

**Approach A1:** Forming  $\Delta^{(n)^{-1}} f^{(n)}$  requires  $\mathcal{O}(2n)$ .  $u^{(n)}$  needs an additional  $\mathcal{O}(n^2)$  flops.  $u^{(n-1)}$  needs only  $\mathcal{O}(3n)$  more. Forming  $\delta_j$  requires  $n$  flops. Each element requires approximately 5 flops (see (3.6)). Thus, since there are  $n - 1$  additional rows, to form all the elements we need  $\mathcal{O}(2.5n^2)$  flops. Stability is checked by inspection only.

**Approach A2:**  $\Delta^{(n)^{-1}} f^{(n)}$  required to find  $\hat{F}(z)$  needs  $\mathcal{O}(2n)$  flops. Getting  $\hat{F}(z)$  (premultiplication of  $\Delta^{(n)^{-1}} f^{(n)}$  by an upper triangular matrix) requires an additional  $\mathcal{O}(n^2)$  flops. The first two rows need  $\mathcal{O}(0.5n)$  plus  $\mathcal{O}(2n)$  flops (there is a deconvolution operation in forming the second row).  $\hat{\delta}_j$  requires  $n$  flops. Next, each element requires approximately 3 flops. Thus, since there are  $n - 1$  additional rows, to form all the elements we need  $\mathcal{O}(1.5n^2)$  flops. Since it suffices to compute only half the entries, actually we need  $\mathcal{O}(0.75n^2)$  flops. Stability is checked by looking at the sum of each row. This requires  $\mathcal{O}(0.5n^2)$  flops.

## V. CONCLUSION AND FINAL REMARKS

If the polynomial's coefficients have numerical values, widely available root locations algorithms may be utilized. However, as mentioned in Section I, tabular methods are indispensable whenever literal parameters are present. Moreover, they may be used to determine the maximum possible parameter(s) perturbation(s) of an initially stable system. In fact, the table developed above provides the required critical stability constraints directly [15]. Also, incorporating the polynomial array technique [6]–[8], one may easily develop systematic procedures that are easily programmable in software.

The tabular method developed above provides a direct check of stability with no recourse to transformations to a more familiar stability region. The desirability of such direct stability checking is apparent from the results developed. This is especially the case when the sampling frequency is high relative to the system bandwidth. The relationship of the tabular method with the Routh–Hurwitz table is also indicated [14]. These constitute the motivation and value of this work.

The note also compares the computational time of the direct method (approach A1), with that of applying BT to the transformed polynomial (approach A2). Both approaches are quadratic in  $n$ . In forming the table entries, approach A2 fares better because only about half the entries are needed. However, due to poor coefficient sensitivity properties, with high sampling frequency, it may not be suitable. Note that, in approach A1, stability is determined by simple inspection.

Only polynomials with real coefficients are dealt with here. Stability of  $q$ -systems with complex coefficients is in [17]. The extension of this to  $\delta$ -systems is under present investigation. This of course is useful in stability investigations of multidimensional  $\delta$ -systems.

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## A Reduced Order Observer Based Controller Design for $H_\infty$ -Optimization

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**Abstract**—In this note the  $H_\infty$  control problem with measurement feedback is investigated. It is well-known that for this problem, in general, we need controllers of the same dynamic order as the given system. However, in the case that some entries of the measurement vector are not noise-corrupted, we show that one can find dynamic compensators of a lower dynamical order. Note that this implies that the standard assumptions on the direct feedthrough matrices, as made in most papers on  $H_\infty$  control, are not satisfied. Our result can be derived by using reduced order observer based controllers.

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## I. INTRODUCTION

The  $H_\infty$  control problem attracted a lot of attention in the last decade. It started with the paper [15]. After that several techniques were developed:

- Interpolation approach: e.g., [8]
- Frequency domain approach: e.g., [4]
- Polynomial approach: e.g., [6]
- $J$ -spectral factorization approach; e.g., [5]
- Time-domain approach: e.g., [3].

The above list is far from complete. In our view the time-domain approach yielded the most intuitive results. Moreover, the conditions were easily checkable: there exists a stabilizing compensator which makes the  $H_\infty$  norm less than 1 if and only if there exist positive semidefinite stabilizing solutions of two algebraic Riccati equations, which satisfy a coupling condition (the spectral radius of their product should be less than 1). However, all the techniques mentioned above had one major drawback. The systems under consideration should satisfy a number of assumptions.

- The subsystem from control to the to be controlled output should not have invariant zeros on the imaginary axis and the direct feedthrough matrix of this system should be injective.
- The subsystem from disturbance to the measurement output should not have invariant zeros on the imaginary axis and the direct feedthrough matrix of this system should be surjective.

Note that identical conditions were assumed in the linear quadratic Gaussian control problem. The above assumptions for the  $H_\infty$  control problem were removed in [10]–[13]. In this note we will assume that the conditions on the invariant zeros are still satisfied but we do not make assumptions on the direct feedthrough matrices. This will be called the singular case (contrary to the regular case).

In general (even without any assumptions) it turns out that if we can find a stabilizing controller which makes the  $H_\infty$  norm less than 1 (a so-called suitable controller) then we can always find a suitable controller of McMillan degree  $n$  (where  $n$  is the McMillan degree of our system). This controller has the standard form of an observer interconnected with a state feedback. In the regular case, the direct feedthrough matrix from the disturbance to the measurement output is surjective and hence we cannot observe any states directly: the measurement of each state is perturbed by the disturbance. On the other hand, in the singular case, we can measure, say  $k$ , states directly without any disturbances. In principle it then suffices to build an observer for the remaining  $n-k$  states which would yield a controller of McMillan degree  $n-k$ . However, the separation principle does not hold in  $H_\infty$  control, in the sense that suitable state feedbacks and observers can not be chosen independently. Moreover, it is in general hard to show formally that we can attain the same level of performance with reduced order observers as we could with full order observers. Therefore, we introduce a transformation which simplifies these problems considerably. In this note we will formalize the above.

In Section II we will give the problem formulation. Then, in Section III, we will present a preliminary factorization needed in the construction of the controller. Finally, in Section IV, we will present our main result and give a constructive method to derive a suitable controller of the required McMillan degree. We conclude in Section V with some concluding remarks.

## II. PROBLEM STATEMENT

Consider the following system

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1x + D_1w, \\ z = C_2x + D_2u, \end{cases} \quad (2.1)$$

where  $x \in \mathfrak{R}^n$  is the state,  $u \in \mathfrak{R}^m$  is the control input,  $w \in \mathfrak{R}^l$  is the unknown disturbance,  $y \in \mathfrak{R}^p$  is the measured output and  $z \in \mathfrak{R}^q$  is the controlled output. The following assumptions are made:

- 1)  $(A, B, C_2, D_2)$  has no invariant zeros on the  $j\omega$  axis, and
- 2)  $(A, E, C_1, D_1)$  has no invariant zeros on the  $j\omega$  axis.

Remember that invariant zeros are points in the complex plane where the Rosenbrock system matrix loses rank.

The basic objective of the  $H_\infty$  control problem is to find a stabilizing compensator of the form

$$\Sigma_{cmp}: \begin{cases} \dot{v} = A_{cmp}v + B_{cmp}y, \\ -u = C_{cmp}v + D_{cmp}y. \end{cases} \quad (2.2)$$

for  $\Sigma$  such that the resulting closed-loop system has an  $H_\infty$  norm strictly less than an, a priori given, bound  $\gamma$ . In [3], [14] it has been shown that if a suitable controller, that is a controller satisfying the above objectives, exists then there always exists a suitable controller of McMillan degree  $n$ . We show in this note that in some cases we can reduce the order of the controller even more without loss of performance. This is described in the following theorem.

**Theorem 2.1:** Let  $\Sigma$  be given by (2.1) such that assumptions a) and b) are satisfied. Let  $\gamma > 0$  be given. The following conditions are equivalent:

- 1) There exists a stabilizing controller of the form (2.2) which, when applied to  $\Sigma$ , yields a closed loop system with  $H_\infty$  norm strictly less than  $\gamma$ .
- 2) There exists a stabilizing controller of McMillan degree

$$n - \text{rank}(C_1 \ D_1) + \text{rank} \ D_1 \leq n$$

which, when applied to  $\Sigma$ , yields a closed loop system with  $H_\infty$  norm strictly less than  $\gamma$ .  $\square$

It is easily seen that we only need to derive this result for  $\gamma = 1$ . The general result can be easily obtained via scaling. Hence, in the remainder of this note, we will always assume that  $\gamma = 1$ .

The implication b)  $\Rightarrow$  a) is trivial. The objective for the rest of this paper is to assume a suitable controller exists satisfying part a) and to give a constructive proof of the existence of a suitable controller satisfying part b).

## III. PRELIMINARY FACTORIZATIONS

In this section, we recall a result from [12], [13]. Let the original system (2.1) be given. For  $P \in \mathfrak{R}^{n \times n}$  we define the following matrix:

$$F(P) := \begin{Bmatrix} A^T P + PA + C_2^T C_2 + \dots PEE^T P & PB + C_2^T D_2 \\ B^T P + D_2^T C_2 & D_2^T D_2 \end{Bmatrix}.$$

If  $F(P) \geq 0$ , we say that  $P$  is a solution of the quadratic matrix inequality. We also define a dual version of this quadratic matrix inequality. For any matrix  $Q \in \mathfrak{R}^{n \times n}$  we define the following matrix:

$$G(Q) := \begin{pmatrix} AQ + QA^T + EE^T + QC_2^T C_2 Q & QC_1^T + ED_1^T \\ C_1 Q + D_1 E^T & D_1 D_1^T \end{pmatrix}.$$

If  $G(Q) \geq 0$ , we say that  $Q$  is a solution of the dual quadratic matrix inequality. In addition to these two matrices, we define two matrix pencils, which play dual roles:

$$L(P, s) := (sI - A - EE^T P - B),$$

$$M(Q, s) := \begin{pmatrix} sI - A - QC_2^T C_2 \\ -C_1 \end{pmatrix}.$$

Finally, we define the following two transfer matrices:

$$G_{ci}(s) := C_2(sI - A)^{-1}B + D_2,$$

$$G_{di}(s) := C_1(sI - A)^{-1}E + D_1.$$

Let  $\rho(M)$  denote the spectral radius of the matrix  $M$ . By  $\text{rank}_{\mathcal{R}(s)} M$  we denote the rank of  $M$  as a matrix with elements in the field of real rational functions  $\mathcal{R}(s)$ . We are now in a position to recall the main result from [13].

**Theorem 3.1:** Consider the system (2.1). Assume that both the system  $(A, B, C_2, D_2)$  as well as the system  $(A, E, C_1, D_1)$  have no invariant zeros on the imaginary axis. Then the following two statements are equivalent.

- 1) For the system (2.1) there exists a time-invariant, finite-dimensional dynamic compensator  $\Sigma_{cmp}$  of the form (2.2) such that the resulting closed-loop system, with transfer matrix  $G_{cl}$ , is internally stable and has  $H_\infty$  norm less than 1, i.e.,  $\|G_{cl}\|_\infty < 1$ .
- 2) There exist positive semi-definite solutions  $P, Q$  of the quadratic matrix inequalities  $F(P) \geq 0$  and  $G(Q) \geq 0$  satisfying  $\rho(PQ) < 1$ , such that the following rank conditions are satisfied
  - a)  $\text{rank } F(P) = \text{rank}_{\mathcal{R}(s)} G_{ci}$ ,
  - b)  $\text{rank } G(Q) = \text{rank}_{\mathcal{R}(s)} G_{di}$ ,
  - c)  $\text{rank} \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci} \quad \forall s \in \mathcal{E}^0 \cup \mathcal{E}^+$ ,
  - d)  $\text{rank} (M(Q, s) G(Q)) = n + \text{rank}_{\mathcal{R}(s)} G_{di} \quad \forall s \in \mathcal{E}^0 \cup \mathcal{E}^+$ .  $\square$

Note that the existence and determination of  $P$  and  $Q$  can be checked by investigating reduced order Riccati equations. For details we refer to [14].

Next, we construct a new system,

$$\Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = A_{P,Q} x_{P,Q} + B_{P,Q} u_{P,Q} + E_{P,Q} w, \\ y_{P,Q} = C_{1,P} x_{P,Q} + D_{P,Q} w, \\ z_{P,Q} = C_{2,P} x_{P,Q} + D_{P,Q} u_{P,Q}, \end{cases} \quad (3.1)$$

where

$$F(P) = \begin{pmatrix} C_{2,P}^T \\ D_P^T \end{pmatrix} (C_{2,P} \quad D_P),$$

$$G(Q) = \begin{pmatrix} E_Q \\ D_{P,Q} \end{pmatrix} (E_Q^T \quad D_{P,Q}^T),$$

and

$$A_{P,Q} := A + EE^T P + (I - QP)^{-1} Q C_{2,P}^T C_{2,P}$$

$$B_{P,Q} := B + (I - QP)^{-1} Q C_{2,P}^T D_P$$

$$E_{P,Q} := (I - QP)^{-1} E_Q$$

$$C_{1,P} := C_1 + D_1 E^T P.$$

It has been shown in [13] that this new system has the following properties:

- 1)  $(A_{P,Q}, B_{P,Q}, C_{2,P}, D_P)$  is right invertible and minimum phase.
- 2)  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$  is left invertible and minimum phase.

Moreover, in [13] the following theorem has been proven.

**Theorem 3.2:** Let an arbitrary compensator  $\Sigma_{cmp}$ , of the form (2.2) be given. The following two statements are equivalent:

- i) The compensator  $\Sigma_{cmp}$  applied to the original system  $\Sigma$  as in (2.1) is internally stabilizing and the resulting closed loop transfer function from  $w$  to  $z$  has  $H_\infty$  norm less than 1.
- ii) The compensator  $\Sigma_{cmp}$  applied to the new system  $\Sigma_{P,Q}$  as in (3.1) is internally stabilizing and the resulting closed loop transfer function from  $w$  to  $z_{P,Q}$  has  $H_\infty$  norm less than 1.  $\square$

We will show that there exists a time-invariant, finite-dimensional dynamic compensator  $\Sigma_{cmp}$  of the form (2.2) and with McMillan degree  $n - \text{rank}[C_1, D_1] + \text{rank}(D_1)$  for  $\Sigma$  such that the resulting closed loop system is internally stable and the closed loop transfer function from  $z$  to  $w$  has  $H_\infty$  norm less than 1. Moreover, we give an explicit construction of such a reduced order compensator. More specific, we design a reduced order observer based control law for  $H_\infty$ -optimization problem. By the above theorem we can devote all our attention to our new system  $\Sigma_{P,Q}$  and design controllers for this system.

#### IV. REDUCED ORDER OBSERVER BASED CONTROLLER DESIGN

In this section, we construct explicitly a reduced order observer based controller of order  $n - \text{rank}[C_1, D_1] + \text{rank}(D_1)$  for  $\Sigma_{P,Q}$ . However, it is evident from Theorem 3.2 that such a controller will yield the same performance when applied to the original system  $\Sigma$ . Note that in  $H_\infty$  control we have, like in for instance Linear Quadratic Gaussian control, controllers which have the structure of a state feedback interconnected with an observer. However, in the  $H_\infty$  control problem the matter is more complicated since we cannot choose the state feedback and the observer independently. This is due to the fact that we have a suboptimal design and because for an  $H_\infty$  observer, the observer gain depends on the part of the state space we would like to estimate.

We will eliminate states which can be directly observed and concentrate on those states which still need to be observed. In order to do this we choose suitable basis. Without loss of generality but for simplicity of presentation, we assume that the matrices  $C_{1,P}$  and  $D_{P,Q}$  are transformed in the following form:

$$C_{1,P} = \begin{bmatrix} 0 & C_{02} \\ I_k & 0 \end{bmatrix} \quad \text{and} \quad D_{P,Q} = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}. \quad (4.1)$$

Thus, the system  $\Sigma_{P,Q}$  as in (3.1) can be partitioned as follows:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{P,Q} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} w, \\ \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & C_{02} \\ I_k & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_0 \\ 0 \end{bmatrix} w, \\ z = C_{2,P} x_{P,Q} + D_{P,Q} u_{P,Q}, \end{cases} \quad (4.2)$$

where  $[x_1^T, x_2^T]^T = x_{P,Q}$  and  $[y_0^T, y_1^T]^T = y_{P,Q}$ . We observe that  $y_1 = x_1$  is already available and need not be estimated. Thus, we need to estimate only the state variable  $x_2$ . We first rewrite the state equation for  $x_1$  in terms of the output  $y_1$  and state  $x_2$  as follows:

$$\dot{y}_1 = A_{11} y_1 + A_{12} x_2 + B_1 u_{P,Q} + E_1 w, \quad (4.3)$$

where  $y_1$  and  $u_{P,Q}$  are known signals. Equation (4.3) can be rewritten as

$$\tilde{y} = A_{12} x_2 + E_1 w = \dot{y}_1 - A_{11} y_1 - B_1 u_{P,Q}. \quad (4.4)$$

Thus, observation of  $x_2$  is made via (4.4) as well as by

$$y_0 = C_{02} x_2 + D_0 w.$$

Now, a reduced order system suitable for estimating the state  $x_2$  is given by

$$\Sigma_r : \begin{cases} \dot{x}_2 = A_{22} x_2 + [A_{21} \quad B_2] \begin{bmatrix} y_1 \\ u_{P,Q} \end{bmatrix} + E_2 w, \\ \begin{bmatrix} y_0 \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} x_2 + \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} w. \end{cases} \quad (4.5)$$

Before we proceed to construct the reduced order observer, we present in the following a key lemma which plays an important role in our design.

**Lemma 4.1:** Let  $\Sigma_{re}$  be denoted the subsystem characterized by matrix quadruple

$$\left( A_{22}, E_2, \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}, \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} \right).$$

Then we have

- 1)  $\Sigma_{re}$  is minimum (nonminimum) phase iff  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$  is minimum (nonminimum) phase.
- 2)  $\Sigma_{re}$  is detectable iff  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$  is detectable.
- 3) Invariant zeros of  $\Sigma_{re}$  are the same as those of  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ .
- 4) Orders of infinite zeros of  $\Sigma_{re}$  are reduced by one from those of  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ .
- 5)  $\Sigma_{re}$  is left invertible iff  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$  is left invertible.  $\square$

*Proof:* We will not proof part 4) since this will not be explicitly needed in the remainder of this paper. For a proof of part 4) we refer to [9].

We have the following relationship between the Rosenbrock system matrices for  $\Sigma_{re}$  and  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$ :

$$\begin{aligned} & \text{rank} \begin{pmatrix} sI - A_{P,Q} & -E_{P,Q} \\ C_{1,P} & D_{P,Q} \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} sI - A_{11} & -A_{12} & -E_1 \\ -A_{21} & sI - A_{22} & -E_2 \\ 0 & C_{02} & D_0 \\ I_k & 0 & 0 \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} sI - A_{22} & -E_2 \\ C_{02} & D_0 \\ A_{12} & E_1 \end{pmatrix} + k. \end{aligned} \quad (4.6)$$

Invariant zeros of  $(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})$  are, by definition, points  $s \in \mathbb{C}$  where the rank in (4.6) drops. Similarly invariant zeros of  $\Sigma_{re}$  are those points  $s \in \mathbb{C}$  where the rank in (4.7). The above equality then immediately implies that the two systems have the same (finite) invariant zeros, which proves part 3).

A system is minimum-phase if and only if all finite invariant zeros are in the open left half plane. Hence, part 3) immediately implies part 1).

A system is left-invertible if and only if the Rosenbrock system matrix has full column rank for all but finitely many  $s \in \mathbb{C}$ . Hence the above equality also immediately proves part 5).

Similarly part 2) is a direct consequence of the following equality:

$$\text{rank} \begin{pmatrix} sI - A_{P,Q} \\ C_{1,P} \end{pmatrix} = \text{rank} \begin{pmatrix} sI - A_{22} \\ C_{02} \\ E_1 \end{pmatrix} + k \quad \blacksquare$$

Now, based on (4.5), we can construct a reduced order observer

of  $x_2$  as,

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}y_1 + B_2u_{P,Q} + K_r \left( \begin{bmatrix} y_0 \\ \hat{y} \end{bmatrix} - \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} \hat{x}_2 \right),$$

and

$$\hat{x}_{P,Q} = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \hat{x}_2 + \begin{bmatrix} I_k \\ 0 \end{bmatrix} y_1,$$

where  $K_r$  is the observer gain matrix for the reduced order system and is chosen such that

$$A_{22} - K_r \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}$$

is asymptotically stable. In order to move the dependency on  $\hat{y}_1$ , let us partition  $K_r = [K_{r0}, K_{r1}]$  to be compatible with the dimensions of the output  $[y_0^T, y_1^T]^T$ . Then (see e.g., [7]), one can define a new variable  $v := \hat{x}_2 - K_{r1}y_1$  and obtain a new dynamic equation,

$$\begin{aligned} \dot{v} &= (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})v + (B_2 - K_{r1}B_1)u_{P,Q} \\ &\quad + [K_{r0}, A_{21} - K_{r1}A_{11}] \\ &\quad + (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})K_{r1} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \end{aligned} \quad (4.8)$$

Thus, by implementing (4.8),  $\hat{x}_2$  can be obtained without generating  $\hat{y}_1$ .

The following theorem, when coupled with Theorem 3.2, yields the proof of the implication a)  $\Rightarrow$  b) of Theorem 2.1.

**Theorem 4.1:** Let  $\Sigma_{P,Q}$  be given described by (4.2). Then there exist for every  $\epsilon > 0$  a state feedback gain  $F$  and a reduced order observer gain matrix  $K_r$  such that the following reduced order observer based controller, as shown in (4.9) at the bottom of the page, when applied to  $\Sigma_{P,Q}$  is internally stabilizing and yields an  $H_\infty$  norm of the closed loop transfer matrix from  $w$  to  $z$  strictly less than  $\epsilon$ .  $\square$

*Remark:* Let  $\Sigma$  be given by (2.1). Assume there exists a stabilizing dynamic compensator of the form (2.2) for  $\Sigma$  (of arbitrary order). In that case there exist matrices  $P$  and  $Q$  satisfying the conditions of Theorem 3.1. We can then define  $\Sigma_{P,Q}$  by (3.1). Via a suitable basis-transformation we can bring  $\Sigma_{P,Q}$  in the form (4.2). The above theorem guarantees the existence of a reduced-order, stabilizing compensator  $\Sigma_{rcmp}$  for the system (4.2) which yields a closed loop system with  $H_\infty$  norm strictly less than 1. After reversing the effect of the basis transformations (which included a basis change for the space of measurements  $y$ ) we find a reduced-order, stabilizing compensator  $\tilde{\Sigma}_{rcmp}$  for the system (3.1) which yields the same  $H_\infty$  norm of the closed loop system and hence this  $H_\infty$  norm is still strictly less than 1. According to Theorem 3.2 this compensator  $\tilde{\Sigma}_{rcmp}$  will also stabilize  $\Sigma$  and result in an  $H_\infty$  norm of the closed loop system strictly less than 1.

One point might need clarification. We can via a stabilizing compensator applied to  $\Sigma_{P,Q}$ , as defined by (3.1), make the  $H_\infty$  norm of the closed loop system arbitrarily small. Theorem 3.2 guarantees that this same compensator when applied to  $\Sigma$  will be stabilizing and will also have an  $H_\infty$  norm strictly less than 1. But you can not say more about the resulting closed loop  $H_\infty$  norm; it might be arbitrarily close to 1. In general it will not be possible to find stabilizing compensators for  $\Sigma$  which make the resulting  $H_\infty$  norm arbitrarily small.

$$\Sigma_{rcmp} : \begin{cases} \dot{v} = (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})v + (B_2 - K_{r1}B_1)u_{P,Q} \\ \quad + [K_{r0}, A_{21} - K_{r1}A_{11} + (A_{22} - K_{r0}C_{02} - K_{r1}A_{12})K_{r1}]y_{P,Q}, \\ u_{P,Q} = -F\hat{x}_{P,Q} = -F \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} v - F \begin{bmatrix} 0 & I_k \\ 0 & K_{r1} \end{bmatrix} y_{P,Q}, \end{cases} \quad (4.9)$$

*Proof:* It is straightforward to verify that the closed-loop system comprising  $\Sigma_{P,Q}$  and the reduced order observer based controller (4.9) is given by

$$\begin{aligned} G_{cl} = & (C_{2,P} - D_P F)(sI - A_{P,Q} + B_{P,Q} F)^{-1} E_{P,Q} \\ & + [(C_{2,P} - D_P F) \\ & \times (sI - A_{P,Q} + B_{P,Q} F)^{-1} B_{P,Q} + D_P] \\ & \times \left[ F_2 \left( sI - A_{22} + K_r \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} \right)^{-1} \left( E_2 - K_r \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} \right) \right], \end{aligned} \quad (4.10)$$

where  $F := (F_1, F_2)$  is partitioned compatibly with the partitioning of  $A$ . Now, it is well known that for left invertible and minimum phase system,

$$\left( A_{22}, E_2, \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix}, \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} \right),$$

(see Lemma 4.1), for any given  $\epsilon_1 > 0$ , there exists a gain matrix  $K_r$  such that

$$\left\| \left( sI - A_{22} + K_r \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} \right)^{-1} \left( E_2 - K_r \begin{bmatrix} D_0 \\ E_1 \end{bmatrix} \right) \right\|_{\infty} < \epsilon_1. \quad (4.11)$$

Moreover, the explicit methods for the construction of such a  $K_r$  are summarized in the appendix. Similarly, for right invertible and minimum phase system  $(A_{P,Q}, B_{P,Q}, C_{2,P}, D_P)$ , for any given  $\epsilon_2 > 0$ , there exist a gain matrix  $F$  such that

$$\|(C_{2,P} - D_P F)(sI - A_{P,Q} + B_{P,Q} F)^{-1}\|_{\infty} < \epsilon_2. \quad (4.12)$$

Again, such a  $F$  can be constructed by dualizing the results of the appendix

In view of (4.10) to (4.12), as well as Theorem 3.2, the result of Theorem 4.1 can easily be verified by some simple algebra. ■

*Remark 4.1:* In the case that the given system  $\Sigma$  is regular (i.e., in additions to the assumptions a) and b), the feedthrough matrices  $D_1$  and  $D_2$  are surjective and injective, respectively), then the controller (4.9) reduces to the well-known full order observer based control design for the regular  $H_{\infty}$ -optimization as given in [3].

## V. CONCLUSION

In this note we presented a technique of finding stabilizing controllers of a dynamical order lower than the dynamical order of the plant which make the  $H_{\infty}$  norm of the closed loop system strictly less than 1. If  $p$  states of a system with McMillan degree  $n$  are measured without noise, then we find a compensator with McMillan degree  $n - p$ .

We think that the technique presented in this paper is quite general and can for instance also be applied to the linear quadratic Gaussian control problem in the case that states are measured without noise.

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## APPENDIX DESIGN ALGORITHMS

For economy of notation, in this appendix we will consider the following system  $\Sigma_a$ ,

$$\Sigma_a: \begin{cases} \dot{\hat{x}} = A\hat{x} + B\hat{u}, \\ \hat{u} = C\hat{x} + D\hat{u}, \end{cases} \quad (A.1)$$

where  $x \in \mathfrak{R}^n$ ,  $u \in \mathfrak{R}^m$  and  $y \in \mathfrak{R}^p$ . Throughout this Appendix we will assume that  $\Sigma_a$  is left invertible and minimum phase. The

goal is to introduce an algorithm of designing for any given  $\epsilon > 0$  a parameterized gain matrix  $K_{\epsilon}(\sigma)$  for which there exists a  $\sigma^*$  such that for all  $\sigma > \sigma^* > 0$ ,  $A - K_{\epsilon}(\sigma)C$  is asymptotically stable and

$$\|[sI_n - A + K_{\epsilon}(\sigma)C]^{-1}[B - K_{\epsilon}(\sigma)D]\|_{\infty} < \epsilon.$$

Without loss of generality but for simplicity of presentation, we assume that matrices  $[C, D]$  and  $[B^T, D^T]^T$  are of maximal rank and matrix  $D$  is in the form of

$$D = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix},$$

where  $m_0$  is the rank of  $D$ . Thus,  $\Sigma_a$  can be partitioned as follows:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + [B_0, B_1] \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \end{bmatrix}, \\ \begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \hat{x} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \end{bmatrix}, \end{cases} \quad (A.2)$$

where  $B_0, B_1$  and  $C_0, C_1$  are partitionings of the matrices  $B$  and  $C$  of appropriate dimension. First we have the following simple observation.

*Observation A.1:* Assume that  $\text{rank}(B_1) > 0$  and let  $A_1 = A - B_0 C_0$ . We have the following:

- 1)  $(A_1, B_1, C_1)$  is left invertible and of minimum phase iff  $(A, B, C, D)$  is left invertible and of minimum phase.
- 2) Invariant zeros of  $(A_1, B_1, C_1)$  are the same as those of  $(A, B, C, D)$ . □

*Proof:* Completely similar to the proof of Lemma 4.1. For details see [1]. ■

In the following we present a design algorithms for the computation of  $K_{\epsilon}(\sigma)$ , based on a cheap control approach or ARE-based design. An alternative method can be based on the asymptotic time-scale and eigenstructure assignment (ATEA) design. In ARE-based design the asymptotic behavior of the fast eigenvalues of  $A - K_{\epsilon}(\sigma)C$  are fixed by the infinite zero structure of the system  $\Sigma_a$ . However, in ATEA design one can assign arbitrarily the asymptotic behavior of these eigenvalues. For a detailed discussion and comparison between ARE-based and ATEA design the interested readers are referred to [9].

## THE CHEAP CONTROL APPROACH

*Step 1)* Solving the following algebraic Riccati equation,

$$A_1 P + P A_1^T - P C_1^T C_1 P + \sigma^2 B_1 B_1^T = 0, \quad (A.3)$$

for the positive solution  $P$ .

*Step 2)* Calculate

$$K_1(\sigma) = P C_1^T.$$

*Step 3)* Let

$$K_{\epsilon}(\sigma) = [B_0, K_1(\sigma)]. \quad (A.4)$$

We have the following lemma.

*Lemma A.1:* Consider a system  $\Sigma_a$  as in (A.1) which is left invertible and which is of minimum phase. Let  $K_{\epsilon}(\sigma)$  be computed via the above algorithm. Then for any given  $\epsilon > 0$ , there exists a  $\sigma^* > 0$  such that for all  $\sigma > \sigma^*$ ,  $A - K_{\epsilon}(\sigma)C$  is asymptotically stable and

$$\|[sI_n - A + K_{\epsilon}(\sigma)C]^{-1}[B - K_{\epsilon}(\sigma)D]\|_{\infty} < \epsilon.$$

*Proof:* Since  $(A_1, B_1, C_1)$  is of minimum phase and left invertible, it is shown in [2] that  $K_1(\sigma)$  calculated in the above procedure has the following properties: as  $\sigma \rightarrow \infty$ ,

$$[sI_n - A_1 + K_1(\sigma)C_1]^{-1}B_1 \rightarrow 0 \text{ pointwise in } s$$

and  $A_1 - K_1(\sigma)C_1$  is asymptotically stable. Hence, for any given  $\epsilon > 0$  there exists  $\sigma^* > 0$  such that for all  $\sigma > \sigma^*$ ,  $A_1 - K_1(\sigma)C_1 = A - K_\epsilon(\sigma)C$  is stable and

$$\|[sI_n - A_1 + K_1(\sigma)C_1]^{-1}B_1\|_\infty < \epsilon,$$

which implies

$$\|[sI_n - A + K_\epsilon(\sigma)C]^{-1}[B - K_\epsilon(\sigma)D]\|_\infty < \epsilon.$$

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## A Large Deviations Analysis of Range Tracking Loops

Amir Dembo and Ofer Zeitouni

**Abstract**—Large deviations theory is applied to the analysis of a discrete time range tracking loop. It is shown that the resulting asymptotics differ from those of the continuous time diffusion limit.

### I. INTRODUCTION AND MAIN RESULTS

In the design of tracking loops in the presence of noise, two performance measures are of utmost importance: the steady state error ("accuracy of the loop") and the time till lock is lost ("stability" of the loop). While the analysis of the loop accuracy may usually be performed by considering a linearized version of the loop, it is well known that this is not a good approach for studying stability. At least when the tracking loop may be modeled as a Markov process, analytic solutions for the latter question exist. Those solutions are hardly ever explicitly computable. Since often one is interested in systems where some natural parameter  $\epsilon$  is small (for example, the "Noise to Signal" ratio, or the bandwidth), an asymptotic study of the stability question, which hopefully yields explicit expressions, is of interest.

In recent years, large deviations methods have been applied extensively to the latter problem. Beginning with the pioneering work of Freidlin and Wentzell [11], it became clear that in many cases the question of loss of lock ("problem of exit from a domain"), which involves longer and longer (in  $\epsilon$ ) time intervals, may be reduced to the analysis of fixed intervals large deviations estimates. Such analysis has been carried out for many Markov processes, and in particular, for diffusion processes (see [7] and, in the context of tracking systems, [6], [8], [12]). It seems that the discrete time version of this problem has not received much attention in the literature. An often used approach, namely the use of the diffusion limit of the discrete time chain as an intermediate step in the exit problem analysis, may lead to completely wrong estimates if the process noise is not Gaussian (see remark c) below).

In this article we focus on a discrete time model for a range tracking loop. As will be clear from our exposition, the approach presented is quite general, but we chose to present it in the simplest possible situation which still captures the main features of the problem. A related discussion and some other examples appear in the book [4].

There exists a vast literature on tracking systems and algorithms. For a guide to the literature, we refer the reader to [1]. The model we discuss here is as follows. By transmitting a pulse  $s(t)$  and analyzing its return from a target  $s(t - \tau)$ , a radar receiver may estimate  $\tau$ , the time it took the pulse to travel to the target and return. Dividing by twice the speed of light, an estimate of the distance to the target is obtained.

A range tracker keeps track of changes in  $\tau$ . Since the range of the target is unknown to the tracker, and fluctuates, it is common to model it, or actually its representation by  $\tau$ , as a random process. In order to keep the analysis simple, and yet to provide a meaningful model, we describe the range of the target as a first order AR process. That

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