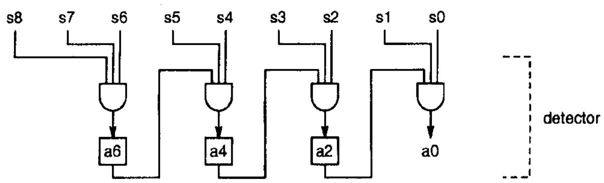


Fig. 2. A 9-b constant time detector.

then we can safely initialize $A$ to 0. By the time $n/2$ clock cycles have passed, $A$ will contain the correct prefix-and of $S$.

## IV. Discussion

The circuits discussed in this paper create an output pulse $k$ clock pulses *after* initialization. If we need an output pulse after *every* $k$ clock pulses, then the counter should be initialized to count to $k-1$, and the output pulse should be used to reinitialize the counter.

A variation of the frequency divider is a counter that produces a fixed sequence of output pulses. Instead of producing an output every $k$ cycles, the counter is required to cycle through $m$ different output values $(K_0, K_1, \cdots, K_{m-1})$. A small modification to the frequency divider in Fig. 1 allows it to perform this function. The values $(K_0, K_1, \cdots, K_{m-1})$ are stored in a memory. Initially, $(C, S) = (0, \overline{K_0})$. Whenever the $i$th output pulse is generated, $(C, S)$ is set to $(0, \overline{K_i})$.

The proofs in Section III assume that every clock pulse causes the counter to increment, but this is not necessary. Sometimes the increment is not tied to the clock, and an input value could be zero. This can easily be accommodated by changing the fixed "1" that is input to the low-order half adder to an input that is zero or one. The output pulse will now be generated when $k$ ones have been received. Constant time detection will still work. Indeed the worst case for constant-time detection occurs when the input is always one — if the input is sometimes zero then there is more time to do the detection.

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# Inner-Outer Factorization of Discrete-Time Transfer Function Matrices

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***Abstract*—The inner-outer factorization of the transfer function matrix of a linear time-invariant system has been an important algebraic problem in a variety of areas in electrical engineering, including systems and control analysis and design. This paper gives an explicit state space-based algorithm for the inner-outer factorization of the transfer function matrix of a general discrete-time linear system. More specifically, the algorithm applies to any discrete-time linear system whose transfer function is proper and stable.**

## I. Introduction and Problem Formulation

The inner-outer factorization of the transfer functions of linear time-invariant systems has been an important algebraic problem in a variety of areas in electrical engineering, including systems and control analysis and design (see, for example, Chen *et al.*, 1993; Francis, 1987; Saberi *et al.*, 1993, and the references therein). For a continuous-time linear system with transfer function matrix $G(s)$ belonging to $\boldsymbol{RH}^\infty$, the set of proper real-rational matrices analytic in $\mathrm{Re}(s) \geq 0$, one would like to find an inner matrix $G_i(s)$ and an outer matrix $G_o(s)$ such that $G(s) = G_i(s)G_o(s)$ holds, where $G_i(s) \in \boldsymbol{RH}^\infty$ is called inner if $G_i^T(-s)G_i(s) = I$ holds and $G_o(s) \in \boldsymbol{RH}^\infty$ is called outer if it has full row rank for every $\mathrm{Re}(s) > 0$. (Alternatively, $G_o(s) \in \boldsymbol{RH}^\infty$ is called outer if it has a right inverse which is analytic in the open right-half plane.) It is also well known that such factorization for continuous-time results in inner and outer factors which are unique up to multiplication by an orthogonal matrix (See Youla, 1961). There are several papers that provide state space-based algorithms for obtaining inner-outer factorization of continuous-time systems. Such algorithms are applicable to transfer function matrices that satisfy certain conditions, such as, the transfer function matrices be either injective or sujective (see Chen and Francis, 1989) and $G(j\omega)$ has a constant rank for all $\omega \in \mathbb{R}$, which implies that the systems have no infinite zeros (see Weiss, 1994).

In this paper we focus on the inner-outer factorization of the transfer function matrices of discrete-time systems. The relevant definitions in discrete time are analogous to those in the continuous-time setting and are given below.

**Definition 1.1.** A discrete-time matrix function $G(z) \in \boldsymbol{RH}^\infty$, the set of proper real-rational matrices analytic in $|z| \geq 1$, is said to be inner if $G^T(z^{-1})G(z) = I$ and outer if it has a right inverse which is analytic outside the unit circle ($|z| > 1$). An inner-outer factorization of a matrix $G(z) \in \boldsymbol{RH}^\infty$ is a factorization

$$G(z) = G_i(z)G_o(z)$$

with $G_i(z)$ an inner matrix and $G_o(z)$ an outer matrix.

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The goal of this brief is to provide a state space-based algorithm for obtaining an inner-outer factorization of the transfer function matrix $G(z) \in \boldsymbol{RH}^\infty$ of a general discrete-time system. Here we make no assumption on $G(z)$.

The brief is organized as follows. Section II gives an explicit method of constructing the inner-outer factorization while Section III gives an illustrative example. Section IV draws the conclusions of our work. Throughout this brief, $A^T$ denotes the transpose of $A, I$ denotes an identity matrix with appropriate dimension. Similarly, $\lambda(A)$ denotes the set of eigenvalues of $A$. The close unit disc is denoted by $\mathbb{C}^{\odot}$ while $\mathbb{C}^{\otimes} := \mathbb{C} \backslash \mathbb{C}^{\odot}$.

## II. Inner-Outer Factorization

In this section, we give simple and explicit expressions for the inner $G_i(z)$, and the outer $G_o(z)$ of the transfer function $G(z)$ of a general discrete-time linear system. First, we need the following preliminaries.

### A. Preliminaries

Consider a linear discrete-time system

$$\Sigma: \begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k) + Du(k). \end{cases} \tag{2.1}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation $U$ and a nonsingular matrix $V$ that render the direct feedthrough matrix $D$ into the following form

$$\overline{D} = UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.2}$$

where $m_0$ is the rank of $D$. Thus the system in (2.1) can be rewritten as

$$\begin{aligned} x(k+1) &= Ax(k) + [B_0 \quad B_1]\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix} \\ \begin{pmatrix} y_0(k) \\ y_1(k) \end{pmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x(k) + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix} \end{aligned} \tag{2.3}$$

where $B_0, B_1, C_0$ and $C_1$ are the matrices of appropriate dimensions. Note that the inputs $u_0$ and $u_1$, and the outputs $y_0$ and $y_1$ are those of the transformed system. Namely,

$$u = V\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = Uy.$$

We recall the following theorem of special coordinate basis (SCB) from Sannuti and Saberi (1987) and Saberi and Sannuti (1990).

**Theorem 2.1** Consider a linear time-invariant system characterized by a quadruple $(A, B, C, D)$. There exist nonsingular transformations $\Gamma_1, \Gamma_2$ and $\Gamma_3$ (nonunique), respectively associated with the state space, output space and input space, such that

$$x = \Gamma_1 \begin{pmatrix} x_c \\ x_a \\ x_b \\ x_d \end{pmatrix}, \quad x_a = \begin{pmatrix} x_a^- \\ x_a^+ \end{pmatrix},$$

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \Gamma_2 \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix}, \quad \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \Gamma_3 \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix} \tag{2.4}$$

and

$$\Gamma_1^{-1}(A - B_0C_0)\Gamma_1 = \begin{bmatrix} A_{cc} & B_cE_{ca}^- & B_cE_{ca}^+ & L_{cb}C_b & L_{bd}C_d \\ 0 & A_{aa}^- & 0 & L_{ab}^-C_b & L_{ad}^-C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+C_b & L_{ad}^+C_d \\ 0 & 0 & 0 & A_{bb} & L_{bd}C_d \\ B_dE_{dc} & B_dE_{da}^- & B_dE_{da}^+ & B_dE_{db} & A_d \end{bmatrix} \tag{2.5}$$

$$\Gamma_1^{-1}[B_0 \quad B_1]\Gamma_3 = \begin{bmatrix} B_{c0} & 0 & B_c \\ B_{a0}^- & 0 & 0 \\ B_{a0}^+ & 0 & 0 \\ B_{b0} & 0 & 0 \\ B_{d0} & B_d & 0 \end{bmatrix} \tag{2.6}$$

$$\Gamma_2^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_1 = \begin{bmatrix} C_{0c} & C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0d} \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix} \tag{2.7}$$

and

$$\Gamma_2^{-1}\begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\Gamma_3 = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.8}$$

where $\lambda(A_{aa}^-) \in \mathbb{C}^{\odot}$ and $\lambda(A_{aa}^+) \in \mathbb{C}^{\otimes}$ are, respectively, those invariant zeros that are inside or on the unit circle and those outside the unit circle; $(A_{cc}, B_c)$ is controllable and is nonexistent if and only if $\Sigma$ is left invertible; $(C_b, A_{bb})$ is observable and is nonexistent if and only if $\Sigma$ is right invertible; and the system characterized by $(A_d, B_d, C_d)$ is invertible with no invariant zeros. Moreover, the pair $(A, B)$ is stabilizable if and only if the pair $(A_x, B_x)$ is stabilizable, where

$$A_x := \begin{bmatrix} A_{aa}^+ & L_{ab}^+C_b & L_{ad}^+C_d \\ 0 & A_{bb} & L_{bd}C_d \\ B_dE_{da}^+ & B_dE_{db} & A_d \end{bmatrix}$$

$$B_x := \begin{bmatrix} B_{a0}^+ & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix}. \tag{2.9}$$

The proof of this theorem can be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). The explicit construction of the SCB utilizes the well-understood Silverman's structural algorithm and is numerically stable. A software package that computes this special coordinate basis is given by Lin *et al.*, (1992).

We are now ready to present our state space-based factorization algorithm.

### B. A State Space-Based Algorithm

Given a $G(z) \in \boldsymbol{RH}^\infty$, its inner-outer factorization can be carried out in the following steps.

**Step 1.** Find any state space realization $(A, B, C, D)$ of $G(z)$ such that $(A, B)$ is stabilizable. We note here that this realization is not required to be a minimal one.

**Step 2.** Transfer the state space realization $(A, B, C, D)$ into its SCB form as described in the previous subsection. Form matrices $A_x$ and $B_x$ as in (2.9). Also form matrices $C_x$ and $D_x$ as follows:

$$C_x := \Gamma_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C_d \\ 0 & C_b & 0 \end{bmatrix}, \quad D_x := \Gamma_2 \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.10}$$

**Step 3.** Let $G_o(z)$ be the transfer function matrix of the quadruple $(A, B, C_{\text{outer}}, D_{\text{outer}})$, i.e.,

$$G_o(z) = C_{\text{outer}}(zI - A)^{-1}B + D_{\text{outer}}. \tag{2.11}$$

Here

$$C_{\text{outer}} := \Gamma_m^{-1}\begin{bmatrix} C_{0c} & C_{0a}^- & C_{0a}^+ + F_{0a}^+ & C_{0b} + F_{0b} & C_{0d} + F_{0d} \\ E_{dc} & E_{da}^- & F_{da}^+ & F_{db} & F_{dd}\end{bmatrix}\Gamma_1^{-1} \tag{2.12}$$

$$D_{\text{outer}} := \Gamma_m^{-1}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0\end{bmatrix}\Gamma_3^{-1} \tag{2.13}$$

where

$$\Gamma_m = (D_x^T D_x + B_x^T P_x B_x)^{-1/2} \tag{2.14}$$

and

$$F_x := \begin{bmatrix} F_{0a}^+ & F_{0b} & F_{0d} \\ F_{da}^+ & F_{db} & F_{dd}\end{bmatrix} = (D_x^T D_x + B_x^T P_x B_x)^{-1}(B_x^T P_x A_x + D_x^T C_x) \tag{2.15}$$

and $P_x$ is the positive definite solution of the algebraic Riccati equation,

$$P_x = A_x^T P_x A_x + C_x^T C_x - (B_x^T P_x A_x + D_x^T C_x)^T \cdot (D_x^T D_x + B_x^T P_x B_x)^{-1}(B_x^T P_x A_x + D_x^T C_x) \tag{2.16}$$

**Step 4.** Let

$$G_i(z) = [D_x + (C_x - D_x F_x)(zI - A_x + B_x F_x)^{-1}B_x]\Gamma_m. \tag{2.17}$$

We have the following main result of this brief.

**Theorem 2.2.** Given a discrete-time transfer function matrix $G(z) \in \boldsymbol{RH}^\infty$. Then an inner-outer factorization of $G(z)$ is given by

$$G(z) = G_i(z)G_o(z)$$

where $G_o(z)$ and $G_i(z)$ are given respectively in Steps 3 and 4 of the above algorithm.

*Proof:* We prove the theorem by showing that

a) $G_i(z)$ as given by (2.17) is inner;
b) $G_o(z)$ as given by (2.11) is outer;
c) $G(z) = G_i(z)G_o(z)$.

To show a), we first note that, for $(A_x, B_x, C_x, D_x)$, the pair $(A_x, B_x)$ is stabilizable, and all the eigenvalues of $A_x$ which are inside or on the unit circle are observable. Hence, it follows from Richardson and Kwong (1986) and Chen et al (1994) that (2.16) has a unique, symmetric and positive definite solution, i.e., $P_x = P_x^T > 0$ such that $A_x - B_x F_x$ is a Schur stable matrix. Hence, $G_i(z) \in \boldsymbol{RH}^\infty$. We next show that $G^T(z^{-1})G(z) = I$ and hence $G_i(z)$ is inner. It follows from (2.15) that

$$D_x^T C_x = (D_x^T D_x + B_x^T P_x B_x)F_x - B_x^T P_x A_x$$

and hence

$$D_x^T(C_x - D_x F_x) = D_x^T C_x - D_x^T D_x F_x = B_x^T(P_x B_x F_x - P_x A_x)$$

Also, it follows from (2.16) that

$$P_x - A_x^T P_x A_x - C_x^T C_x + F_x^T(D_x^T D_x + B_x^T P_x B_x)F_x = 0$$

Hence, we have

$$\begin{aligned}
&G_i^T(z^{-1})G_i(z) \\
&= \Gamma_m^T[B_x^T(z^{-1}I - A_x^T + F_x^T B_x^T)^{-1} \cdot (C_x^T - F_x^T D_x^T) + D_x^T] \\
&\quad \times [D_x + (C_x - D_x F_x) \cdot (zI - A_x + B_x F_x)^{-1}B_x]\Gamma_m \\
&= \Gamma_m^T(D_x^T D_x + B_x^T P_x B_x)\Gamma_m \\
&\quad + \Gamma_m^T[B_x^T(z^{-1} - A_x^T + F_x^T B_x^T)^{-1} \cdot (F_x^T B_x^T P_x - A_x^T P_x)B_x \\
&\quad + B_x^T(z^{-1}I - A_x^T + F_x^T B_x^T)^{-1}(C_x^T - F_x^T D_x^T) \cdot (C_x - D_x F_x)(zI - A_x + B_x F_x)^{-1}B_x \\
&\quad - B_x^T P_x B_x + B_x^T(P_x B_x F_x - P_x A_x) \cdot (zI - A_x + B_x F_x)^{-1}B_x]\Gamma_m \\
&= I + \Gamma_m^T B_x^T(z^{-1}I - A_x^T + F_x^T B_x^T)^{-1} \cdot [-P_x + A_x^T P_x A_x + C_x^T C_x \\
&\quad - F_x^T D_x^T C_x - C_x^T D_x F_x - A_x^T P_x B_x F_x - F_x^T B_x^T P_x A_x + F_x^T B_x^T P_x B_x F_x + F_x^T D_x^T D_x F_x] \\
&\quad \times (zI - A_x + B_x F_x)^{-1}B_x\Gamma_m \\
&= I + \Gamma_m B_x^T(z^{-1}I - A_x^T + F_x^T B_x^T)^{-1} \cdot [-P_x + A_x^T P_x A_x + C_x^T C_x - F_x^T(D_x^T D_x + B_x^T P_x B_x)F_x] \\
&\quad \cdot (zI - A_x + B_x F_x)^{-1}B_x\Gamma_m \\
&= I
\end{aligned}$$

We now proceed to show that $G_o(z)$ is outer. Noting that $A$ is Schur stable, we see that $G_o(z) \in \boldsymbol{RH}^\infty$. The fact that $G_o(z)$ is right invertible follows from the fact that the matrix $D_o$ is of full row rank, which also implies that $(A, B, C_{\text{outer}}, D_{\text{outer}})$ has no infinite zeros. It remains to show that the right inverse of $G_o(z)$ is analytic for $|z| > 1$, which is equivalent to the fact that all the invariant zeros of $(A, B, C_{\text{outer}}, D_{\text{outer}})$ are inside or on the unit circle. To this end, without loss of generality, we assume that the given system $\Sigma$ is in the form of SCB as described by Theorem 2.1. It follows from the properties of SCB that the invariant zeros of $(A, B, C_{\text{outer}}, D_{\text{outer}})$ are exactly the same as the input decoupling zeros of $(\overline{A}, \overline{B})$, where

$$\overline{A} := A - B_0 C_0 - \begin{bmatrix} B_{c0} & 0 \\ B_{a0}^- & 0 \\ B_{a0}^+ & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & F_{0a}^+ & F_{0b} & F_{0d} \\ E_{dc} & E_{da}^- & F_{da}^+ & F_{db} & F_{dd}\end{bmatrix} = \begin{bmatrix} A_{cc} & B_c E_{ca}^- & \star \\ 0 & A_{aa}^- & \star \\ 0 & 0 & A_x - B_x F_x \end{bmatrix}$$

and

$$\overline{B} = \begin{bmatrix} B_c \\ 0 \\ 0 \end{bmatrix}$$

Since $(A_{cc}, B_c)$ is controllable, it is obvious that the input decoupling zeros of $(\overline{A}, \overline{B})$, or the invariant zeros of $(A, B, C_{\text{outer}}, D_{\text{outer}})$, are given by $\lambda(A_{aa}^-) \cup \lambda(A_x - B_x F_x)$, and hence are all inside or on the unit circle due to the fact that all the eigenvalues of $A_{aa}^-$ are inside or on the unit circle and that $A_x - K_x C_x$ is a Schur stable matrix.

Finally, we show that $G(z) = G_i(z)G_o(z)$. Let us define

$$\overline{B}_0 := \begin{bmatrix} B_{c0} \\ B_{a0}^- \\ B_{a0}^+ \\ B_{b0} \\ B_{d0} \end{bmatrix}, \quad \overline{B}_d := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ B_d \end{bmatrix}$$

$$\begin{aligned}
\overline{C}_0 &:= [C_{c0} \quad C_{a0}^- \quad C_{a0}^+ \quad C_{b0} \quad C_{d0}] \\
\overline{C}_{F0} &:= [0 \quad 0 \quad F_{0a}^+ \quad F_{0b} \quad F_{0d}] \\
\overline{C}_{F1} &:= [E_{dc} \quad E_{da}^- \quad F_{da}^+ \quad F_{db} \quad F_{dd}] \\
\overline{C}_b &:= [0 \quad 0 \quad 0 \quad C_b \quad 0] \\
\overline{C}_d &:= [0 \quad 0 \quad 0 \quad 0 \quad C_d] \\
\Phi_x(z) &:= (zI - A_x + B_xF_x)^{-1}, \\
\overline{\Phi}_x(z) &:= [0 \quad 0 \quad \Phi_x(z)]
\end{aligned}$$

and (see the first matrix at the bottom of the page). It then follows that

$$\begin{aligned}
D_x\Gamma_m C_{\text{outer}} &= \Gamma_2 \begin{bmatrix} \overline{C}_0 + \overline{C}_{F0} \\ 0 \\ 0 \end{bmatrix} \\
D_x\Gamma_m D_{\text{outer}} &= D \\
\Phi_x(z)B_x\Gamma_m D_{\text{outer}} &= \overline{\Phi}_x(z)\Gamma_1^{-1}B \\
\Phi_x(z)B_x &= \overline{\Phi}_x(z)[\overline{B}_0 \quad \overline{B}_d] \\
\Gamma_m^{-1}C_{\text{outer}} &= \left(\begin{bmatrix} \overline{C}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \overline{C}_{F0} \\ \overline{C}_{F1} \end{bmatrix}\right)\Gamma_1^{-1} \\
(C_x - D_xF_x)[0 \quad I] &= \Gamma_2 \begin{bmatrix} -\overline{C}_{F0} \\ \overline{C}_d \\ \overline{C}_b \end{bmatrix}
\end{aligned}$$

and

$$C = \Gamma_2 \begin{bmatrix} \overline{C}_0 \\ \overline{C}_d \\ \overline{C}_b \end{bmatrix} \Gamma_1^{-1}$$

We then have

$$\begin{aligned}
&G_i(z)G_o(z) \\
&= [D_x + (C_x - D_xF_x)(zI - A_x + B_xF_x)^{-1}B_x] \\
&\quad \cdot \Gamma_m[C_{\text{outer}}(zI - A)^{-1}B + D_{\text{outer}}] \\
&= D_x\Gamma_m^{-1}C_{\text{outer}}(zI - A)^{-1}B + D_x\Gamma_m D_{\text{outer}} \\
&\quad + (C_x - D_xF_x)\Phi_x(z)B_x\Gamma_m C_{\text{outer}}(zI - A)^{-1}B \\
&\quad + (C_x - D_xF_x)\Phi_x(z)B_x\Gamma_m D_{\text{outer}} \\
&= \Gamma_2 \begin{bmatrix} \overline{C}_0 + \overline{C}_{F0} \\ 0 \\ 0 \end{bmatrix} \Gamma_1^{-1}(zI - A)^{-1}B \\
&\quad + (C_x - D_xF_x)\Phi_x(z)B_x\Gamma_m C_{\text{outer}}(zI - A)^{-1}B \\
&\quad \cdot (C_x - D_xF_x)\overline{\Phi}_x(z)\Gamma_1^{-1}B + D \\
&= \left[\Gamma_2 \begin{bmatrix} \overline{C}_0 + \overline{C}_{F0} \\ 0 \\ 0 \end{bmatrix} \Gamma_1^{-1} + (C_x - D_xF_x)\Phi_x(z) \right. \\
&\quad \cdot B_x\Gamma_m C_{\text{outer}} + (C_x - D_xF_x)\overline{\Phi}_x(z) \\
&\quad \left. \cdot \Gamma_1^{-1}(zI - A) \right] \\
&\quad \times (zI - A)^{-1}B + D \\
&= \left[\Gamma_2 \begin{bmatrix} \overline{C}_0 + \overline{C}_{F0} \\ 0 \\ 0 \end{bmatrix} \Gamma_1^{-1} + (C_x - D_xF_x)\overline{\Phi}_x(z) \right. \\
&\quad \cdot \left([\overline{B}_0 \quad \overline{B}_d]\left(\begin{bmatrix} \overline{C}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \overline{C}_{F0} \\ \overline{C}_{F1} \end{bmatrix}\right)\Gamma_1^{-1} \right. \\
&\quad \left.\left. + \Gamma_1^{-1}(zI - A)\right)\right](zI - A)^{-1}B + D \\
&= \left[\Gamma_2 \begin{bmatrix} \overline{C}_0 + \overline{C}_{F0} \\ 0 \\ 0 \end{bmatrix} \Gamma_1^{-1} + (C_x - D_xF_x)\overline{\Phi}_x(z) \right. \\
&\quad \left. \cdot (\overline{B}_0\overline{C}_{F0} + \overline{B}_d\overline{C}_{F1} + \Phi(z))\Gamma_1^{-1} \right](zI - A)^{-1}B \\
&\quad + D \\
&= \left[\Gamma_2 \begin{bmatrix} \overline{C}_0 + \overline{C}_{F0} \\ 0 \\ 0 \end{bmatrix} \Gamma_1^{-1} + (C_x - D_xF_x) \right. \\
&\quad \left. \cdot [0 \quad I]\Gamma_1^{-1} \right](zI - A)^{-1}B + D \\
&= \left[\Gamma_2 \begin{bmatrix} \overline{C}_0 + \overline{C}_{F0} \\ 0 \\ 0 \end{bmatrix} \Gamma_1^{-1} + \Gamma_2 \begin{bmatrix} -\overline{C}_{F0} \\ \overline{C}_d \\ \overline{C}_b \end{bmatrix} \Gamma_1^{-1} \right] \\
&\quad \cdot (zI - A)^{-1}B + D \\
&= \Gamma_2 \begin{bmatrix} \overline{C}_0 \\ \overline{C}_d \\ \overline{C}_b \end{bmatrix} \Gamma_1^{-1}(zI - A)^{-1}B + D \\
&= C(zI - A)^{-1}B + D \\
&= G(z) \qquad \blacksquare
\end{aligned}$$

## III. An Example

In this section we present an example that illustrates our results. Consider the transfer function matrix $G(z)$ of a system $\Sigma$ character-

$$\Phi(z) = \begin{bmatrix} zI - A_{cc} & -B_cE_{ca}^- & -B_cE_{ca}^+ & -L_{cb}C_b & -L_{bd}C_d \\ 0 & zI - A_{aa}^- & 0 & -L_{ab}^-C_b & -L_{ad}^-C_d \\ 0 & 0 & zI - A_{aa}^+ & -L_{ab}^+C_b & -L_{ad}^+C_d \\ 0 & 0 & 0 & zI - A_{bb} & -L_{bd}C_d \\ -B_dE_{dc} & -B_dE_{da}^- & -B_dE_{da}^+ & -B_dE_{db} & zI - A_d \end{bmatrix}$$

$$C_{\text{outer}} = \begin{bmatrix} -0.001944 & 0.001944 & 0.206749 & -0.158263 & 0.033402 \\ -0.323808 & 0.323808 & -0.307107 & 0.825649 & 1.164639 \end{bmatrix}$$

ized by a matrix quadruple $(A, B, C, D)$ with

$$A = \begin{bmatrix} 0.5 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1.1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ -0.2 & 0.2 & -0.2 & -0.1 & 0.1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is simple to verify that $(A, B)$ is stabilizable and that the above system is neither left nor right invertible with two invariant zeros at $z = 0$ and $z = 1.1$ and one infinite zero. Moreover, it is in the form of SCB. Following the factorization algorithm proposed in Section III, we obtain

$$A_x = \begin{bmatrix} 1.1 & 0 & 1 \\ 0 & 1 & 1 \\ -0.2 & -0.1 & -0.1 \end{bmatrix}, \quad B_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

With these data, solving (2.16), we obtain

$$P_x = \begin{bmatrix} 0.208722 & -0.163321 & 0.026426 \\ -0.163321 & 2.745830 & 1.597356 \\ 0.026426 & 1.597356 & 2.621380 \end{bmatrix}$$

$$\Gamma_m = \begin{bmatrix} 0.909655 & -0.005462 \\ -0.005462 & 0.617683 \end{bmatrix},$$

$$F_x = \begin{bmatrix} 0.189747 & -0.148474 & 0.024024 \\ -0.190824 & 0.510854 & 0.719196 \end{bmatrix}$$

and (see matrix at the bottom of the previous page).

$$D_{\text{outer}} = \begin{bmatrix} 1.099376 & 0.009721 & 0 \\ 0.009721 & 1.619038 & 0 \end{bmatrix}$$

and hence

$$G_i(z) = [D_x + (C_x - D_x F_x)(zI - A_x + B_x F_x)^{-1} B_x]\Gamma_m^{-1}$$

and

$$G_o(z) = C_{\text{outer}}(zI - A)^{-1}B + D_{\text{outer}}$$

It is easy to verify that $G_i(z) \in \boldsymbol{RH}^\infty$ with $G_i^T(z^{-1})G_i(z) = I$ and hence is an inner, $G_o(z) \in \boldsymbol{RH}^\infty$ is right invertible and of minimum-phase with four invariant zeros at $\{0, 0, 0.381966, 0.909091\}$ and hence is an outer, and $G_i(z)G_o(z) = G(z)$.

## IV. Conclusion

An explicit and simple expression for the inner-outer factorization of the transfer function matrix of a *general* discrete time system has been obtained in this paper. Such a factorization is useful in several applications in control thoery, including loop transfer recovery and $H_2$ optimal control.

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# On the Design of Positive Real Functions

Jiří Gregor

***Abstract*— Construction of all rational positive real functions with a given denominator is described. Examples showing how to respect various requirements on the degrees of the resulting polynomials are given.**

## I. Introduction

Positive real functions are analytic functions, which map the open right half plane onto itself and the positive real half-axis onto itself. For any positive real function $f \in \mathcal{B}$ there is $\lim_{s\to\infty} \frac{f(s)}{s} = \mu, \mu \geq 0$. Therefore if $f \in \mathcal{B}$ is a rational function, then

$$f(s) = \mu s + \frac{q(s)}{p(s)}, \quad \mu \geq 0, \deg q \leq \deg p$$

and $p$, $q$ are (nonstrict) Hurwitz polynomials (i.e., poles and zeros on the imaginary axis are allowed).

In [2] the following problem has been addressed. Given a strictly Hurwitz polynomial $p_n \in \mathcal{H}^S$ of degree $n$, find all polynomials $q \in \mathcal{H}^S$ with $deg\, q \leq n$ such that $q/p$ is a positive real function

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