The Discrete-Time H_∞ Control Problem with Strictly Proper Measurement Feedback

Anton A. Stoorvogel, Ali Saberi, and Ben M. Chen

***Abstract*—This paper is concerned with the discrete-time H_∞ control problem with strictly proper measurement feedback. We derive necessary and sufficient conditions for the existence of a strictly proper compensator which achieves a given H_∞ norm bound. Note that contrary to the continuous time, in discrete time there might exist a suitable proper compensator but no suitable strictly proper compensator. Finally, we given an explicit formula for one controller which achieves this bound.**

I. Introduction and Problem Formulation

The H_∞ control problem has been studied extensively in literature over the past decade. It was first introduced in continuous time (see e.g., [5], [6], [12], [13]) and later in discrete time (see e.g., [1], [7], [9], [14]). For a more extensive reference list, we refer to two recent books [2] and [15]. In a recent paper [16], we have studied the discrete-time H_∞ control problem with measurement feedback under a fairly general setting. We have obtained a set of necessary and sufficient conditions under which an H_∞ norm bound can be achieved by an internally stabilizing measurement feedback controller. Moreover, the structure of such a controller is also explicitly given. The controller given in [16] is in general nonstrictly proper. In this note, we would like to derive a set of necessary and sufficient conditions for the discrete time H_∞ control problem with strictly proper measurement feedback controllers. This problem has been studied before in [17]. Our assumptions are much weaker, however, and we are able to impose arbitrary restrictions on the direct feedthrough matrix of the controller.

There are several reasons for restricting our attention to strictly proper compensators. Nonstrictly proper compensators can have a lack of robustness regarding discarded parasitic dynamics (see [8]). Also in the study of a simultaneous H_2/H_∞ control problem we have to restrict attention to strictly proper compensators (see [10]). A last example is the sampled data control problem (where we design a discrete time controller for a continuous time plant which are connected via sample and hold devices, see, e.g., [3], [4]). Here also requirements on the direct feedthrough matrix of the controller come in very naturally. In fact, in [4], a rather special block structure was required for the direct-feedthrough matrix. We will show that our technique also enables us to treat that case. It is interesting to note that for the continuous time H_∞ optimal control problem, the conditions for satisfying an H_∞ norm bound whether we require strictly proper controllers or when we allow for general proper controllers are the same (except when the open-loop direct feedthrough matrix from disturbance to output is too large). For the discrete time problem, they are different in general.

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The notation in this note will be fairly standard. By $\mathcal{N}$ and $\mathcal{R}$ we denote the natural numbers and the real numbers, respectively. By σ we denote the shift

$$(\sigma x)(k) := x(k+1) \quad \forall k \in \mathcal{N}$$

$\text{rank}_{\mathcal{K}}$ denotes the rank as a matrix with entries in the field $\mathcal{K}$. By $\mathcal{R}(z)$ we denote the field of real rational functions. By $X^\dagger$ we denote the Moore–Penrose inverse of the matrix X. Finally, by $\rho(X)$ we denote the spectral radius of the matrix X.

We consider the following time-invariant system

$$\Sigma: \begin{cases} \sigma x = Ax + Bu + Ew, \\ y = C_1 x + D_{12} w, \\ z = C_2 x + D_{21} u + D_{22} w \end{cases} \tag{1.1}$$

where for all $k \in \mathcal{N}$, $x(k) \in \mathcal{R}^n$ is the state $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^l$ is the measurement, $w(k) \in \mathcal{R}^q$ is the unknown disturbance, and $z(k) \in \mathcal{R}^p$ is the output to the controlled. A, B, E, C_1, C_2, D_{12}, D_{21}, and D_{22} are matrices of appropriate dimension.

If we apply a dynamic feedback law $u = Fy$ to Σ then the closed-loop system with zero initial conditions defines a convolution operator $\Sigma_{cl,F}$ from w to z. We seek a feedback law $u = Fy$ which is internally stabilizing and which minimizes the $\mathcal{L}_2$-induced operator norm of $\Sigma_{cl,F}$ over all internally stabilizing feedback laws. We will investigate dynamic feedback laws of the form

$$\Sigma_F: \begin{cases} \sigma p = Kp + Ly, \\ u = Mp + Ny. \end{cases} \tag{1.2}$$

We will say that the dynamic compensator Σ_F, given by (1.2), is internally stabilizing when applied to the system Σ, described by (1.1), if the following matrix is asymptotically stable

$$\begin{pmatrix} A + BNC_1 & BM \\ LC_1 & K \end{pmatrix} \tag{1.3}$$

i.e., all its eigenvalues lie in the open unit disc. Denote by G_F the closed-loop transfer matrix. The $\mathcal{L}_2$-induced operator norm of the convolution operator $\Sigma_{cl,F}$ is equal to the H_∞ norm of the transfer matrix G_F and is given by

$$\|G_F\|_\infty := \sup_{\theta \in [0, 2\pi]} \|G_F(e^{i\theta})\| = \sup_w \left\{ \frac{\|z\|_2}{\|w\|_2} \middle| w \in \mathcal{L}_2^l,\ w \neq 0 \right\}$$

where the $\mathcal{L}_2$-norm is given by

$$\|p\|_2 := \left(\sum_{k=0}^{\infty} p^T(k) p(k) \right)^{1/2}$$

and where $\|\cdot\|$ denotes the largest singular value. We shall refer to the norm $\|G_F\|_\infty$ as the H_∞ norm of the closed-loop system.

In this paper we will derive necessary and sufficient conditions for the existence of a strictly proper dynamic compensator Σ_F which is internally stabilizing and which is such that the closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$. By scaling the plant we can thus, in principle, find the infimum of the H_∞ norm of the closed-loop system over all stabilizing controllers. This will involve a search procedure. Furthermore, if a stabilizing Σ_F exists which makes the H_∞ norm of the closed-loop system less than one, then we derive an explicit formula for one particular F satisfying these requirements.

In the development of our main result we will need the concept of invariant zero. Recall that z_0 is called an invariant zero of the system (A, B, C, D) if

$$\operatorname{rank}_{\mathcal{R}}\begin{pmatrix} z_0 I - A & -B \\ C & C \end{pmatrix} < \operatorname{rank}_{\mathcal{R}(z)}\begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}.$$

II. Main Results

In this section, we will derive sets of necessary and sufficient conditions under which two special structured H_∞ γ-suboptimal controllers exist. The first controller structure considered is a strictly proper one, i.e., its direct feedthrough term $N = 0$, while the second structure has a direct feedthrough matrix N that must satisfy some arbitrary constraints. We first recall in the following a theorem from [16].

Theorem 2.1: Consider system (1.1). Assume that (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the unit circle. The following statements are equivalent.

1) There exists a dynamic compensator Σ_F of the form (1.2) such that the resulting closed-loop system is internally stable and the closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$.
2) There exist symmetric matrices $P \geq 0$ and $Q \geq 0$ such that

a) We have $R > 0$ where

$$V := B^T PB + D_{21}^T D_{21},$$

$$R := I - D_{22}^T D_{22} - E^T PE + (E^T PB + D_{22}^T D_{21})V^\dagger (B^T PE + D_{21}^T D_{22}).$$

b) P satisfies the discrete algebraic Riccati equation

$$P = A^T PA + C_2^T C_2 - \begin{pmatrix} B^T PA + D_{21}^T C_2 \\ E^T PA + D_{22}^T C_2 \end{pmatrix}^T \cdot G(P)^\dagger \begin{pmatrix} B^T PA + D_{21}^T C_2 \\ E^T PA + D_{22}^T C_2 \end{pmatrix} \quad (2.1)$$

where

$$G(P) := \begin{pmatrix} D_{21}^T D_{21} & D_{21}^T D_{22} \\ D_{22}^T D_{21} & D_{22}^T D_{22} - I \end{pmatrix} - \begin{pmatrix} B^T \\ E^T \end{pmatrix} P (B \quad E). \quad (2.2)$$

c) For all $z \in \mathcal{C}$ with $|z| \geq 1$, we have the equation found at the bottom of the page.

d) We have $S > 0$ where

$$W := D_{12} D_{12}^T + C_1 Q C_1^T,$$

$$S := I - D_{22} D_{22}^T - C_2 Q C_2^T + (C_2 Q C_1^T + D_{22} D_{12}^T) W^\dagger (C_1 Q C_2^T + D_{12} D_{22}^T).$$

e) Q satisfies the following discrete algebraic Riccati equation

$$Q = AQA^T + EE^T - \begin{pmatrix} C_1 QA^T + D_{12} E^T \\ C_2 QA^T + D_{22} E^T \end{pmatrix}^T \cdot H(Q)^\dagger \begin{pmatrix} C_1 QA^T + D_{12} E^T \\ C_2 QA^T + D_{22} E^T \end{pmatrix} \quad (2.3)$$

where

$$H(Q) := \begin{pmatrix} D_{12} D_{12}^T & D_{12} D_{22}^T \\ D_{22} D_{12}^T & D_{22} D_{22}^T - I \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Q \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^T. \quad (2.4)$$

f) For all $z \in \mathcal{C}$ with $|z| \geq 1$, we have

$$\operatorname{rank}_{\mathcal{R}}\begin{pmatrix} zI - A & AQC_1^T + ED_{12}^T & AQC_2^T + ED_{22}^T \\ -C_1 & C_1 QC_1^T + D_{12} D_{12}^T & C_1 QC_2^T + D_{12} D_{22}^T \\ -C_2 & C_2 QC_1^T + D_{22} D_{12}^T & C_2 QC_2^T + D_{22} D_{22}^T - I \end{pmatrix} = n + q + \operatorname{rank}_{\mathcal{R}(z)} C_1 (zI - A)^{-1} E + D_{12}.$$

g) $\rho(PQ) < 1$. □

The above result finds a characterization for the existence of a general proper compensator. Conditions a) and b) have the standard form except that some of the inverses are replaced by generalized inverses. Condition c) is nothing else than the requirement that P must be a stabilizing solution of the Riccati equation. A similar comment can be made about conditions d)–f). In [16] we showed how to reduce these very general algebraic Riccati equations appearing in the above lemma to classical Riccati equations which can be solved using standard techniques.

In the following theorem, we find an additional condition which together with the conditions of the previous theorem guarantees the existence of a suitable strictly proper dynamic compensator.

Theorem 2.2: Consider system (1.1). Assume that (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the unit circle. The following statements are equivalent.

1) There exists a strictly proper dynamic compensator Σ_F of the form (1.2) with $N = 0$ such that the resulting closed-loop system is internally stable and the closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$.
2) There exist symmetric matrices $P \geq 0$ and $Q \geq 0$ such that the conditions a)–g) are satisfied. Moreover

$$D_{22,P} D_{22,P}^T + C_{2,P} (I - QP)^{-1} P C_{2,P}^T < I \quad (2.5)$$

where

$$A_x := A - BV^\dagger [B^T PA + D_{21}^T C_2],$$

$$C_x := C_2 - D_{21} V^\dagger [B^T PA + D_{21}^T C_2],$$

$$C_{2,P} := (V^{1/2})^\dagger (B^T PA + D_{21}^T C_2 + [B^T PE + D_{21}^T D_{22}] \cdot R^{-1} [E^T PA_x + D_{22}^T C_x]),$$

$$D_{22,P} := (V^{1/2})^\dagger (B^T PE + D_{21}^T D_{22}) R^{-1/2}.$$

Suppose we allow for a direct feedthrough matrix N which is not arbitrary but has to satisfy some additional requirements, say $N \in \mathcal{V}$. We have the following theorem. We would like to note that, under more restrictive assumptions and for one particular set $\mathcal{V}$, a similar result was derived in [4], [11] using a frequency domain approach.

Theorem 2.3: Consider system (1.1). Let $\mathcal{V}$ be some arbitrary subset of $\mathcal{R}^{p \times m}$. Assume that (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the unit circle. The following statements are equivalent.

1) There exists a strictly proper dynamic compensator Σ_F of the form (1.2) with $N \in \mathcal{V}$ such that the resulting closed-loop

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$$\operatorname{rank}_{\mathcal{R}}\begin{pmatrix} zI - A & -B & -E \\ B^T PA + D_{21}^T C_2 & B^T PB + D_{21}^T D_{21} & B^T PE + D_{21}^T D_{22} \\ E^T PA + D_{22}^T C_2 & E^T PB + D_{22}^T D_{21} & E^T PE + D_{22}^T D_{22} - I \end{pmatrix} = n + q + \operatorname{rank}_{\mathcal{R}(z)} C_2 (zI - A)^{-1} B + D_{21}.$$

system is internally stable and the closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$.

2) There exist symmetric matrices $P \geq 0$ and $Q \geq 0$ such that the conditions a)–g) are satisfied. Moreover there exists $N \in \mathcal{V}$ such that

$$\|D_{22,P,Y} + D_{21,P,Y} N D_{12,P,Y}\|_\infty < 1 \tag{2.6}$$

where we use the definitions of Theorems 2.1 and 2.2 together with

$$\begin{aligned} Y &:= (I - QP)^{-1}Q, \\ C_{1,P} &:= C_1 + D_{12}R^{-1}(E^T P A_x + D_{22}^T C_x), \\ D_{12,P} &:= D_{12}R^{-1/2}, \\ W_P &:= D_{12,P}D_{12,P}^T + C_{1,P} Y C_{1,P}^T, \\ S_P &:= I - D_{22,P}D_{12,P}^T - C_{2,P} Y C_{2,P}^T \\ &\quad + (C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T) \\ &\quad \cdot W_P^\dagger (C_{1,P} Y C_{2,P}^T + D_{12,P} D_{22,P}^T), \\ D_{21,P} &:= V^{1/2}, \\ D_{12,P,Y} &:= W_P^{1/2} \\ D_{21,P,Y} &:= S_P^{-1/2} D_{21,P} \\ D_{22,P,Y} &:= S_P^{-1/2}(C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T)(W_P^{1/2})^\dagger. \end{aligned}$$

III. Proofs of Main Results

Before we proceed to prove our results, we first recall a system transformations from [16]. Assume that part 2 of Theorem 2.1 is satisfied, which guarantees that $Y := (I - QP)^{-1}Q \geq 0$. We define an auxiliary system $\Sigma_{P,Y}$

$$\Sigma_{P,Y}: \begin{cases} \sigma x_{P,Y} = A_{P,Y} x_{P,Y} + B_{P,Y} u_{P,Y} + E_{P,Y} w_{P,Y}, \\ y_{P,Y} = C_{1,P} x_{P,Y} + D_{12,P,Y} w_{P,Y}, \\ z_{P,Y} = C_{2,P,Y} X_{P,Y} + D_{21,P,Y} u_{P,Y} + D_{22,P,Y} w_{P,Y} \end{cases} \tag{3.1}$$

where we use the definitions of Theorems 2.1, 2.2, and 2.3 together with

$$\begin{aligned} A_P &:= A + ER^{-1}(E^T P A_x + D_{22}^T C_x), \\ E_P &:= ER^{-1/2}, \\ A_y &:= A_P - (A_P Y C_{1,P}^T + E_P D_{12,P}^T) W_P^\dagger C_{1,P} \\ E_y &:= E_P - (A_P Y C_{1,P}^T + E_P D_{12,P}^T) W_P^\dagger D_{12,P} \\ A_{P,Y} &:= A_P + (A_y Y C_{2,P}^T + E_y D_{22,P}^T) S_P^{-1} C_{2,P} \\ C_{2,P,Y} &:= S_P^{-1/2} C_{2,P} \\ B_{P,Y} &:= B + (A_y Y C_{2,P}^T + E_y D_{22,P}^T) S_P^{-1} D_{21,P} \\ E_{P,Y} &:= (A_P Y C_{1,P}^T + E_P D_{12,P}^T + [A_Y Y C_{2,P}^T + E_y D_{22,P}^T] S_P^{-1} \\ &\quad \cdot [C_{2,P} Y C_{1,P}^T + D_{22,P} D_{12,P}^T])(W_P^{1/2})^\dagger. \end{aligned}$$

The rest of the matrices are as defined in Theorems 2.1 and 2.2 and the matrices P and Q satisfy the conditions of Theorem 2.1.

It is shown in [16] that a compensator is internally stabilizing and makes the H_∞ norm of the closed-loop system less than one for the system Σ if and only if the same compensator is internally stabilizing and makes the H_∞ norm of the closed-loop system less than one for our transformed system $\Sigma_{P,Y}$. Moreover, $\Sigma_{P,Y}$ has a very special property:

> There exists an internally stabilizing compensator which makes the closed-loop transfer matrix equal to zero, i.e., w does not have any effect on the output of the system z. This property of $\Sigma_{P,Y}$ has a special name: "the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS) is solvable."

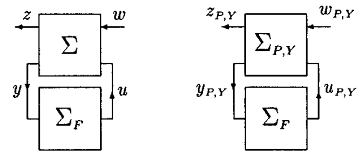

Fig. 1.

We now recall a key lemma from [16].

Lemma 3.1: Let P satisfy Theorem 2.1.a)–c). Moreover let an arbitrary linear time-invariant finite-dimensional compensator Σ_F be given, described by (1.2). Consider the following two systems, where the system on the left is the interconnection of (1.1) and (1.2) and the system on the right is the interconnection of (3.1) and (1.2) as shown in Fig. 1. Then the following statements are equivalent.

1) The system on the left is internally stable, and its transfer matrix from w to z has H_∞ norm less than one.
2) The system on the right is internally stable, and its transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ has H_∞ norm less than one. □

The above lemma tells us that we can completely focus our attention on the new system $\Sigma_{P,Y}$. To construct controllers for this new system, the following lemma from [16] shows the existence of a suitable state feedback and a suitable output injection.

Lemma 3.2: There exists a matrix F_0 such that if we define

$$F_{1,P,Y} := -D_{21,P}^\dagger C_{2,P} + (I + D_{21,P}^\dagger D_{21,P}) F_0$$

then we have

1) $A_{P,Y} + B_{P,Y} E_{1,P,Y}$ is stable,
2) $C_{2,P,Y} + D_{21,P,Y} F_{1,P,Y} = 0$.

Moreover, there exist a matrix K_0 such that if we define

$$K_{1,P,Y} = -E_{P,Y} D_{12,P,Y}^\dagger + K_0(I - D_{12,P,Y} D_{12,P,Y}^\dagger)$$

then we have

1) $A_{P,Y} + K_{1,P,Y} C_{1,P}$ is stable,
2) $E_{P,Y} + K_{1,P,Y} D_{12,P,Y} = 0$. □

Now, we are ready to prove our results.

First the implication $1 \Rightarrow 2$ in Theorem 2.2. In view of Theorem 2.1, we only need to prove the additional conditions (2.5). Since, however, there exists a strictly proper compensator for Σ (and hence for $\Sigma_{P,Y}$) which results in an H_∞ norm strictly less than one, it is straightforward that we must have $\|D_{22,P,Y}\| < 1$. This is equivalent to condition (2.5).

The implication $2 \Rightarrow 1$ of Theorem 2.2 follows from the following lemma which constructs a strictly proper compensator that makes the H_∞ norm of the closed-loop transfer matrix from w to z less than one, provided that the conditions in part 2 of Theorem 2.2 are satisfied.

Lemma 3.3: Assume the condition in part 2 of Theorem 2.2 are satisfied. Let Σ_F be given by

$$\Sigma_F: \begin{cases} \sigma p = K_{P,Y} p + L_{P,Y} y_{P,Y}, \\ u_{P,Y} = M_{P,Y} p \end{cases} \tag{3.2}$$

where

$$\begin{aligned} M_{P,Y} &:= F_{1,P,Y} \\ L_{P,Y} &:= -K_{1,P,Y} \\ K_{P,Y} &:= A_{P,Y} + B_{P,Y} M_{P,Y} + K_{1,P,Y} C_{1,P}. \end{aligned}$$

Then the interconnection of Σ_F and Σ is internally stable and the closed-loop transfer matrix from w to z has H_∞ norm strictly less than one. □

Proof: It is trivial to check that if Σ_F is applied to $\Sigma_{P,Y}$, the closed-loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is equal to $D_{22,P,Y}$, and we know that this matrix has a norm strictly less than one. Moreover, from the stability conditions in Lemma 3.2 it is also immediate that this compensator is internally stabilizing. Therefore the results follow from Lemma 3.1. □

Again the implication $1 \Rightarrow 2$ in Theorem 2.3 is straightforward. Clearly we must be able to make the closed-loop direct-feedthrough matrix from w to z less than one via a static output feedback from our restricted set $\mathcal{V}$.

We will now prove the reverse implication. We first apply a preliminary static output feedback $u = Ny + v$ where N is such that (2.6) is satisfied. After this preliminary output feedback we apply the compensator as constructed in Lemma 3.3. It is easily checked that the resulting closed-loop system will achieve our objectives. Combining the preliminary feedback with this strictly proper compensator then yields the desired (and admissible) compensator for the original system Σ.

IV. Conclusion

We have shown in this paper that if we impose arbitrary constraints on the direct feedthrough matrix then we can still find necessary and sufficient conditions for the existence of a stabilizing compensator which achieves the desired H_∞ norm bound. A suitable choice for the direct feedthrough matrix is obtained via a related static optimization problem. In particular we found necessary and sufficient conditions for the existence of a suitable strictly proper compensator.

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Harmonic Generation in Adaptive Feedforward Cancellation Schemes

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Abstract—**The paper investigates the generation of harmonics in adaptive feedforward cancellation schemes. Specifically, it is shown that an algorithm designed to reject a certain number of frequency components may in fact be capable of reducing higher-order harmonics as well. This effect originates in the time-variation of the adaptive parameters. A consequence is that the adaptive system may perform better than the algorithm with the fixed, ideal parameters. Surprisingly, the response to higher-order harmonics can be calculated precisely, using a Laplace transform analysis. The origin of the numerical procedure is in the equivalence between the adaptive feedforward cancellation scheme and a control scheme based on the internal model principle. The implication of the results on design and implementation issues is discussed.**

I. Introduction

Periodic disturbances occur in a variety of engineering applications. In data storage systems, for example, the eccentricity of the track on a disk requires a periodic movement of the read/write head at the frequency of rotation of the disk. This disturbance is particularly large if the disk is removable, such as in compact disc players. In electric motors, the so-called cogging torque (in dc motors) and detent torque (in stepper motors) create torque pulsations at the frequency of rotation of the motor, due to the tendency of permanent magnets to align themselves along directions of minimum reluctance.

In some cases, the frequency of the disturbance is not precisely known. In many others (including those mentioned above), the fundamental frequency of the disturbance originates from some variable that is independently regulated and/or easily measurable. Unless the disturbance is purely sinusoidal, the harmonics also have to be compensated for, with magnitudes and phases unknown. Because of the low-pass properties of physical systems, at most a handful of harmonics needs to be considered in general.

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