Exact Computation of the Infimum in H_∞-Optimization Via Output Feedback

Ben M. Chen, *Student Member, IEEE,* Ali Saberi, *Member, IEEE,* and Uy-Loi Ly

Abstract—**This paper presents a simple and noniterative procedure for the computation of the exact value of the infimum in the standard H_∞-optimal control with output feedback. The problem formulation is general and does not place any restrictions on the direct feedthrough terms between the control input and the controlled output variables, and between the disturbance input and the measurement output variables. The method is applicable to systems that satisfy 1) the transfer function from the control input to the controlled output is right-invertible and has no invariant zeros on the $j\omega$ axis and, 2) the transfer function from the disturbance to the measurement output is left-invertible and has no invariant zeros on the $j\omega$ axis.**

I. Introduction

Over the past decade one has witnessed a proliferation of literature on H_∞-optimal control since it was first introduced by Zames [1]. The main focus of the work has been and continues to be on the formulation of the problem for robust multivariable control and its solution. Since the original formulation of the H_∞ problem in [1], a great deal of work has been on the solution of this problem. Practically all research results of early years involved a mixture of time-domain and frequency-domain techniques [2]–[4]. Recently, considerable attention has been focused on purely time-domain methods based on algebraic Riccati equations (ARE) [5]–[14]. Along this line of research, connections are also made between H_∞-optimal control and differential games [15]. Typically in ARE approaches to H_∞-optimal control problems, the achieved design solution is suboptimal in the sense that the H_∞-norm of the closed-loop system transfer function from the disturbances to the controlled outputs is less than a prescribed value. For the regular case,[1] the existence of suboptimal state (output) feedback laws is formulated in terms of the existence of a stabilizing positive semidefinite solutions(s) for one (two) "indefinite" algebraic Riccati equation(s) and the satisfaction of a coupling condition for the case of output feedback. A recent paper by Stoorvogel [10] has shown that conditions for the existence of suboptimal output feedback laws for general singular case (i.e., not a regular case) can be expressed in terms of the existence of solutions to two quadratic matrix inequalities. Solutions of these inequalities must also satisfy two rank conditions and a coupling condition. The latter condition requires that the spectral radius of the product of the two solutions to be smaller than a certain prior given upper bound. Their results are general and elegant. In their formulation, no assumptions are made on the direct feedthrough matrices between the control inputs and the controlled outputs, and between the disturbance inputs and the measurement outputs. Their conditions are very intuitive and reminiscent of the dissipation inequality in singular linear quadratic optimal control.

In this paper, we address the problem of computing the infimum in H_∞-optimization for the output feedback case. The ARE-based approach to this problem provides simply an iterative scheme of approximating the infimum (denoted here by γ_o^*) of the H_∞-norm of the closed-loop transfer function using output feedback compensators. For example, in the regular case and utilizing the results of [5], an iterative procedure for approximating γ_o^* would proceed as follows: one starts with a value of γ and determines whether $\gamma > \gamma_o^*$ by solving two "indefinite" algebraic Riccati equations and checking the positive semidefiniteness and stabilizing properties of these solutions. In the case where such positive semidefinite solutions exist and satisfy a *coupling condition*, then we have $\gamma > \gamma_o^*$ and one simply repeats the above steps using a smaller value of γ. In principle, one can approximate the infimum γ_o^* to within any degree of accuracy in this manner. However, this search procedure is exhaustive and can be very costly. More significantly, due to the possible high-gain occurrence as γ gets close to γ_o^*, numerical solutions for these ARE's can become highly sensitive and ill-conditioned. This difficulty also arises in the *coupling condition*. Namely, as γ decreases evaluation of the *coupling condition* would generally involve finding eigenvalues of stiff matrices. These numerical difficulties are likely to be more severe for problems associated with the singular case. So in general, the iterative procedure for the computation of γ_o^* based on ARE's is not reliable and thus should not be used to determine the infimum γ_o^*.

The subject of this paper is to provide an alternate simple and noniterative method of computing γ_o^* without solving any ARE or quadratic matrix inequalities. Our algorithm is

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B. M. Chen and A. Saberi are with the School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752.

U.-L. Ly is with the Department of Aeronautics and Astronautics, FS-10, University of Washington, Seattle, WA 98195.

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[1] Regular case refers to a system where the feedthrough matrix from the disturbance to the measurement output is surjective and the feedthrough matrix from the control input to the controlled output is injective.

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applicable to systems that satisfy 1) the transfer function from the control input to the controlled output is right-invertible and has no invariant zeros on the $j\omega$ axis and, 2) the transfer function from the disturbance to the measurement output is left-invertible and has no invariant zeros on the $j\omega$ axis. However, we make no assumptions on the feedthrough matrices from the control input to the controlled output and from the disturbance to the measured output. Our results provide basically an extension of the well-known one-block problem for the singular case. Our algorithm has been implemented efficiently in a MATLAB-software environment for numerical solutions.

The outline of this paper is as follows. In Section II, we introduce the problem statement. In Section III we provide some preliminaries on the special coordinate basis (SCB) [16], [17] and its properties for nonstrictly proper systems, and the main results of Stoorvogel [10] in notations consistent with the problem statement of Section II. The SCB transformation and Stoorvogel's theorem are both instrumental in the derivation of the main results given in Section IV for the exact computation of γ_o^* and the conclusions are given in Section V.

Throughout this paper we shall adopt the following conventions and notations:

A'	Transpose of A.
I	An identity matrix of appropriate dimension.
$\mathbb{R}$	The set of real numbers.
$\mathscr{C}$	Whole complex plane.
$\mathscr{C}^-$	Open left-half complex plane.
$\mathscr{C}^+$	Open right-half complex plane.
$\mathscr{C}^o$	Imaginary axis $j\omega$.
$\sigma_{\max}(A)$	Maximum singular value of A.
$\lambda(A)$	The set of eigenvalues of A.
$\lambda_{\max}(A)$	Maximum eigenvalue of A where $\lambda(A) \subset \mathbb{R}$.
$\rho(A)$	The spectral radius of A.
$Ker(V)$	Kernal of V.
$Im(V)$	Image of V.

We refer to the linear dynamical system

$$\dot{x} = Ax + Bu, \; y = Cx + Du \tag{1.1}$$

as the system (A, B, C, D). We also refer to $T_{yu}(s) = C(sI - A)^{-1}B + D$ as the transfer function matrix of the system (A, B, C, D) between the input u and the output y. For any real rational matrix $T(s)$

$$\|T(s)\|_\infty := \sup\{\sigma_{\max}[T(j\omega)] : \omega \in \mathbb{R}\} \tag{1.2}$$

then $\|T(s)\|_\infty$ coincides with the L_∞-norm of $T(s)$ if $T(s)$ is proper and has no poles in $\mathscr{C}^\circ$, and with the H_∞-norm of $T(s)$ if it is proper and stable.

II. Problem Formulation

Let us consider the following linear system:

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x \qquad + D_1 w, \\ z = C_2 x + D_2 u \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $w \in \mathbb{R}^p$ is the disturbance, $y \in \mathbb{R}^r$ is the measured output available for feedback control and $z \in \mathbb{R}^q$ is the controlled output. Let $T_{zw}(s)$ denote the transfer function matrix from the disturbance w to the controlled output z. The standard H_∞-optimal control problem is concerned with the construction of stabilizing feedback control-laws that minimize the H_∞-norm of $T_{zw}(s)$. We consider three different classes of control-laws: static-state feedback, dynamic-state feedback and dynamic-output feedback laws. Furthermore, we denote the infimum of the H_∞-norm achieved under these three classes of feedback laws as γ_s^*, γ_d^*, and γ_o^*, respectively. Namely

$\gamma_s^* := \inf\{\|T_{zw}(s)\|_\infty$ where $u(s) = Fx(s)$ for any F which internally stabilizes the system of (2.1), i.e., $A + BF$ is a stability matrix$\}$;

$\gamma_d^* := \inf\{\|T_{zw}(s)\|_\infty$ where $u(s) = F_s(s)x(s)$ for any proper transfer function matrix $F_s(s)$ which internally stabilizes the system of (2.1)$\}$;

$\gamma_o^* := \inf\{\|T_{zw}(s)\|_\infty$ where $u(s) = F_o(s)y(s)$ for any proper transfer function matrix $F_o(s)$ which internally stabilizes the system of (2.1)$\}$.

Zhou and Khargonekar in [13] have shown that $\gamma_d^* = \gamma_s^*$ which also implies that $\gamma_s^* \le \gamma_o^*$. It is also well-known that in general γ_o^* is not equal to γ_s^*. In a recent paper [18] we presented a noniterative algorithm for the exact computation of γ_s^*. In this paper, we will present a simple noniterative procedure to compute exact γ_o^* for the output feedback case.

One of the key components of our method is to put the problem in a SCB introduced in [16], [17] which exhibits explicitly the finite and infinite zero structures of the system. The other component utilizes the results of Stoorvogel [10].

III. Preliminaries

In the following section we shall recall the definition of the SCB for a linear time-invariant nonstrictly proper system [17], and the theorem of Stoorvogel [10]. Such a coordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structures of a given system as well as other system geometric properties. The results of Stoorvogel provide conditions for the existence of an H_∞-norm bound solution in the output feedback case. They are both instrumental in the derivation of the method described in Section IV.

A. Special Coordinate Basis

In the following, we recapitulate the main results in a theorem and some properties of the special coordinate basis while leaving detailed derivation and proofs to be found in [16], [17]. Consider the system described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew, \\ z &= Cx + Du. \end{aligned} \tag{3.1}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a nonsingular matrix V that put the direct feedthrough matrix

D into the following form

$$\overline{D} = UDV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

where r is the rank of D. Without loss of generality one can assume that the matrix D in (3.1) has the form as shown in (3.2). Thus the system in (3.1) can be rewritten as

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} B_0 & B_1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + Ew, \\ \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \end{aligned} \tag{3.3}$$

where B_0, B_1, C_0, and C_1 are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely

$$u = V \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = Uz.$$

Note that the H_∞-norm of the system transfer function $T_{zw}(s)$ is unchanged when we apply an orthogonal transformation on the output z, and under any nonsingular transformations on the states and control inputs. We have the following main theorem.

Theorem 3.1: There exist nonsingular transformations Γ_s, Γ_o, and Γ_i such that

$$x = \Gamma_s \left[(x_a^+)', (x_a^-)', x_b', x_c', x_f' \right]',$$

$$[z_0', z_1']' = \Gamma_o [z_0', z_f', z_b']', \quad [u_0', u_1']' = \Gamma_i [u_0', u_f', u_c']'$$

and

$$\Gamma_s^{-1}(A - B_0 C_0)\Gamma_s = \begin{bmatrix} A_{aa}^+ & 0 & L_{ab}^+ C_b & 0 & L_{af}^+ C_f \\ 0 & A_{aa}^- & L_{ab}^- C_b & 0 & L_{af}^- C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf} C_f \\ B_c E_{ca}^+ & B_c E_{ca}^- & L_{cb} C_b & A_{cc} & L_{cf} C_f \\ B_f E_{fa}^+ & B_f E_{fa}^- & B_f E_{fb} & B_f E_{fc} & A_{ff} \end{bmatrix}, \tag{3.4}$$

$$\Gamma_s^{-1} \begin{bmatrix} B_0 & B_1 \end{bmatrix} \Gamma_i = \begin{bmatrix} B_{0a}^+ & 0 & 0 \\ B_{0a}^- & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix}, \tag{3.5}$$

$$\Gamma_o^{-1} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \Gamma_s = \begin{bmatrix} C_{0a}^+ & C_{0a}^- & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix},$$

$$\Gamma_0^{-1} D \Gamma_i = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

where the pair (A_{cc}, B_c) is controllable, the pair (A_{bb}, C_b) is observable, and the subsystem (A_{ff}, B_f, C_f) is invertible with no invariant zeros.

The proof of this theorem can be found in reference [16], [17]. We also note that the output transformation Γ_o is of form

$$\Gamma_o = \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{or} \end{bmatrix}. \tag{3.7}$$

In what follows, we state some important properties of the SCB which are pertinent to our present work. For further details regarding SCB and its properties, interested readers are referred to [19].

Property 3.1: The given system (A, B, C, D) is right-invertible iff x_b and hence z_b are nonexistent, left-invertible iff x_c and hence u_c are nonexistent, invertible iff both x_c and x_b are nonexistent.

Property 3.2: Invariant zeros of (A, B, C, D) are the eigenvalues of A_{aa}^- and A_{aa}^+. Moreover, the stable and unstable invariant zeros of (A, B, C, D) are the eigenvalues of A_{aa}^- and A_{aa}^+, respectively.

Property 3.3: The pair (A, B) is stabilizable if and only if $(A_{\text{con}}, B_{\text{con}})$ is stabilizable where

$$A_{\text{con}} = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix}, B_{\text{con}} = \begin{bmatrix} B_{0a}^+ & L_{af}^+ \\ B_{0b} & L_{bf} \end{bmatrix}. \tag{3.8}$$

Property 3.4: If the system (A, B, C, D) is stabilizable and right-invertible, i.e., x_b is nonexistent, then the pair $(A_{aa}^+, [B_{oa}^+, L_{af}^+])$ is controllable.

There are interconnections between the SCB and various invariant and almost invariant geometric subspaces. To establish these interconnections, let us define the following subspaces:

- $\mathscr{V}^g(A, B, C, D)$—the maximal subspace of $\mathbb{R}^n$ which is $(A + BF)$-invariant and contained in $Ker(C + DF)$ such that the eigenvalues of $(A + BF) | \mathscr{V}^g$ are contained in $\mathscr{C}_g \subseteq \mathscr{C}$ for some F.
- $\mathscr{S}^g(A, B, C, D)$—the minimal $(A + KC)$-invariant subspace of $\mathbb{R}^n$ containing $Im(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n / \mathscr{S}^g$ are contained in $\mathscr{C}_g \subseteq \mathscr{C}$ for some K.

For the cases that $\mathscr{C}_g = \mathscr{C}$, $\mathscr{C}_g = \mathscr{C}^-$ and $\mathscr{C}_g = \mathscr{C}^\circ \cup \mathscr{C}^+$, we replace the index g in $\mathscr{V}^g$ and $\mathscr{S}^g$ by "$*$", "$-$", and "$+$", respectively. We list in the following the geometrical interpretations of some state vector components of SCB.

Property 3.5:

1) $x_a^- \oplus x_a^+ \oplus x_c$ spans $\mathscr{V}^*(A, B, C, D)$.
2) $x_a^- \oplus x_c$ spans $\mathscr{V}^-(A, B, C, D)$.
3) $x_a^+ \oplus x_c$ spans $\mathscr{V}^+(A, B, C, D)$.
4) $x_c \oplus x_f$ spans $\mathscr{S}^*(A, B, C, D)$.
5) $x_a^- \oplus x_c \oplus x_f$ spans $\mathscr{S}^+(A, B, C, D)$.
6) $x_a^+ \oplus x_c \oplus x_f$ spans $\mathscr{S}^-(A, B, C, D)$.

B. Stoorvogel's Theorem

We recall in this section a main theorem of Stoorvogel [10] that will play an important role in our present work. Before we introduce the theorem, let us define the following quadratic matrices:

$$F_\gamma(P(\gamma)) := \begin{bmatrix} A'P(\gamma) + P(\gamma)A + C_2'C_2 + \gamma^{-2}P(\gamma)EE'P(\gamma) & P(\gamma)B + C_2'D_2 \\ B'P(\gamma) + D_2'C_2 & D_2'D_2 \end{bmatrix} \tag{3.9}$$

and

$$G_\gamma(Q(\gamma)) := \begin{bmatrix} AQ(\gamma) + Q(\gamma)A' + EE' + \gamma^{-2}Q(\gamma)C_2'C_2Q(\gamma) & Q(\gamma)C_1' + ED_1' \\ C_1Q(\gamma) + D_1E' & D_1D_1' \end{bmatrix}. \tag{3.10}$$

It should be noted that the above matrices are dual of each other. In addition to these two matrices, we define two polynomial matrices whose role is again completely dual

$$L(P(\gamma),s) := \begin{bmatrix} sI - A - \gamma^{-2}EE'P(\gamma) & -B \end{bmatrix}, \tag{3.11}$$

and

$$M(Q(\gamma),s) := \begin{bmatrix} sI - A - \gamma^{-2}Q(\gamma)C_2'C_2 \\ -C_1 \end{bmatrix}. \tag{3.12}$$

Now we are ready to introduce the theorem of Stoorvogel [10]. We have the following theorem.

Theorem 3.2: Consider the system (2.1). Assume that (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros in $\mathscr{C}^o$. Then the following statements are equivalent:

1) There exists a linear time-invariant finite-dimensional proper dynamic compensator $F_o(s)$ such that by applying $u(s) = F_o(s)y(s)$ in (2.1) the resulting closed-loop system is internally stable. Moreover, the H_∞-norm of the closed-loop transfer function from the disturbance input w to the controlled output z is less than γ.

2) There exist positive semidefinite solutions $P(\gamma)$, $Q(\gamma)$ of the quadratic matrix inequalities $F_\gamma(P(\gamma)) \geq 0$ and $G_\gamma(Q(\gamma)) \geq 0$ satisfying $\rho[P(\gamma)Q(\gamma)] < \gamma^2$, such that the following rank conditions are satisfied:

a) $\operatorname{rank}\{F_\gamma(P(\gamma))\} = \operatorname{normrank}\{G_2(s)\}$,

b) $\operatorname{rank}\{G_\gamma(Q(\gamma))\} = \operatorname{normrank}\{G_1(s)\}$,

c) $\operatorname{rank}\begin{bmatrix} L(P(\gamma),s) \\ F_\gamma(P(\gamma)) \end{bmatrix} = n + \operatorname{normrank}\{G_2(s)\}, \quad \forall s \in \mathscr{C}^o \cup \mathscr{C}^+$,

d) $\operatorname{rank}[M(Q(\gamma),s), G_\gamma(Q(\gamma))] = n + \operatorname{normrank}\{G_1(s)\}, \forall s \in \mathscr{C}^o \cup \mathscr{C}^+$

where $G_1(s) = C_1(sI - A)^{-1}E + D_1$, $G_2(s) = C_2(sI - A)^{-1}B + D_2$ and "normrank" denotes the rank of a matrix with entries in the field of rational functions.

Proof: See Stoorvogel [10].

IV. Computational Algorithm for γ_o^*

In this section we give a simple and noniterative procedure for determining γ_o^*. The method is applicable to the general system of (2.1) satisfying the following two sets of basic assumptions.

Assumption A: The system (A, B, C_2, D_2) is stabilizable, right-invertible, and has no invariant zeros in $\mathscr{C}^o$.

Assumption B: The system (A, E, C_1, D_1) is detectable, left-invertible, and has no invariant zeros in $\mathscr{C}^o$.

The algorithm for γ_o^* involves the computation of two nonnegative scalars γ_P^* and γ_Q^* which are, respectively, the infima in H_∞-optimization of the system Σ and its dual, where in each case the measurement output is replaced by the system state. Computation of γ_P^* and γ_Q^* provides the necessary preliminary for the computation of γ_o^*.

The following Sections IV-A and IV-B deal with the definition and computation of γ_P^* and γ_Q^*, respectively, while in Section IV-C we present our main theorem regarding the computation of γ_o^*.

A. Computation of γ_P^*

We define the nonnegative scalar γ_P^* as the infimum of H_∞-optimization for the system

$$\Sigma_P : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = x, \\ z = C_2x + D_2u. \end{cases} \tag{4.1}$$

By definition, γ_P^* is clearly equal to γ_s^*. However, we use the terms γ_P^* and γ_Q^* in the next section to conform with the notation in matrix inequalities of Stoorvogel's theorem. In what follows, we apply the procedure of [18] for γ_s^* to the system Σ_P in the computation of γ_P^*. It involves the following steps.

Step 1: Transform the system (A, B, C_2, D_2) into the SCB described in Section III. To all submatrices and transformations in the SCB of Σ_P, we append the subscript "$_P$" to signify their relation to the system Σ_P. Next, we compute

$$\Gamma_{s_P}^{-1}E = \left[(E_{a_P}^+)' \quad (E_{a_P}^-)' \quad (E_{c_P})' \quad (E_{f_P})' \right]'. \tag{4.2}$$

Note that the component associated with x_b is missing since x_b is nonexistent for a right-invertible system.

Step 2: If the system (A, B, C_2, D_2) is of nonminimum phase[2] then solve the following Lyapunov equations

$$A_{aa_P}^+S_P + S_P(A_{aa_P}^+)' = \left[B_{0a_P}^+, L_{af_P}^+\Gamma_{or_P}^{-1} \right]\left[B_{0a_P}^+, L_{af_P}^+\Gamma_{or_P}^{-1} \right]', \tag{4.3}$$

$$A_{aa_P}^+T_P + T_P(A_{aa_P}^+)' = E_{a_P}^+(E_{a_P}^+)' \tag{4.4}$$

for S_P and T_P. Existence and uniqueness of these solutions follow from the fact that $\lambda(A_{aa_P}^+) \in \mathscr{C}^+$ (i.e., $-A_{aa_P}^+$ is a

[2] A system is said to be of nonminimum phase if at least one of its invariant zeros is in the closed right-half plane, otherwise it is said to be of minimum phase.

stable matrix) since the eigenvalues of $A^+_{aa_P}$ are the right-half plane invariant zeros of the system (A, B, C_2, D_2). Moreover, from the property 3.4 of Section III, the pair $(A^+_{aa_P}, [B^+_{0a_P}, L^+_{af_P}\Gamma^{-1}_{or_P}])$ is controllable when the system (A, B, C_2, D_2) is stabilizable and right-invertible. The solution S_P of (4.3) is therefore positive definite and hence invertible.

Step 3: The scalar γ_P^* is given by

$$\gamma_P^* = \begin{cases} \sqrt{\lambda_{\max}(T_P S_P^{-1})} \\ \quad \text{if } (A, B, C_2, D_2) \text{ is of nonminimum phase}, \\ 0 \\ \quad \text{if } (A, B, C_2, D_2) \text{ is of minimum phase}. \end{cases} \tag{4.5}$$

Here, we note that the eigenvalues of $(T_P S_P^{-1})$ are real and nonnegative.[3]

Theorem 4.1: Consider the system Σ_P given by (4.1). Then under the Assumption A

1) γ_P^* is the infimum of H_∞-optimization for Σ_P;

2) for $\gamma > \gamma_P^*$, the positive semidefinite matrix $P(\gamma)$ given by

$$P(\gamma) = (\Gamma_{s_P}^{-1})' \begin{bmatrix} P_a^+(\gamma) & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{s_P}^{-1} \tag{4.6}$$

where

$$P_a^+(\gamma) = \begin{cases} (S_P - \gamma^{-2} T_P)^{-1} \\ \quad \text{if } (A, B, C_2, D_2) \text{ is of nonminimum phase}, \\ 0 \\ \quad \text{if } (A, B, C_2, D_2) \text{ is of minimum phase} \end{cases} \tag{4.7}$$

is the unique solution of the matrix inequality $F_\gamma(P(\gamma)) \geq 0$ and satisfies both rank conditions a) and c) of Theorem 3.2. Moreover, such a solution $P(\gamma)$ does not exist when $\gamma < \gamma_P^*$.

Proof: See [18].

Remark 4.1: Note that part 2) of the above theorem implies that $\gamma_o^* \geq \gamma_P^*$.

The next lemma provides the necessary and sufficient conditions for $\gamma_P^* = 0$.

Lemma 4.1: $\gamma_P^* = 0$ iff $\operatorname{Im}(E) \subseteq \mathscr{S}^+(A, B, C_2, D_2)$.

Proof: See [18].

B. Computation of γ_Q^*

As in the definition of γ_P^*, the nonnegative scalar γ_Q^* is defined as the infimum in H_∞-optimization for the dual system

$$\Sigma_Q: \begin{cases} \dot{x} = A'x + C_1'u + C_2'w, \\ y = x, \\ z = E'x + D_1'u. \end{cases} \tag{4.8}$$

The determination of γ_Q^* follows exactly the procedure described in Section IV-A for the computation of γ_P^* but it now applies to the subsystem Σ_Q of (4.8). For completeness and to properly define matrices required in the computation γ_o^* of Section IV-C and in our main theorem of Section IV-D, we reiterate here the three steps involved in the computation of γ_Q^*

Step 1: Transform the system (A', C_1', E', D_1') into the special coordinate basis SCB described in Section III. Again, we add here the subscript "$_Q$" to all submatrices and transformations in the SCB of the system Σ_Q. Next, we compute

$$\Gamma_{s_Q}^{-1} C_2' = \left[(E^+_{a_Q})' \quad (E^-_{a_Q})' \quad (E_{c_Q})' \quad (E_{f_Q})' \right]'. \tag{4.9}$$

Note that the component associated with x_b is missing since x_b is nonexistent for a right-invertible system (A', C_1', E', D_1'). This comes from the Assumption B that the system (A, E, C_1, D_1) be left-invertible.

Step 2: If the system (A, E, C_1, D_1) is of nonminimum phase, then solve the following Lyapunov equations

$$A^+_{aa_Q} S_Q + S_Q (A^+_{aa_Q})' = \left[B^+_{0a_Q} \, L^+_{af_Q} \Gamma^{-1}_{or_Q} \right] \cdot \left[B^+_{0aQ}, L^+_{af_Q} \Gamma^{-1}_{or_Q} \right]', \tag{4.10}$$

$$A^+_{aa_Q} T_Q + T_Q (A^+_{aa_Q})' = E^+_{a_Q} (E^+_{a_Q})' \tag{4.11}$$

for S_Q and T_Q. As in the computation of γ_P^* of Section IV-A, these solutions are also unique due to the fact that the system (A', C_1', E', D_1') is stabilizable and right-invertible and has no invariant zeros on the $j\omega$ axis. Moreover, S_Q is positive definite and hence invertible.

Step 3: The scalar γ_Q^* is given by

$$\gamma_Q^* = \begin{cases} \sqrt{\lambda_{\max}(T_Q S_Q^{-1})} \\ \quad \text{if } (A, E, C_1, D_1) \text{ is of nonminimum phase}, \\ 0 \\ \quad \text{if } (A, E, C_1, D_1) \text{ is of minimum phase}. \end{cases} \tag{4.12}$$

We note that the eigenvalues of $(T_Q S_Q^{-1})$ are also real and nonnegative [20].

Theorem 4.2: Consider the system Σ_Q given by (4.8). Then under the Assumption B,

1) γ_Q^* is the infimum of H_∞-optimization for Σ_Q;

2) for $\gamma > \gamma_Q^*$, the positive semidefinite matrix $Q(\gamma)$ given by

$$Q(\gamma) = (\Gamma_{s_Q}^{-1})' \begin{bmatrix} Q_a^+(\gamma) & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{s_Q}^{-1} \tag{4.13}$$

where

$$Q_a^+(\gamma) = \begin{cases} (S_Q - \gamma^{-2} T_Q)^{-1} \\ \quad \text{if } (A, E, C_1, D_1) \text{ is of nonminimum phase}, \\ 0 \\ \quad \text{if } (A, E, C_1, D_1) \text{ is of minimum phase} \end{cases} \tag{4.14}$$

is the unique solution of the matrix inequality $G_\gamma(Q(\gamma)) \geq 0$

[3] It is shown in [20] that AB has as many positive, zero, and negative eigenvalues as A if A is Hermitian and B is Hermitian and positive definite.

and satisfies both rank conditions b) and d) of Theorem 3.2. Moreover, such a solution $Q(\gamma)$ does not exist when $\gamma < \gamma_Q^*$.

Proof: This is a dual version of Theorem 4.1.

Remark 4.2: Note that part 2) of the above theorem also implies that $\gamma_o^* \geq \gamma_Q^*$.

Again, analogous to Lemma 4.1 we have the following.

Lemma 4.2: $\gamma_Q^* = 0$ iff $\mathscr{V}^+(A, E, C_1, D_1) \subseteq Ker(C_2)$.

Proof: This is a dual version of Lemma 4.1.

C. Computation of γ_o^*

In this section we provide our main results on a simple and noniterative procedure for the computation of the exact value of γ_o^*. First, we reformulate the computation of γ_o^* in the following lemma.

Lemma 4.3. Let $\gamma_{PQ}^* = \max\{\gamma_P^*, \gamma_Q^*\}$. Then

$$\gamma_o^* = \inf\{\gamma \in (\gamma_{PQ}^*, \infty) : f(\gamma) < \gamma^2\}. \tag{4.15}$$

where $f(\gamma) = \rho[P(\gamma)Q(\gamma)]$, and $P(\gamma)$ and $Q(\gamma)$ are given by (4.6) and (4.13), respectively.

Proof: It follows from Remarks 4.1 and 4.2 that $\gamma_o^* \geq \gamma_{PQ}^*$. Next, given any $\hat{\gamma} \in (\gamma_{PQ}^*, \infty)$ such that $f(\hat{\gamma}) < \hat{\gamma}^2$, i.e., $\rho[P(\hat{\gamma})Q(\hat{\gamma})] < \hat{\gamma}^2$, then such $P(\hat{\gamma})$ and $Q(\hat{\gamma})$ as given by (4.6) and (4.13) satisfy the conditions of Theorem 3.2. Hence, $\hat{\gamma} > \gamma_o^*$. This concludes our proof. ■

Straightforward computation of γ_o^* can be done via an iterative search algorithm which involves in each step the multiplication of two matrices $P(\gamma)$ and $Q(\gamma)$ of dimensions $n \times n$ and the determination of the spectral radius of the product $P(\gamma)Q(\gamma)$. This iterative search is costly and usually involves computation of eigenvalues of stiff matrices since the product $P(\gamma)Q(\gamma)$ could become ill-conditioned as γ approaches γ_{PQ}^* from above.[4] Hence, the overall procedure tends to be ill-conditioned.

In contrast to the above iterative procedure, here we present an elegant well-conditioned noniterative algorithm for the exact computation of γ_o^*. First, we derive an explicit expression for $f(\gamma)$ using (4.6) and (4.13). Let us denote $n_{a_P}^+$ and $n_{a_Q}^+$ the numbers of nonminimum-phase invariant zeros of the systems (A, B, C_2, D_2) and (A, E, C_1, D_1), respectively. If $\min\{n_{a_P}^+, n_{a_Q}^+\} > 0$, then we can partition the product of the inverses of the SCB state transformations as follows

$$\Gamma_{s_P}^{-1}\left(\Gamma_{s_Q}^{-1}\right)' = \begin{bmatrix} \Gamma & \star \\ \star & \star \end{bmatrix} \tag{4.16}$$

where Γ is of dimension $n_{a_P}^+ \times n_{a_Q}^+$.

Then it is straightforward to show that the scalar function $f(\gamma)$ is given by

$$f(\gamma) = \begin{cases} \lambda_{\max}\left[\left(S_P - \gamma^{-2}T_P\right)^{-1}\Gamma\left(S_Q - \gamma^{-2}T_Q\right)^{-1}\Gamma'\right] \\ \qquad \text{if } \min\{n_{a_P}^+, n_{a_Q}^+\} > 0, \\ 0 \\ \qquad \text{if } \min\{n_{a_P}^+, n_{a_Q}^+\} = 0. \end{cases} \tag{4.17}$$

[4] Note that as γ gets close to γ_P^*, $P(\gamma)$ contains the inverse of an almost singular submatrix and, similarly $Q(\gamma)$ contains the inverse of an almost singular submatrix as γ close to γ_Q^* [see equations (4.6) and (4.13)].

The function $f(\gamma)$ of (4.17) is a well-defined mapping from (γ_{PQ}^*, ∞) to $[0, \infty)$. Its evaluation is from the minimum eigenvalue of a matrix of dimension $n_{a_P}^+ \times n_{a_P}^+$, which is normally of a much smaller dimension than the original product $P(\gamma)Q(\gamma)$. We establish some important properties of the function $f(\gamma)$ in the following observation.

Observation 4.1: $f(\gamma)$ is a continuous nonnegative nonincreasing function of γ on (γ_{PQ}^*, ∞).

Proof: When $\min\{n_{a_P}^+, n_{a_Q}^+\} = 0, f(\gamma) = 0$ for all $\gamma \in (\gamma_{PQ}^*, \infty)$ and the result is trivial. For the case where $\min\{n_{a_P}^+, n_{a_Q}^+\} > 0$, we first show that $P_a^+(\gamma) = (S_P - \gamma^{-2}T_P)^{-1}$ is nonincreasing, i.e., if $\gamma_2 > \gamma_1$ then $P_a^+(\gamma_2) \leq P_a^+(\gamma_1)$. Recall that $S_P > 0$ and $T_P \geq 0$, we have for all $\gamma_2 > \gamma_1 > \gamma_{PQ}^*$

$$\left(\gamma_1^{-2} - \gamma_2^{-2}\right)T_P \geq 0$$

which implies that

$$S_P - \gamma_1^{-2}T_P \leq S_P - \gamma_2^{-2}T_P.$$

Hence,

$$P_a^+(\gamma_2) \leq P_a^+(\gamma_1), \qquad \text{for } \gamma_2 > \gamma_1.$$

Following the same procedure as above, one can show that $Q_a^+(\gamma) = (S_Q - \gamma^{-2}T_Q)^{-1}$ is nonincreasing. This implies that $\Gamma Q_a^+(\gamma)\Gamma'$ is also nonincreasing. Then clearly, $f(\gamma)$ is a continuous nonnegative nonincreasing function of γ on (γ_{PQ}^*, ∞). ■

The function $f(\gamma)$ defined above can be extended as a mapping from $[\gamma_{PQ}^*, \infty)$ to $[0, \infty)$ by setting $f(\gamma_{PQ}^*) = \lim_{\gamma \to \gamma_{PQ}^*} f(\gamma)$. It follows from Observation 4.1 that the limit $f(\gamma_{PQ}^*)$ exists and could be finite or infinite.

Before stating our main result of this section regarding the computation of γ_o^*, we need to establish several important observations and a lemma.

Observation 4.2: $f(\gamma) = \gamma^2$ has either no solution or a unique solution in the interval (γ_{PQ}^*, ∞).

Proof: The result follows from Observation 4.1 and the fact that γ^2 is strictly increasing for positive γ. ■

Lemma 4.4: If $f(\gamma) = \gamma^2$ has no solution in the interval (γ_{PQ}^*, ∞) then γ_o^* is equal to γ_{PQ}^*. Otherwise, γ_o^* is equal to the unique solution of $f(\gamma) = \gamma^2$ in the interval (γ_{PQ}^*, ∞).

Proof: $f(\gamma) = \gamma^2$ has no solution in the interval (γ_{PQ}^*, ∞) implies that $f(\gamma) < \gamma^2$ for all $\gamma \in (\gamma_{PQ}^*, \infty)$ and hence according to Lemma 4.3, $\gamma_o^* = \gamma_{PQ}^*$. On the other hand, it is obvious that γ_o^* is equal to the unique solution of $f(\gamma) = \gamma^2$ when such a solution exists. ■

At a first glance, it seems that the solution of $f(\gamma) = \gamma^2$ would involve the rooting of a highly nonlinear algebraic equation in γ. Actually, its solution can be achieved in one-step. Namely the problem of solving $f(\gamma) = \gamma^2$, if such a solution exists in the interval (γ_{PQ}^*, ∞), can be converted to the problem of calculating the maximum eigenvalue of a constant matrix. In fact, we also show that, when $f(\gamma) = \gamma^2$ has no solution in the interval (γ_{PQ}^*, ∞), the maximum eigenvalue of this matrix is equal to γ_{PQ}^*, which is γ_o^* as well. Let us define

$$N(\gamma) := \begin{cases} (S_P - \gamma^{-2}T_P)^{-1}\Gamma(S_Q - \gamma^{-2}T_Q)^{-1}\Gamma' - \gamma^2 I \\ \qquad \text{if } \min\{n_{a_P}^+, n_{a_Q}^+\} > 0, \\ -\gamma^2 I \\ \qquad \text{if } \min\{n_{a_P}^+, n_{a_Q}^+\} = 0 \end{cases} \tag{4.18}$$

and

$$M := \begin{cases} \begin{bmatrix} T_P S_P^{-1} + \Gamma S_Q^{-1}\Gamma' S_P^{-1} & -\Gamma S_Q^{-1} \\ -T_Q S_Q^{-1}\Gamma' S_P^{-1} & T_Q S_Q^{-1} \end{bmatrix} \\ \qquad \text{if } n_{a_P}^+ > 0 \text{ and } n_{a_Q}^+ > 0, \\ T_P S_P^{-1} \\ \qquad \text{if } n_{a_P}^+ > 0 \text{ and } n_{a_Q}^+ = 0, \\ T_Q S_Q^{-1} \\ \qquad \text{if } n_{a_P}^+ = 0 \text{ and } n_{a_Q}^+ > 0, \\ 0 \\ \qquad \text{if } n_{a_P}^+ = 0 \text{ and } n_{a_Q}^+ = 0. \end{cases} \tag{4.19}$$

We have the following observations on the matrices M and $N(\gamma)$.

Observation 4.3: Eigenvalues of M are real and nonnegative.

Proof: It is trivial when $\min\{n_{a_P}^+, n_{a_Q}^+\} = 0$. For the case where $\min\{n_{a_P}^+, n_{a_Q}^+\} > 0$, we have

$$\begin{aligned} \lambda[M] &= \lambda\left\{\begin{bmatrix} I & 0 \\ 0 & T_Q \end{bmatrix}\begin{bmatrix} T_P + \Gamma S_Q^{-1}\Gamma' & -\Gamma S_Q^{-1} \\ -S_Q^{-1}\Gamma' & S_Q^{-1} \end{bmatrix}\begin{bmatrix} S_P^{-1} & 0 \\ 0 & I \end{bmatrix}\right\} \\ &= \lambda\left\{\begin{bmatrix} S_P^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & T_Q \end{bmatrix}\begin{bmatrix} T_P + \Gamma S_Q^{-1}\Gamma' & -\Gamma S_Q^{-1} \\ -S_Q^{-1}\Gamma' & S_Q^{-1} \end{bmatrix}\right\} \\ &= \lambda\left\{\begin{bmatrix} S_P^{-1} & 0 \\ 0 & T_Q \end{bmatrix}\begin{bmatrix} T_P + \Gamma S_Q^{-1}\Gamma' & -\Gamma S_Q^{-1} \\ -S_Q^{-1}\Gamma' & S_Q^{-1} \end{bmatrix}\right\}. \end{aligned} \tag{4.20}$$

Now, it is trivial to verify that both submatrices in (4.20) are symmetric and positive semidefinite. Then using the result of [20] (i.e., Theorem 3), it is simple to show that the eigenvalues of M are real and nonnegative. ■

Observation 4.4:

1) $N(\gamma)$ has real eigenvalues for all $\gamma \in (\gamma_{PQ}^*, \infty)$.

2) $\lambda_{\max}[N(\gamma)] = f(\gamma) - \gamma^2$ is continuous and strictly decreasing on (γ_{PQ}^*, ∞).

Proof: Again, it is trivial when $\min\{n_{a_P}^+, n_{a_Q}^+\} = 0$. For the case where $\min\{n_{a_P}^+, n_{a_Q}^+\} > 0$, we have the following.

1) It is straightforward to show that $(S_P - \gamma^{-2}T_P)^{-1} > 0$ and $(S_Q - \gamma^{-2}T_Q)^{-1} > 0$ for all $\gamma \in (\gamma_{PQ}^*, \infty)$. Hence, all the eigenvalues of $N(\gamma)$ are real for $\gamma \in (\gamma_{PQ}^*, \infty)$.

2) It follows from Observation 4.1. ■

Observation 4.5: If $\min\{n_{a_P}^+, n_{a_Q}^+\} > 0$, then the roots of $\det\{N(\gamma)\} = 0$ are real. Moreover, the largest root of $\det\{N(\gamma)\} = 0$ in the interval (γ_{PQ}^*, ∞) is equal to $\sqrt{\lambda_{\max}(M)}$.

Proof: Using the definition of $N(\gamma)$ in (4.18), we have

$$\begin{aligned} \det[N(\gamma)] &= (-1)^{n_{a_P}^+}\det\left[\gamma^2 I - (S_P - \gamma^{-2}T_P)^{-1} \cdot\Gamma(S_Q - \gamma^{-2}T_Q)^{-1}\Gamma'\right] \\ &= \frac{(-1)^{n_{a_P}^+}}{\det[S_P - \gamma^{-2}T_P]} \cdot \det\left[\gamma^2 S_P - T_P - \gamma^2\Gamma(\gamma^2 S_Q - T_Q)^{-1}\Gamma'\right] \\ &= \frac{(-1)^{n_{a_P}^+}}{\det[S_P - \gamma^{-2}T_P]\det[\gamma^2 S_Q - T_Q]} \cdot \det\begin{bmatrix} \gamma^2 S_P - T_P & \Gamma \\ \gamma^2\Gamma' & \gamma^2 S_Q - T_Q \end{bmatrix} \\ &= \frac{(-1)^{n_{a_P}^+}\det[S_P]\det[S_Q]}{\det[S_P - \gamma^{-2}T_P]\det[\gamma^2 S_Q - T_Q]} \cdot \det[\gamma^2 I - M]. \end{aligned} \tag{4.21}$$

Now it is simple to see that the roots of $\det[N(\gamma)] = 0$ are real since all the roots of $\det[\gamma^2 S_P - T_P] = 0$, $\det[\gamma^2 S_Q - T_Q] = 0$ and $\det[\gamma^2 I - M] = 0$ are real. Moreover, it follows from (4.5) and (4.12) that $\det[S_P - \gamma^{-2}T_P] \neq 0$ and $\det[\gamma^2 S_Q - T_Q] \neq 0$ for all $\gamma \in (\gamma_{PQ}^*, \infty)$. Hence the largest root of $\det[N(\gamma)] = 0$ in (γ_{PQ}^*, ∞) is equal to the largest root of $\det[\gamma^2 I - M] = 0$, which is equal to $\sqrt{\lambda_{\max}(M)}$. ■

The main result of this section is summarized in the following theorem.

Theorem 4.3:

$$\gamma_o^* = \sqrt{\lambda_{\max}(M)}$$

where M is defined in (4.19).

Proof: The result is obvious for the case where $\min\{n_{aP}^+, n_{aQ}^+\} = 0$. In what follows, we proceed to prove our claim for the case where $\min\{n_{aP}^+, n_{aQ}^+\} > 0$.

First, we will show that γ_o^* is equal to the largest root of $\det[N(\gamma)] = 0$ when $f(\gamma) = \gamma^2$ has a unique solution in the interval (γ_{PQ}^*, ∞). It is simple to observe that $\det[N(\gamma_o^*)] = 0$ since $\lambda_{\max}[N(\gamma_o^*)] = f(\gamma_o^*) - (\gamma_o^*)^2 = 0$. Now suppose that there exists a γ_1 such that $\det[N(\gamma_1)] = 0$ and $\gamma_1 > \gamma_o^*$. This implies that there exists an eigenvalue of $N(\gamma_1)$, say $\lambda_i[N(\gamma_1)]$, such that $\lambda_i[N(\gamma_1)] \neq \lambda_{\max}[N(\gamma_1)]$ and $\lambda_i[N(\gamma_1] = 0$. Thus, we have

$$\lambda_{\max}[N(\gamma_1)] > \lambda_i[N(\gamma_1)] = 0 = \lambda_{\max}[N(\gamma_o^*)]$$

contradicting the findings in Observation 4.4 that $\lambda_{\max}[N(\gamma)]$ must be a nonincreasing function. Hence, γ_o^* is the largest root of $\det[N(\gamma)] = 0$ and it is equal to $\sqrt{\lambda_{\max}(M)}$ as shown in Observation 4.5.

Now we consider the situation when $f(\gamma) = \gamma^2$ has no solution in the interval (γ_{PQ}^*, ∞). In this case, clearly we have $\gamma_o^* = \gamma_{PQ}^*$ and $0 \leq f(\gamma_{PQ}^*) \leq (\gamma_{PQ}^*)^2$. The last inequality and the definition of $N(\gamma)$ in (4.18) imply that $-(\gamma_{PQ}^*)^2 \le \lambda_i[N(\gamma_{PQ}^*)] \le 0$. Thus the determinant of $N(\gamma_{PQ}^*)$ is bounded. Evaluating (4.21) at $\gamma = \gamma_{PQ}^*$, we have

$$\det\left[N(\gamma_{PQ}^*)\right] \det\left[S_P - (\gamma_{PQ}^*)^{-2} T_P\right] \det\left[(\gamma_{PQ}^*)^2 S_Q - T_Q\right] = (-1)^{n_{aP}^+} \det[S_P] \det[S_Q] \det\left[(\gamma_{PQ}^*)^2 I - M\right]. \tag{4.22}$$

Note that from (4.5) and (4.12) and the definition of γ_{PQ}^*, we have

$$\det\left[S_P - (\gamma_{PQ}^*)^{-2} T_P\right] \det\left[(\gamma_{PQ}^*)^2 S_Q - T_Q\right] = 0$$

and since $\det[N(\gamma_{PQ}^*)]$ is bounded, it follows from (4.22) that $\det[\gamma_{PQ}^{*2} I - M] = 0$ or $(\gamma_{PQ}^*)^2$ is an eigenvalue of M. Furthermore since $\det[N(\gamma)] = 0$ and similarly $\det[\gamma^2 I - M] = 0$ do not have a root in (γ_{PQ}^*, ∞), hence $\gamma_{PQ}^* = \sqrt{\lambda_{\max}(M)}$. ■

We have the following interesting lemma and corollaries.

Lemma 4.5: $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$ iff $\Gamma = 0$, where Γ is as defined in (4.16).

Proof: It is simple to show that

$$\mathcal{V}^+(A, E, C_1, D_1) = \operatorname{Im}\left\{\left(\Gamma_{sQ}^{-1}\right)' \begin{bmatrix} I_{n_{aQ}^+} \\ 0 \end{bmatrix}\right\} \quad \text{and}$$

$$\mathcal{S}^+(A, B, C_2, D_2) = \operatorname{Ker}\left\{\left[I_{n_{aP}^+} \quad 0\right] \Gamma_{sP}^{-1}\right\}.$$

Hence, it is straightforward to verify that $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$ iff

$$\left[I_{n_{aP}^+} \quad 0\right] \Gamma_{sP}^{-1} \left(\Gamma_{sQ}^{-1}\right)' \begin{bmatrix} I_{n_{aQ}^+} \\ 0 \end{bmatrix} = \Gamma = 0. \quad ■$$

Corollary 4.1: $\gamma_o^* = \gamma_{PQ}^*$ if $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$.

Proof: It follows from Theorem 4.3 and Lemma 4.5. ■

The condition $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$ is not necessary for $\gamma_o^* = \gamma_{PQ}^*$. In fact, $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$ becomes a necessary and sufficient condition for $\gamma_o^* = \gamma_{PQ}^*$ when $\gamma_{PQ}^* = 0$ as seen in the following corollary.

Corollary 4.2: Assume that $\gamma_{PQ}^* = 0$. Then $\gamma_o^* = 0$ iff $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$.

Proof: The sufficient part of this corollary follows from Corollary 4.1. We prove the converse part by contradiction. Suppose that $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$ or $\Gamma \neq 0$. It follows from Lemmas 4.1 and 4.2 that with the assumption $\gamma_{PQ}^* = 0$, i.e., $\gamma_P^* = \gamma_Q^* = 0$, we have $T_P = 0$ and $T_Q = 0$. Thus

$$M = \begin{bmatrix} \Gamma S_Q^{-1} \Gamma' S_P^{-1} & -\Gamma S_Q^{-1} \\ 0 & 0 \end{bmatrix}$$

and $\gamma_o^* = \sqrt{\lambda_{\max}(M)} > 0$. This is a contradiction. Hence the result follows. ■

An interesting question in H_∞-optimization problem is under what conditions the infimum in H_∞-optimization via output feedback be equal to that achieved using state feedback. In the following theorem, we provide a necessary and sufficient condition under which $\gamma_o^* = \gamma_s^*$.

Theorem 4.4: Consider the system Σ given by (2.1). Assume that both Assumptions A and B hold. Then $\gamma_o^* = \gamma_s^*$ iff

$$\lambda_{\max}(M) = \begin{cases} \lambda_{\max}(T_P S_P^{-1}) \\ \qquad \text{if } (A, B, C_2, D_2) \text{ is of nonminimum phase}, \\ 0 \\ \qquad \text{if } (A, B, C_2, D_2) \text{ is of minimum phase}. \end{cases}$$

■

Proof: It follows from Theorems 4.1 and 4.3. ■

Corollary 4.3:

1) If (A, E, C_1, D_1) is of minimum phase, then $\gamma_o^* = \gamma_s^*$.

2) If (A, B, C_2, D_2) is of minimum phase, then $\gamma_o^* = \gamma_s^*$ iff $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \operatorname{Ker}(C_2)$.

3) If both (A, B, C_2, D_2) and (A, E, C_1, D_1) are of nonminimum phase, then $\gamma_o^* = \gamma_s^*$ if $\mathcal{V}^+(A, E, C_1, D_1) \subseteq \mathcal{S}^+(A, B, C_2, D_2)$ and $\gamma_Q^* \le \gamma_P^*$.

Proof: Using the fact $\gamma_P^* = \gamma_s^*$, cases 1) and 2) are obvious in view of Theorem 4.3 and Lemma 4.2, while case 3) follows directly from Corollary 4.1. ■

V. Conclusion

In this paper, we have presented a simple and noniterative algorithm for the computation of the infimum in the standard H_∞-optimization problem using output feedback. We have shown that this infimum is equal to the square root of the maximum eigenvalue of a constant matrix that can be easily obtained from the data of the system Σ. Our results are obtained under the assumptions that the two subsystems Σ_P and Σ_Q are right- and left-invertible, respectively, and they do not have invariant zeros on $j\omega$ axis. The proposed algorithm for computing the infimum is applicable to the general case of singular H_∞-optimization problem where no restrictions have been placed on the direct feedthrough matrices from the control input to the controlled output, and from the disturbance to the measurement output. Our current research effort is directed toward removing some of the assumptions imposed in this paper on Σ_P and Σ_Q.

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Ben M. Chen (S'89) was born in Fuqing, Fujian, China, on November 25, 1963. He received the B.S. degree in mathematics and computer science from Amoy University, Xiamen, China, in July 1983, and the M.S. degree in electrical engineering from Gonzaga University, Spokane, Washington, in May 1988.

From July 1983 to March 1986 he worked as a software engineer in the South-China Computer Corporation, Guangzhou, China. From September 1986 to May 1988 he held a Presidential scholarship at Gonzaga University and a Cardinal Yu-Pin scholarship from the Sino-American Amity Fund, Inc., New York. He is currently completing the Ph.D. degree in the School of Electrical Engineering and Computer Science, Washington State University, Pullman. His current research interests are in robust and optimal control theory, linear system theory, and computer aided design for control systems.

Ali Saberi (S'79-M'79-S'80-M'83) is currently working in the School of Electrical Engineering and Computer Science, Washington State University, Pullman, Washington.

Uy-Loi Ly received the B.Sc. degree in aeronautics and mathematics from the University of California, Davis, in 1975 and the M.Sc. degree in applied mechanics, and the Engineer degree in aeronautics from the California Institute of Technology, Pasadena, in 1976 and 1977, respectively. From 1980 to 1982, he returned to complete the Ph.D. degree at Stanford University, Stanford, CA, under the guidance of Prof. A. E. Bryson and R. H. Cannon in the Department of Aeronautics and Astronautics.

From 1977 to 1988 he was with the Boeing Company as a Principal Engineer in the Flight Controls Research organization responsible for the development of practical design methods for the synthesis and analysis of multivariable control systems. Since 1988, he has been on the faculty of the Department of Aeronautics and Astronautics at the University of Washington, Seattle, WA. His current interests are in the research and practical application of multivariable control theory to aerospace related problems.