Construction and Parameterization of all Static and Dynamic H_2-Optimal State Feedback Solutions, Optimal Fixed Modes, and Fixed Decoupling Zeros

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Abstract—This paper considers an H_2 optimization problem via state feedback. The class of problems dealt with here are general singular type which have a left invertible transfer matrix function from the control input to the controlled output. This class subsumes the regular H_2 optimization problems. The paper constructs and parameterizes all the static and dynamic H_2 optimal state feedback solutions. Moreover, all the eigenvalues of an optimal closed-loop system are characterized. All optimal closed-loop systems share a set of eigenvalues which are termed here as the *optimal fixed modes*. Every H_2 optimal controller must assign among the closed-loop eigenvalues the set of *optimal fixed modes*. This set of *optimal fixed modes* includes a set of *optimal fixed decoupling zeros* which shows the minimum absolutely necessary number and locations of pole-zero cancellations present in any H_2 optimal design. It is shown that both the sets of *optimal fixed modes* and optimal fixed decoupling zeros do not vary depending upon whether the static or the dynamic controllers are used.

I. INTRODUCTION

OPTIMIZATION theory is one of the corner stones of modern control theory. In a typical control design, the given specifications are at first transformed into a performance index, and then control laws are sought which would minimize some norm, say H_2 or H_∞ norm, of the performance index. This paper focuses on the H_2 optimal control theory using state feedback controllers. H_2 optimal control theory was heavily studied in 1960's and early 1970's as a Linear Quadratic Gaussian (LQG) optimal control problem in which the performance index

consists of an integral of a quadratic function of errors in controller output variables as well as in control variables. The development of H_2 optimal control theory in the above mentioned LQG setting can be found in most graduate text books on control. In recent years, H_2 optimal control problems have been considered in a different setting than the traditional LQG setting. The interest in this new setting is to minimize the H_2 norm of a transfer matrix function from an exogenous disturbance to the controlled output of a given linear time-invariant system by an appropriate selection of a controller. In this setting, most of the works are confined to so called *regular* problems (see e.g., [4] and the references therein). However, some of the recent works (see e.g., [5], [17], and [18]) pay attention to singular problems as well.

It is known that, in general, for either a regular or a singular problem, an optimal control is not necessarily unique. Then, for a variety of reasons, one would like to characterize or construct the set of all H_2 optimal controllers. This is typically true since, in practice, minimization of the H_2 norm of a chosen transfer matrix function is not necessarily always a sole design goal. Several other secondary considerations also come into play in the final stages of a controller design. Recent papers on some mixed H_2 and H_∞ norm minimization problems (see e.g., [1], [2], [10], and [12]) are examples of such a design philosophy. In the mixed setting of H_2 and H_∞ norm minimization, knowing the parameterization of all H_2 optimal controllers plays a significant role. Next, a query that arises in general in parameterizing all the H_2 optimal controllers is about the architecture of a controller. In this connection, the most desired architecture from the practical point of view is the use of static state feedback. However, in the general mixed H_2 and H_∞ control problems, as demonstrated by an example of [12], a static state feedback controller may not exist while a dynamic state feedback controller may exist. This implies that one needs to study dynamic state feedback controllers as well. Thus, one of the goals of this paper is to characterize the set of

all H_2 optimal static as well as dynamic state feedback controllers.

Having recognized the need to characterize the set of all H_2 optimal state feedback controllers, let us now look at what has been accomplished so far in the literature. A recent work by Stoorvogel, Saberi, and Chen [17] investigated the H_2 control problems in their most general form without making any assumptions on the given plant, and unified all the existing results. However, [17] succeeds only partially in characterizing and constructing the set of all optimal controllers. To date, no complete characterization of the set of all H_2 optimal static state feed back controllers exists. Regarding the set of all H_2 optimal dynamic state feed back controllers, the only characterization that exists so far is the work of Rotea and Khargonekar [12] which, however, considers only the class of *regular* problems. Thus the construction of the set of all optimal state feedback controllers of either static or dynamic type, is still an open research problem for the case of general systems whether they are *regular* or *singular*. One of our intentions here is to solve this problem.

In recent years, two other aspects of optimal designs have sprouted out. These aspects are of much concern and need to be examined carefully. The first aspect relates to some inherent pole-zero cancellations present in any optimal design, not necessarily H_2 norm minimization but other optimal designs such as H_∞ designs. That is, every optimal controller inherits an inverse of certain part of the given plant dynamics. Some of the H_2 optimal controllers cancel *all* the stable invariant zeros of the plant, while some others cancel out only *some* stable invariant zeros. A primary concern of a designer is not to have pole-zero cancellations close to the imaginary axis. Besides the pole-zero cancellations, the second aspect of concern relates to placing some of the closed-loop eigenvalues at the mirror images of some unstable invariant zeros which are close to the imaginary axis. Some optimal controllers induce the mirror images of *all* the unstable invariant zeros among the closed-loop eigenvalues, and some others do not. In other words, in general, it is not necessary to induce the mirror images of *all* the unstable invariant zeros among the closed-loop eigenvalues. In fact, it turns out that there exists a set of fixed complex numbers which are among the closed-loop eigenvalues under any H_2 optimal state feedback law. Such complex numbers can be called as the *optimal fixed modes*. The set of *optimal fixed modes* includes also a set of *optimal fixed decoupling zeros* (both input and output decoupling zeros) of the closed-loop system. The set of *optimal fixed decoupling zeros* indeed contain as its elements those closed-loop eigenvalues which should be involved in pole-zero cancellations whatever may be the H_2 optimal control one uses. Obviously, there is a definite need to study and to construct the set of *optimal fixed modes* along with identifying the included set of *optimal fixed decoupling zeros*, in order to help a designer ascertain what can and what cannot be assigned as closed-loop eigenvalues while still preserving optimality. Such a study has not yet been undertaken in

the literature for a general problem, although Stoorvogel *et al.* [17] identified the set of optimal fixed modes for a particular case. Thus, another intention of this paper is to undertake the study of optimal fixed modes as well as optimal fixed decoupling zeros for regular or singular problems.

In view of the above discussion, we can summarize our goals in this paper as follows. Given an H_2 optimal control problem, the first goal is to construct or to characterize the set of all optimal state feedback controllers of either static or dynamic type. For each such a set of optimal controllers, the next goal is to characterize the set of *optimal fixed modes* which every optimal controller must assign among the closed-loop eigenvalues; and to identify the set of *optimal fixed decoupling zeros* which is a subset of the set of *optimal fixed modes*, and which shows the minimum absolutely necessary number and locations of pole-zero cancellations in any H_2 optimal design. We confine ourselves here with general singular H_2 optimization problems where the transfer matrix function from the control input to the controlled output is left invertible. This class subsumes the regular H_2 optimization problems. An H_2 optimization problem where the transfer matrix function from the control input to the controlled output is not left invertible, can however be converted to an equivalent H_2 optimization problem where the transfer matrix function from the control input to the controlled output is in fact left invertible. We do not show this conversion here as it requires clarification of some technical issues that need lengthy discussions.

The paper is organized as follows. Section II gives a clear mathematical statement of the problem, while Section III deals with several needed preliminary results. Section IV considers static feedback controllers. The heart of it is a step by step development of an algorithm what we appropriately term as algorithm '*Optimal Gains and Fixed Modes*' or OGFM to be short. The algorithm OGFM takes as its input parameters a set of five matrices which characterize the given H_2 optimal control problem. After ensuring that an optimal static state feedback controller exists, the OGFM algorithm proceeds to construct explicitly, the set of all H_2 optimal static state feedback gains, and the corresponding set of all H_2 optimal *fixed modes*. The set of all H_2 optimal *fixed decoupling zeros* is also identified. Section V considers the case of dynamic controllers. Here, the well known Q-parameterization technique is used to characterize all the possible H_2 optimal solutions. It is also shown here that the set of *optimal fixed modes* and *optimal fixed decoupling zeros* do not vary depending upon whether the static or the dynamic controllers are used. Finally, Section VI draws the conclusions of our work.

Throughout the paper, A' denotes the transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\mathbb{C}$, $\mathbb{C}^-$, $\mathbb{C}^0$, and $\mathbb{C}^+$ respectively denote the whole complex plane, the open left-half complex plane, the imaginary axis, and the open right-half complex plane. A matrix is said to be stable if all its

eigenvalues are in $\mathbb{C}^-$. Ker$[V]$ and Im$[V]$ denote respectively the kernel and the image of V. Given a stable transfer function $G(s)$, as usual, its H_2 norm is defined by

$$\|G\|_2^2 := \int_0^\infty \text{Trace}\{G'(-j\omega)G(j\omega)\}\, d\omega.$$

Also, RH^2 denotes the set of real-rational transfer functions which are stable and strictly proper. RH^∞ denotes the set of real-rational transfer functions which are stable and proper.

II. PROBLEM STATEMENT

Consider the following system Σ characterized by,

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = x \\ z = Cx + Du + Gw \end{cases} \quad (2.1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^l$ is the unknown disturbance and $z \in \mathbb{R}^q$ is the controlled output. Also, consider an arbitrary proper controller

$$u = F(s)x. \quad (2.2)$$

A controller $u = F(s)x$ is said to be admissible if it provides internal stability of the resulting closed-loop system. Let $T_{zw}(F)$ denote the closed-loop transfer function from w to z after applying an admissible controller $u = F(s)x$ to Σ. Then the H_2-optimization state feedback problem for Σ is to find an admissible state feedback controller $F(s)$ which minimizes $\|T_{zw}(F)\|_2$.

For future use, let us define a system Σ_*,

$$\Sigma_*: \begin{cases} \dot{x} = Ax + Bu, \\ z = Cx + Du. \end{cases} \quad (2.3)$$

The following definitions will be convenient in the sequel.

Definition 2.1 (The regular H_2 optimization problem): A regular H_2 optimization state feedback problem refers to a problem in which the given system Σ satisfies, i) D is injective, and ii) the system Σ_* has no invariant zeros on the $j\omega$ axis.

Definition 2.2 (The singular H_2 optimization problem): A singular H_2 optimization state feedback problem refers to a problem in which the given system Σ does not satisfy either one or both of the conditions i) and ii) in Definition 2.1.

Definition 2.3 (The infimum of H_2 optimization): For a given system Σ, the infimum of the H_2 norm of the closed-loop transfer function $T_{zw}(F)$ over all the stabilizing proper controllers $F(s)$ is denoted by γ^*, namely

$$\gamma^* := \inf\{\|T_{zw}(F)\|_2 \mid u = F(s)x \text{ internally stabilizes } \Sigma\}. \quad (2.4)$$

Definition 2.4 (The H_2 optimal controller): A stabilizing proper controller $F(s)$ is said to be an H_2 optimal controller if $\|T_{zw}(F)\|_2 = \gamma^*$. The sets of all optimal static and dynamic state feedback controllers are respectively denoted by F_s^* and F_d^*. Obviously, $F_s^* \subset F_d^*$.

Definition 2.5 (The H_2 optimal fixed modes): A scalar $\lambda \in \mathbb{C}^-$ is said to be an H_2 optimal fixed mode if λ is a pole of the closed-loop system for every H_2 optimal controller of a particular type, say static or dynamic, that one uses. The sets of all the H_2 optimal fixed modes corresponding to the static and the dynamic controllers are respectively denoted by Ω_s^* and Ω_d^*.

Definition 2.6 (The H_2 optimal fixed decoupling zeros): A scalar $\lambda \in \mathbb{C}^-$ is said to be an H_2 optimal fixed decoupling zero if λ is either an input or an output decoupling zero (or both) [11] of the closed-loop system for every H_2 optimal controller of a particular type, say static or dynamic, that one uses. The sets of all the H_2 optimal fixed decoupling zeros corresponding to the static and the dynamic controllers are respectively denoted by Λ_s^* and Λ_d^*.

Throughout this paper, without loss of generality, we assume that the direct feedthrough matrix from w to z is zero, i.e., $G \equiv 0$. Otherwise, it is simple to verify that the closed-loop system comprising Σ and any arbitrary stabilizing proper controller $F(s)$ is nonstrictly proper and hence $\gamma^* = \infty$. Certainly, in this paper, we are not interested in the case of γ^* being infinity and as such G is assumed zero. We also assume that Σ_* is left invertible.

The goals of this paper are:
1) To present explicit design methods to determine the sets of static and dynamic optimal controllers F_s^* and F_d^*.
2) To determine the sets of optimal fixed modes Ω_s^* and Ω_d^*.
3) To determine the sets of optimal fixed decoupling zeros Λ_s^* and Λ_d^*.

III. PRELIMINARIES

A. A Special Coordinate Basis (SCB)

In this subsection we recall the special coordinate basis for a linear time-invariant nonstrictly proper system [13]. Such a coordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structures of a given system as well as other system geometric properties. It is instrumental in the derivation of the method described in Section IV.

Consider the system Σ_* as in (2.3). It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a nonsingular matrix V that render the direct feedthrough matrix D into the following form,

$$\overline{D} = UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \quad (3.1)$$

where m_0 is the rank of D. Without loss of generality one can assume that the matrix D in equation (2.3) has the form as shown in (3.1). Thus the system in (2.3) can be rewritten as

$$\begin{cases} \dot{x} = Ax + \begin{bmatrix} B_0 & B_1 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\ \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \end{cases} \quad (3.2)$$

where B_0, B_1, C_0, and C_1 are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely,

$$u = V\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = Uz.$$

We have the following theorem.

Theorem 3.1 (SCB): Let Σ_* be left invertible. Then, there exist nonsingular transformations Γ_s, Γ_o and Γ_i such that

$$x = \Gamma_s\left[(x_a^-)', (x_a^0)', (x_a^+)', x_b', x_f'\right]',$$

$$[z'_0, z'_1]' = \Gamma_o[z_0^*, z_f', z_b']', \qquad [u'_0, u'_1]' = \Gamma_i[u'_0, u'_f]'$$

and

$$\Gamma_s^{-1}(A - B_0 C_0)\Gamma_s$$

$$= \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & L_{af}^- C_f \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & L_{af}^0 C_f \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & L_{af}^+ C_f \\ 0 & 0 & 0 & A_{bb} & L_{bf} C_f \\ B_f E_{fa}^- & B_f E_{fa}^0 & B_f E_{fa}^+ & B_f E_{fb} & A_{ff} \end{bmatrix},$$

(3.3)

$$\Gamma_s^{-1}[B_0, B_1]\Gamma_i = \begin{bmatrix} B_{a0}^- & 0 \\ B_{a0}^0 & 0 \\ B_{a0}^+ & 0 \\ B_{b0} & 0 \\ B_{f0} & B_f \end{bmatrix}, \qquad (3.4)$$

$$\Gamma_o^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix}, \quad (3.5)$$

and

$$\Gamma_o^{-1}\begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\Gamma_i = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (3.6)$$

where $\lambda(A_{aa}^-) \in \mathbb{C}^-$, $\lambda(A_{aa}^0) \in \mathbb{C}^0$, $\lambda(A_{aa}^+) \in \mathbb{C}^+$, (A_{bb}, C_b) is observable and the subsystem characterized by (A_{ff}, B_f, C_f) is invertible with no invariant zeros.

The proof of this theorem can be found in [13], [14]. We also note that the output transformation Γ_o is of the form,

$$\Gamma_o = \begin{bmatrix} I_{m_0} & 0 \\ 0 & \Gamma_{or} \end{bmatrix}. \qquad (3.7)$$

In what follows, we state some important properties of the SCB which are pertinent to our present work.

Property 3.1: System Σ_* is invertible if and only if x_b is nonexistent.

Property 3.2: $\lambda(A_{aa}^-) \cup \lambda(A_{aa}^0) \cup \lambda(A_{aa}^+)$ are the invariant zeros [9] of Σ_*. We note that $\lambda(A_{aa}^-)$ are the stable

(left hand s-plane) and $\lambda(A_{aa}^+)$ are the unstable (right hand s-plane) invariant zeros of Σ_*, while $\lambda(A_{aa}^0)$ are on the imaginary axis.

Property 3.3: The pair (A, B) is stabilizable if and only if (A_{con}, B_{con}) is stabilizable where

$$A_{con} = \begin{bmatrix} A_{aa}^0 & 0 & L_{ab}^0 C_b \\ 0 & A_{aa}^+ & L_{ab}^+ C_b \\ 0 & 0 & A_{bb} \end{bmatrix},$$

$$B_{con} = \begin{bmatrix} B_{a0}^0 & L_{af}^0 \\ B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}. \qquad (3.8)$$

Let us recall the following definition of weakly unobservable subspace [6] and [19].

Definition 3.1: We define the stabilizable weakly unobservable subspace $\mathscr{V}_g(\Sigma_*)$ as the largest subspace $\mathscr{V}$ for which there exists a mapping F such that the following subspace inclusions are satisfied:

$$(A + BF)\mathscr{V} \subseteq \mathscr{V}, \qquad (3.9)$$

$$(C + DF)\mathscr{V} = \{0\}, \qquad (3.10)$$

and such that $A + BF \mid \mathscr{V}$ is asymptotically stable.

Property 3.4: x_a^- spans $\mathscr{V}_g(\Sigma_*)$.

B. Existence of Optimal Controllers

In this section, we recall from [17] the necessary and sufficient conditions under which an optimal state feedback control law of either static or dynamic type exists. The conditions under which an optimal controller exists are formulated in terms of an auxiliary system Σ_{au} constructed from the data of the given H_2 optimization problem. The auxiliary system Σ_{au} is as given below:

$$\Sigma_{au}: \begin{cases} \dot{x}_P = Ax_P + Bu_P + Ew_P, \\ y_P = x_P, \\ z_P = C_P x_P + D_P u_P. \end{cases} \qquad (3.11)$$

Here C_P and D_P satisfy

$$F(P) = \begin{bmatrix} C'_P \\ D'_P \end{bmatrix}\begin{bmatrix} C_P & D_P \end{bmatrix}, \qquad (3.12)$$

where

$$F(P) := \begin{bmatrix} A'P + PA + C'C & PB + C'D \\ B'P + D'C & D'D \end{bmatrix}, \quad (3.13)$$

and where P is the largest solution of the matrix inequality $F(P) \geq 0$. It is known that under the condition that (A, B) is stabilizable, such a solution P exists and is unique. Later on in Section IV, a procedure for the computation of such a P is given.

Consider a controller $\mathsf{F}(s)$ and let it be applied to both Σ and Σ_{au}. That is, let $u = \mathsf{F}(s)x$ and $u_P = \mathsf{F}(s)x_P$. A key idea of [17] is that a controller $\mathsf{F}(s)$ solves the given H_2 optimization problem when applied to Σ if and only if it solves a Disturbance Decoupling Problem with internal

Stability (DDPS) when applied to Σ_{au}. We recall the following result.

Lemma 3.1: The following two statements are equivalent:

i) The controller $u = F(s)x$ when applied to the given system Σ is internally stabilizing and the resulting closed-loop transfer function from w to z is strictly proper and has the H_2 norm γ^*.

ii) The controller $u_P = F(s)x_P$ when applied to the new system Σ_{au} is internally stabilizing and the resulting closed-loop transfer function from w_P to z_P is strictly proper and has the H_2 norm 0.

Proof: See [16]. ∎

The above lemma and the existing literature on DDPS, enabled [17] to generate the following theorem.

Theorem 3.2: Consider the given system Σ as in (2.1), and the auxiliary system Σ_{au} as in (3.11). Define a subsystem Σ_P of Σ_{au} as that characterized by the quadruple (A, B, C_P, D_P). We have the following results:

1) (Existence of an optimal static state feedback controller): The infimum, γ^*, can be attained by a static stabilizing state feedback controller if and only if the pair (A, B) is stabilizable and $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$.

2) (Existence of an optimal proper dynamic state feedback controller): The infimum, γ^*, can be attained by a proper dynamic stabilizing state feedback controller if and only if the pair (A, B) is stabilizable and $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$.

Moreover, the infimum, γ^*, is given by

$$\gamma = \sqrt{\mathrm{Trace}\,(E'PE)}, \qquad (3.14)$$

where P is the maximal solution of the matrix inequality $F(P) \geq 0$ for $F(P)$ as in (3.12).

Proof: See [17]. ∎

Remark 3.1: If Σ satisfies the conditions of regular case, then it is simple to verify that $\mathcal{V}_g(\Sigma_P) = \mathbb{R}^n$, and thus the condition $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$ is automatically satisfied. Hence, an optimal static and a proper dynamic state feedback controller for a regular H_2 problem always exist whenever the pair (A, B) is stabilizable.

Remark 3.2: An algorithm for verifying the existence conditions of Theorem 3.2 are incorporated in the algorithm *OGFM* discussed in the next section.

IV. DESIGN OF OPTIMAL STATIC STATE FEEDBACK CONTROLLERS

In this section, we present a design procedure to determine the set of all optimal static state feedback controllers F_s^* and the resulting set of optimal fixed modes Ω_s^* as well as the set of optimal fixed decoupling zeros Λ_s^*. Clearly, in view of Theorem 3.2, the infimum, γ^*, can be attained by a static state feedback law if and only if (A, B) is stabilizable and $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Let us look at two extreme cases of the condition $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. For the case of $E = 0$, that is for the case when no exogenous disturbance affects the system, every controller that guarantees the internal stability of the resulting closed-loop system is obviously an H_2 optimal controller. The other
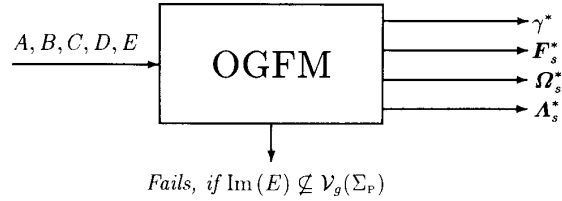

Fig. 1. A block diagram interpretation of *OGFM*.

extreme case, namely when $\mathrm{Im}(E) = \mathcal{V}_g(\Sigma_P)$, is referred to in [17] as the 'worst case' since it corresponds to the situation when a disturbance signal can affect the dynamics of the given system in the worst way while still satisfying the geometric subspace inclusion condition $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. For this worst case, the set of all H_2 optimal controllers has been constructed by [17]. For the general case when $\mathrm{Im}(E)$ is strictly included in $\mathcal{V}_g(\Sigma_P)$, the set of all H_2 optimal controllers includes the corresponding set for the worst case. Construction of all the H_2 optimal static feedback controllers, the set of otpimal fixed modes as well as the set of optimal fixed decoupling zeros, has not been done so far in the literature, and the goal of this section is to do exactly that.

A basic component of our design procedure to construct all the static state feedback controllers, is an algorithm called '*Optimal Gains and Fixed Modes*' which is abbreviated henceforth as *OGFM*. As depicted in Fig. 1, the matrix quintuple (A, B, C, D, E) is a set of input parameters to *OGFM*, while the outputs of *OGFM* are, a) the set of all H_2 static optimal state feedback controllers F_s^*, b) the set of all H_2 optimal fixed modes Ω_s^*, c) the set of all H_2 optimal fixed decoupling zeros Λ_s^*, and d) the infimum, γ^*. Besides these, *OGFM* also calculates the maximal solution P of the inequality $F(P) \geq 0$, and checks whether the condition, $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$, is satisfied by the given problem or not. The leading component of the algorithm is the isolation of a pair of matrices (A_z, B_z) from the input data (A, B, C, D, E). A gain matrix F_z which renders $A_z - B_z F_z$ asymptotically stable, is a parameter by appropriately varying which the entire set of static optimal feedback gains is constructed. In what follows, a step-by-step description of the algorithm *OGFM* is given. The aim of the description here is to present a step by step computation of various matrices. The explanation of intuition behind these computations and the proofs of certain existence statements made in the algorithm are delegated to the appendix. A key tool used for all the main calculations in *OGFM* is the construction of appropriate SCB's of some subsystems. A software tool box for constructing SCB's is given by Lin *et al.* [8].

Step 1 (Computation of the Pair (A_z, B_z)): In this step, we compute a pair (A_z, B_z) which leads to the parameterization of the set of all optimal static state feedback gains. Our computations are divided into several substeps.

Step 1(a) (Construction of the SCB of Σ_*): Transform the subsystem Σ_* into the SCB as given in (3.3) to (3.6) of

Section III. For future development, let us compute

$$\Gamma_s^{-1}E = \left[(E_a^-)', (E_a^0)', (E_a^+)', (E_b)', (E_f)'\right]'$$

and define $E_x = [(E_a^+)' \ (E_b)']'$.

Step 1(b) (Construction of the Subsystem Σ_P): An explicit construction of the subsystem Σ_P is pursued in this step. Define a matrix quadruple,

$$A_x := \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix}, \qquad B_x := \begin{bmatrix} B_{a0}^+ & L_{af}^+ \\ B_{b0} & L_{bf} \end{bmatrix}$$

$$C_x := \Gamma_o \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_b \end{bmatrix}, \qquad D_x := \Gamma_o \begin{bmatrix} I & 0 \\ 0 & C_f C_f' \\ 0 & 0 \end{bmatrix} \quad (4.1)$$

and partition

$$\Gamma_o' \Gamma_o = \begin{bmatrix} (\Gamma_P)^2 & \star \\ \star & \star \end{bmatrix}, \quad (4.2)$$

where Γ_P is of dimension $(\dim z_0 + \dim z_f) \times (\dim z_0 + \dim z_f)$, and $\star$ denotes a matrix of not much interest to us. Then solve the following algebraic Riccati equation,

$$P_x A_x + A_x' P_x + C_x' C_x$$
$$- (P_x B_x + C_x' D_x)(D_x' D_x)^{-1}(B_x' P_x + D_x' C_x) = 0 \quad (4.3)$$

for $P_x > 0$. Note that such a solution P_x always exists (see for example [17]). Define

$$\begin{bmatrix} F_{a0}^+ & F_{b0} \\ F_{a1}^+ & F_{b1} \end{bmatrix} := (D_x' D_x)^{-1}(B_x' P_x + D_x' C_x) \quad (4.4)$$

and

$$C_P := \Gamma_P \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ + F_{a0}^+ & C_{0b} + F_{b0} & C_{0f} \\ 0 & 0 & F_{a1}^+ & F_{b1} & C_f \end{bmatrix} \Gamma_s^{-1},$$

$$D_P := \Gamma_P \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_i^{-1}. \quad (4.5)$$

As explained in the proof of Lemma A.1 of the Appendix, the above defined matrices C_P and D_P satisfy (3.12). Thus the subsystem Σ_P is characterized by (A, B, C_P, D_P) where C_P and D_P are as in (4.5). Moreover, the maximum solution of $F(P) \geq 0$ is given by,

$$P = (\Gamma_s^{-1})' \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P_x & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Gamma_s^{-1}. \quad (4.6)$$

Thus,

$$\gamma^* = \sqrt{\text{Trace}(E'PE)}. \quad (4.7)$$

Step 1(c) (Construction of the SCB of Σ_P): Here we transform the system Σ_P into a SCB as given in Section III. We note that, as shown in Appendix, Σ_P is invertible and does not have any invariant zeros in $\mathbb{C}^+$. To all the submatrices and transformations in the SCB of Σ_P, we append a subscript $_P$ to signify their relation to the system Σ_P. To facilitate the construction of the SCB of Σ_P, first

compute the orthogonal transformation matrix U_P and a nonsingular transformation matrix V_P such that

$$BV_P = \begin{bmatrix} B_{0P} & B_{1P} \end{bmatrix}, \qquad U_P C_P = \begin{bmatrix} C_{0P} \\ C_{1P} \end{bmatrix}$$

and

$$U_P D_P V_P = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \quad (4.8)$$

where m_0 is the rank of D_P. Then construct the nonsingular transformations Γ_{sP}, Γ_{iP} and Γ_{oP} such that

$$\Gamma_{sP}^{-1}(A - B_{0P}C_{0P})\Gamma_{sP} = \begin{bmatrix} A_{aaP}^- & 0 & L_{afP}^- C_{fP} \\ 0 & A_{aaP}^0 & L_{afP}^0 C_{fP} \\ B_{fP}E_{faP}^- & B_{fP}E_{faP}^0 & A_{ffP} \end{bmatrix},$$

$$\Gamma_{sP}^{-1}[B_{0P}, B_{1P}]\Gamma_{iP} = \begin{bmatrix} B_{a0P}^- & 0 \\ B_{a0P}^0 & 0 \\ B_{f0P} & B_{fP} \end{bmatrix},$$

$$\Gamma_{oP}^{-1}\begin{bmatrix} C_{0P} \\ C_{1P} \end{bmatrix}\Gamma_{sP} = \begin{bmatrix} C_{0aP}^- & C_{0aP}^0 & C_{0fP} \\ 0 & 0 & C_{fP} \end{bmatrix}$$

and

$$\Gamma_{oP}^{-1}\begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\Gamma_{iP} = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

From the property of SCB (e.g., Property 3.4), it is simple to see that $\text{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$ implies that

$$\Gamma_{sP}^{-1}E = \begin{bmatrix} E_{aP}^- \\ 0 \\ 0 \end{bmatrix}. \quad (4.9)$$

If $\Gamma_{sP}^{-1}E$ is not of the form (4.9), the infimum γ^* is not attainable and the procedure of *OGFM* stops at this point. Otherwise it continues to the next step.

Step 1(d) (Decomposition of Σ_{au}): In this step, Σ_{au} is decomposed into two parts, one part being controllable via the disturbance w and the other not. Consider the pair (A_{aaP}^-, E_{aP}^-). This pair need not be controllable, that is, the disturbance w with its coefficient matrix as E_{aP}^- need not affect all the modes of A_{aaP}^-. Compute a nonsingular transformation T_{aP} such that

$$T_{aP}^{-1}A_{aaP}^-T_{aP} = \begin{bmatrix} A_{aaP}^{11} & A_{aaP}^{12} \\ 0 & A_{aaP}^{22} \end{bmatrix} \quad \text{and} \quad T_{aP}^{-1}E_{aP}^- = \begin{bmatrix} E_{aP}^1 \\ 0 \end{bmatrix},$$

where the pair (A_{aaP}^{11}, E_{aP}^1) is completely controllable. Also, let us partition

$$T_{aP}^{-1}L_{afP}^- = \begin{bmatrix} L_{afP}^1 \\ L_{afP}^2 \end{bmatrix}, \qquad T_{aP}^{-1}B_{a0P}^- = \begin{bmatrix} B_{a0P}^1 \\ B_{a0P}^2 \end{bmatrix}$$

and

$$E_{faP}^-T_{aP} = \begin{bmatrix} E_{faP}^1 & E_{faP}^2 \end{bmatrix}, \qquad C_{0aP}^-T_{aP} = \begin{bmatrix} C_{0aP}^1 & C_{0aP}^2 \end{bmatrix}.$$

Finally, form the matrices A_z and B_z as follows:

$$A_z := \begin{bmatrix} A_{aaP}^{22} & 0 & L_{afP}^2 C_{fP} \\ 0 & A_{aaP}^0 & L_{afP}^0 C_{fP} \\ B_{fP} E_{faP}^2 & B_{fP} E_{faP}^0 & A_{ffP} \end{bmatrix},$$

$$B_z := \begin{bmatrix} B_{a0P}^2 & 0 \\ B_{a0P}^0 & 0 \\ B_{f0P} & B_{fP} \end{bmatrix}. \tag{4.10}$$

Again, from the property of SCB (e.g., Property 3.3), it is simple to verify that the pair (A_z, B_z) is stabilizable if and only if the pair (A, B) is stabilizable. Thus, whenever (A, B) is stabilizable, a gain F_z exists such that $\lambda(A_z - B_z F_z) \subseteq \mathbb{C}^-$.

Step 2 (Parameterization and Construction of the Sets F_s^*, Ω_s^* and Λ_s^*): In this step, F_s^*, Ω_s^*, and Λ_s^* are parameterized in terms of F_z which renders $A_z - B_z F_z$ asymptotically stable. Let us define the set

$$F_z := \{ F_z | \lambda(A_z - B_z F_z) \subseteq \mathbb{C}^- \}. \tag{4.11}$$

Let us also partition $F_z \in F_z$ to be compatible with the partitions of A_z and B_z as,

$$F_z = \begin{bmatrix} F_{a0P}^2 & F_{a0P}^0 & F_{f0P} \\ F_{afP}^2 & F_{afP}^0 & F_{ffP} \end{bmatrix}. \tag{4.12}$$

Let

$$F = -V_P \Gamma_{iP}$$

$$\cdot \begin{bmatrix} C_{0aP}^1 & C_{0aP}^2 + F_{a0P}^2 & C_{0aP}^0 + F_{a0P}^0 & C_{0fP} + F_{f0P} \\ E_{faP}^1 & F_{afP}^2 & F_{afP}^0 & F_{ffP} \end{bmatrix} T_{sP}^{-1},$$

$$\tag{4.13}$$

where

$$T_{sP} = \Gamma_{sP} \begin{bmatrix} T_{aP} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Also, let

$$F_s^* := \{ F \in \mathbb{R}^{m \times n} | F \text{ is given by } (4.13) \text{ with } F_z \in F_z \}, \tag{4.14}$$

$$\Omega_s^* := \lambda(A_{aaP}^{11}) \cup \{ \text{input decoupling zeros of } (A_z, B_z) \}, \tag{4.15}$$

and

$$\Lambda_s^* := \{ \lambda(A_{aaP}^{11}) \cap \lambda(A_{aa}^-) \}$$

$$\cup \{ \text{input decoupling zeros of } (A_z, B_z) \}. \tag{4.16}$$

This concludes the description of *OGFM*.

We have the following theorem.

Theorem 4.1: Consider the given system Σ as in (2.1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Then we

have:

1) (Optimal static state feedback controllers): Any member of the set F_s^* is an optimal state feedback controller, i.e., the state feedback law $u = Fx$ where F is of the form (4.13) with $F_z \in F_z$, when applied to Σ is stabilizing and the closed-loop H_2 norm is equal to γ^*. Conversely, any state feedback law $u = Fx$ which is stabilizing and yields a closed-loop H_2 norm equal to γ^* is such that F is of the form (4.13) with $F_z \in F_z$.

2) (Optimal fixed modes): The set of H_2 optimal fixed modes under a static state feedback is given by Ω_s^*. That is, any optimal static state feedback controller must assign the elements of Ω_s^* among the closed-loop eigenvalues. The rest of the closed-loop eigenvalues can be assigned arbitrarily in $\mathbb{C}^-$ as long as they are symmetric with respect to the real axis, by an appropriate selection of a static state feedback controller from F_s^*.

3) (Optimal fixed decoupling zeros): The set of H_2 optimal fixed decoupling zeros under a static state feedback is given by Λ_s^*. That is, regardless of the choice of F from F_s^*, the absolutely minimum number and locations of pole-zero cancellations in the optimal closed-loop transfer functions are given by the set Λ_s^*.

4) (Other pole-zero cancellations in optimal closed-loops): For any $F \in F_s^*$, define $\Lambda^{idz}(F) := \lambda(A + BF)/\Omega_s^*$. Then for any $\lambda \in \Lambda^{idz}(F)$, λ is an input decoupling zero of the closed-loop system comprising of Σ and the static state feedback controller $u = Fx$. Moreover, by varying F over the set F_s^*, the elements of $\Lambda^{idz}(F)$ can be assigned arbitrarily in $\mathbb{C}^-$ as long as they are symmetric with respect to the real axis.

Proof: See Appendix A.   ∎

Remark 4.1: If $\mathrm{Im}(E) = \mathcal{V}_g(\Sigma_P)$, then Ω_s^* consists of:
1) All the stable invariant zeros of Σ_*,
2) All the mirror image of the unstable invariant zeros of Σ_*, and
3) Some fixed locations in the open left half plane which can be calculated using the system data of Σ_*. In fact, these locations are given by $\lambda(-A_{bb} - SC_b'C_b)$, where S is the unique positive definite solution of the following algebraic Riccati equation (ARE),

$$SA_{bb}' + A_{bb}S + SC_b'C_bS - B_{b0}B_{b0}' - L_{bf}L_{bf}' = 0. \tag{4.17}$$

On the other hand, if $\mathrm{Im}(E) = \{0\}$, Ω_s^* contains only the input decoupling zeros of Σ_*. In general, only *some* but not *all* the stable invariant zeros and only *some* but not *all* the mirror image of the unstable invariant zeros of Σ_* are contained in Ω_s^* if $\mathrm{Im}(E)$ is strictly included in $\mathcal{V}_g(\Sigma_P)$.

The importance of Theorem 4.1 cannot be over emphasized. The knowledge of the entire set of optimal feedback controllers F_s^*, makes it easier to take into account design criteria other than H_2 optimality. Also, the set of fixed modes Ω_s^* clearly points out what every optimal closed-loop system must include among its eigenvalues. The set of fixed decoupling zeros Λ_s^* includes both the fixed input and output decoupling zeros of the closed-loop

system, and it shows the *minimum absolutely necessary number and locations* of pole-zero cancellations present in any H_2 optimal design. We emphasize that an arbitrarily chosen H_2 optimal design may induce pole-zero cancellations beyond that given by the set Λ_s^*. Thus having the knowledge of Ω_s^* and Λ_s^*, a designer can easily ascertain whether any unwanted pole-zero cancellations or as a matter of fact any unwanted closed-loop eigenvalues, are to be necessarily involved or not in a final design.

The following example illustrates our results.

Example 4.1: Consider a given system Σ characterized by

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -0.001 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0.001 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is simple to verify that the given system Σ_* is invertible and of nonminimum phase with one infinite zero of order 1 and four invariant zeros at $\{-1, -0.001, 1, 0.001\}$. Moreover, it is already in the form of SCB. Following Step 1(b) of *OGFM*, we obtain

$$C_P = \begin{bmatrix} 0 & 0 & 0 & 0.002 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D_P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0.002 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and thus

$$\gamma^* = \sqrt{\text{Trace}\,(E'PE)} = \sqrt{2}.$$

Using the software package of Lin *et al.* [8] to compute the SCB for Σ_P and following the algorithm of *OGFM*, we obtain,

$$\Gamma_{iP} = \Gamma_{oP} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Gamma_{sP} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -0.4472 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.8944 & 0 & 0 & 1 \end{bmatrix},$$

$$\Gamma_{sP}^{-1}(A - B_{0P}C_{0P})\Gamma_{sP}$$

$$= \begin{bmatrix} -1 & 0.8944 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -2.2361 \\ 0 & 0 & -0.001 & -0.002 & 0 \\ 0 & 0 & 0 & -0.001 & 0 \\ 1 & 0.4472 & 0 & 0 & 2 \end{bmatrix},$$

$$\Gamma_{sP}^{-1}B\Gamma_{iP} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Gamma_{sP}^{-1}E = \begin{bmatrix} 1 & 0 \\ 0 & -2.2361 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Gamma_{oP}^{-1}C_P\Gamma_{sP} = \begin{bmatrix} 0 & 0 & 0 & 0.002 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Gamma_{oP}^{-1}D_P\Gamma_{iP} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_{aaP}^- = \begin{bmatrix} -1 & 0.8944 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.001 & -0.002 \\ 0 & 0 & 0 & -0.001 \end{bmatrix},$$

$$E_{aP}^- = \begin{bmatrix} 1 & 0 \\ 0 & -2.2361 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_{aP} = I_4,$$

$$A_{aaP}^{11} = \begin{bmatrix} -1 & 0.8944 \\ 0 & -1 \end{bmatrix}, \quad E_{aP}^1 = \begin{bmatrix} 1 & 0 \\ 0 & -2.2361 \end{bmatrix},$$

$$A_{aaP}^{22} = \begin{bmatrix} -0.001 & -0.002 \\ 0 & -0.001 \end{bmatrix},$$

$$A_z = \begin{bmatrix} -0.001 & -0.002 & 0 \\ 0 & -0.001 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B_z = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we get the set of static feedback controllers F_s^* as,

$$F_s^* := \Bigg\{ F \,\Big|\, F =$$

$$-\begin{bmatrix} 0 & 0 & F_{z11} & F_{z12} + 0.002 & F_{z13} \\ 1 & 0.4472 & F_{x21} & F_{z22} & F_{z23} \end{bmatrix}\Gamma_{sP}^{-1} \Bigg\},$$

$$\tag{4.18}$$

where

$$F_z = \begin{bmatrix} F_{z11} & F_{z12} & F_{z13} \\ F_{z21} & F_{z22} & F_{z23} \end{bmatrix}$$

is such that $\lambda(A_z - B_zF_z) \in \mathbb{C}^-$. We also have,

$$\Omega_s^* = \{-1, -1\} \quad \text{and} \quad \Lambda_s^* = \{-1\}.$$

We note that the set F_s^* is parameterized in terms of F_z which assigns $\lambda(A_z - B_zF_z)$ in $\mathbb{C}^-$. Now, let us pick two optimal static state feedback controllers having gains, F_1 and F_2,

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & -0.002 & 0 \\ -1 & 0 & -5 & 0 & -3 \end{bmatrix}$$

and

$$F_2 = \begin{bmatrix} 0 & 4.9011 & 0 & -5.1011 & 0 \\ -1 & 0 & -5 & 0 & -3 \end{bmatrix}.$$

$$\lambda(A + BF_1) = \{-1, -1, -1, -0.001, -0.001\}$$

and

$$\lambda(A + BF_2) = \{-1, -1, -1, -0.1, -0.1\}.$$

The gain F_1 induces a closed-loop eigenvalue at -0.001 in order to cancel the stable invariant zero at -0.001, and it also assigns as a closed-loop eigenvalue the mirror image of the unstable invariant zero at 0.001. Consequently, the closed-loop system with F_1 as a state feedback gain, inherits a double eigenvalue at -0.001. This double eigenvalue at -0.001, for obvious reasons, is not practically acceptable. On the other hand, F_2 does not induce closed-loop eigenvalues close to the imaginary axis, and as such is perhaps practically acceptable. Furthermore, the optimal closed-loop transfer function is given by

$$T_{zw}^*(s) = \begin{bmatrix} 0 & 0 \\ 0 & -2/(s+1) \end{bmatrix}.$$

It is simple to verify that $\|T_{zw}^*\|_2 = \sqrt{2}$.   □

As formalized in Theorem 4.1, the algorithm *OGFM* constructs the set of all static state feedback controllers F_s^*. An important question that arises next is under what conditions F_s^* is a singleton. The following proposition gives these conditions.

Proposition 4.1 (Uniqueness of a Static State Feedback Solution): Consider the given system Σ as in (2.1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\text{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Then, an H_2 optimal static state feedback law is unique if and only if Σ_* satisfies the conditions of regular case as in Definition 2.1 and the pair $(A - BD_P^{-1}C_P, E)$ is completely controllable. Moreover, under these conditions:

1) $F_s^* = \{-D_P^{-1}C_P\}$ which is a singleton;
2) $\Lambda_s^* = \{$stable invariant zeros of $\Sigma_*\}$;
3) $\Omega_s^* = \lambda(A - BD_P^{-1}C_P)$ which is the union of all the stable invariant zeros of Σ_*, all the mirror images of the unstable invariant zeros of Σ_*, and $\lambda(-A_{bb} - SC_b'C_b)$, where S is the unique positive definite solution of the ARE (4.17).

Proof: It is simple to see under the conditions given in the proposition that the matrices A_z and B_z as in (4.10) of algorithm *OGFM* are nonexistent. Thus the result is obvious from the construction procedure of *OGFM*.   ∎

Remark 4.2: An interesting case is when the problem is regular and when $\text{Im}(E) = \mathbb{R}^n$. In this case, since $(A - BD_P^{-1}C_P, E)$ is completely controllable, an H_2 optimal static state feedback solution is always unique. However, as Proposition 4.1 alludes, there are cases when a solution to the H_2 optimization problem is unique even if $\text{Im}(E) \neq \mathbb{R}^n$.

V. DESIGN OF OPTIMAL DYNAMIC STATE FEEDBACK CONTROLLERS

In this section, we characterize all the possible H_2 optimal dynamic state feedback control laws using the well-known Q-parameterization technique. From Lemma 3.1, it follows that there exists an H_2 optimal state feed-

back law for Σ if and only if there exists a state feedback law which when applied to Σ_{au} of (3.11) achieves disturbance decoupling. Then, in view of the necessary and sufficient conditions given in Theorem 3.2 under which the disturbance decoupling problem with internal stability (DDPS) of Σ_{au} is solvable, we know that whenever an optimal solution to the original system exists, there exits a constant gain F such that $A + BF$ is stable and that

$$(C_P + D_P F)(sI - A - BF)^{-1} E \equiv 0. \qquad (5.1)$$

Next, following the results of Rotea and Khargonekar [12], it can be shown easily that any proper dynamic controller $F(s)$ that stabilizes the system Σ_{au} (which is not necessarily regular) can be written in the following form,

$$\begin{cases} \dot{\xi} = (A + BF)\xi + By_1, \\ u = Fx_P + y_1, \end{cases} \qquad (5.2)$$

where

$$y_1 = Q(s)(x_P - \xi) \qquad (5.3)$$

for some proper and stable $Q(s)$, i.e., $Q(s) \in \text{RH}^\infty$, with appropriate dimensions. The following theorem qualifies $Q(s)$ so that the controller $F(s)$ is H_2 optimal for the given system Σ.

Theorem 5.1: Consider the given system Σ as in (2.1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\text{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Define a set Q as,

$$Q := \{Q(s) \in \text{RH}^\infty | Q(s) = W(s)(I - EE^+)(sI - A - BF),$$
$$W(s) \in \text{RH}^2 \}, \qquad (5.4)$$

where E^+ is the generalized inverse of E, i.e., $EE^+E = E$. Then a proper controller $F(s)$ stabilizes Σ and achieves the infimum, γ^*, i.e., $F(s) \in F_d^*$, if and only if $F(s)$ can be written in the form of (5.2) and (5.3) for some $Q(s) \in Q$.

Proof: In view of Lemma 3.1, it is sufficient to prove Theorem 5.1 by showing that $F(s)$ when applied to Σ_{au} achieves disturbance decoupling with internal stability if and only if it can be written in the form of (5.2) and (5.3) for some $Q(s) \in Q$. Let a matrix quadruple (A_q, B_q, C_q, D_q) correspond to a state space realization of $Q(s)$. After some simple algebraic manipulations, it follows then that the controller (5.2) and (5.3) when applied to Σ_{au} yields the closed-loop transfer function from w_P to z_P as,

$$T_{z_P w_P}(F) = C_c(sI - A_c)^{-1} B_c, \qquad (5.5)$$

where

$$A_c = \begin{bmatrix} A + BF & BC_q & BD_q \\ 0 & A_q & B_q \\ 0 & 0 & A + BF \end{bmatrix}, \qquad B_e = \begin{bmatrix} E \\ 0 \\ E \end{bmatrix}, \qquad (5.6)$$

and

$$C_c = [C_P + D_P F \quad D_P C_q \quad D_P D_q]. \qquad (5.7)$$

Then it is simple to verify that

$$T_{z_p w_p}(F) = (C_P + D_P F)(sI - A - BF)^{-1} E$$

$$+ \left[(C_P + D_P F)(sI - A - BF)^{-1} B + D_P \right]$$

$$\cdot Q(s)(sI - A - BF)^{-1} E$$

$$= \left[(C_P + D_P F)(sI - A - BF)^{-1} B + D_P \right]$$

$$\cdot Q(s)(sI - A - BF)^{-1} E. \tag{5.8}$$

By the assumption that Σ_* is left invertible, it follows from Lemma A.1 that Σ_P is invertible and so does $(C_P + D_P F)(sI - A - BF)^{-1} B + D_P$. Hence, it is simple to see that $T_{z_p w_p}(F) = 0$ if and only if

$$Q(s)(sI - A - BF)^{-1} E = 0. \tag{5.9}$$

We next show that (5.9) is equivalent to $Q(s) \in \mathbf{Q}$. We note that if $Q(s) \in \mathbf{Q}$, it is trivial to see that $Q(s)(sI - A - BF)^{-1} E = W(s)(I - EE^\dagger)E = 0$. Hence, (5.9) holds.

Conversely, if (5.9) holds, we let $W(s) = Q(s)(sI - A - BF)^{-1} \in \mathbf{RH}^2$. Then we have $W(s)(I - EE^\dagger) = Q(s)(sI - A - BF)^{-1}$, which implies that $Q(s) = W(s)(I - EE^\dagger)(sI - A - BF) \in \mathbf{Q}$. This completes the proof of Theorem 5.1. ∎

We have the following result regarding the uniqueness of a dynamic state feedback solution.

Proposition 5.1 (Uniqueness of a Dynamic State Feedback Solution): Consider the given system Σ as in (2.1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\mathrm{Im}(E) \subseteq \mathscr{V}_g(\Sigma_P)$. Then, an H_2 optimal dynamic state feedback law is unique if and only if Σ_* satisfies the conditions of regular case as in Definition 2.1 and $\mathrm{Im}(E) = \mathbb{R}^n$. Moreover, under these conditions:

1) $F_d^* = \{ -D_P^{-1} C_P \}$ which is a singleton;
2) $\Lambda_s^* = \{$stable invariant zeros of $\Sigma_*\}$;
3) $\Omega_s^* = \lambda(A - BD_P^{-1} C_P)$ which is the union of all the stable invariant zeros of Σ_*, all the mirror images of the unstable invariant zeros of Σ_*, and $\lambda(-A_{bb} - SC_b'C_b)$, where S is the unique positive definite solution of the ARE (4.17).

Proof: The fact that F_d^* is a singleton implies that $Q(s) \equiv 0$. By (5.4), we have $\mathrm{Im}(E) = \mathbb{R}^n$. It is then simple to show that this and the condition $\mathrm{Im}(E) \subseteq \mathscr{V}_g(\Sigma_P)$, imply that Σ_* satisfies the conditions of regular case. Next, the converse part is obvious. Also, the remaining results follow directly from Proposition 4.1. ∎

The following remarks are in order.

Remark 5.1:

1) It is simple to verify that, in connection with the H_2 optimal dynamic state feedback control laws, our results are equivalent to those of Rotea and Khargonekar [12] if Σ is regular.

2) It is interesting to note that the condition under which an optimal dynamic state feedback controller is unique, is stronger than the condition for which a static optimal control law is unique (see Proposition 4.1).

Remark 5.2: It is easy to show that an H_2 optimal closed-loop transfer function from z to w, denoted here as $T_{zw}^*(s)$, is unique whatever may be the type of optimal controller used, i.e., whether the controller is an element of F_s^* or F_d^*. Moreover, $T_{zw}^*(s)$ is given by

$$T_{zw}^*(s) = C^*(sI - A^*)^{-1} B^*, \tag{5.10}$$

where

$$A^* = A_x - B_x D_x'(D_x' D_x)^{-1}(B_x' P_x + D_x' C_x)$$

$$B^* = E_x, \qquad C^* = C_x - D_x D_x'(D_x' D_x)^{-1}(B_x' P_x + D_x' C_x),$$

and where A_x, B_x, C_x, D_x, E_x, and P_x are as defined in Step 1 of *OGFM* algorithm. It is simple to verify that the set of poles of the irreducible transfer function $T_{zw}^*(s)$ is equal to Ω_s^*/Λ_s^*, i.e., the optimal fixed modes that are not the optimal fixed decoupling zeros.

Next, the following theorem shows that the optimal fixed modes and the fixed decoupling zeros remain unchanged regardless of what type of controller is used.

Theorem 5.2: Consider the given system Σ as in (2.1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\mathrm{Im}(E) \subseteq \mathscr{V}_g(\Sigma_P)$. Then we have i) $\Omega_d^* = \Omega_s^*$, and ii) $\Lambda_d^* = \Lambda_s^*$.

Proof: See Appendix A. ∎

VI. CONCLUSIONS

An H_2 optimization problem in a general setting while using either a static or a dynamic state feedback controller is considered. All the static and dynamic H_2 optimal state feedback solutions are explicitly constructed and parameterized. Moreover, the necessary and sufficient conditions for the uniqueness of an H_2 optimal solution for both the cases of static and dynamic state feedback controllers, are established. Also, for a general problem, it turned out that all the optimal controllers must include certain fixed modes among the closed-loop eigenvalues, and moreover, must inherently involve certain pole-zero cancellations. The set of *optimal fixed modes* which is a set of complex numbers that every optimal state feedback controller must assign among the closed-loop eigenvalues, is identified and constructed. Similarly, the set of *optimal fixed decoupling zeros* which shows the minimum absolutely necessary number and locations of pole-zero cancellations present in any H_2 optimal solution, is identified and constructed. It is also shown that the sets of *optimal fixed modes* and *optimal fixed decoupling zeros* do not vary depending upon whether the static or the dynamic controllers are used. Also, although an H_2 optimal control law need not be unique, the resulting H_2 optimal closed-loop transfer function is shown to be unique.

A constructional algorithm called *OGFM* plays a central role in our development. Given a matrix quintuple which characterizes the given optimal control problem, the algorithm *OGFM* constructs, a) the set of all optimal static state feedback gains, b) the set of *optimal fixed modes*, c) the set of *optimal fixed decoupling zeros*, and d) the infimum, γ^*. Besides these, *OGFM* also checks whether the geometric subspace inclusion condition,

$\text{Im}(E) \subseteq \mathscr{V}_g(\Sigma_P)$, is satisfied by the given problem or not. All these tasks of *OGFM* require the, computation of the maximal solution P of the inequality $F(P) \geq 0$, construction of Σ_P and its special coordinate basis (SCB), as well as construction of $\mathscr{V}_g(\Sigma_P)$. A key step of *OGFM* is the isolation of a matrix pair (A_z, B_z) from the given input data. The gain F_z which renders $A_z - B_z F_z$ asymptotically stable, acts as a parameter by appropriately varying which one obtains the entire set of static feedback gains. On the other hand, for the case of dynamic state feedback controllers, the well known Q-parameterization technique is instrumental in characterizing all the possible H_2 optimal solutions.

APPENDIX
A. PROOFS OF THEOREMS 4.1 AND 5.2

In this appendix, we prove certain facts mentioned in the algorithm *OGFM* and Theorems 4.1 and 5.2. The concepts behind the algorithm *OGFM* are rooted in Lemma 3.1 which enables the study of an H_2 optimal controller problem for the given system Σ as a study of a DDPS problem for the new system Σ_{au}. This implies that one can construct all the H_2 optimal controllers by constructing all the solutions of a DDPS problem for Σ_{au}. To do so, one constructs first Σ_P and then Σ_{au}. By studying Σ_{au}, one recognizes that a certain subsystem Σ_z, namely the one characterized by the pair (A_z, B_z), is not affected by the disturbance at all and as such can be separated from the rest of Σ_{au}. Thus, letting F_z as the gain which stabilizes Σ_z, and taking into account the interconnections between Σ_z and the rest of Σ_{au}, one can easily parameterize the set of all static state feedback controllers F_s^* that solves the DDPS for Σ_{au}.

Before proceeding to the proof of Theorem 4.1, we first introduce the following lemma which details the properties of the subsystem Σ_P of Σ_{au}.

Lemma A.1: Consider the given system Σ as in (2.1). Assume that (A, B) is stabilizable. Then the subsystem Σ_P characterized by (A, B, C_P, D_P) with C_P and D_P as in (3.12), has the following properties:
1) Σ_P is invertible.
2) Σ_P has the same infinite zero structure as Σ_*,
3) Σ_P has a total number of $n - n_f$ invariant zeros, where n_f is the number of infinite zeros of Σ_*. The invariant zeros of Σ_P are given by:
   a) the stable invariant zeros of Σ_*;
   b) the $j\omega$ axis invariant zeros of Σ_*;
   c) the mirror images of the unstable invariant zeros of Σ_*; and
   d) $\lambda(-A_{bb} - SC_b'C_b)$, where $S > 0$ is the unique solution of the ARE (4.17).

Proof: See [17]. Also, see [3] and [7] for the dual version of the above lemma. $\square$

We next introduce the following lemma.

Lemma A.2: Consider a left invertible system Σ_* characterized by (A, B, C, D). Then all the unobservable modes of $(A + BF, C + DF)$ for any appropriate dimensional F, are the invariant zeros of Σ_*.

Proof: Let λ be an unobservable mode of $(A + BF, C + DF)$ and let $v \neq 0$ be the corresponding right eigenvector of $A + BF$, i.e., $(\lambda I - A - BF)v = 0$ and $(C + DF)v = 0$. Then it is simple to verify that

$$\begin{bmatrix} v \\ Fv \end{bmatrix} \neq 0 \quad \text{and} \quad \begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} v \\ Fv \end{bmatrix} = 0.$$

Hence, by definition, λ is an invariant zero of Σ_* as Σ_* is assumed to be left invertible. $\square$

Now we are ready to prove the theorems.

A.1. Proof of Theorem 4.1

Part 1: It is straightforward to verify by some simple calculations that for any F as in (4.13), we have $(C_P + D_P F)(sI - A - BF)^{-1}E \equiv 0$. Hence, it follows from Lemma 3.1 that the control law $u = Fx$ with F as in (4.13) attains the infimum γ^*.

Conversely, if a state feedback $u = Fx$ achieves the infimum, it follows from Lemma 3.1 that F is such that $A + BF$ is stable and $(C_P + D_P F)(sI - A - BF)^{-1}E \equiv 0$. Without loss of generality but for simplicity of presentation, we assume that Σ_P is in the form of SCB with A_{aaP}^- partitioned in Step 1(d) of *OGFM*. Let us define $\mathscr{V}_s := \langle A + BF \mid \text{Im}(E) \rangle$, i.e., $\mathscr{V}_s$ is the smallest $(A + BF)$-invariant subspace containing $\text{Im}(E)$. Thus, $\mathscr{V}_s \subseteq \text{Ker}(C_P + D_P F)$ and by definition $\mathscr{V}_s \subseteq \mathscr{V}_g(\Sigma_P)$, which is given by

$$\mathscr{V}_g(\Sigma_P) = \text{span} \left\{ \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Hence, there exists a similarity transformation T such that

$$T^{-1}(A + BF)T = \begin{bmatrix} A_{Pcc} & A_{Pc\bar{c}} \\ 0 & A_{P\bar{c}\bar{c}} \end{bmatrix}, \quad T^{-1}E = \begin{bmatrix} E_{Pc} \\ 0 \end{bmatrix},$$
$$\tag{A.1}$$

and

$$(C_P + D_P F)T = \begin{bmatrix} 0 & C_{P\bar{c}} \end{bmatrix}, \quad \mathscr{V}_s = \text{span} \left\{ T \begin{bmatrix} I \\ 0 \end{bmatrix} \right\},$$
$$\tag{A.2}$$

where (A_{Pcc}, E_{Pc}) is completely controllable. It is now straightforward to verify that T can be chosen as the following form,

$$T = \begin{bmatrix} T_* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{A.3}$$

where T_* is of dimension $\dim \mathscr{V}_g(\Sigma_P) \times \dim \mathscr{V}_g(\Sigma_P)$. Let

$$F = -\begin{bmatrix} C_{0P} \\ 0 \end{bmatrix} - \begin{bmatrix} F_{a0P}^1 & F_{a0P}^2 & F_{a0P}^0 & F_{f0P} \\ F_{afP}^1 & F_{afP}^2 & F_{afP}^0 & F_{ffP} \end{bmatrix}. \tag{A.4}$$

We note that (A.1)–(A.4) imply that

$$T_*^{-1} \begin{bmatrix} A_{aaP}^{11} - B_{a0P}^1 F_{a0P}^1 & A_{aaP}^{12} - B_{a0P}^1 F_{a0P}^2 \\ -B_{a0P}^2 F_{a0P}^1 & A_{aaP}^{22} - B_{a0P}^2 F_{a0P}^2 \end{bmatrix} T_*$$

$$= \begin{bmatrix} A_{Pcc} & A_{Pc\bar{c}}^* \\ 0 & A_{P\bar{c}\bar{c}}^* \end{bmatrix}, \quad \text{(A.5)}$$

$$T_*^{-1} \begin{bmatrix} E_{aP}^1 \\ 0 \end{bmatrix} = \begin{bmatrix} E_{Pc} \\ 0 \end{bmatrix}, \quad \text{(A.6)}$$

and

$$\begin{bmatrix} F_{a0P}^1 & F_{a0P}^2 \\ B_{fP}(E_{faP}^1 - F_{afP}^1) & B_{fP}(E_{faP}^2 - F_{afP}^2) \end{bmatrix} T_* = [0 \quad \star], \quad \text{(A.7)}$$

where again $\star$ denotes a matrix of not much interest. Here we note that (A.5)–(A.7) imply that the system characterized by the matrix triple,

$$\left(\begin{bmatrix} A_{aaP}^{11} & A_{aaP}^{12} \\ 0 & A_{aaP}^{22} \end{bmatrix}, \begin{bmatrix} E_{aP}^1 \\ 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} F_{a0P}^1 & F_{a0P}^2 \\ B_{fP}(E_{faP}^1 - F_{afP}^1) & B_{fP}(E_{faP}^2 - F_{afP}^2) \end{bmatrix} \right), \quad \text{(A.8)}$$

does not have any infinite zeros. Then the controllability of the pair (A_{aaP}^{11}, E_{aP}^1) implies that $F_{a0P}^1 \equiv 0$. Noting that B_{fP} is injective, it follows that $F_{afP}^1 = E_{faP}^1$ and thus F must be of the form given by (4.13). □

Part 2: It follows simply from the construction algorithm *OGFM*. □

Part 3: In view of the algorithm *OGFM*, it is trivial to verify that for any $F \in F_s^*$, the set of input decoupling zeros of the corresponding closed-loop system come from $\lambda(A_z^c)$, where $A_z^c := A_z - B_z F_z$. Also, it is obvious that only the uncontrollable modes, or the input decoupling zeros, of (A_z, B_z) are in the set Λ_s^*, while the rest of the eigenvalues of A_z^c are not in the set Λ_s^* as they can be assigned arbitrarily in $\mathbb{C}^-$ by appropriately selecting the static state feedback controller F from the set F_s^*. On the other hand, it follows from the construction algorithm *OGFM* and the properties of SCB that the stable invariant zeros of Σ_* contained in $\lambda(A_{aaP}^{11})$, i.e., $\lambda(A_{aaP}^{11}) \cap \lambda(A_{aa}^-)$, are the output decoupling zeros of the closed-loop system for any $F \in F_s^*$. Also, in view of Lemma A.2, we know that the rest elements of $\lambda(A_{aaP}^{11})$ cannot be the output decoupling zeros of the closed-loop system for any $F \in F_s^*$. Moreover, we note that $\lambda(A_{aaP}^{11})$ cannot be the input decoupling zeros either for any $F \in F_s^*$ as (A_{aaP}^{11}, E_{aP}^1) is controllable. Hence, the result follows. □

Part 4: It is trivial (see also the proof of Part 3). ∎

A.2. Proof of Theorem 5.2

By the definitions, it is trivial to see that $\Omega_d^* \subseteq \Omega_s^*$ and $\Lambda_d^* \subseteq \Lambda_s^*$.

Conversely, let us consider any given optimal dynamic controller with the state space realization,

$$F(s): \begin{cases} v = A_{cmp} v + B_{cmp} x_P, \\ u = C_{cmp} v + D_{cmp} x_P, \end{cases} \quad \text{(A.9)}$$

i.e., $F(s)$ is such that $T_{z_P} w(F) = 0$. Then it is simple to verify that

$$u = F_d x_d = \begin{bmatrix} C_{cmp} & D_{cmp} \end{bmatrix} x_d \quad \text{(A.10)}$$

is an optimal static state feedback law for the following auxiliary system,

$$\Sigma_{aud}: \begin{cases} x_d = A_d x_d + B_d u + E_d w, \\ z_P = C_d x_d + D_d u, \end{cases} \quad \text{(A.11)}$$

where

$$x_d = \begin{bmatrix} v \\ x_P \end{bmatrix}, \quad A_d = \begin{bmatrix} A_{cmp} & B_{cmp} \\ 0 & A \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad E_d = \begin{bmatrix} 0 \\ E \end{bmatrix},$$

and

$$C_d = \begin{bmatrix} 0 & C_P \end{bmatrix}, \quad D_d = D_P.$$

We first observe that the input decoupling zeros of the pair (A, B) are also among the input decoupling zeros of (A_d, B_d). Next, without loss of generality, we again assume that the matrix quadruple (A, B, C_P, D_P) is in the form of SCB with A_{aaP}^- partitioned as in Step 1(d) of *OGFM*, i.e., we have

$$A_d - \begin{bmatrix} 0 \\ B_{0P} \end{bmatrix} \begin{bmatrix} 0 & C_{0P} \end{bmatrix}$$

$$= \begin{bmatrix} A_{cmp} & B_{cmp}^1 & B_{cmp}^2 & B_{cmp}^0 & B_{cmp}^f \\ 0 & A_{aaP}^{11} & A_{aaP}^{12} & 0 & L_{afP}^1 C_{fP} \\ 0 & 0 & A_{aaP}^{22} & 0 & L_{afP}^2 C_{fP} \\ 0 & 0 & 0 & A_{aaP}^0 & L_{afP}^0 C_{fP} \\ 0 & B_{fP} E_{fP}^1 & B_{fP} E_{fP}^2 & B_{fP} E_{fP}^0 & A_{ffP} \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 & 0 \\ B_{a0P}^1 & 0 \\ B_{a0P}^2 & 0 \\ B_{a0P}^0 & 0 \\ B_{f0P} & B_{fP} \end{bmatrix}, \quad E_d = \begin{bmatrix} 0 \\ E_{aP}^1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 0 & C_{0aP}^1 & C_{0aP}^2 & C_{0aP}^0 & C_{0fP} \\ 0 & 0 & 0 & 0 & C_{fP} \end{bmatrix},$$

$$D_d = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then following the results of Sannuti and Saberi [14] (e.g., Appendix A.2) and some simple algebra, one can compute a nonsingular state transformation Γ_d such that

$$\Gamma_d^{-1}\left(A_d - \begin{bmatrix} 0 \\ B_{0P} \end{bmatrix}\begin{bmatrix} 0 & C_{0P} \end{bmatrix}\right)\Gamma_d = \begin{bmatrix} A_{cmp}^{a+} & 0 & 0 & 0 & \star & L_{afP}^{a+}C_{fP} \\ 0 & A_{cmp}^{a-} & \star & \star & 0 & L_{afP}^{a-}C_{fP} \\ 0 & 0 & A_{aaP}^{11} & A_{aaP}^{12} & 0 & L_{afP}^{1}C_{fP} \\ 0 & 0 & 0 & A_{aaP}^{22} & 0 & L_{afP}^{2}C_{fP} \\ 0 & 0 & 0 & 0 & A_{aaP}^{0} & L_{afP}^{0}C_{fP} \\ 0 & 0 & B_{fP}E_{faP}^{1} & B_{fP}E_{faP}^{2} & B_{fP}E_{faP}^{0} & A_{ffP}^{*} \end{bmatrix},$$

$$\Gamma_d^{-1}B_d = \begin{bmatrix} B_{a0P}^{a+} & 0 \\ B_{a0P}^{a-} & 0 \\ B_{a0P}^{1} & 0 \\ B_{a0P}^{2} & 0 \\ B_{a0P}^{0} & 0 \\ B_{fP}^{*} & B_{fP} \end{bmatrix}, \qquad \Gamma_d^{-1}E_d = \begin{bmatrix} 0 \\ 0 \\ E_{aP}^{1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$C_d\Gamma_d = \begin{bmatrix} C_{0aP}^{a+} & C_{0aP}^{a-} & C_{0aP}^{1} & C_{0aP}^{2} & C_{0aP}^{0} & C_{0fP}^{*} \\ 0 & 0 & 0 & 0 & 0 & C_{fP} \end{bmatrix},$$

where $\lambda(A_{cmp}^{a+}) \subset \mathbb{C}^+ \cup \mathbb{C}^0$ and $\lambda(A_{cmp}^{a-}) \subset \mathbb{C}^-$. Moreover, the transformed system is in the form of SCB. Then following the same line of reasoning as in Part 1 of Appendix A.1, it can be shown that $T_{z_pw}(F) = 0$ implies that

$$F_d\Gamma_d = \begin{bmatrix} C_{cmp} & D_{cmp} \end{bmatrix}\Gamma_d$$

$$= \begin{bmatrix} \star & \star & C_{0aP}^{1} & \star & \star & \star \\ \star & \star & E_{faP}^{1} & \star & \star & \star \end{bmatrix}$$

and hence $\Omega_s^* \subseteq \lambda(A_d + B_dF_d)$. Similarly, following the same arguments as in Part 3 of Appendix A.1, one can show that the elements of Λ_s^* are among the decoupling zeros of the closed-loop system. Since $F(s)$ can be any member of F_d^*, it follows that $\Omega_s^* \subseteq \Omega_d^*$ and $\Lambda_s^* \subseteq \Lambda_d^*$. This completes the proof of Theorem 5.2. ∎

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