Adaptive estimation and rejection of unknown sinusoidal disturbances through measurement feedback for a class of non-minimum phase non-linear MIMO systems

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SUMMARY

This paper develops an adaptive estimation method to estimate unknown disturbances in a class of non-minimum phase non-linear MIMO systems. The unknown disturbances are generated by an unknown linear exosystem. The frequencies, phases and amplitudes of the disturbances are unknown, the only available information of the disturbances is the number of distinctive frequencies. The system considered in this paper is a class of MIMO non-linear systems in the output feedback form which can be non-minimum phase. The proposed estimation algorithm provides exponentially convergent estimates of system states, unknown disturbances in the system and frequencies of the disturbances characterized by the eigenvalues of the exosystem. Moreover, based on the stabilization controller for the disturbance free system, the estimates of the disturbances are used to solve the disturbance rejection problem. The unknown disturbances are compensated completely with the stability of the whole closed-loop system. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: adaptive estimation; disturbance estimation; disturbance rejection; non-minimum-phase systems; non-linear systems

1. INTRODUCTION

Disturbance rejection problem has attracted great research interests in control theory and applications because there are various deterministic and random disturbances in engineering systems. In many disturbance rejection problems, especially for deterministic disturbance rejection, the disturbances are assumed to be known. For example, in non-linear output regulation problem [1], the disturbances to be rejected are assumed to be generated by a known linear exosystem. However, in most practical engineering systems, the disturbances may be unknown to us, that is, the frequencies, phases and amplitudes of the disturbances are unknown. The unknown disturbances make the disturbance rejection problem more challenging and thus

more interesting. A series of results have been published on the sinusoidal disturbance rejection problem for disturbances with unknown frequencies, see, for example, References [2–6], etc. Two algorithms, a direct and an indirect one, are presented in Reference [2] for disturbance compensation for stable linear time invariant systems. The indirect one estimates the disturbance frequency first and then compensates it. Only the direct one ensures the complete compensation or asymptotic rejection of disturbances with unknown frequencies. The algorithm proposed in Reference [4] ensures robust compensation of unknown disturbances for linear systems. For non-linear systems, a result for strict feedback non-linear system is reported in Reference [5] based on full state feedback. With adaptive internal model, semi-global non-linear output regulation problem is solved in Reference [6] using output feedback. More recently, global disturbance rejection with stabilization is reported in Reference [7] for non-linear systems in output feedback form. We noted that both References [6, 7] consider the minimum phase non-linear systems. For non-minimum phase non-linear systems, an adaptive estimation algorithm is proposed in Reference [8] to estimate the unknown sinusoidal disturbances for single-input and single-output (SISO) systems.

In this paper, we extend the results of Reference [8] to a class of non-minimum phase non-linear multi-input and multi-output (MIMO) systems. Our disturbance rejection scheme is an indirect one, the unknown disturbances are estimated separately, and then the estimates of the disturbances are used to solve the disturbance rejection problem. In the estimation stage, a new filter and adaptive law is designed to extract the contribution of the disturbances to the states and to estimate the disturbances and their frequencies. The estimation starts from the contribution to the output of the system, from which the disturbance characterization such as frequencies can be obtained. Based on this estimation and by transforming the given system into a so-called special co-ordinate basis (SCB) (see e.g., References [9, 10]), the contributions to other states can then be calculated. Finally, the unknown disturbances are reconstructed. To extract the contribution of the disturbances to the state from all output channels of the system, different from the SISO case, the filter is designed to a multi-input one. In the proposed estimation algorithm, only the number of distinct frequencies in the disturbances is required to be known. But, there is no restriction on the number or the range of disturbance frequencies. The estimates of the disturbance and frequencies converge to their ideal value exponentially. It is interesting that, after transforming the given non-linear system into special co-ordinate basis, the non-linear system is in the so-called output feedback form which has been extensively studied in the literature. The geometric conditions for transforming an affine non-linear system into a non-linear system in output feedback form are given in References [11, 12]. In the past two decades, various control problems have been investigated for the non-linear system in output feedback form, such as, global stabilization [11], adaptive output feedback control [13], non-linear output regulation [6, 14], unknown disturbance rejection [7, 15], and so on.

Stabilization of non-minimum phase non-linear systems is itself a very challenging problem. Until now, there is no general control design method to achieve global stabilization for the systems considered in this paper even when the systems are disturbance free. In the SISO cases, only a few results are available. For example, a global stabilization result is reported in Reference [16], and a semi-global stabilization result in Reference [17]. Therefore, to solve the disturbance rejection with stabilization problem, it is reasonable to assume that the stabilization problem for the non-minimum phase non-linear system is solvable for the corresponding disturbance free system. Thus, as an application of our adaptive estimation algorithm, we show that the estimates of the disturbances can be used to reject the disturbance

effectively under the assumption that the disturbance free system is stabilizable (Assumption A4). It should be noted that, apart from disturbance rejection, estimation and reconstruction of unknown disturbances have their own importance for detection and monitoring. It was until fairly recently that a global convergent estimation algorithm was proposed for estimation of a single frequency of the stand alone sinusoidal signal [18]. And, more recently, an algorithm was proposed to estimate multiple frequencies from a sinusoidal signal using adaptive observers [19].

The paper is organized as follows. Section 2 gives the problem formulation and preliminary analysis. Section 3 deals with the disturbance estimation and rejection. We first focus on estimating the unknown disturbances using some adaptive schemes and then following with an application of disturbance estimation to the disturbance rejection problem. An example on a benchmark problem is given in Section 4 to illustrate our adaptive estimation algorithm and rejection scheme. Finally, we draw some concluding remarks in Section 5.

2. PROBLEM FORMULATION AND PRELIMINARIES

We consider a MIMO non-linear system characterized by

$$\dot{\bar{z}} = \bar{A}\bar{z} + \bar{\phi}(\bar{y}) + \bar{B}\bar{u} + \bar{E}w \tag{1}$$

$$\bar{y} = \bar{C}\bar{z} \tag{2}$$

$\bar{z} \in \mathbb{R}^n$ is the state vector, $\bar{u} \in \mathbb{R}^m$ the control input, $\bar{y} \in \mathbb{R}^m$ the system output. $\bar{\phi}$ is a known non-linear smooth vector field in $\mathbb{R}^n$ with $\bar{\phi}(0) = 0$. $w \in \mathbb{R}^s$ is the disturbance generated by an unknown exosystem

$$\dot{w} = Sw, \quad w(0) = w_0 \tag{3}$$

with unknown $S \in \mathbb{R}^{s \times s}$. In this work, we propose to solve the following two problems.

Disturbance Estimation Problem: Estimate the disturbance w, the state $\bar{z}$ and the unknown disturbance frequencies characterized by the eigenvalues of S.

Disturbance Rejection Problem: Use the estimates obtained in the disturbance estimation problem to design a feedback control such that the closed-loop system is stable and the output of the system converges to zero, i.e. $\lim_{t \to \infty} y(t) = 0$.

To solve the above disturbance estimation problem and disturbance rejection problem, we make the following assumptions.

Assumption A1: $(\bar{A}, \bar{B}, \bar{C})$ is invertible[‡] and having no invariant zeros on the imaginary axis.

Assumption A2: The eigenvalues of S are distinct and located on the imaginary axis. The initial state w_0 is such that all the frequency components in the disturbance system are fully excited. Furthermore, the state w of the exosystem is observable for the output $\bar{y}$.

Assumption A3: $(\bar{A}, \bar{C})$ is detectable.

[‡] $(\bar{A}, \bar{B}, \bar{C})$ is said to be invertible if there exist two rational matrix functions of s, say $L(s)$ and $R(s)$, such that $L(s)H(s) = I_m$ and $H(s)R(s) = I_m$ where $H(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$.

Assumption A4: Consider the disturbance free system

$$\dot{\bar{z}} = \bar{A}\bar{z} + \bar{\phi}(\bar{y}) + \bar{B}\bar{u} \tag{4}$$

$$\bar{y} = \bar{C}\bar{z} \tag{5}$$

there exists a control law of the form

$$\dot{v} = f(v, \bar{y}) \tag{6}$$

$$\bar{u} = h(v, \bar{y}) \tag{7}$$

such that the closed-loop system (4)–(7) is asymptotically stable. Moreover, there exists a Lyapunov function $V(\bar{x})$ with $\bar{x} = \mathrm{col}(\bar{z}, v)$ such that

$$\alpha_1(\|\bar{x}\|) \leqslant V(\bar{x}) \leqslant \alpha_2(\|\bar{x}\|) \tag{8}$$

$$\frac{\partial V(\bar{x})}{\partial \bar{z}}(\bar{A}\bar{z} + \bar{\phi}(\bar{y}) + \bar{B}h(v, \bar{y})) + \frac{\partial V(\bar{x})}{\partial v}f(v, \bar{y}) \leqslant -\alpha_3(\|\bar{x}\|) \tag{9}$$

$$c_1 \left\| \frac{\partial V(\bar{x})}{\partial \bar{x}} \right\|^{c_2} \leqslant \alpha_3(\|\bar{x}\|) \tag{10}$$

where α_i, $i = 1, 2, 3$, are class $\mathcal{K}_\infty$ functions[§] and c_i, $i = 1, 2$, are positive reals with $c_2 > 1$.

Since $(\bar{A}, \bar{B}, \bar{C})$ is invertible, it follows from the result of the special co-ordinate basis (see, e.g. Reference [9]) that, there exist non-singular state, output and input transformations

$$\bar{z} = \Gamma_s z, \quad \bar{y} = \Gamma_o y, \quad \bar{u} = \Gamma_i u \tag{11}$$

which transform system (1)–(2) into

$$\dot{z} = Az + \phi(y) + Bu + Ew \tag{12}$$

$$y = Cz \tag{13}$$

where $\phi(y) = \Gamma_s^{-1}\bar{\phi}(\Gamma_o y)$ and $E = \Gamma_s^{-1}\bar{E}$, and

$$A = \Gamma_s^{-1}\bar{A}\Gamma_s = \begin{bmatrix} A_d & B_d E_z \\ L_z C_d & A_z \end{bmatrix} \tag{14}$$

$$B = \Gamma_s^{-1}\bar{B}\Gamma_i = \begin{bmatrix} B_d \\ 0 \end{bmatrix} \tag{15}$$

$$C = \Gamma_o^{-1}\bar{C}\Gamma_s = [C_d \quad 0] \tag{16}$$

[§]A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to the class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r \to \infty} \alpha(r) = \infty$, the function is said to belong to the class $\mathcal{K}_\infty$.

where

$$A_d = \text{blkdiag}\{A_1, \ldots, A_m\} + B_d E_d + L_d C_d$$

$$B_d = \text{blkdiag}\{B_1, \ldots, B_m\}$$

$$C_d = \text{blkdiag}\{C_1, \ldots, C_m\}$$

with

$$A_i = \begin{bmatrix} 0 & I_{r_i-1} \\ 0 & 0 \end{bmatrix}_{r_i \times r_i}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{r_i \times 1}, \quad C_i = [1 \quad 0]_{1 \times r_i}$$

for $i = 1, \ldots, m$, where $r_1, \ldots, r_m$ are some positive integers with $r_1 + \cdots + r_m = r$. A_z, E_z, E_d, L_z and L_d are known matrices with proper dimensions.

Moreover, under Assumptions A1 and A2, there exist X and U such that (see, e.g. Reference [1])

$$XS = AX + BU + E \tag{17}$$

$$0 = CX \tag{18}$$

Then, the state transformation $x = z - Xw$ for system (12)–(13) yields

$$\dot{x} = Ax + \phi(y) + B(u - \mu) \tag{19}$$

$$y = Cx \tag{20}$$

where

$$\mu = Uw \tag{21}$$

$$\dot{w} = Sw \tag{22}$$

Hence, without loss of generality, we investigate the disturbance estimation problem and the disturbance rejection problem for system (19)–(22) and assume that the triple (A, B, C) is in the form of the special co-ordinate basis given in (14)–(16). Note that the transformations in (11) are all non-singular, and Assumption A4 is free of disturbance w, system (19)–(22) also satisfies Assumptions A1–A4 if system (1)–(3) does.

Remark 2.1
Assumption A3 is obviously necessary. Assumptions A1 and A2 are to avoid the overlap between the poles of the internal zero dynamics of the given system and those of the exosystem. It is clear that system (19)–(20) has a vector relative degree (defined in Reference [20]) $\{r_1, \ldots, r_m\}$ with $r = r_1 + \cdots + r_m$. Splitting $x = \text{col}(x_d, x_z)$ with $x_d \in \mathbb{R}^r$ and $x_z \in \mathbb{R}^{n-r}$, the zero dynamics of system (19)–(20) is given by

$$\dot{x}_z = A_z x_z$$

By Assumption A1, A_z has no eigenvalues on the imaginary axis. However, Assumption A1 can be relaxed to that the eigenvalues of A_z are distinct with the ones of S, which can be seen in our practical example in Section 4.

Remark 2.2
The restriction on S in Assumption A2 is quite standard in the non-linear output regulation problem (Reference [1]). However, unlike the neutral stable assumptions on the exosystem in References [1, 20, 21], the amplitudes, phases and frequencies of the disturbances are unknown. What we only know is the dimension of S. The dimension of S decides the number of independent frequencies in the disturbances. In case there is a degeneration of independent frequencies in the disturbance due to the initial state w_0, the exosystem can be reduced in dimension such that the disturbance is fully excited in the reduced order. Therefore, for a disturbance with known number of independent frequencies, Assumption A2 does not impose a restriction on the initial state of the exosystem.

Remark 2.3
Even without disturbance, stabilization problem of non-minimum phase non-linear systems is a challenging problem itself. To utilize the obtained disturbance estimate in the disturbance estimation problem to solve the disturbance rejection problem, Assumption A4 is given. According to the Inverse Lyapunov Theorem [22], (8) and (9) are automatically satisfied if the closed-loop system is asymptotically stable. (10) is always satisfied if the closed-loop system is exponentially stable. However, there exist systems that the conditions in Assumption A4 are all satisfied, but the systems are not exponentially stable [22].

Under Assumption A3, there exists $K \in \mathbb{R}^{n \times m}$ such that $A + KC$ is Hurwitz. Without considering the disturbance in (19), we can design a state observer as

$$\dot{p} = (A + KC)p + \phi(y) + Bu - Ky \tag{23}$$

where $p \in \mathbb{R}^n$. Assumption A2 and the design of K imply that $A + KC$ and S have exclusive eigenvalues. Therefore, given S, there exists an unique solution $Q \in \mathbb{R}^{n \times s}$ for the following Sylvester equation:

$$QS = (A + KC)Q + BU \tag{24}$$

Then, defining

$$q(w) = Qw \tag{25}$$

gives

$$\dot{q} = (A + KC)q + B\mu \tag{26}$$

Moreover, the state variable x of (19) can be expressed as

$$x = p - q + \epsilon \tag{27}$$

where p and q are generated from the observer (23) and the filter (26), respectively, and ϵ satisfies

$$\dot{\epsilon} = (A + KC)\epsilon \tag{28}$$

That is, the state estimation is solved if q is available. Unfortunately, since S is unknown, we cannot obtain Q from Equation (24), also the filer (26) cannot be implemented due to the unknown disturbance μ. To solve the disturbance estimation problem, we will develop an adaptive estimation algorithm. To this end, we introduce a reformulation of the exosystem (3). Let $\{F, G\}$ be any controllable pair with $F \in \mathbb{R}^{s \times s}$ Hurwitz and $G \in \mathbb{R}^{s \times m}$. Consider the following

Sylvester equation:

$$MS - FM = GCQ \tag{29}$$

We claim that (29) has a non-singular solution $M \in \mathbb{R}^{s \times s}$. In fact, since $\{F, G\}$ is controllable and there is no overlap between the eigenvalues of F and S, we just need to prove that $\{S, CQ\}$ is observable. Using (3), (23), (25), (27) and (28), we have

$$\dot{p} = Ap - KC\epsilon + KCQw + \phi(y) + Bu$$

$$\dot{\epsilon} = (A + KC)\epsilon$$

$$\dot{w} = Sw$$

$$y = Cp + C\epsilon - CQw$$

Noting that

$$\begin{bmatrix} \lambda I - A & KC & -KCQ \\ 0 & \lambda I - A - KC & 0 \\ 0 & 0 & \lambda I - S \\ C & C & -CQ \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 & K \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \lambda I - A - KC & 0 & 0 \\ 0 & \lambda I - A - KC & 0 \\ 0 & 0 & \lambda I - S \\ C & C & -CQ \end{bmatrix}$$

under Assumption A2, w is observable from y, it is necessary that (S, CQ) is observable.

Now, introducing a state transform of the exosystem

$$\eta = Mw \tag{30}$$

we have

$$\dot{\eta} = (F + GCQM^{-1})\eta \coloneqq F_0\eta \tag{31}$$

In the new co-ordinate η, q and μ can be expressed, respectively, as

$$q = QM^{-1}\eta, \quad \mu = UM^{-1}\eta$$

Let $\psi_j^{di} \in \mathbb{R}^{s \times 1}$, $i = 1, \ldots, m$, $j = 1, \ldots, r_m$, and $\psi_j^z \in \mathbb{R}^{s \times 1}$, $j = 1, \ldots, n - r$ be the columns of $(QM^{-1})^{\mathrm{T}}$, and $\psi_j^u \in \mathbb{R}^{s \times 1}$ be the columns of $(UM^{-1})^{\mathrm{T}}$, then

$$q = QM^{-1}\eta \coloneqq [\psi_1^{d1}, \ldots, \psi_{r_1}^{d1}, \ldots, \psi_1^{dm}, \ldots, \psi_{r_m}^{dm}, \psi_1^z, \ldots, \psi_{n-r}^z]^{\mathrm{T}}\eta \tag{32}$$

$$\mu = UM^{-1}\eta \coloneqq [\psi_1^u, \ldots, \psi_m^u]^{\mathrm{T}}\eta \tag{33}$$

For convenience, in the following we denote:

$$\psi^x = [\psi^d \quad \psi^z] \tag{34}$$

$$\psi^u = [\psi_1^u, \ldots, \psi_m^u] \tag{35}$$

with

$$\psi^d = [\psi_1^{d1}, \ldots, \psi_{r_1}^{d1}, \ldots, \psi_1^{dm}, \ldots, \psi_{r_m}^{dm}] \tag{36}$$

$$\psi^z = [\psi_1^z, \ldots, \psi_{n-r}^z] \tag{37}$$

Splitting K, L_d, E_d and E_z as follows:

$$K = \begin{bmatrix} K_d \\ K_z \end{bmatrix} = \begin{bmatrix} K_{d1}^1 & \cdots & K_{d1}^m \\ \vdots & \vdots & \vdots \\ K_{dm}^1 & \cdots & K_{dm}^m \\ K_z^1 & \cdots & K_z^m \end{bmatrix}, \quad L_d = \begin{bmatrix} L_{d1}^1 & \cdots & L_{d1}^m \\ \vdots & \vdots & \vdots \\ L_{dm}^1 & \cdots & L_{dm}^m \end{bmatrix}$$

where $K_{dj}^i \in \mathbb{R}^{r_j \times 1}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant m$, $K_z^i \in \mathbb{R}^{(n-r) \times 1}$ for $1 \leqslant i \leqslant m$, $L_{dj}^i \in \mathbb{R}^{r_j \times 1}$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m$, and

$$E_d = \begin{bmatrix} E_d^1 \\ \vdots \\ E_d^m \end{bmatrix}, \quad E_z = \begin{bmatrix} E_z^1 \\ \vdots \\ E_z^m \end{bmatrix}$$

where $E_d^i \in \mathbb{R}^{1 \times r}$ and $E_z^i \in \mathbb{R}^{1 \times (n-r)}$ for $1 \leqslant i \leqslant m$. Then, substituting (32) and (33) into (26) gives, for $i = 1, \ldots, m$,

$$(\psi_j^{di})^{\mathrm{T}} F_0 = (\psi_{j+1}^{di})^{\mathrm{T}} + \sum_{k=1}^{m} (L_{di}^k(j) + K_{di}^k(j))(\psi_1^{dk})^{\mathrm{T}}, \quad j = 1, \ldots, r_i - 1 \tag{38}$$

$$(\psi_{r_i}^{di})^{\mathrm{T}} F_0 = (\psi_i^u)^{\mathrm{T}} + \sum_{k=1}^{m} (L_{di}^k(r_i) + K_{di}^k(r_i))(\psi_1^{dk})^{\mathrm{T}} + E_d^i(\psi^d)^{\mathrm{T}} + E_z^i(\psi^z)^{\mathrm{T}} \tag{39}$$

where $L_{di}^k(j)$ and $K_{di}^k(j)$ are the jth item of the vectors L_{di}^k and K_{di}^k, respectively, and

$$(\psi^z)^{\mathrm{T}} F_0 = (L_z + K_z) C_d(\psi^d)^{\mathrm{T}} + A_z(\psi^z)^{\mathrm{T}} \tag{40}$$

Noting that $(L_z + K_z)C_d(\psi^d)^{\mathrm{T}}$ only involves in $\psi_1^{d1}, \psi_1^{d2}, \ldots, \psi_1^{dm}$, if F_0 is known, we can solve out ψ^d, ψ^z and ψ^u from (38) to (40) in terms of $\psi_1^{d1}, \psi_1^{d2}, \ldots, \psi_1^{dm}$. In fact, using the notation $\otimes$ for the Kronecker product of matrices, and vec($\cdot$) for the vector obtained by rolling out the

column vectors of a matrix, we can obtain from (40) that

$$\text{vec}(\psi^z) = \Sigma^{-1}\text{vec}((\psi^d)C_d^{\mathrm{T}}(L_z + K_z)^{\mathrm{T}}) \tag{41}$$

where

$$\Sigma = I_{n-r} \otimes F_0^{\mathrm{T}} - A_z \otimes I_s$$

and then ψ^d and ψ^u can be obtained from (38) and (39), respectively. In the next section, we will propose an adaptive estimation algorithm to solve the disturbance estimation problem, that is, to estimate η, and $\psi_1^{d1}, \psi_1^{d2}, \ldots, \psi_1^{dm}$, and then we can estimate x, μ and F_0.

Remark 2.4
Using (29), we have

$$F_0 = F + GCQM^{-1} = MSM^{-1}$$

that is, the eigenvalues of F_0 are the same as those of S. Thus, under Assumptions A1 and A2, all the eigenvalues of F_0 are exclusively different from the ones of A_z, which guarantees Σ is non-singular. Moreover, we can estimate the disturbance frequencies characterized by the eigenvalues of S through the estimation of F_0.

3. DISTURBANCE ESTIMATION AND REJECTION

From the previous analysis, it is clear that q and μ can be estimated or evaluated if η, and $\psi_1^{d1}, \psi_1^{d2}, \ldots, \psi_1^{dm}$ are available. In this section, we will develop an adaptive estimation algorithm to estimate η, and $\psi_1^{d1}, \psi_1^{d2}, \ldots, \psi_1^{dm}$. To this end, denote

$$\psi_p = \psi^x C^{\mathrm{T}} = [\psi_1^{d1}, \psi_1^{d2}, \ldots, \psi_1^{dm}]$$

and consider the following filters and adaptive law:

$$\dot{\xi} = F\xi + G(Cp - y) \tag{42}$$

$$\dot{\zeta} = F\zeta + G\hat{\psi}_p^{\mathrm{T}}\xi \tag{43}$$

$$\dot{\hat{\psi}}_p = \Gamma\xi(\xi - \zeta)^{\mathrm{T}}PG \tag{44}$$

where $\hat{\psi}_p \in \mathbb{R}^{s \times m}$ is the estimate of ψ_p, Γ is a positive definite matrix, and P is the positive definite solution of

$$PF + F^{\mathrm{T}}P = -2I_s \tag{45}$$

Then, we have the following result.

Lemma 3.1
There exist some positive real constants d_ξ, d_ψ, λ_ξ and λ_ψ such that

$$\|\eta(t) - \xi(t)\| \leqslant d_\xi e^{-\lambda_\xi t} \tag{46}$$

$$\|\psi_p - \hat{\psi}_p(t)\| \leqslant d_\psi e^{-\lambda_\psi t} \tag{47}$$

Proof
Define $e_\xi = \eta - \xi$, (31) and (42) gives

$$\dot{e}_\xi = F e_\xi + G C \epsilon \tag{48}$$

Combining (48) and (28), we have

$$\begin{bmatrix} \dot{e}_\xi \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} F & GC \\ 0 & (A + KC) \end{bmatrix} \begin{bmatrix} e_\xi \\ \epsilon \end{bmatrix} \tag{49}$$

Noting that F and $A + KC$ are Hurwitz, it is clear that there exist positive reals d_ξ and λ_ξ such that (46) is satisfied.

To establish the convergence of $\hat{\psi}_p$, we define

$$e = \xi - \zeta \tag{50}$$

then, we have

$$\dot{e} = Fe + G \psi_p^T e_\xi - GC\epsilon + G \tilde{\psi}_p^T \xi \tag{51}$$

where $\tilde{\psi}_p = \psi_p - \hat{\psi}_p$. Denote

$$\bar{e} = [e^T \quad e_\xi^T \quad \epsilon^T]^T$$

Then, by (44), (49) and (51), we can arrange the adaptive system in the following form:

$$\dot{\bar{e}} = A_a \bar{e} + \Omega(t)^T \text{vec}(\tilde{\psi}_p) \tag{52}$$

$$\text{vec}(\dot{\tilde{\psi}}_p) = -\Gamma_e \Omega(t) \bar{P} \bar{e} \tag{53}$$

where

$$A_a = \begin{bmatrix} F & G\psi_p^T & -GC \\ 0 & F & GC \\ 0 & 0 & A + KC \end{bmatrix}, \quad \Omega(t) = \begin{bmatrix} \begin{bmatrix} \xi & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \xi \end{bmatrix} G^T & 0 & 0 \end{bmatrix},$$

$$\Gamma_e = \begin{bmatrix} \Gamma & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Gamma \end{bmatrix} \quad \text{and} \quad \bar{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & \gamma_1 P & 0 \\ 0 & 0 & \gamma_2 P_\epsilon \end{bmatrix}$$

where γ_1 and γ_2 are sufficient large positive reals to be selected later and P_ϵ is the positive definite matrix satisfying

$$P_\epsilon(A + KC) + (A + KC)^T P_\epsilon = -2I_n$$

Let

$$\bar{Q} = -(\bar{P}A_a + A_a^{\mathrm{T}}P) = \begin{bmatrix} 2I_s & -PG\psi_p^{\mathrm{T}} & PGC \\ -\psi_p G^{\mathrm{T}}P & 2\gamma_1 I_s & -\gamma_1 PGC \\ C^{\mathrm{T}}G^{\mathrm{T}}P & -\gamma_1 C^{\mathrm{T}}G^{\mathrm{T}}P & 2\gamma_2 I_n \end{bmatrix} \tag{54}$$

It is clear that we can make $\bar{Q}$ positive definite by choosing a sufficient large γ_1 and then a sufficient large γ_2. Define

$$V_e(\bar{e}, \tilde{\psi}_p) = \bar{e}^{\mathrm{T}}\bar{P}\bar{e} + (\mathrm{vec}(\tilde{\psi}_p))^{\mathrm{T}}\Gamma_e(\mathrm{vec}(\tilde{\psi}_p)) \tag{55}$$

then we have

$$\dot{V}_e(\bar{e}, \tilde{\psi}_p) = -\bar{e}\bar{Q}\bar{e} \tag{56}$$

Therefore, $\bar{e}$ and $\tilde{\psi}_p$ are bounded.

To establish the convergence of $\tilde{\psi}_p$, we need the consistent excitation condition of $\Omega(t)$. From the definition of η in the previous section, it can be seen that η is persistently excited, i.e. there exist two positive reals T and γ_3 such that

$$\int_t^{t+T} \eta(\tau)\eta(\tau)^{\mathrm{T}}\,\mathrm{d}\tau \geqslant \gamma_3 I_s > 0 \quad \forall t \geqslant 0 \tag{57}$$

With

$$\int_t^{t+T} \Omega(\tau)\Omega(\tau)^{\mathrm{T}}\,\mathrm{d}\tau = \|G\|^2 \int_t^{t+T} \begin{bmatrix} \xi & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \xi \end{bmatrix} \begin{bmatrix} \xi^{\mathrm{T}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \xi^{\mathrm{T}} \end{bmatrix}\,\mathrm{d}\tau$$

$$= \|G\|^2 \int_t^{t+T} \begin{bmatrix} (\eta - e_\xi)(\eta - e_\xi)^{\mathrm{T}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\eta - e_\xi)(\eta - e_\xi)^{\mathrm{T}} \end{bmatrix}\,\mathrm{d}\tau$$

and the fact that η is bounded and e_ξ convergence to zero exponentially, we can conclude that there exist a $t_0 >$ and a γ_4 with $0 < \gamma_4 < \gamma_3\|G\|^2$ such that

$$\int_t^{t+T} \Omega(\tau)\Omega(\tau)^{\mathrm{T}}\,\mathrm{d}\tau \geqslant \gamma_4 I > 0 \quad \forall t \geqslant t_0 > 0$$

Since $\bar{e}(t_0)$ and $\tilde{\psi}(t_0)$ are bounded, we apply Lemma B.2.3 [23] to obtain that $\mathrm{col}(\bar{e}, \mathrm{vec}(\tilde{\psi}_p)) = 0$ is a globally exponentially stable equilibrium point for the adaptive system (52)–(53), which implies (47). □

Remark 3.1
In the proof of Lemma 3.1, we have argued that the positive real numbers γ_1 and γ_2 can be set to large enough values, for the convenience of establishing the positive define matrix $\bar{Q}$. It should

be noted that even though they appear in $\bar{P}$, they do not affect the choice of Γ in the adaptive law in (44).

With the estimates of ψ_p and η ready, we can obtain the estimate of F_0 by using

$$F_0 = F + GCQM^{-1} = F + GC(\psi^x)^{\mathrm{T}} = F + G\psi_p^{\mathrm{T}}$$

and then of ψ_j^{di}, $i = 1, \ldots, m$, $j = 2, \ldots, r_i$, ψ^z and ψ^u from (38), (41) and (39), respectively, and finally of the state x and the disturbance μ. Specifically,

$$\hat{F}_0 = F + G\hat{\psi}_p^{\mathrm{T}} \tag{58}$$

Denote

$$\hat{\psi}^x = [\hat{\psi}_1^{d1}, \ldots, \hat{\psi}_{r_1}^{d1}, \ldots, \hat{\psi}_1^{dm}, \ldots, \hat{\psi}_{r_m}^{dm}, \hat{\psi}_1^z, \ldots, \hat{\psi}_{n-r}^z]$$

$$\hat{\psi}^u = [\hat{\psi}_1^u, \ldots, \hat{\psi}_m^u]$$

then using (38) and (41), we have, for $i = 1, \ldots, m$,

$$\hat{\psi}_j^{di} = \hat{F}_0^{\mathrm{T}} \hat{\psi}_{j-1}^{di} - \sum_{k=1}^m (L_{di}^k(j-1) + K_{di}^k(j-1)) \hat{\psi}_1^{dk}, \quad j = 2, \ldots, r_i \tag{59}$$

and

$$\mathrm{vec}(\hat{\psi}^z) = \frac{|\hat{\Sigma}|}{\sigma + |\hat{\Sigma}|^2} \mathrm{adj}(\hat{\Sigma}) \mathrm{vec}(\hat{\psi}_p(L_z + K_z)^{\mathrm{T}}) \tag{60}$$

with

$$\hat{\Sigma} = I_{n-r} \otimes \hat{F}_0^{\mathrm{T}} - A_z \otimes I_s$$

$$\dot{\sigma} = -\lambda_\sigma \sigma, \quad \sigma(0) = \sigma_0$$

for some positive reals λ_σ and σ_0, where $|\cdot|$ and $\mathrm{adj}(\cdot)$ denote the determinant and the adjoint matrix of a matrix, respectively.

Then, by (39), we have, for $i = 1, \ldots, m$

$$\hat{\psi}_i^u = \hat{F}_0^{\mathrm{T}} \hat{\psi}_{r_i}^{di} - \sum_{k=1}^m (L_{di}^k(r_i) + K_{di}^k(r_i)) \hat{\psi}_1^{dk} - \hat{\psi}^d (E_d^i)^{\mathrm{T}} - \hat{\psi}^z (E_z^i)^{\mathrm{T}} \tag{61}$$

Finally, the state and disturbance estimation are given by

$$\hat{x} = p - (\hat{\psi}^x)^{\mathrm{T}} \xi \tag{62}$$

$$\hat{\mu} = (\hat{\psi}^u)^{\mathrm{T}} \xi \tag{63}$$

Moreover, the properties of the estimation of (58), (62) and (63) are described by the following theorem.

Theorem 3.1
Under Assumptions A1–A3, the disturbance estimation problem is solved by (58), (62) and (63) based on the filter and adaptive law (42), (43) and(44). Moreover, there exist positive real

constants λ_x, d_x, λ_μ, d_μ, λ_F and d_F such that

$$\|x(t) - \hat{x}(t)\| \leqslant d_x e^{-\lambda_x t} \tag{64}$$

$$\|\mu(t) - \hat{\mu}(t)\| \leqslant d_\mu e^{-\lambda_\mu t} \tag{65}$$

$$\|F_0 - \hat{F}_0(t)\| \leqslant d_F e^{-\lambda_F t} \tag{66}$$

Proof

From the previous description, we have shown that, under Assumptions A1–A3, we can estimate F, x and μ by (58), (62) and (63), respectively, based on the filter and adaptive law (42), (43) and (44). We say that an estimate is an exponentially convergent estimate if the estimation error is bounded by a decaying exponential function. Thus, we need to show that $\hat{x}$, $\hat{\mu}$ and $\hat{F}_0$ are exponentially convergent estimates. Let $\tilde{F}_0 = F_0 - \hat{F}_0$, we have

$$\|\tilde{F}_0\| = \|G\tilde{\psi}_p^{\mathrm{T}}\| \leqslant \|G\|\|\tilde{\psi}_p^{\mathrm{T}}\| \leqslant \|G\| d_\psi e^{-\lambda_\psi t}$$

That is, $\hat{F}_0$ is an exponentially convergent estimate. Let $\tilde{\psi} = \psi - \hat{\psi}$, and denote

$$[\tilde{\psi}_1^{d1}, \ldots, \tilde{\psi}_{r_1}^{d1}, \ldots, \tilde{\psi}_1^{dm}, \ldots, \tilde{\psi}_{r_m}^{dm}, \tilde{\psi}_1^{z}, \ldots, \tilde{\psi}_{n-r}^{z}]$$
$$= [\psi_1^{d1} - \hat{\psi}_1^{d1}, \ldots, \psi_{r_1}^{d1} - \hat{\psi}_{r_1}^{d1}, \ldots, \psi_1^{dm} - \hat{\psi}_1^{dm}, \ldots, \psi_{r_m}^{dm} - \hat{\psi}_{r_m}^{dm},$$
$$\psi_1^{z} - \hat{\psi}_1^{z}, \ldots, \psi_{n-r}^{z} - \hat{\psi}_{n-r}^{z}]$$

then we have, for $i = 1, \ldots, m$,

$$\|\tilde{\psi}_j^{di}\| \leqslant \|F_0\|\|\tilde{\psi}_{j-1}^{di}\| + \|\hat{\psi}_{j-1}^{di}\|\|\tilde{F}_0\| + \sum_{k=1}^{m}(\|L_{di}^k(j) + K_{di}^k(j)\|)\|\tilde{\psi}_1^{dk}\| \tag{67}$$

Note that $\hat{\psi}_p$ and $\hat{F}_0$ are exponentially convergent estimates, and $\hat{\psi}_p$ is bounded, by using (67) recursively for $j = 2, \ldots, r_i$ and repeat for $i = 1, \ldots, m$, we can conclude that $\hat{\psi}_j^{di}$, for $i = 1, \ldots, m$, and $j = 2, \ldots, r_i$, are exponentially convergent estimates.

Now consider

$$\text{vec}(\psi^z) - \text{vec}(\hat{\psi}^z) = \frac{|\hat{\Sigma}|^2(\text{adj}(\Sigma)\text{vec}(\psi_p(L_z + K_z)^{\mathrm{T}}) - \text{adj}(\hat{\Sigma})\text{vec}(\hat{\psi}_p(L_z + K_z)^{\mathrm{T}}))}{|\Sigma|(\sigma + |\hat{\Sigma}^2|)}$$

$$+ \frac{|\hat{\Sigma}|(|\hat{\Sigma}| - |\Sigma|)\text{adj}(\hat{\Sigma})\text{vec}(\hat{\psi}_p(L_z + K_z)^{\mathrm{T}})}{|\Sigma|(\sigma + |\hat{\Sigma}^2|)}$$

$$+ \frac{\sigma\text{adj}(\Sigma)\text{vec}(\psi_p(L_z + K_z)^{\mathrm{T}})}{|\Sigma|(\sigma + |\hat{\Sigma}^2|)} \tag{68}$$

It is clear that $|\hat{\Sigma}|$ and $\text{adj}(\hat{\Sigma})\text{vec}(\hat{\psi}_p(L_z + K_z)^{\mathrm{T}})$ are exponentially convergent estimates of $|\Sigma|$ and $\text{adj}(\Sigma)\text{vec}(\psi_p(L_z + K_z)^{\mathrm{T}})$, respectively, because they are functions of the elements of $\hat{\psi}_p$ obtained by multiplications and additions. Moreover, σ is a decaying exponential function. Thus, by (68), we can show that $\hat{\psi}^z$ is an exponentially convergent estimate of ψ^z. Therefore, we have shown that $\hat{\psi}^x$ is an exponentially convergent estimate.

Furthermore, we have, for $i = 1, \ldots, m$,

$$\psi_i^u - \hat{\psi}_i^u = F_0 \tilde{\psi}_{r_i}^{di} + \tilde{F}_0 \hat{\psi}_{r_i}^{di} - \sum_{k=1}^{m} (L_{di}^k(r_i) + K_{di}^k(r_i)) \tilde{\psi}_1^{dk} - \tilde{\psi}^d (E_d^i)^{\mathrm{T}} - \tilde{\psi}^z (E_z^i)^{\mathrm{T}} \qquad (69)$$

Using the similar reasoning as that of (67), we can conclude that $\hat{\psi}^u$ is also an exponentially convergent estimate.

Finally, from

$$\|x - \hat{x}\| = \|\epsilon - (\psi^x)^{\mathrm{T}} \eta + (\hat{\psi}^x)^{\mathrm{T}} \xi\| \leqslant \|\epsilon\| + \|\psi^x\| \|\eta - \xi\| + \|\hat{\psi}^x - \psi^x\| \|\eta\|$$

$$\|\mu - \hat{\mu}\| = \|(\psi^u)^{\mathrm{T}} \eta - (\hat{\psi}^u)^{\mathrm{T}} \xi\| \leqslant \|\psi^u\| \|\eta - \xi\| + \|\psi^u - \hat{\psi}^u\| \|\xi\|$$

we can conclude that $\hat{x}$ and $\hat{\mu}$ are also exponentially convergent estimates of x and μ. $\qquad \square$

Next, based on the control law designed without considering the disturbance, we can solve the disturbance rejection problem by using the disturbance estimate obtained above.

Theorem 3.2
Under Assumptions A1–A4, the control law

$$u = h(v, y) + \hat{\mu} \qquad (70)$$

$$\dot{v} = f(v, y) \qquad (71)$$

solves the disturbance rejection problem.

Proof
Under Assumptions A1–A3, by Theorem 3.1, we can obtain the estimate $\hat{\mu}$ of the disturbance μ such that

$$\|\tilde{\mu}(t)\| = \|\mu(t) - \hat{\mu}(t)\| \leqslant d_\mu e^{-\lambda_\mu t}$$

for some real constants d_μ and λ_μ. Thus, we can construct a first-order system

$$\dot{\bar{\mu}} = -\lambda_{\bar{\mu}} \bar{\mu}, \quad \bar{\mu}(0) = \bar{\mu}_0$$

such that $|\bar{\mu}(t)| \geqslant \|\tilde{\mu}\|$. Let $\bar{x} = \mathrm{col}(x, v)$, and define a Lyapunov function candidate

$$W(\bar{x}, \bar{\mu}) = V(\bar{x}) + c_3 |\bar{\mu}|^{c_4}$$

where c_3 and c_4 are positive real constants with $c_4 = c_2/(c_2 - 1)$. Then

$$\dot{W} = \frac{\partial V(\bar{x})}{\partial x}(Ax + \phi(y) + B(h(v, y) + \hat{\mu} - \mu)) + \frac{\partial V(\bar{x})}{\partial v} f(v, y) - c_3 c_4 \lambda_{\bar{\mu}} |\bar{\mu}|^{c_4}$$

$$\leqslant - \alpha_3(\|\bar{x}\|) - c_3 c_4 \lambda_{\bar{\mu}} |\bar{\mu}|^{c_4} + \left\| \frac{\partial V(\bar{x})}{\partial \bar{x}} \right\| \|B\| \|\tilde{\mu}\|$$

$$\leqslant - \alpha_3(\|\bar{x}\|) - c_3 c_4 \lambda_{\bar{\mu}} |\bar{\mu}|^{c_4} + \frac{c_1}{2} \left\| \frac{\partial V(\bar{x})}{\partial \bar{x}} \right\|^{c_2} + \frac{\|B\|^{c_4}}{c_4 \left(\frac{c_1 c_2}{2} \right)^{c_4/c_2}} \|\bar{\mu}\|^{c_4}$$

Let

$$c_3 = \frac{2\|B\|^{c_4}}{\lambda_{\bar{\mu}} c_4^2 \left(\frac{c_1 c_2}{2} \right)^{c_4/c_2}}$$

and using (10), we have

$$\dot{W} \leqslant -\tfrac{1}{2}(\alpha_3(\|\bar{x}\|) + c_3 c_4 |\bar{\mu}|^{c_4})$$

that is, the extended system with state $(\bar{x}, \bar{\mu})$ is asymptotically stable, which implies $\lim_{t\to\infty} x(t) = 0$. Moreover, $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} Cx(t) = 0$. ◻

4. A BENCHMARK PROBLEM

In this section, we will consider a benchmark problem, i.e. disturbance rejection problem of the rotational/translational Actuator (RTAC) system. The model of the RTAC system, as shown in Figure 1, consists of a translation cart of mass M connected to a wall by a spring of stiffness k, and a rotational actuator mounted on the cart. The rotational actuator consists of a proof mass of mass m and centroidal moment of inertia I mounted at a fixed distance d from its centre of rotation. The control torque N is applied to the rotational proof mass, while the disturbance f is applied to the translational cart. The normalized motion equation of the RTAC system is given by [1, 24, 25]

$$\ddot{\varsigma} + \varsigma = \varepsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + w \tag{72}$$

$$\ddot{\theta} = -\varepsilon\ddot{\varsigma}\cos\theta + \upsilon \tag{73}$$

where ς is the normalized displacement of the cart, θ the angular position of the eccentric mass, w the normalized disturbance, υ the normalized control input. The coupling between the translational and rotational motion is captured by the parameter ε which is defined by

$$\varepsilon = \frac{md}{\sqrt{(I + md^2)(M + m)}}$$

Let $y = \theta$ and

$$z_1 = \theta, \quad z_2 = \dot{\theta}, \quad z_3 = \varsigma + \varepsilon\sin\theta, \quad z_4 = \dot{\varsigma} + \varepsilon\dot{\theta}\cos\theta$$

the state space representation of (72) and (73) is given by
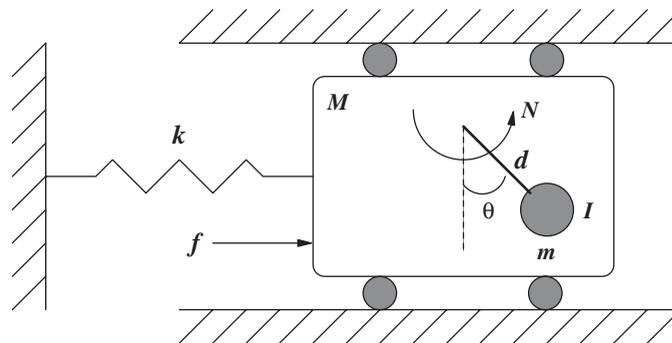
$$\dot{z}_1 = z_2 \tag{74}$$

Figure 1. Model of the RTAC system.

$$\dot{z}_2 = z_3 - \frac{\varepsilon \cos z_1}{1 - \varepsilon^2 \cos^2 z_1} w + u \tag{75}$$

$$\dot{z}_3 = z_4 \tag{76}$$

$$\dot{z}_4 = -z_3 + \varepsilon \sin(y) + w \tag{77}$$

$$y = z_1 \tag{78}$$

where

$$u = \frac{\varepsilon \cos z_1}{1 - \varepsilon^2 \cos^2 z_1}(z_3 - (1 + z_2^2)\varepsilon \sin z_1) + \frac{1}{1 - \varepsilon^2 \cos^2 z_1}\upsilon - z_3 \tag{79}$$

Assume that the disturbance applied on the RTAC system is generated by the following linear equation:

$$\dot{w}_1 = \omega w_2 \tag{80}$$

$$\dot{w}_2 = -\omega w_1 \tag{81}$$

$$w = w_1 \tag{82}$$

where the frequency ω of the disturbance is unknown. Then the state transformation

$$x_1 = z_1 - \frac{1}{1 - \omega^2}w_1, \quad x_2 = z_2 - \frac{\omega}{1 - \omega^2}w_2, \; x_3 = z_3, \quad x_4 = z_4 \tag{83}$$

transform the system (74)–(78) into the following form:

$$\dot{x} = Ax + B(u - \mu) + \phi(y) \tag{84}$$

$$y = Cx \tag{85}$$

where $x = \mathrm{col}(x_1, x_2, x_3, x_4)$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}}, \quad \phi(y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varepsilon \sin y \end{bmatrix}$$

and

$$\mu = \left(\frac{\varepsilon}{1 - \varepsilon^2} - \frac{1}{1 - \omega^2}\right)w$$

The zero dynamics of system (84)–(85) is given by

$$\dot{x}_z = A_z x_z = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_z$$

The eigenvalues $\pm 1j$ of A_z are on the imaginary axis. However, our algorithm still works provided that $\omega \neq 1$. In fact, the disturbance free system of (84)–(85) is asymptotically

stabilizable by

$$\dot{v} = (A + KC)v + \phi(y) - Ky + Bh(v, y)$$

$$u = h(v, y)$$

where

$$h(v, y) = -2v_2 - 2(y + \arctan v_4) + \frac{2(v_3 - \varepsilon \sin y) + v_4 - \varepsilon v_2 \cos y}{1 + v_4^2} + \frac{2v_4(v_3 - \varepsilon \sin y)^2}{(1 + v_4^2)^2}$$

$$\frac{\varepsilon(v_4 - v_3 + \varepsilon \sin y)(\sin y + \sin(\arctan v_4))}{y + \arctan v_4} - \frac{\varepsilon \, v_2 v_4 \cos y}{y + \arctan v_4}$$

and

$$K = [-10 \ -34 \ -40 \ 10]^T$$

Moreover, let $\hat{\psi}_p \in \mathbb{R}^{2 \times 1}$ be the estimate of $\psi_p = \psi_1^{d1}$, then $\hat{\psi}_p$ can be obtain the following filter and adaptive law:

$$\dot{p} = (A + KC)p + \phi(y) - Ky + Bu \tag{86}$$

$$\dot{\xi} = F\xi + G(Cp - y) \tag{87}$$

$$\dot{\zeta} = F\zeta + G\hat{\psi}_p^T \xi \tag{88}$$

$$\dot{\hat{\psi}}_p = \Gamma \xi (\xi - \zeta)^T PG \tag{89}$$

where

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 6000 & 0 \\ 0 & 6000 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

Furthermore, we can calculate $\hat{F}_0$, $\hat{\psi}_2^{d1}$, $\hat{\psi}^z$ and $\hat{\psi}^u$ as follows:

$$\hat{F}_0 = F + G\hat{\psi}_p^T$$

$$\hat{\psi}_2^{d1} = \hat{F}_0^T \hat{\psi}_1^{d1} + 10\hat{\psi}_1^{d1}$$

$$\text{vec}(\hat{\psi}^z) = \text{col}(\hat{\psi}_1^z, \hat{\psi}_2^z) = \frac{|\hat{\Sigma}|}{\sigma + |\Sigma|^2} \text{adj}(\hat{\Sigma}) \text{vec}(\hat{\psi}_p[-40 \ 10])$$

$$\hat{\psi}^u = \hat{F}_0^T \hat{\psi}_2^{d1} + 34\hat{\psi}_1^{d1} - \hat{\psi}_1^z$$

where

$$\hat{\Sigma} = I_2 \otimes \hat{F}_0 - A_z \otimes I_2$$
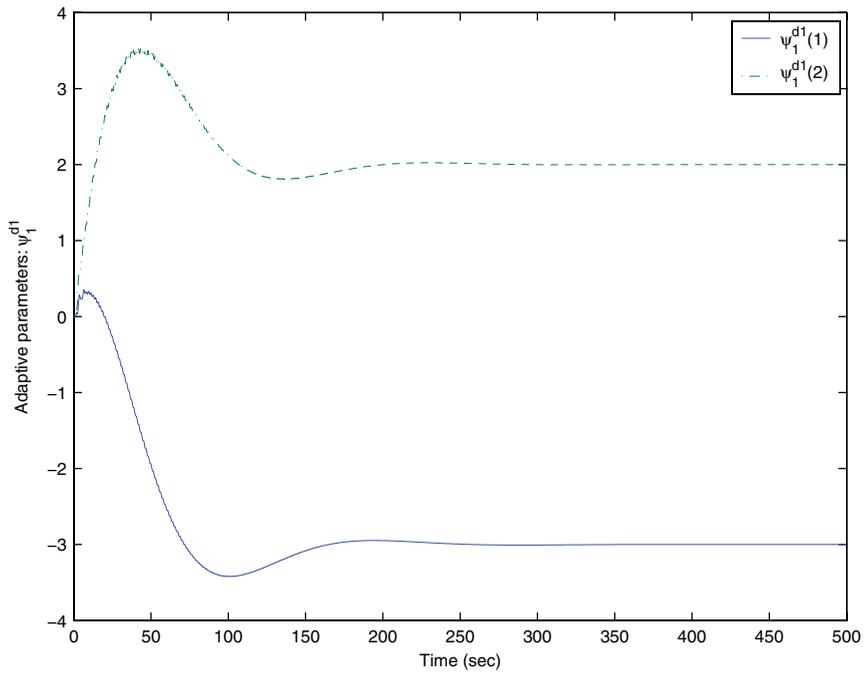$$\dot{\sigma} = -3\sigma, \quad \sigma(0) = 1$$

Figure 2. Estimation of ψ_p for the RTAC system.
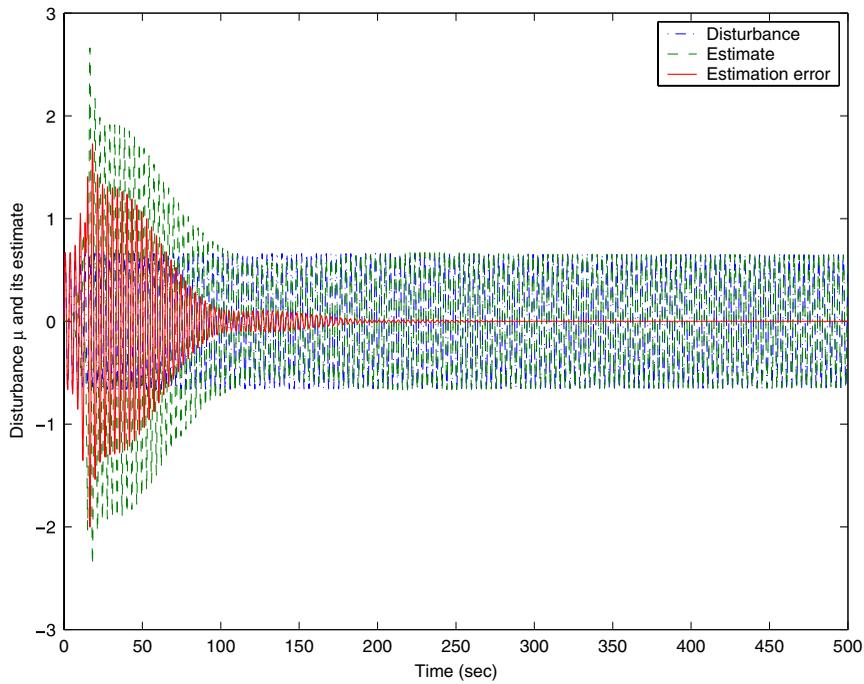

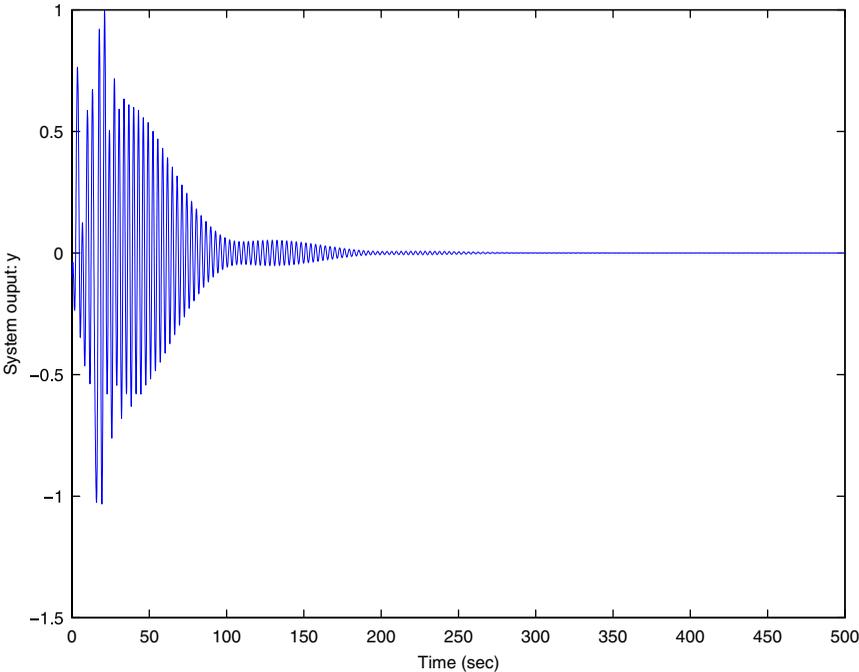Figure 3. Estimation of μ for the RTAC system.

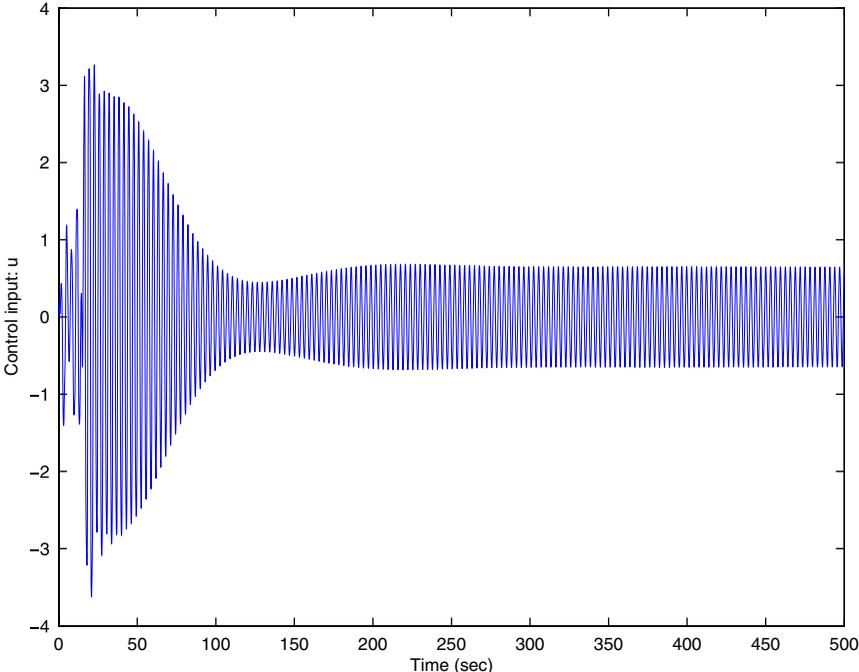Figure 4. System output of the RTAC system.

Figure 5. Control input of the RTAC system.

Finally, the disturbance rejection problem is solved by

$$u = h(v, y) + \hat{\mu} = h(v, y) + (\hat{\psi}^u)^T \xi \tag{90}$$

The simulation results are shown in Figures 2–5 with $\varepsilon = 0.3$, $\omega = 2$ and $w_0 = [0 \ 1]^T$. As shown in Figure 2, ψ_p is estimated by $\hat{\psi}_p = [-3 \ 2]^T$

$$\lambda(F + G\hat{\psi}_p^T) = \pm 2j$$

Thus, we can know that the estimate frequency of the disturbance is exactly $\omega = 2$. The disturbance estimation of μ and their estimation error are shown in Figure 3, which shows that the estimation error convergence asymptotically to zero. The system output and control input are shown in Figures 4 and 5, respectively. The control law (90) rejects the disturbance μ effectively.

5. CONCLUSIONS

An indirect disturbance rejection scheme is proposed in this paper. Firstly, an adaptive estimation algorithm is developed to estimate the unknown disturbances, the system state and the frequencies of the disturbances. The algorithm can deal with both minimum phase and non-minimum phase non-linear MIMO systems in output feedback form. Secondly, by combining the estimates of the disturbances and the stabilization control law designed without considering the disturbances, the unknown disturbances are asymptotically rejected or completely compensated.

REFERENCES

1. Huang J. *Nonlinear Output Regulation: Theory and Applications*. SIAM: Philadelphia, 2004.
2. Bodson M, Douglas SC. Adaptive algorithm for the rejection of sinusoidal disturbances with unknown frequencies. *Automatica* 1997; **33**(10):2213–2221.
3. Bodson M, Sacks A, Khosla P. Harmonic generation in adaptive feedforward cancellation schemes. *IEEE Transactions on Automatic Control* 1994; **39**(4):1939–1944.
4. Marino R, Santosuosso GL, Tomei P. Robust adaptive compensation of biased sinusoidal disturbances with unknown frequencies. *Automatica* 2003; **39**:1755–1761.
5. Nikiforov VO. Adaptive nonlinear tracking with complete compensation of unknown disturbances. *European Journal of Control* 1998; **4**:132–139.
6. Serrani A, Isidori A, Marconi L. Semiglobal nonlinear output regulation with adaptive internal model. *IEEE Transactions on Automatic Control* 2001; **46**(8):1178–1194.
7. Ding Z. Global stabilization and disturbance suppression of a class of nonlinear systems with uncertain internal model. *Automatica* 2003; **39**(3):471–479.
8. Ding Z. Adaptive estimation and rejection of unknown sinusoidal disturbance in a class of non-minimum phase nonlinear systems. *IEE Proceedings Control Theory and Applications*, to appear. A preliminary conference version for the estimation part is presented in the 16th IFAC World Congress, Prague, July, 2005.
9. Chen BM, Lin Z, Shamash Y. *Linear Systems Theory: A Structural Decomposition Approach*. Birkhäuser: Boston, 2004.
10. Sannuti P, Saberi A. A special coordinate basis of multivariable linear systems—finite and infinite zero structure, squaring down and decoupling. *International Journal of Control* 1987; **45**:1655–1704.
11. Marino R, Tomei P. Global adaptive output feedback control of nonlinear systems. Part I: linear parameterization. *IEEE Transactions on Automatic Control* 1993; **38**(1):17–32.
12. Marino R, Tomei P. *Nonlinear Control Design—Geometric, Adaptive and Robust*. Prentice-Hall: Englewood Cliffs, NJ, 1995.

13. Kanellakopoulos I, Kokotović PV, Morse AS. Adaptive output feedback control of systems with output non-linearities. *IEEE Transactions on Automatic Control* 1992; **37**:1266–1282.
14. Ding Z. Universal output regulation for nonlinear systems in output feedback form. In *Proceedings of the IEEE 41st Conference on Decision and Control*, Las Vegas, U.S.A., 2002; 3837–3842.
15. Marino R, Tomei P. Adaptive tracking and disturbance rejection for uncertain nonlinear systems. *IEEE Transactions on Automatic Control* 2005; **50**(1):90–95.
16. Ding Z. Backstepping stabilization of nonlinear systems with a nonminimum phase zero. In *Proceedings of 40th IEEE Conference on Decision and Control*, Orlando, U.S.A., 2001; 85–86.
17. Ding Z. Semiglobal stabilization of a class of nonminimum phase nonlinear output feedback systems. *IEE Proceedings Control Theory and Applications*, 2005; **152**(4):460–464.
18. Hsu L, Ortega R, Damm G. A global convergent frequency estimator. *IEEE Transactions on Automatic Control* 1999; **44**:698–713.
19. Marino R, Tomei P. Global estimation of n unknown frequencies. *IEEE Transactions on Automatic Control* 2002; **47**:1324–1328.
20. Isidori A. *Nonlinear Control Systems*. Springer: London, 1995.
21. Byrnes CI, Priscoli FD, Isidori A. *Regulation of Uncertain Nonlinear Systems*. Birkhäuser: Boston, 1997.
22. Khalil HK. *Nonlinear Systems* (3rd edn). Prentice-Hall: NJ, 2002.
23. Marino R, Tomei P. *Nonlinear Control Design*: *Geometric*, *Adaptive*, *and Robust*. Prentice-Hall: London, 1995.
24. Bupp RT, Bernstein DS, Coppola VT. A benchmark problem for nonlinear control design. *International Journal of Robust and Nonlinear Control* 1998; **8**:307–310.
25. Bupp RT, Bernstein DS, Coppola VT. Experimental implementation of integrator backstepping and passive nonlinear controllers on the RTAC testbed. *International Journal of Robust and Nonlinear Control* 1998; **8**:435–457.