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On Improving Transient Performance in Tracking Control for a Class of Nonlinear Discrete-Time Systems With Input Saturation

Yingjie He, Ben M. Chen, and Weiyao Lan

***Abstract*—Quick response and small overshoot are two desired transient performances of target tracking control. While most of the design schemes compromise between these two performances, we try to achieve both simultaneously for the tracking control of a class of nonlinear discrete-time systems with input saturation by using a composite nonlinear feedback (CNF) control technique. The closed-loop system with improved transient performance preserves the stability of the nonlinear part of the partially linear composite system.**

***Index Terms*—Discrete-time systems, input saturation, nonlinear systems, tracking control, transient performance.**

I. INTRODUCTION

The tracking control problems, such as target tracking [4] and output regulation [8], are extensively studied in the literature. Settling time and overshoot are two important transient performance indices, and quick response and small overshoot are desirable in the most of the target tracking control problems. However, it is well known that, in general, quick response results in a large overshoot. Thus, most of the design schemes have to make a tradeoff between these two transient performance indices. To improve the transient performance, Lin *et al.* [14] proposed a composite nonlinear feedback (CNF) control technique for a class of second-order linear systems. The CNF control law consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping rate for quick response, while the nonlinear feedback part is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot. Turner *et al.* [22] later extended the results of [14] to higher order and multiple-input systems under a restrictive assumption on the system. However, both [14] and [22] considered only the state feedback case. Recently, Chen *et al.* [2] have developed a CNF control design to a more general class of systems with measurement feedback, and successfully applied the technique to solve a hard-disk servo problem. The extension of this idea to general linear continuous multiple-input–multiple-output (MIMO) systems is found in [6]. The CNF control techniques for linear discrete-time systems can be found in [7] and [23].

This note aims to design a CNF control law for discrete-time partially linear composite systems with input saturation. The results for its continuous-time counterpart have been reported in [12]. In the last two decades, the nonlinear control problems for partially linear composite systems have been extensively studied by many researchers such as [10], [15], [19], and [20], to name just a few. It was shown in [19] that a nonlinear system which is zero-input globally asymptotically stable (GAS) will preserve its GAS property if its input decreases to zero with a very fast exponential rate. However, a bad transient performance may destroy the stability of the nonlinear part before the output rapidly decays to zero. This is also true for discrete-time systems since the intersampling behavior is equivalent to the response of a continuous-time system with unchanging input. Based on the linear part of the composite system, the CNF control is designed such that the closed-loop system has desired performances, e.g., quick response and small overshoot. Moreover, we show that the closed-loop system with improved transient performance preserves the stability of the nonlinear part of the partially linear composite system.

II. PROBLEM FORMULATION

Consider a partially linear composite discrete-time systems with input saturation Σ characterized by

$$\xi(k+1) = f(\xi(k), x(k), y(k)), \qquad \xi(0) = \xi_0 \tag{1}$$

$$x(k+1) = Ax(k) + B\mathrm{sat}(u(k)), \qquad x(0) = x_0 \tag{2}$$

$$y(k) = Cx(k) \tag{3}$$

where $(\xi, x) \in \mathbb{R}^m \times \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are, respectively, the state, control input, and control output of the given system Σ, f is a continuous function, A, B, and C are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \mathrm{sign}(u)\min(|u|, u_{\max}) \tag{4}$$

where $u_{\max}$ is the maximum amplitude of the control channel.

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A. Tracking Control Problem

Let r be the amplitude of the reference step input; design a state feedback control law

$$u(k) = Fx(k) + Gr \tag{5}$$

such that the closed-loop system consisting of (1)–(3) and (5) has the following two properties:

1) the trajectory $(\xi(k), x(k))$ exists and is bounded for all $k = 0, 1, 2, \ldots$;
2) $\lim_{k\to\infty} y(k) = r$.

To solve the previously defined tacking control problem, the following two assumptions are required:

A1) (A, B) is controllable;
A2) (A, B, C) is right invertible and has no invariant zeros at $z = 1$.

Moreover, to guarantee the stability of the nonlinear part of the system, some restriction on the nonlinear system (1) is necessary. Specifically, according to [8, Th. 1.9], under assumption A1) and A2), there exist unique $x_e \in \mathbb{R}^n$ and $u_e \in \mathbb{R}$ such that

$$\begin{aligned} 0 &= Ax_e + Bu_e \\ 0 &= Cx_e - r. \end{aligned}$$

Define $\tilde{x}(k) = x(k) - x_e$, the nonlinear system (1) can be expressed as

$$\xi(k+1) = f(\xi(k), x_e + \tilde{x}(k), r + C\tilde{x}(k)) =: f_1(\xi(k), \tilde{x}(k)) \tag{6}$$

Without loss of generality, we assume $f_1(0,0) = 0$. In fact, if $f_1(\xi^*, 0) = \xi^*$ with $\xi^* \neq 0$, the state transformation $\tilde{\xi}(k) = \xi(k) - \xi^*$ gives

$$\tilde{\xi}(k+1) = f_1(\tilde{\xi}(k) + \xi^*, \tilde{x}(k)) - \xi^* =: \tilde{f}_1(\tilde{\xi}(k), \tilde{x}(k))$$

then, we have $\tilde{f}_1(0,0) = 0$. Furthermore, we assume the following:

A3) There exist a continuous function $V_\xi : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ with $V_\xi(0) = 0$, and $\mathcal{K}_\infty$-functions α_1, α_2 and α_3 such that

$$\alpha_1(\|\xi\|) \leq V_\xi(\xi) \leq \alpha_2(\|\xi\|) \tag{7}$$
$$V_\xi(f_1(\xi, 0)) - V_\xi(\xi) \leq -\alpha_3(\|\xi\|) \tag{8}$$

for all $\xi \in \mathbb{R}^m$.

Remark 2.1: It is well understood in the literature that assumption A1) and A2) are quite standard. Assumption A3) simply says that the nonlinear system (1) is GAS when the system output y tracks exactly the step command input r, i.e., $\xi(k+1) = f(\xi(k), x_e, r)$ is GAS.

The objective of this note is to improve the transient performance of the tracking control problem by using a CNF control law

$$u(k) = Fx(k) + Gr + u_N(k). \tag{9}$$

The state feedback gain F is designed to make the closed-loop system with small damping ratio to get the property of quick response. $u_N(k)$ is designed to change the damping ratio of the closed-loop system to reduce the overshoot. The design of the CNF control law is detailed in Section III.

Before going to the design of the CNF control law, we introduce a preliminary lemma on the property of the nonlinear systems which are zero-input GAS.

Lemma 2.1: Consider the nonlinear discrete-time control system of the form

$$\xi(k+1) = f_1(\xi(k), \eta(k)) \tag{10}$$

which satisfies assumption A3). Then, given any $\gamma > 0$ and $\beta > 0$, there exists a scalar $0 < a < 1$ such that for any

$$|\eta(k)| \leq \beta \cdot a^k, \qquad k \geq 0 \tag{11}$$

the solution $\xi(k)$ of (10) is bounded for all $k \geq 0$ provided that $\xi(0) \in \Omega_\gamma := \{\xi \ : \ \|\xi\| \leq \gamma\}$. For such a triple (a, γ, β), we say that a is good for (γ, β).

Proof: The proof of Lemma 2.1 follows the similar lines of reasoning as in [19, Th. 4.1]. Noting that $V_\xi(\xi)$ is a continuous positive–definite function, we let

$$c = \max\{V_\xi(\xi) \ : \ \xi \in \Omega_\gamma\} \tag{12}$$

for any given $\gamma > 0$. Since $V_\xi(\xi)$ and $f_1(\xi, \eta)$ are continuous, there exists a constant $h > 0$ such that, for all $\xi(k) \in \Omega_\gamma$ and $|v| \leq \beta$, where $\beta > 0$ is any positive real

$$\begin{aligned} |V_\xi(\xi(k+1)) - V_\xi(\xi(k))| &= |V_\xi(f_1(\xi(k), v)) - V_\xi(\xi(k))| \\ &\leq h. \end{aligned} \tag{13}$$

Denote $\xi(k, \xi_0, v)$ to be the solution of

$$\xi(k+1) = f_1(\xi(k), v(k)), \qquad \xi(0) = \xi_0.$$

Let τ be the smallest integer such that $\xi(\tau, \xi_0, v) \notin \Omega_\gamma$ but $\xi(\tau - 1, \xi_0, v) \in \Omega_\gamma$ for any $\xi_0 \in \Omega_\gamma$ and $|v(k)| \leq \beta$. Then, for every solution $\xi(k)$ of (10) under any input such that $|\eta(k)| \leq \beta$ and $\xi(0) \in \Omega_\gamma$

$$V_\xi(\xi(k)) \leq c + h, \qquad 0 \leq k \leq \tau. \tag{14}$$

By (8)

$$V_\xi(f_1(\xi, 0)) - V_\xi(\xi) \leq -\alpha_3(\|\xi\|) < 0$$

for all $\xi \in \Omega_{[c,c+h]} := \{\xi : c \leq V_\xi(\xi) \leq c + h\}$. Then, by the continuity of $V_\xi(\xi)$ and $f_1(\xi, v)$, there exists an $\alpha > 0$ such that

$$V_\xi(f_1(\xi, v)) - V_\xi(\xi) \leq 0 \tag{15}$$

for all $|v| \leq \alpha$ and $\xi \in \Omega_{[c,c+h]}$. Now, select an $0 < a < 1$ such that

$$\beta \cdot a^\tau \leq \alpha. \tag{16}$$

If η is an input satisfying (11) and $\xi(k)$ is the solution of (10) with $\xi(0) \in \Omega_\gamma$, we claim that

$$V_\xi(\xi(k)) \leq c + h, \qquad k \geq 0. \tag{17}$$

In fact, we have proved that $V_\xi(\xi(k)) \leq c + h$ for $0 \leq k \leq \tau$. For $k = \tau + 1$, if $V_\xi(\xi(\tau)) \leq c$, by (13)

$$V_\xi(\xi(\tau+1)) \leq V_\xi(\xi(\tau)) + h \leq c + h.$$

On the other hand, if $c < V_\xi(\xi(\tau)) \leq c + h$, then by (15), we have

$$V_\xi(\xi(\tau+1)) = V_\xi(f_1(\xi(\tau), \eta(\tau))) \leq V_\xi(\xi(\tau)) \leq c + h \tag{18}$$

since $|\eta(\tau)| \leq \alpha$. Thus, (17) can be concluded by induction, and this completes the proof of Lemma 2.1. ■

III. Design of the CNF Control Law

In this section, we proceed to develop a CNF control technique for the case when all the state variables of the linear part of the plant Σ are measurable. The design will be done in four steps described in the following step-by-step design procedure which is a natural extension of the results of [2].

Step s.1) Under assumption A3), given any $\gamma > 0$ and $\beta > 0$, find a $0 < a < 1$ such that a is good for (γ, β).

Step s.2) Design a linear feedback law

$$u_{\rm L}(k) = Fx(k) + Gr \tag{19}$$

where $r \in \mathbb{R}$ is the step reference. The state feedback gain matrix $F \in \mathbb{R}^{1\times n}$ is chosen such that

1) $A + BF$ is Schur;
2) for a given positive–definite matrix $W \in \mathbb{R}^{n\times n}$

$$\sqrt{1 - \frac{\lambda_{\min}(W)}{\lambda_{\max}(P)}} \le a \tag{20}$$

where $P > 0$ is the solution of the following Lyapunov equation:

$$P = (A + BF)'P(A + BF) + W; \tag{21}$$

3) the transfer function of the resulting closed-loop system, i.e., $C(zI - A - BF)^{-1}B$, has certain desired properties, e.g., having a small dominating damping ratio;

and, the feedforward gain $G \in \mathbb{R}$ is given by

$$G := [C(I - A - BF)^{-1}B]^{-1}. \tag{22}$$

Here, we note that G is well defined because $A+BF$ is Schur, and (A, B, C) is right invertible and has no invariant zeros at $z = 1$, which implies $(A + BF, B, C)$ is right invertible and has no invariant zeros at $z = 1$ (see, e.g., [3, Th. 3.8.1]).

Step s.3) Compute

$$H := [I + F(I - A - BF)^{-1}B]G \tag{23}$$

and

$$x_e := G_e r := (I - A - BF)^{-1}BGr. \tag{24}$$

The nonlinear feedback control law u_N is given by

$$u_N(k) = \rho(r, y(k), y(0))B'P(A + BF)(x(k) - x_e) \tag{25}$$

where $\rho(r, y(k), y(0))$ is some nonpositive function, locally Lipschitz in $y(k)$. The choice of this nonlinear function will be discussed at the end of this section.

Step s.4) The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller

$$\begin{aligned} u(k) &= u_{\rm L}(k) + u_N(k) \\ &= Fx(k) + Gr \\ &\quad + \rho(r, y(k), y(0))B'P(A + BF)(x(k) - x_e). \end{aligned} \tag{26}$$

This completes the design of the CNF controller.

Theorem 3.1: Consider the given system (1)–(3) satisfying assumptions A1)–A3). Define

$$\mathcal{N} := \left\{ x \in \mathbb{R}^n : \|x\| \le \beta \left(\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} \right)^{1/2} \right\}. \tag{27}$$

For any $\delta \in (0, 1)$, let $c_\delta > 0$ be the largest positive scalar satisfying the following property:

$$|Fx| \le (1 - \delta)u_{\max} \tag{28}$$

and

$$\boldsymbol{X}_\delta := \left\{ x \in \mathbb{R}^n : x'Px \le c_\delta \right\} \subset \mathcal{N} \tag{29}$$

for all $x \in \boldsymbol{X}_\delta$. Then, for any nonpositive function $\rho(r, y(k), y(0))$, locally Lipschitz in $y(k)$ and $|\rho(r, y(k), y(0))| \le \rho^* := 2(B'PB)^{-1}$, the solution of the closed-loop system under the CNF control law (26) is bounded for all $k \ge 0$, provided that the initial states $\xi(0) = \xi_0$, $x_0 = x(0)$, and r satisfy

$$\begin{aligned} &\xi_0 \in \Omega_\gamma \quad \tilde{x}_0 = \tilde{x}(0) := (x_0 - x_e) \in \boldsymbol{X}_\delta \\ &|Hr| \le \delta u_{\max}. \end{aligned} \tag{30}$$

Moreover, the system output y tracks asymptotically the step command input of amplitude r.

Proof: Consider the closed-loop systems (1)–(3) and (26)

$$\xi(k + 1) = f(\xi(k), x(k), y(k)) \tag{31}$$

$$\begin{aligned} x(k + 1) = {} & Ax(k) + B\text{sat}(Fx(k) + Gr \\ & + \rho(r, y(k), y(0))B'P(A + BF)(x(k) - x_e)) \end{aligned} \tag{32}$$

$$y(k) = Cx(k). \tag{33}$$

Let $\tilde{x}(k) = x(k) - x_e$; then, the closed-loop system (31)–(33) can be expressed as

$$\xi(k + 1) = f_1(\xi(k), \tilde{x}(k)) \tag{34}$$

$$\tilde{x}(k + 1) = (A + BF)\tilde{x}(k) + Bw \tag{35}$$

where

$$\begin{aligned} w = {} & \text{sat}(F\tilde{x}(k) + Hr \\ & +\rho(r, y(k), y(0))B'P(A + BF)\tilde{x}(k)) - F\tilde{x}(k) - Hr. \end{aligned} \tag{36}$$

Define a Lyapunov function $V(\tilde{x}) = \tilde{x}'P\tilde{x}$; then, we have

$$\lambda_{\min}(P)\|\tilde{x}\|^2 \le V(\tilde{x}) \le \lambda_{\max}(P)\|\tilde{x}\|^2 \tag{37}$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of P, respectively. Then

$$\begin{aligned} \triangle V(\tilde{x}(k)) &= V(\tilde{x}(k + 1)) - V(\tilde{x}(k)) \\ &= -\tilde{x}'(k)W\tilde{x}(k) + 2\tilde{x}'(k)(A + BF)'PBw(k) \\ &\quad + w'(k)B'PBw(k). \end{aligned}$$

It has been shown in [23] that

$$2\tilde{x}'(k)(A + BF)'PBw(k) + w'(k)B'PBw(k) \le 0 \tag{38}$$

for all $\tilde{x} \in \boldsymbol{X}_\delta$, $|Hr| \le \delta u_{\max}$, and $-\rho^* \le \rho(r, y(k), y(0)) \le 0$. Thus

$$\triangle V(\tilde{x}(k)) = V(\tilde{x}(k + 1)) - V(\tilde{x}(k)) \le -\tilde{x}'(k)W\tilde{x}(k) \le 0 \tag{39}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (35). Thus, the solution of (35) is bounded for all $k \ge 0$ and $\tilde{x}_0 \in \boldsymbol{X}_\delta$. Nothing that $x(k) = x_e + \tilde{x}(k)$, $x(k)$ is bounded for all $k \ge 0$ if x_0 satisfies (30).

To show the boundedness of the solution ξ of (31), it suffices to show that $\|\tilde{x}(k)\| \le \beta \cdot a^k$. To this end, by [18, Lemma 13.2], we have $0 < \lambda_{\min}(W) \le \lambda_{\max}(P)$ and $V(\tilde{x}(k + 1)) \le \varrho \cdot V(\tilde{x}(k))$ where $\varrho = 1 - (\lambda_{\min}(W))/(\lambda_{\max}(P))$.

Therefore, we get

$$V(\tilde{x}(k + 1)) \le \varrho^{k+1} \cdot V(\tilde{x}(0)) \tag{40}$$

and, then

$$\lambda_{\min}(P) \cdot \|\tilde{x}(k+1)\|^2 \leq \varrho^{k+1} \cdot \lambda_{\max}(P) \cdot \|\tilde{x}(0)\|^2 \tag{41}$$

so that

$$\|\tilde{x}(k)\| \leq \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \cdot \|\tilde{x}(0)\| \cdot (\sqrt{\varrho})^k \qquad \forall k \geq 0. \tag{42}$$

Finally, note that $\sqrt{\varrho} \leq a$

$$\|\tilde{x}(k)\| \leq \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \cdot \|\tilde{x}(0)\| \cdot a^k \leq \beta \cdot a^k \tag{43}$$

for all $\tilde{x}(0) \in \boldsymbol{X}_\delta$. By Lemma 2.1, the solution of (31) is bounded for all $k \geq 0$ and $\xi_0 \in \Omega_\gamma$.

Moreover, noting that $W > 0$, all trajectories of (35) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state x_0 and the step command input r that satisfy (30), we have

$$\lim_{k\to\infty} x(k) = x_e \tag{44}$$

which implies

$$\begin{aligned}\lim_{k\to\infty} y(k) &= C \lim_{k\to\infty} x(k) = Cx_e \\ &= C(I - A - BF)^{-1}BGr = G^{-1}Gr = r.\end{aligned} \tag{45}$$

This completes the proof of Theorem 3.1. ■

The key component in designing the CNF controllers is the selection of ρ and W. The freedom in choosing the nonlinear function ρ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output y approaches the target reference. Since the main purpose of adding the nonlinear part to the CNF controller is to speed up the settling time and to reduce the overshoot, or, equivalently, to contribute a significant value to the control input when the tracking error $r - y$ is small, it is appropriate for us to select a nonlinear function such that the nonlinear part will be in action when the control signal is far away from its saturation level, and thus, it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the linear part of the plant, i.e., (2), and the CNF control law (26) can be expressed as

$$\tilde{x}(k+1) = (A + BF)\tilde{x}(k) + \rho BB'P(A + BF)\tilde{x}(k). \tag{46}$$

It is clear that eigenvalues of the closed-loop system in (46) can be changed by the nonlinear function ρ. Assuming that $y(0) \neq r$ (for the trivial case, when $y = r$, there is no need to add any nonlinear gain to the control), we propose the following nonlinear gain:

$$\begin{aligned}\rho(r, y(k), y(0)) = \frac{-\kappa_1 (B'PB)^{-1}}{|y(0) - r|^{\kappa_2}} \left| |y(k) - r|^{\kappa_2} - |y(0) - r|^{\kappa_2} \right|, \\ 0 \leq \kappa_1 \leq 1\end{aligned} \tag{47}$$

where $|\rho(r, y(k), y(0))|$ starts from 0 and gradually increases to a final gain of $\kappa_1 (B'PB)^{-1}$ as y approaches to the target reference r. The parameter κ_2 is used to determine the speed of change in ρ. It can be shown that the closed-loop poles of (46) are related to the invariant zeros of an auxiliary system characterized by

$$\begin{aligned}G_{\text{aux}}(z) &:= C_{\text{aux}}(zI - A_{\text{aux}})^{-1}B_{\text{aux}} \\ &:= B'P(zI - A - BF)^{-1}B\end{aligned} \tag{48}$$

which is obviously stable, and which was shown in [7] to be a square, invertible, and uniform rank system with one infinite zero of order 1 and with $n - 1$ stable invariant zeros. In fact, if we select $\kappa_1 = 1$, the closed-loop poles of (46) in the steady state when $y = r$ are precisely given by the invariant zeros of $G_{\text{aux}}(z)$ together with additional one at $z = 0$. Generally, the invariant zeros of $G_{\text{aux}}(z)$ can be preassigned by the appropriate choice of W, which can also be selected using a trial-and-error approach by limiting it to be in a diagonal matrix and adjusting its diagonal weights through simulation. We refer interested readers to [7] for detail.

Remark 3.1: The previously described method can be extended to the output regulation problem [9], [11]. Specifically, consider the system

$$\xi(k+1) = f(\xi(k), x(k), e(k)) \tag{49}$$

$$x(k+1) = Ax(k) + B\text{sat}(u(k)) + Ev(k) \tag{50}$$

$$e(k) = Cx(k) + Qv(k) \tag{51}$$

where $v \in \mathbb{R}^s$ is the reference and/or disturbance input generated by the following exo-system

$$v(k+1) = Sv(k). \tag{52}$$

Assume that (49)–(51) and (52) satisfy assumption A1) and the following assumptions hold:

A4) all the eigenvalues of S are simple and located on the unit circle;

A5) there exist matrices Π and Γ such that they solve the linear matrix equations

$$\Pi S = A\Pi + B\Gamma + E \tag{53}$$

$$0 = C\Pi + Q; \tag{54}$$

A6) there exist a continuous function $V : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, $\mathcal{K}_\infty$-functions α_1, α_2, and α_3 such that

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|)$$
$$V(f(\xi, \Pi v, 0)) - V(\xi) \leq -\alpha_3(\|\xi\|)$$

for all $\xi \in \mathbb{R}^m$ and v is the solution of (52).

Then, a CNF control law is given by

$$u(k) = Fx(k) + Gv(k) + \rho(e(k), e(0))B'P(A+BF)(x(k) - \Pi v(k))$$

where F and P are the same as designed in Step s.2), $-2(B'PB)^{-1} \leq \rho(e(k), e(0)) \leq 0$, and $G = \Gamma - F\Pi$.

IV. Illustrative Examples

Example 1 (Target Tracking Problem): Consider a system characterized by

$$\xi_1(k+1) = \frac{4\xi_1(k)\xi_2(k)}{1 + \xi_1^2(k)\xi_2^2(k)}\xi_1(k) \tag{55}$$

$$\xi_2(k+1) = y(k) - r \tag{56}$$

$$x(k+1) = Ax(k) + B\ \text{sat}\ (u) \tag{57}$$

$$y(k) = Cx(k) \tag{58}$$

with $u_{\max} = 1$, where A, B, and C are shown in the equation at the bottom of the page. The nonlinear part (55) and (56) of the system is taken from [21]. It is shown in [21] that the nonlinear part (55) and (56)

$$A = \begin{bmatrix} 0.9950 & 0.0998 & 0.0050 & 0.0002 & 0.0000 \\ -0.0998 & 0.9950 & 0.0997 & 0.0050 & 0.0002 \\ 0 & 0 & 0.9950 & 0.0998 & 0.0050 \\ 0 & 0 & -0.0998 & 0.9950 & 0.0998 \\ 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix} \quad B = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0002 \\ 0.0050 \\ 0.1000 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

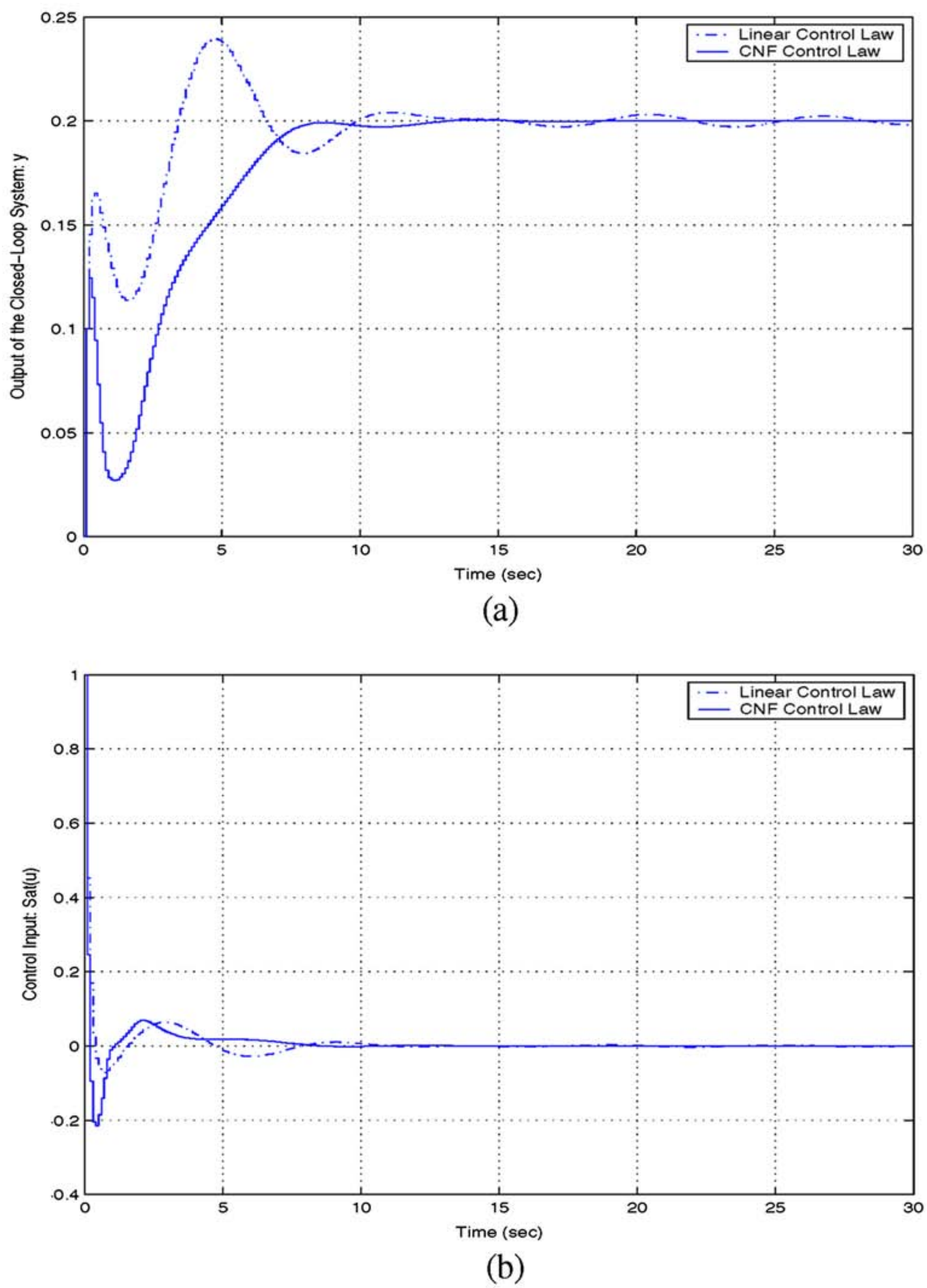

Fig. 1. Target tacking control: linear versus CNF control. (a) System output of the closed-loop system. (b) Control input of the closed-loop system.

is globally exponential stable when the tracking error $y-r=0$, that is, assumption A3) is satisfied. However, even the tracking error converges exponentially to zero with $y(k)-r=\xi_2(0)(1/2)^{k+1}$, the nonlinear part of the system is unstable when $\xi_1(0)\neq 0$, and $\xi_2(0)=\xi_1^{-1}(0)$. However, let $\gamma=1$ and $\beta=1$, for any $0<a<1$; we can verify that a is good for (γ,β).

The tracking target is a step function with amplitude $r=0.2$. Our aim is to design an appropriate CNF controller with state feedback to improve the transient performance of the closed-loop system while maintaining the exponential stability of the nonlinear part of the system. It is not difficult to verify that assumptions A1) and A2) are also satisfied for the system (55)–(58). A linear feedback control law is first designed by using the low-gain feedback technique [13]. We obtain a linear control law $u_{\rm L}(k)=Fx(k)+Gr$ with

$$F=[0.6101 \quad -0.0005 \quad -0.3135 \quad -4.1350 \quad -7.2667]$$
$$G=5.0571.$$

The nonlinear function $\rho(r,y(k),y(0))$ is chosen as in (47) with $\kappa_1=0.25$ and $\kappa_2=16$. Finally, the CNF control law is given by

$$u(k)=Fx(k)+Gr+\rho(r,y(k),y(0))B'P(A+BF)(x(k)-x_e) \tag{59}$$

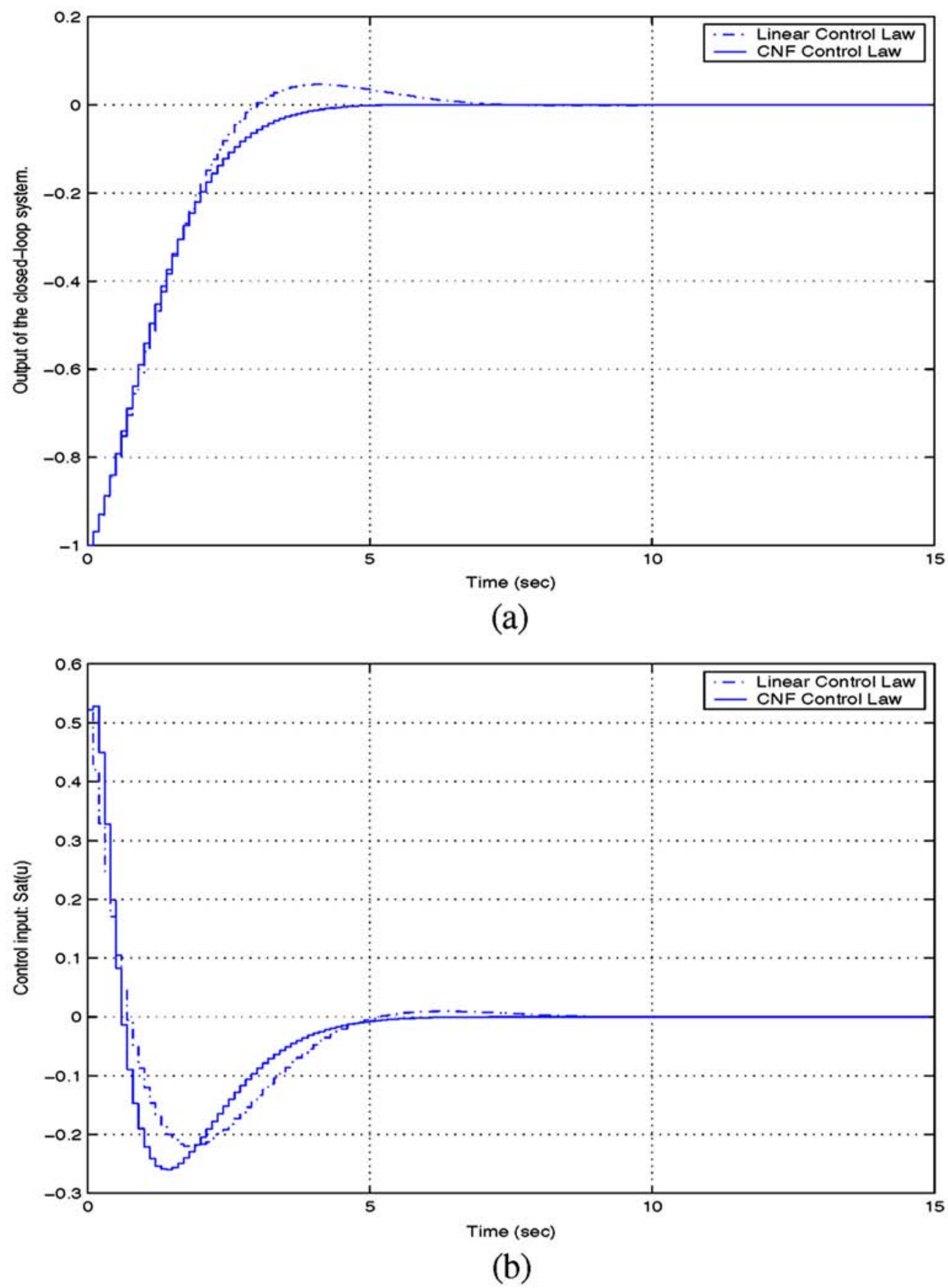

Fig. 2. Output regulation for RTAC system. (a) System output of the closed-loop system. (b) Control input of the closed-loop system.

where $x_e = (I - A - BF)^{-1} BG\, r$ and P is the positive–definite solution of $P = (A+BF)' P (A+BF) + I_5$. Using Simulink in Matlab, we obtain the simulation result in Fig. 1, which is done under the following initial conditions: $\xi_1(0) = -0.8$, $\xi_2(0) = 0.5$, and $x(0) = 0$. The simulation result shows that the control law with the nonlinear components, i.e., the CNF controller, improved the transient performance significantly. Specifically, the output of the closed-loop system under the CNF control law converges to r quickly in 15 s. However, under the linear control law, more than 30 s are required for the output y convergence to r. Moreover, the overshoot under the linear control is 19.73%, but for the CNF control it is only 0.27%.

Example 2 (RTAC Benchmark Problem): We consider a regulation problem for a benchmark problem on a rotational/translational actuator (RTAC) system proposed in [1] (see also [8]). The normalized motion equation of the RTAC system is given by (see [1] and [8])

$$\ddot{\varsigma} + b\dot{\varsigma} + \varsigma = \epsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) \tag{60}$$

$$\ddot{\theta} = -\epsilon\ddot{\varsigma}\cos\theta + \upsilon \tag{61}$$

where ς is the normalized displacement of the cart, θ the angular position of the eccentric mass, υ the normalized control input, ϵ the coupling between the translational and rotational motion, and b the coefficient of viscous friction for motion of the cart.

Let $y_1 = \theta, y_2 = \dot{\theta}, \xi_1 = \varsigma + \epsilon\sin\theta$, and $\xi_2 = \dot{\varsigma} + \epsilon\dot{\theta}\cos\theta$, and using a prestate feedback

$$\upsilon = \epsilon\cos y_1 \left(\xi_1 - \left(1 + y_2^2\right)\epsilon\sin y_1\right) - (1 - \epsilon^2\cos^2 y_1)u \tag{62}$$

the state–space representation of (60) and (61) is given by

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -b\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}(\epsilon \sin y_1 + \epsilon b y_2 \cos y_1) \quad (63)$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u. \quad (64)$$

Letting $b = 0.1$, $\epsilon = 0.2$, and sampling period $T = 0.1$ and considering the input saturation $u_{\max} = 1$, we can obtain the discrete-time model of (63) and (64) by using zero-order hold method as follows:

$$\begin{bmatrix}\xi_1(k+1)\\ \xi_2(k+1)\end{bmatrix} = \begin{bmatrix}0.9950 & 0.0993\\ -0.0993 & 0.9851\end{bmatrix}\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\end{bmatrix} + \begin{bmatrix}0.0050\\ 0.0993\end{bmatrix}\epsilon(\sin y_1(k) + b y_2(k)\cos y_1(k)) \quad (65)$$

$$\begin{bmatrix}y_1(k+1)\\ y_2(k+1)\end{bmatrix} = \begin{bmatrix}1.0000 & 0.1000\\ 0 & 1.0000\end{bmatrix}\begin{bmatrix}y_1(k)\\ y_2(k)\end{bmatrix} + \begin{bmatrix}0.0050\\ 0.1000\end{bmatrix}\mathrm{sat}(u(k)). \quad (66)$$

Our objective is to design a CNF control law for the systems (65) and (66) to regulate the output y_1 to zero quickly without any overshoot. To this end, let $r = 0$. It is easy to verify that the systems (65) and (66) satisfy assumption A1)–A3). A linear feedback gain is obtained by using the integral of the time multiplied by the absolute value of the error (ITAE) prototype design [5] which gives

$$F = [-0.9318 \quad -1.3643] \quad (67)$$

Then, the CNF control law is given by

$$u(k) = Fx(k) + \rho(r, y(k), y(0))B'P(A + BF)x(k) \quad (68)$$

where P is the positive–definite solution of $P = (A + BF)'P(A + BF) + I_2$ and $\rho(r, y(k), y(0))$ is given by (47) with $y = y_1$, $\kappa_1 = 0.1$, and $\kappa_2 = 5$. Let the initial conditions $\xi_1(0) = -0.2$, $\xi_2(0) = 0.3$, $y_1(0) = -1$, and $y_2(0) = 0.3$; the simulation result is shown in Fig. 2. Under the ITAE linear control law, the output of the closed-loop system is regulated to zero with visible overshoot. However, the CNF control law regulates the output of the closed-loop system to zero without any overshoot. Moreover, the rise time under the CNF control law is almost the same as the one under linear control law.

V. Conclusion

We have extended the so-called CNF control techniques for linear input-saturated discrete-time systems to a class of single-input–single-output (SISO) partially linear composite discrete-time systems with actuator saturation. The closed-loop system is able to track step function signals, yet the whole system is stable. It has been shown that the transient performance is improved comparing to normal linear approaches. Both CNF and linear controllers avoid adverse effect of peaking phenomenon. Further extension to MIMO case can be established similarly by provoking the results of CNF control for linear MIMO discrete-time systems (see [7]). We note that it might be interesting and challenging to extend our result to sampled-data nonlinear systems. Viewing the plant (1)–(3) as the approximate discrete-time model of the sampled-data model, it is possible to solve the problem by using the results on the framework for controller design of sampled-data nonlinear systems via their approximate discrete-time models [16], [17]. Indeed, this is one of our future works.

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