A_x as n_{xQ}, and S_x of (3.7) and T_x of (3.8) as S_{xQ} and T_{xQ}, respectively. Again, for later use in §4, we rename E_x and B_{x1} respectively as E_{xQ} and B_{x1Q}.

Step C. Partition

$$\Gamma_{sP}^{-1}(\Gamma_{sQ}^{-1})' = \begin{bmatrix}\star & \star\\ \star & \Gamma\end{bmatrix} \tag{3.64}$$

where Γ is a $n_{xP}\times n_{xQ}$ matrix, and define a constant matrix

$$M = \begin{bmatrix} T_{xP}S_{xP}^{-1}+\Gamma S_{xQ}^{-1}\Gamma' S_{xP}^{-1} & -\Gamma S_{xQ}^{-1} \\ -T_{xQ}S_{xQ}^{-1}\Gamma' S_{xP}^{-1} & T_{xQ}S_{xQ}^{-1}\end{bmatrix} \tag{3.65}$$

Step D. The infimum γ^* is then given by

$$\gamma^* = (\lambda_{\max}(M))^{1/2} \tag{3.66}$$

Proof of the Algorithm: Once the result for the full information case is established, the proof of this algorithm is similar to the one given by Chen *et al.* (1992a, 1992b). □

As was pointed out by Stoorvogel *et al.* (1994), for discrete-time H_∞ control, the infimum for the full information problem is, in general, different from that of the full state feedback problem. For the state feedback case, i.e. $C_1 = I$ and $D_{12} = 0$, we note that the subsystem (A, E, C_1, D_{12}) is always free of invariant zeros (and hence free of unit circle invariant zeros) and left invertible. Thus, as long as (A, B, C_2, D_{21}) is free of unit circle invariant zeros and satisfies assumption (A1), one can apply the above algorithm to get the infimum, γ^*. As is reported in Chen (1995b), for this special case, Γ_{sQ}, n_{xQ}, S_{xQ} and T_{xQ} in Step B of the above algorithm can be directly obrained using the following simple procedure. Compute a non-singular transformation Γ_{sQ} such that

$$\Gamma_{sQ}'E = \begin{bmatrix}0\\ \hat{E}\end{bmatrix} \tag{3.67}$$

where $\hat{E}$ is a $n_{xQ}\times n_{xQ}$ non-singular matrix. Then S_{xQ} and T_{xQ} are respectively given by

$$S_{xQ} = (\hat{E}^{-1})'\hat{E}^{-1} \quad \text{and} \quad T_{xQ} = 0 \tag{3.68}$$

and hence

$$\gamma^* = [\lambda_{\max}(T_{xP}S_{xP}^{-1}+\Gamma S_{xQ}^{-1}\Gamma' S_{xP}^{-1})]^{1/2} \tag{3.69}$$

Again, the following remark deals with the case when the direct feedthrough term from the disturbance to the controlled output of (1.1) is non-zero, i.e. $D_{22} \neq 0$.

Remark 3.2: For the case when $D_{22} \neq 0$, the assumptions (A1) and (A2) should be replaced by the conditions given in Remark 3.1, which is associated with the full information system of (3.62), and a similar set of conditions as in that remark but for the full information system of (3.63). Then our procedure would again carry through and yield a correct result. Note that if (A, B, C_2, D_{21}) is right invertible and (A, E, C_1, D_{12}) is left invertible, then all these conditions will be automatically satisfied. The result will then reduce to that of Chen (1995 b). □

3.3. *An illustrative example*

In this subsection, we will use a numerical example to illustrate our computational procedures developed in the previous subsections. We consider a given system characterized by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is can be verified that (A, B) is controllable and (A, B, C_2, D_{21}) is neither right nor left invertible, and is of non-minimum phase with two invariant zeros at 0 and 2, respectively. Moreover, it is already in the form of the special coordinate basis as given in Theorem 2.1, and assumption (A1) is satisfied as $E_b = 0$.

Case 1: The full information problem. We first consider the computation of γ^* for the full information case. Following the algorithm in §3.1, we obtain

$$\Gamma_s = I_5, \quad n_x = 3$$

$$A_x = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$C_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{A}_x = \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.50 & -0.50 & 0.50 \\ -0.25 & 0.75 & -0.25 \end{bmatrix}$$

$$\tilde{B}_x = \begin{bmatrix} 0.3125 & -0.1875 \\ -0.6250 & 1.3750 \\ 0.4375 & -1.0625 \end{bmatrix}, \quad \tilde{E}_x = \begin{bmatrix} 0.125 \\ 0.750 \\ -0.625 \end{bmatrix}$$

and

$$\tilde{C}_x = C_x, \quad \tilde{D}_x = \begin{bmatrix} 1{\cdot}000 & 0{\cdot}000 \\ 0{\cdot}250 & -0{\cdot}750 \\ -0{\cdot}125 & 0{\cdot}375 \end{bmatrix}, \quad \tilde{D}_{22} = \begin{bmatrix} 0{\cdot}00 \\ -0{\cdot}50 \\ 0{\cdot}25 \end{bmatrix}$$

It is simple to verity that $(\tilde{A}_x, \tilde{B}_x, \tilde{C}_x, \tilde{D}_x)$ is left invertible with two invariant zeros at 1 and 1/3, respectively. Solving Riccati equations (3.5) and (3.6), we obtain

$$\tilde{S}_x = \begin{bmatrix} 0{\cdot}227615 & -0{\cdot}207890 & 0{\cdot}019725 \\ -0{\cdot}207890 & 1{\cdot}202254 & -1{\cdot}005636 \\ 0{\cdot}019725 & -1{\cdot}005636 & 1{\cdot}014089 \end{bmatrix}$$

$$\tilde{T}_x = \begin{bmatrix} 0{\cdot}09375 & -0{\cdot}062500 & 0{\cdot}031250 \\ -0{\cdot}06250 & 0{\cdot}041667 & -0{\cdot}020833 \\ 0{\cdot}03125 & -0{\cdot}020833 & 0{\cdot}010417 \end{bmatrix}$$

Finally, we get

$$S_x = \begin{bmatrix} 0{\cdot}562306 & -0{\cdot}145898 & -0{\cdot}145898 \\ -0{\cdot}145898 & 0{\cdot}618034 & -0{\cdot}381966 \\ -0{\cdot}145898 & -0{\cdot}381966 & 0{\cdot}618034 \end{bmatrix}, \quad T_x = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\gamma^* = 0{\cdot}934173$$

Case 2: The full state feedback problem. Following the algorithm and the simplified procedure for the state feedback problem given in §3.2, we obtain those matrices as in the full information case and

$$\Gamma_{sQ} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad n_{xQ} = 1$$

$$S_{xQ} = 1, \quad T_{xQ} = 0, \quad \Gamma = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\gamma^* = 3{\cdot}181043$$

Case 3: The output feedback problem. Now, we consider the computation of γ^* for the given system with an output measurement characterized by

$$C_1 = [0 \quad 0 \quad 0 \quad 0 \quad 1], \quad D_{12} = 0$$

It can be shown that (A, C_1) is detectable and (A, E, C_1, D_{12}) is invertible with three invariant zeros at 0, 0·618 and $-1{\cdot}618$, respectively, and one infinite zero of order 2. Hence, Assumption (A2) is automatically satisfied. Following the algorithm of §3.2, we obtain those matrices as in the full information case and

$$\Gamma_{sQ} = \begin{bmatrix} -0{\cdot}455504 & -0{\cdot}563227 & 0{\cdot}191811 & 0 & 0 \\ -0{\cdot}455504 & 0{\cdot}226419 & 0{\cdot}191811 & 0 & 0 \\ 0{\cdot}737020 & -0{\cdot}366354 & 0{\cdot}118545 & 0 & 0 \\ 0{\cdot}173987 & 0{\cdot}703162 & -0{\cdot}502167 & 1 & 0 \\ 0{\cdot}107530 & -0{\cdot}053450 & 0{\cdot}812523 & -4 & 1 \end{bmatrix}, \quad n_{xQ} = 3$$

$$S_{xQ} = \begin{bmatrix} 23{\cdot}027553 & 3{\cdot}772507 & -2{\cdot}331538 \\ 3.772507 & 1 & 0 \\ -2.331538 & 0 & 1 \end{bmatrix}$$

$$T_{xQ} = \begin{bmatrix} 3{\cdot}359675 & 0 & -1{\cdot}440970 \\ 0 & 0 & 0 \\ -1{\cdot}440970 & 0 & 2 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 2{\cdot}331538 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = \left[\begin{matrix} 52{\cdot}087460 & 76{\cdot}552500 & 66{\cdot}462330 \\ 92{\cdot}575462 & 138{\cdot}464005 & 120{\cdot}137767 \\ 28{\cdot}034442 & 42{\cdot}124612 & 36{\cdot}888544 \\ 19{\cdot}202703 & 29{\cdot}289490 & 24{\cdot}966581 \\ 0 & 0 & 0 \\ -46{\cdot}978714 & -70{\cdot}777088 & -61{\cdot}6869177 \end{matrix}\right.$$
$$\left.\begin{matrix} -0{\cdot}959053 & 2{\cdot}618034 & -4{\cdot}236068 \\ -1{\cdot}653030 & 5{\cdot}236068 & -7{\cdot}854102 \\ -0{\cdot}693977 & 2{\cdot}618034 & -2{\cdot}618034 \\ 0 & 0 & -1{\cdot}440970 \\ 0 & 0 & 0 \\ 0{\cdot}959053 & -3{\cdot}618034 & 4.236068 \end{matrix}\right]$$

and finally

$$\gamma^* = 15{\cdot}16907$$

4. Solvability conditions for the discrete-time disturbance decoupling

We now present a set of necessary and sufficient conditions under which the well-known disturbance decoupling problem for the discrete-time systems is solvable. Again, as in the previous section, we will first assume that the D_{22} matrix in the given system of (1.1) is equal to zero. We will tackle the case when $D_{22} \neq 0$ later in the final remark. We first have the following result for the full information case.

Theorem 4.1: *Consider the given system of* (1.1) *with,* $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$, $D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$ *and* $D_{22} = 0$. *Then the following two statements are equivalent.*

(1) *There exists a controller of the form* (1.2) *such that the closed loop system is asymptotically stable and such that the closed loop transfer function from w to z is equal to* 0.

(2) (A, B) *is stabilizable and* $\operatorname{Im}(E) \subseteq \mathscr{V}^-(A, B, C_2, D_{21}) + B\operatorname{Ker}(D_{21})$.

Proof: In view of the proof of the algorithm in §3.1, we note that the existence of a stabilizing proper controller for the given system (1.1) with $C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}$ and $D_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$, such that the resulting closed loop transfer function from w to z is equal to 0, is equivalent to the existence of a stabilizing proper controller for the auxiliary system (3.10), such that the resulting closed loop transfer function from w_x to z_x is equal to 0. It is also equivalent to the existence of a stabilizing proper controller for the continuous-time auxiliary system (3.11), such that the resulting closed loop transfer function from $\tilde{w}_x$ to $\tilde{z}_x$ is equal to 0. Again, following the proof of the alogrithm in §3.1, it is clear that the latter and hence all the above three statements are equivalent to the existence of a stabilizing proper controller for the following auxiliary system

$$\left.\begin{aligned} \dot{\hat{x}} &= \hat{A}_x \hat{x}_x + \hat{B}_x \tilde{v}_x + \hat{E}_x \tilde{w}_x \\ \tilde{y}_x &= \begin{pmatrix} 0 \\ I \end{pmatrix} \hat{x}_x \qquad + \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{w}_x \\ \tilde{z}_x &= \hat{C}_x \hat{x}_x + \hat{D}_x \tilde{v}_x \end{aligned}\right\} \tag{4.1}$$

where $\hat{A}_x$, $\hat{B}_x$, $\hat{E}_x$, $\hat{C}_x$ and $\hat{D}_x$ are as defined from (3.45) to (3.49). Due to the fact that $(\hat{A}_x, \hat{B}_x, \hat{C}_x, \hat{D}_x)$ is left invertible with no infinite zeros and has only unstable invariant zeros, we have $\mathscr{V}^-(\hat{A}_x, \hat{B}_x, \hat{C}_x, \hat{D}_x) + \hat{B}_x \operatorname{Ker}(\hat{D}_x) = \{0\}$. It then follows from the result of Stoorvogel and van der Woude (1991) that the existence of a stabilizing proper controller for (4.1) which solves the disturbance decoupling problem, is equivalent to $(\hat{A}, \hat{B})$ being stabilizable and $\hat{E}_x = 0$, i.e. $E_a^+ = 0$, $E_b = 0$ and $E_d^* = 0$, where E_d^* is defined in (3.34). What we need to show next is to prove the fact that $\hat{E}_x = 0$, which implies, and is also implied by, $T_x = 0$, if and only if $\operatorname{Im}(E) \subseteq \mathscr{V}^-(A, B, C_2, D_{21}) + B\operatorname{Ker}(D_{21})$, which is equivalent to $\operatorname{Im}(E_x) \subseteq \operatorname{Im}(B_{x1})$. In view of the structures of $\hat{E}_x$, E_x and B_{x1}, it is sufficient to show that $E_d^* = 0$ if and only if $\operatorname{Im}(E_d) \subseteq \operatorname{Im}(B_d)$. Clearly, by the definition of E_d^* in (3.34), i.e.

$$E_d^* = E_d - B_d[C_d(A_{dd}+I)^{-1} B_d]^{-1} C_d(A_{dd}+I)^{-1} E_d \tag{4.2}$$

if $E_d^* = 0$, then we have

$$E_d = B_d[C_d(A_{dd}+I)^{-1} B_d]^{-1} C_d(A_{dd}+I)^{-1} E_d \tag{4.3}$$

Hence, E_d is in the range space of B_d. Conversely, if E_d is in the range space of B_d, i.e. $E_d = B_d X$ for some appropriate X, then we have

$$E_d^* = B_d X - B_d[C_d(A_{dd}+I)^{-1} B_d]^{-1} C_d(A_{dd}+I)^{-1} B_d X = 0 \tag{4.4}$$

This completes the proof of Theorem 4.1. □

The following theorem is for the general output feedback case.

Theorem 4.2: *Consider the given system of* (1.1) *with* $D_{22}=0$. *Then the following two statements are equivalent.*

(1) *There exists a controller of the form* (1.2) *such that the closed loop system is asymptotically stable and the closed loop transfer function from* w *to* z *is equal to* 0.

(2) (A,B) *is stabilizable,* (A,C_1) *is detectable, and*
  - (*a*) $\operatorname{Im}(E)\subseteq \mathscr{V}^-(A,B,C_2,D_{21})+B\operatorname{Ker}(D_{21})$,
  - (*b*) $\operatorname{Ker}(C_2)\supseteq \mathscr{S}^-(A,E,C_1,D_{12})\cap C_1^{-1}(D_{12})$,
  - (*c*) $\mathscr{S}^-(A,E,C_1,D_{12})\subseteq \mathscr{V}^-(A,B,C_2,D_{21})$.

Proof: The first statement is equivalent to (A,B) being stabilizable, (A,C_1) detectable, and $\gamma^*=0$. Hence $\lambda_{\max}(M)=0$, and the infima for the auxiliary systems (3.62) and (3.63) should be zero as well. The proof then follows from Theorem 4.1 that $T_{x\mathrm{P}}=0$, which is equivalent to condition 2(*a*), and $T_{x\mathrm{Q}}=0$, which is equivalent to condition 2(*b*). Furthermore, M reduces to

$$M=\begin{bmatrix}\Gamma S_{x\mathrm{Q}}^{-1}\Gamma' S_{x\mathrm{P}}^{-1} & -\Gamma S_{x\mathrm{Q}}^{-1}\\ 0 & 0\end{bmatrix} \tag{4.5}$$

Since both $S_{x\mathrm{P}}$ and $S_{x\mathrm{Q}}$ are positive definite, it is obvious that $\lambda_{\max}(M)=0$ if and only if $\Gamma=0$. Using a similar argument as in Chen *et al.* (1992 a), one can show that $\Gamma=0$ if and only if condition 2(*c*) holds.

Conversely, if conditions 2(*a*)–2(*c*) hold, it is obvious that $T_{x\mathrm{P}}=0$, $T_{x\mathrm{Q}}=0$ and $\Gamma=0$ and hence $\gamma^*=0$. In fact, it is quite straightforward to construct a controller of the form (1.2) that yields a zero closed loop transfer function from w to z. □

Remark 4.1: The following remarks are in order.

(1) It is interesting to note that the necessary and sufficient conditions for the solvability of a discrete-time disturbance decoupling problem are identical to its continuous-time counterpart as given by Stoorvogel and van der Woude (1991). However, as noted in the next item, one can test these conditions without computing any geometric subspaces using our approach.

(2) It follows from the proofs of Theorem 4.1 and Theorem 4.2 that the solvability conditions given in Theorem 4.2 are equivalent to the following

$$\operatorname{Im}(E_{x\mathrm{P}})\subseteq \operatorname{Im}(B_{x1\mathrm{P}}),\quad \operatorname{Im}(E_{x\mathrm{Q}})\subseteq \operatorname{Im}(B_{x1\mathrm{Q}})\quad \text{and}\quad \Gamma=0 \tag{4.6}$$

where $E_{x\mathrm{P}}$, $B_{x1\mathrm{P}}$, $E_{x\mathrm{Q}}$, $B_{x1\mathrm{Q}}$ and Γ are defined in Step A to Step C of §3.2. Obviously, the above conditions are computationally simple to verify.

(3) For the case that $D_{22}\neq 0$, following the result of Stoorvogel and van der Woude (1991), one can show that the solvability of the disturbance decoupling problem implies the existence of a matrix S such that

$$D_{22}+D_{21}SD_{12}=0 \tag{4.7}$$

Next we apply a pre-output feedback $u=Sy+v$ to the system of (1.1). The new system we thus obtain will have a zero direct feedthrough matrix from w to z.

□

5. Conclusions

We have presented in this paper non-iterative procedures that compute the H_∞ optimization infimum γ^* for a class of systems that satisfy certain geometric conditions, and a set of solvability conditions under which a solution to the general discrete-time disturbance decoupling problem exists. The algorithms for computing γ^* involve only the solutions of some γ independent algebraic Riccati equations and Lyapunov equations and the computation of the maximum eigenvalue of a constant matrix. We should also note that the geometric conditions imposed on the given systems are not essential and can be relaxed. This can be shown by some numerical examples. Hence, it leaves some room for improving our results. On the other hand, the solvability conditions of the disturbance decoupling problem, which are also necessary and sufficient, can be easily tested without computing any geometric subspaces. Note that the numerical computation of geometric subspaces is, in general, quite difficult.

References

Basar, T. S., and Bernard, P., 1989, *Differential Games and Applications*, Lecture Notes in Control and Information Science, Vol. 119 (Berlin: Springer-Verlag).

Chen, B. M., 1995a, A simple algorithm for the stable/unstable decomposition of a linear discrete-time system. *International Journal of Control*, **61**, 255–260; 1995b, Exact computation of infimum in discrete-time H_∞-optimization using measurement feedback. *Proceedings of the 13th IFAC Triennial World Congress*, San Francisco, California, U.S.A., July, Vol. G, pp. 151–156.

Chen, B. M., Saberi, A. and Ly, U., 1992a, Exact computation of the infimum in H_∞-optimization via output feedback. *IEEE Transactions on Automatic Control*, **37**, 70–78; 1992b, A non-iterative method for computing the infimum in H_∞-optimization. *International Journal of Control*, **56**, 1399–1418.

Chen, B. M., Saberi, A., and Shamash, Y., 1994, A non-recursive method for solving the general discrete-time Riccati equations related to the H_∞ control problem. *International Journal of Robust and Nonlinear Control*, **4**, 503–519.

Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., 1989, State-space solutions to standard H_2 and H_∞-control problems. *IEEE Transactions on Automatic Control*, **34**, 831–847.

Francis, B. A., 1987, *A Course in H_∞ Control Theory*. Lecture Notes in Control and Information Sciences, Vol. 88 (Berlin: Springer-Verlag).

Glover, K., 1984, All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds. *International Journal of Control*, **39**, 1115–1193.

Kailath, T., 1980, *Linear Systems* (Englewood Cliffs, New Jersey, U.S.A.: Prentice Hall).

Khargonekar, P., Petersen, I. R., and Rotea, M. A., 1988, H_∞-optimal control with state feedback. *IEEE Transactions on Automatic Control*, **33**, 786–788.

Kimura, H., 1989, Conjugation, interpolation and model-matching in H_∞. *International Journal of Control*, **49**, 269–307.

Lin, Z., Saberi, A., and Chen, B. M., 1991, *Linear System Toolbox* (Seattle, Washington, U.S.A.: AJ Control).

Richardson, T. J. and Kwong, R. H., 1986, On positive definite solutions to the algebraic Riccati equations. *Systems and Control Letters*, **7**, 99–104.

Saberi, A., and Sannuti, P., 1990, Squaring down of non-strictly proper systems. *International Journal of Control*, **51**, 621–629.

Sannuti, P., and Saberi, A., 1987, A special coordinate basis of multivariable linear systems-Finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, **45**, 1655–1704.

Stoorvogel, A. A., 1992, *The H_∞ Control Problem: a State Space Approach* (Englewood Cliffs, New Jersey, U.S.A.: Prentice Hall).

Stoorvogel, A. A., and van der Woude, J. W., 1991. The disturbance decoupling problem with measurement feedback and stability for systems with direct feedthrough matrices, *Systems & Control Letters*, **17**, 217–226.

Stoorvogel, A. A., Saberi, A., and Chen, B. M., 1994, The discrete-time H_∞ control problem with measurement feedback. *International Journal of Robust and Nonlinear Control*, **4**, 457–479.

Weiland, S., and Willems, J. C., 1989, Almost disturbance decoupling with internal stability. *IEEE Transactions on Autmoatic Control*, **34**, 277–286.

Zames, G., 1981, Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, **26**, 301–320.