$$= A + \gamma^{-2}B_1B_1^T(X_\infty + Y) + LC_2 - \gamma^{-2}\beta^2 LC_2P(I + \gamma^{-2}\beta^2 YP)^{-1}Y$$
$$= A + \gamma^{-2}B_1B_1^T(X_\infty + Y) + LC_2 + \gamma^{-2}\beta^2 LR_0L^TY \tag{33}$$

which is asymptotically stable because $Y \geq 0$ is a stabilizing solution of (20). Therefore P is a stabilizing solution of (21). The fact that $P \geq 0$ follows from $R_0 > 0$. This completes the proof of Corollary 1.2.

The following theorem shows that Bernstein and Haddad's necessary condition for full-order mixed H_2 and H_∞ control, as given in [2, Theorem 3.1], *is also sufficient*.

Theorem 2: Given $\gamma > 0$ and plant $F(s)$ as described in Fig. 1, with $E_{2\infty} = \beta E_2$ ($R_{2\infty} = \beta^2 R_2$) for some real β. The full-order controller $H(s)$ maintains the internal stability of the closed-loop system and minimizes the cost function J_B of (1) subject to the H_∞ norm constraint (6), *if and only if* there exist stabilizing solutions $Q \geq 0$, $P \geq 0$, and $\hat{Q} \geq 0$ such that

$$0 = AQ + QA^T + V_1 + Q[\gamma^{-2}R_{1\infty} - C^TV_2^{-1}C]Q \tag{34}$$

$$0 = (A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty}) + E_1^TE_1 - C_C^TR_2C_c \tag{35}$$

$$0 = (A + \gamma^{-2}QR_{1\infty} + BC_c)\hat{Q} + \hat{Q}(A + \gamma^{-2}QR_{1\infty} + BC_c)^T + \gamma^{-2}\hat{Q}[R_{1\infty} + \beta^2 C_c^TR_2C_c]\hat{Q} + QC^TV_2^{-1}CQ \tag{36}$$

and the controller satisfies

$$H(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{37}$$

with

$$A_c = A - B_cC + BC_c + \gamma^{-2}QR_{1\infty} \tag{38}$$

$$B_c = QC^TV_2^{-1} \tag{39}$$

$$C_c = -R_2^{-1}B^TP(I + \beta^2\gamma^{-2}\hat{Q}P)^{-1} \tag{40}$$

and the minimal performance index J_B of (1) may be computed with

$$\tilde{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q} \\ \hat{Q} & \hat{Q} \end{bmatrix}. \tag{41}$$

Proof: Substituting $R_{2\infty} = \beta^2 R_2$ into (12) gives (40). Q and $\hat{Q}$ being stabilizing solutions is concluded from Lemma 2. Paraphrasing the proof of Corollary 1.2 proves that P is a stabilizing solution. The necessity has been proved by Bernstein and Haddad [2], and also follows from Lemma 1. The sufficiency follows from Corollary 1.1. This completes the proof of Theorem 2.

Remark: It has been proved by Bernstein and Haddad [2] that $\hat{Q}$ in the necessary condition of Theorem 2 is positive definite. The same can be concluded for Lemma 1. Since in the scope of this note, semidefiniteness can be tightened to definiteness in sufficient conditions and Y is the dual of $\hat{Q}$, all $Y \geq 0$ may be replaced by $Y > 0$ and $\hat{Q} \geq 0$ may be replaced by $\hat{Q} > 0$ in every lemma, theorem, and corollary.

III. Conclusions

The sufficient condition for Doyle, Zhou, and Bodenheimer's mixed H_2 and H_∞ full-order optimal control is shown to be the dual of the necessary condition for Bernstein and Haddad's. Therefore, both conditions are proved to be necessary and sufficient.

References

[1] M. A. Rotea and P. P. Khargonekar, "H^2-optimal control with an H^∞-constraint, the State feedback case," in *Proc. 1990 Amer. Contr. Conf.*, San Diego, CA, May 23–25, 1990.

[2] D. S. Bernstein and W. M. Haddad, "LQG Control with H^∞ performance bound: A Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. 34, no. 3, p. 293, 1989.

[3] J. C. Doyle, K. Zhou, and B. Bodenheimer, "Optimal control with mixed H_2 and H^∞ performance objectives," in *Proc. 1989 Amer. Contr. Conf.*, Pittsburgh, PA, 1989, p. 2065.

[4] D. Mustafa and K. Glover, "Controllers which satisfy H_∞-norm bound and maximize an entropy integral," in *Proc. 1988 Conf. Decision Contr.*, Austin, TX, 1988, p. 959.

[5] H.-H. Yeh, S. S. Banda, S. A. Heise, and A. C. Bartlett, "Robust control design with real-parameter uncertainty and unmodelled dynamics," *J. Guidance Contr. Dynam.*, to be published.

Explicit Expressions for Cascade Factorization of General Nonminimum Phase Systems

Ben M. Chen, Ali Saberi, and P. Sannuti

***Abstract*—In this note explicit expressions for two different cascade factorizations of any detectable left invertible nonminimum phase system are given. The first one is a well known minimum phase/all-pass factorization by which all nonminimum phase zeros of a transfer function $G(s)$ are collected into an all-pass factor $V(s)$, and $G(s)$ is written as $G_m(s)V(s)$ where $G_m(s)$ is considered as a minimum phase image of $G(s)$. The second one is a new cascade factorization by which $G(s)$ is rewritten as $G_M(s)U(s)$ where $U(s)$ collects all "awkward" zeros including all nonminimum phase zeros of $G(s)$. Both $G_m(s)$ and $G_M(s)$ retain the given infinite zero structure of $G(s)$. Further properties of $G_m(s)$, $G_M(s)$, $V(s)$, and $U(s)$ are discussed. These factorizations are useful in several applications including loop transfer recovery.**

I. Introduction

Cascade factorizations of nonminimum phase systems have been used extensively in the literature. The so called minimum phase/all-pass factorization plays a significant role in several applications, the prominent among them being singular filtering [2], [9] and cheap and singular optimal LQ control [10]. Recently, it played a significant role in loop transfer recovery as well [15]. Traditionally such a factorization has been found by spectral factorization techniques [11], [13], [14]. The role that the minimum phase/all-pass factorization plays in the control literature as well as various methods available for such a factorization are well documented by Shaked [8]. Also, Shaked gives a new method of obtaining such a factorization for the controllable, observable, and left invertible systems. Shaked's method yields valid results when all the invariant

Manuscript received April 27, 1990; revised November 9, 1990. The work of B. M. Chen and A. Saberi was supported in part by the National Science Foundation under Grant ECS 8618953 and by the Boeing Commercial Airplane Group.

B. M. Chen and A. Saberi are with the School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 91164-2752.

P. Sannuti is with the Department of Electrical and Computer Engineering, Rutgers University, Piscataway, NJ 08855-0909.

IEEE Log Number 9104404.

zeros of the given system are simple. It has some minor problems when the invariant zeros are not simple due to an erroneous definition of the pseudo zero directions [4], [12] as explained in [6]. However, such difficulties can be possibly resolved by using appropriate definition of pseudo zero directions.

This note provides explicit expressions for two different cascade factorizations of any nonminimum phase system. The first one is same as the traditional minimum phase/all-pass factorization. The second one, a natural extension of the first, is a new cascade factorization which seems to have several promising applications. To be more specific, let us consider a detectable left invertible nonminimum phase system Σ characterized by the matrix triple (A, B, C)

$$\dot{x} = Ax + Bu, \; y = Cx \tag{1.1}$$

where the state vector $x \in \mathbb{R}^n$, output vector $y \in \mathbb{R}^p$, and input vector $u \in \mathbb{R}^m$. Without loss of generality, assume that B and C are of maximal rank. Let the transfer function of Σ be $G(s)$. Then the minimum phase/all-pass factorization, $G(s)$ is expressed as

$$G(s) = G_m(s)V(s) \tag{1.2}$$

such that

$$G(s)G^H(s) = G_m(s)G_m^H(s)$$

and $V(s)$ in an all-pass factor satisfying

$$V(s)V^H(s) = I. \tag{1.3}$$

Here $(\cdot)^H$ denotes the Hermitian paraconjugate of $(\cdot)$. We construct later on a matrix B_m such that a system Σ_m characterized by the matrix triple (A, B_m, C) has the intended transfer function $G_m(s)$. Also, the invariant zeros of Σ_m are those minimum phase invariant zeros of Σ and the mirror images of nonminimum phase invariant zeros of Σ. On the other hand, in loop transfer recovery and perhaps in other applications, one does not necessarily require a true minimum phase image of Σ. What is required is a model which retains the infinite zero structure of Σ and whose invariant zeros can appropriately be assigned to some desired locations in the open left-half s-plane. With this point in view, we develop here a new factorization of the form

$$G(s) = G_M(s)U(s). \tag{1.4}$$

Here $G_M(s)$ is the transfer function matrix of a system Σ_M characterized by the matrix triple (A, B_M, C). An explicit expression for B_M is given later on. Furthermore, Σ_M has the same infinite zero structure as that of Σ, it is of minimum phase having all its invariant zeros at desired locations and is left invertible. On the other hand, $U(s)$ is square, stable, invertible and is asymptotically all-pass in the sense that

$$U(s)U^H(s) \to I \qquad \text{as } |s| \to \infty. \tag{1.5}$$

We emphasize in the following, some important attributes of the method of factorization that is to be presented.

1) The method assumes only detectability and left invertibility of Σ, i.e., Σ need not be controllable and observable.

2) Guided by one's application, one can either seek one of the two cascade factorizations, a traditional minimum phase/all-pass factorization (1.2), and a new general cascade factorization (1.4). In (1.2), $G_m(s)$ has a particular invariant zero structure dictated by the given system Σ while $V(s)$ is an all-pass transfer function matrix. On the other hand, (1.4) provides flexibility to have a chosen invariant zero structure for $G_M(s)$ but $U(s)$, although square, invertible, and stable, is only asymptotically all-pass.

3) Both factorizations given here retain the infinite zero structure of Σ. This is crucial in several applications.

The note is organized as follows. Sections II and III, respectively, give explicit methods of constructing the traditional minimum phase/all-pass factorization and the new general cascade factorization. Section IV draws conclusions of our work. Throughout this note, A' denotes the transpose of A, I denotes an identity matrix with approapriate dimension. Similarly, $\lambda(A)$ denotes the set of eigenvalues of A. The open left and right s-planes are, respectively, denoted by $\mathscr{C}^-$ and $\mathscr{C}^+$.

II. Minimum Phase/All-Pass Factorization

In this section, we give a simple and precise expression for the minimum phase image $G_m(s)$, and the all-pass factor $V(s)$ of Σ. We first transform the given system Σ into the form of a special coordinate basis (SCB) which displays both the finite and infinite zero structures of Σ explicitly ([7], also see the details of SCB in the appendix A). That is, there exist nonsingular transformations Γ_1, Γ_2, and Γ_3 such that

$$x = \Gamma_1 \tilde{x}, \; y = \Gamma_2 \tilde{y}, \; u = \Gamma_3 \tilde{u}$$

and

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \; \tilde{y} = \tilde{C}\tilde{x}$$

where $(\tilde{A}, \tilde{B}, \tilde{C})$ are of the form

$$\tilde{A} = \Gamma_1^{-1} A \Gamma_1 = \begin{bmatrix} A_{aa}^+ & 0 & L_{ab}^+ C_b & L_{af}^+ C_f \\ 0 & A_{aa}^- & L_{ab}^- C_b & L_{af}^- C_f \\ 0 & 0 & A_{bb} & L_{bf} C_f \\ B_f E_a^+ & B_f E_a^- & B_f E_b & A_{ff} \end{bmatrix},$$

$$\tilde{B} = \Gamma_1^{-1} B \Gamma_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_f \end{bmatrix},$$

$$\tilde{C} = \Gamma_2^{-1} C \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 \end{bmatrix}, \text{ and } B_f' B_f = I. \tag{2.1}$$

Here $\lambda(A_{aa}^+)$ and $\lambda(A_{aa}^-)$ are, respectively, the nonminimum phase and minimum phase invariant zeros of Σ. Also, we note that the pair (A_{aa}^+, E_a^+) is observable whenever Σ is detectable. We have the following theorem.

Theorem 2.1: Consider a detectable, left invertible, and nonminimum phase system Σ whose nonminimum phase invariant zeros lie in the open right-half s-plane. Then

1) The minimum phase image of $\Sigma(A, B, C)$ is $\Sigma_m(A, B_m, C)$ having the transfer function $G_m(s) = C(sI - A)^{-1}B_m$, where

$$B_m = \Gamma_1 \tilde{B}_m \Gamma_3^{-1} = \Gamma_1 \begin{bmatrix} K_a^+ \\ 0 \\ 0 \\ B_f \end{bmatrix} \Gamma_3^{-1}, \tag{2.2}$$

$$K_a^+ = P^{-1} E_a^{+\prime} \Gamma_3' \Gamma_3 \tag{2.3}$$

and where P is the solution of the Lyapunov equation

$$A_{aa}^{+\prime} P + P A_{aa}^+ = E_a^{+\prime} \Gamma_3' \Gamma_3 E_a^+. \tag{2.4}$$

Moreover, $\Sigma_m(A, B_m, C)$ is also left invertible and has the same infinite zero structure as $\Sigma(A, B, C)$;

2) the stable all-pass factor of Σ is given as

$$V(s) = \Gamma_3 \left[I - E_a^+ (sI - A_{aa}^+ + K_a^+ E_a^+)^{-1} K_a^+ \right] \Gamma_3^{-1}. \tag{2.5}$$

Proof: See Appendix B.

We demonstrate the above results by the following example.

Example 2.1: Consider a square and invertible system Σ characterized by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and $C = B'$. As seen easily, Σ has a transfer function $G(s)$

$$G(s) = \frac{s-1}{s^5 - 5s^4 + 8s^3 - 5s^2 + s + 1} \cdot \begin{bmatrix} (s-2)(s^2 - s + 1) & (s-1)^2 \\ (s-1)^2 & s(s-1)(s-2) \end{bmatrix}.$$

Also, Σ is controllable and observable and has an invariant zero at $s = 1$. Furthermore, it is easy to verify that Σ is already in the SCB form with

$$A_{aa}^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_a^{+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus we obtain

$$K_a^{+} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 4 \end{bmatrix}, \quad B_m = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$V(s) = \begin{bmatrix} \dfrac{s-1}{s+1} & 0 \\ 0 & \dfrac{(s-1)^2}{(s+1)^2} \end{bmatrix},$$

and

$$G_m(s) = \frac{s+1}{s^5 - 5s^4 + 8s^3 - 5s^2 + s + 1} \cdot \begin{bmatrix} (s-2)(s^2 - s + 1) & (s-1)(s+1) \\ (s-1)^2 & s(s+1)(s-2) \end{bmatrix}.$$

III. A Generalized Cascade Factorization

Whenever some invariant zeros of Σ lie on the $j\omega$ axis, no minimum phase image of Σ can be obtained by any means. In what follows, we introduce a generalized cascade factorization of a given system Σ, which is a natural extension of the minimum phase/all-phase factorization discussed above. The given nonminimum phase and left invertible system is decomposed as

$$G(s) = G_M(s)U(s). \tag{3.1}$$

Here $G_M(s)$ is of minimum phase left invertible and has the same infinite zero structure as that of Σ while $U(s)$ is a square, invertible, and stable transfer function which is asymptotically all-pass. All the invariant zeros of $G_M(s)$ are in a desired set $\mathscr{C}_d \in \mathscr{C}^-$. If the given system Σ is only detectable but not observable, the set $\mathscr{C}_d$ includes all the unobservable but stable eigenvalues of Σ. In this way, all the awkward or unwanted invariant zeros of Σ (say, those in the right-half s-plane or close to $j\omega$ axis) need not be included in $G_M(s)$. Such a generalized cascade factorization has a major application in loop transfer recovery design. For instance, by applying the loop transfer recovery procedure to $G_M(s)$, one has the capability to shape the over all loop transfer recovery error over some frequency band or in some subspace of interest while placing the eigenvalues of the observer corresponding to some awkward invariant zeros of Σ at any desired locations [6].

Let us assume that the given system Σ has been transformed into the form of SCB as in (2.1). Let us also assume that in the SCB formulation, x_a is decomposed into x_a^- and x_a^+ such that the eigenvalues of A_{aa}^+ contain all the awkward invariant zeros of Σ. We have the following theorem.

Theorem 3.1: Consider a left invertible and nonminimum phase system Σ whose awkward invariant zeros are observable and are dumped in $\lambda(A_{aa}^+)$. One can then construct a generalized cascade factorization (3.1) such that

1) the minimum phase counterpart of $\Sigma(A, B, C)$ is $\Sigma_M(A, B_M, C)$ having the transfer function $G_M(s) = C(sI - A)^{-1}B_M$, where

$$B_M = \Gamma_1 \tilde{B}_M \Gamma_3^{-1} = \Gamma_1 \begin{bmatrix} K_a^+ \\ 0 \\ 0 \\ B_f \end{bmatrix} \Gamma_3^{-1} \tag{3.2}$$

and

$$G_M(s) = C(sI - A)^{-1} B_M. \tag{3.3}$$

Here K_a^+ is specified such that $\lambda(A_{aa}^+ - K_a^+ E_a^+)$ are in the desired locations in $\mathscr{C}^-$ with desired admissible eigenvectors [5]. Moreover, $\Sigma_M(A, B_M, C)$ is also left invertible and has the same infinite zero structure as $\Sigma(A, B, C)$;

2) the stable factor $U(s)$ is given as

$$U(s) = \Gamma_3 [I - E_a^+ (sI - A_{aa}^+ + K_a^+ E_a^+)^{-1} K_a^+] \Gamma_3^{-1}. \tag{3.4}$$

Moreover,

$$U^{-1}(s) = \Gamma_3 [I + E_a^+ (sI - A_{aa}^+)^{-1} K_a^+] \Gamma_3^{-1} \tag{3.5}$$

and $U(s)$ is asymptotically all-pass, i.e.,

$$U(s)U^H(s) \to I \qquad \text{as } |s| \to \infty.$$

Proof: See Appendix C.

We illustrate next this generalized factorization on an example.

Example 3.1: Consider a system Σ as given in [15] and characterized by

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 & -1.25 \\ -2.5 & -2.5 \\ 0.3 & 1.25 \\ 1.5 & 3.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

with

$$G(s) = \begin{bmatrix} \dfrac{-0.2(s-1)}{(s+1)(s+0.2)} & \dfrac{1}{(s+1)(s+0.2)} \\ \dfrac{-(s-1)}{(s+1)(s+0.2)} & \dfrac{(s+3)}{(s+1)(s+0.2)} \end{bmatrix}.$$

This system is square and invertible with two invariant zeros at $s = 1$ and $s = 2$. The minimum phase image and the all-pass factor of Σ are obtained as

$$B_m = \begin{bmatrix} 0.7353 & -0.8088 \\ 1.4706 & -1.6176 \\ -0.9353 & 0.8088 \\ -2.4706 & 2.6176 \end{bmatrix},$$

$$G_m(s) = \begin{bmatrix} \dfrac{-0.2(s+3.9412)}{(s+1)(s+0.2)} & \dfrac{0.6470}{(s+1)(s+0.2)} \\ \dfrac{-(s+2.1765)}{(s+1)(s+0.2)} & \dfrac{s+2.2941}{(s+1)(s+0.2)} \end{bmatrix}$$

and

$$V(s) = \begin{bmatrix} \dfrac{(s-1)(s-0.9414)}{(s+1)(s+2)} & \dfrac{-1.7646}{s+2} \\ \dfrac{-1.7646(s-1)}{(s+1)(s+2)} & \dfrac{s+0.9414}{s+2} \end{bmatrix}.$$

The following is a cascade factorization of Σ:

$$B_M = \begin{bmatrix} 0.5 & 0 \\ 3.75 & -3.75 \\ -0.7 & 0 \\ -4.75 & 4.75 \end{bmatrix},$$

$$G_M(s) = \begin{bmatrix} \dfrac{-0.2(s+3)}{(s+1)(s+0.2)} & 0 \\ \dfrac{-(s+4)}{(s+1)(s+0.2)} & \dfrac{s+4}{(s+1)(s+0.2)} \end{bmatrix},$$

and

$$U(s) = \begin{bmatrix} \dfrac{s-1}{s+3} & \dfrac{-5}{s+3} \\ \dfrac{s-1}{(s+3)(s+4)} & \dfrac{s^2+s-11}{(s+3)(s+4)} \end{bmatrix}.$$

It is simple to see that $G_M(s)$ has two invariant zeros at $s = -3$ and $s = -4$.

IV. Conclusion

Explicit and precise expressions for two different cascade factorizations of a detectable and left invertible nonminimum phase system having a transfer function matrix $G(s)$ are given. In a traditional minimum phase/all-pass factorization $G(s) = G_m(s)V(s)$. On the other hand, in a new cascade factorization, $G(s) = G_M(s)U(s)$. Both $G_m(s)$ and $G_M(s)$ retain the infinite zero structure of $G(s)$. The invariant zeros of $G_m(s)$ are those minimum phase invariant zeros of $G(s)$ and the mirror images of nonminimum phase invariant zeros of $G(s)$, where as the invariant zeros of $G_M(s)$ can be assigned as desired in $\mathscr{C}^-$. $V(s)$ is an all-pass transfer function matrix, where as $U(s)$, although square, invertible and stable, is only asymptotically all-pass.

Most of the existing solutions to the factorization problem have some kind of difficulties when the invariant zeros of the given systems are not distinct. Our solution, however, does not have such a problem. Moreover, the computations involved in our method are rather simple. For example, the required computations for the transformation matrices of the special coordinate basis can easily be done by either one of the software packages, one in L-A-S [1] and the other in Matlab [3].

Appendix A
A Special Coordinate Basis

We recall in this appendix a special coordinate basis (SCB) of a linear time-invariant system [7]. Such a SCB has a distinct feature of explicitly and precisely displaying the infinite as well as the finite zero structure (i.e., the invariant zeros and zero directions), of a given system. We summarize in the following the theorem and some properties of SCB while the detailed derivation and proof can be found in [7].

Theorem A.1 (SCB): Consider a system Σ as in (1.1). Then, there exist nonsingular transformations Γ_1, Γ_2, and Γ_3, an integer $m_u \le m$, and integer indexes q_i, $i = 1$ to m_u, such that

$$x = \Gamma_1 \tilde{x},\ y = \Gamma_2 \tilde{y},\ u = \Gamma_3 \tilde{u}$$

$$\tilde{x} = [x_a', x_b', x_c', x_f']',\ x_a = [(x_a^+)', (x_a^-)']'$$

$$x_f = [x_1', x_2', \cdots, x_{m_u}']'$$

$$\tilde{y} = [y_f', y_b']',\ y_f = [y_1, y_2, \cdots, y_{m_u}]'$$

$$\tilde{u} = [u_f', u_c']',\ u_f = [u_1, u_2, \cdots, u_{m_u}]'$$

$$\dot{x}_a^+ = A_{aa}^+ x_a^+ + L_{af}^+ y_f + L_{ab}^+ y_b \tag{A.1}$$

$$\dot{x}_a^- = A_{aa}^- x_a^- + L_{af}^- y_f + L_{ab}^- y_b \tag{A.2}$$

$$\dot{x}_b = A_{bb} x_b + L_{bf} y_f,\ y_b = C_b x_b \tag{A.3}$$

$$\dot{x}_c = A_{cc} x_c + L_{cf} y_f + L_{cb} y_b + B_c [E_{ca}^+ x_a^+ + E_{ca}^- x_a^- + u_c] \tag{A.4}$$

and for each $i = 1$ to m_u

$$\dot{x}_i = A_{q_i} x_i + L_i y_f + B_{q_i}\left[u_i + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c + \sum_{j=1}^{m_u} E_{ij} x_j\right] \tag{A.5}$$

$$y_i = C_{q_i} x_i,\ y_f = C_f x_f. \tag{A.6}$$

Here the states x_a^+, x_a^-, x_b, x_c, and x_f are, respectively, of dimension n_a^+, n_a^-, n_b, n_c, and $n_f = \sum_{i=1}^{m_u} q_i$ while x_i is of dimension q_i for each $i = 1$ to m_u. The control vectors u_f and u_c are, respectively, of dimension m_u and $m_c = m - m_u$ while the output vectors y_f and y_b are, respectively, of dimension $p_f = m_u$ and $p_b \le n_b$. The matrices A_{q_i}, B_{q_i}, and C_{q_i} have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i - 1} \\ 0 & 0 \end{bmatrix},\ B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_{q_i} = [1, 0, \cdots, 0]. \tag{A.7}$$

(Obviously, for the case when $q_i = 1$, $A_{q_i} = 0$, $B_{q_i} = 1$, and $C_{q_i} = 1$.) Furthermore, we have $\lambda(A_{aa}^+) \in \mathscr{C}^+$, $\lambda(A_{aa}^-) \in \mathscr{C}^-$, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable.

In what follows, we state some important properties of the SCB which are pertinent to our present note.

Property A.1: The given system Σ is right invertible iff x_b and hence y_b are nonexistent ($n_b = 0$, $p_b = 0$), left invertible iff x_c and hence u_c are nonexistent ($n_c = 0$, $m_c = 0$), invertible iff both x_b and x_c are nonexistent.

Property A.2: Minimum phase and nonminimum phase invariant zeros of Σ are, respectively, the eigenvalues of A_{aa}^- and A_{aa}^+.

Property A.3: Σ is detectable iff the pair (A_{aa}^+, E_a^+) is observable.

Property A.4: Let $\bar{q}_j$ be an integer such that $\bar{q}_j$ elements of q_i, $i = 1$ to m_u, are equal to j. Also, let k be an integer such that $\bar{q}_j = 0$ for all $j > k$. Then there are $j\bar{q}_j$ number of infinite zeros of order j, for $j = 1$ to k. Also, noting that

$$\sum_{j=1}^{k} j\bar{q}_j = \sum_{i=1}^{m_u} q_i = n_f$$

the total number of infinite zeros of all orders is n_f.

Appendix B
Proof of Theorem 2.1

We first note that since $-A_{aa}^{+}$ is stable and since the pair (A_{aa}^{+}, E_a^{+}) is observable, (2.4) has a unique, symmetric, and positive definite solution, i.e., $P = P' > 0$. Let us now show that $A_{aa}^{+} - K_a^{+}E_a^{+}$ is a stable matrix. By examining (2.3) and (2.4), we have

$$A_{aa}^{+} - K_a^{+}E_a^{+} = A_{aa}^{+} - P^{-1}E_a^{+\prime}\Gamma_3'\Gamma_3E_a^{+} = P^{-1}(-A_{aa}^{+\prime})P.$$

Hence, $A_{aa}^{+} - K_a^{+}E_a^{+}$ is indeed a stable matrix. Now we are ready to prove that Σ_m is of minimum phase, left invertible, and has the same infinite zero structure as Σ. Without loss of generality, we assume that Σ is in the form of SCB of Appendix A. Thus Σ_m can be rewritten as

$$\dot{x}_a^{+} = A_{aa}^{+}x_a^{+} + L_{af}^{+}y_f + L_{ab}^{+}y_b + K_a^{+}u$$
$$\dot{x}_a^{-} = A_{aa}^{-}x_a^{-} + L_{af}^{-}y_f + L_{ab}^{-}y_b$$
$$\dot{x}_b = A_{bb}x_b + L_{bf}y_f,\; y_b = C_bx_b$$
$$\dot{x}_f = A_fx_f + L_fy_f + B_f\left[u + E_a^{+}x_a^{+} + E_a^{-}x_a^{-} + E_bx_b + E_fx_f\right],\; y_f = C_fx_f.$$

Let us now define a new state variable

$$x_a^{m} = x_a^{+} - K_a^{+}B_f'x_f.$$

Since $B_f'B_f = I$, it is then straightforward to verify that

$$\dot{x}_a^{m} = (A_{aa}^{+} - K_a^{+}E_a^{+})x_a^{m} - K_a^{+}E_a^{-}x_a^{-} + L_{ab}^{+}y_b - K_a^{+}E_bx_b + (L_{af}^{+} - K_a^{+}B_f'L_f)y_f + (A_{aa}^{+}K_a^{+}B_f' - K_a^{+}E_f - K_a^{+}E_a^{+}K_a^{+}B_f' - K_a^{+}B_f'A_f)x_f$$

and

$$\dot{x}_f = A_fx_f + L_fy_f + B_f\left[u + E_a^{+}x_a^{m} + E_a^{-}x_a^{-} + E_bx_b + (E_f + K_a^{+}B_f')x_f\right].$$

Then it follows from the results of [7] that there exists a nonsingular transformation T such that

$$[x_a^{m\prime}, x_a^{-\prime}, x_b', x_f']' = T[\bar{x}_a^{m\prime}, x_a^{-\prime}, x_b', x_f']'$$

and

$$\dot{\bar{x}}_a^{m} = (A_{aa}^{+} - K_a^{+}E_a^{+})\bar{x}_a^{m} - K_a^{+}E_a^{+}x_a^{-} + L_{af}^{m}y_f + L_{ab}^{m}y_b$$
$$\dot{x}_a^{-} = A_{aa}^{-}x_a^{-} + L_{af}^{-}y_f + L_{ab}^{-}y_b$$
$$\dot{x}_b = A_{bb}x_b + L_{bf}y_f,\; y_b = C_bx_b$$
$$\dot{x}_f = A_fx_f + L_fy_f + B_f\left[u + E_a^{+}\bar{x}_a^{m} + E_a^{-}x_a^{-} + E_b^{m}x_b + E_f^{m}x_f\right],\; y_f = C_fx_f \tag{B.1}$$

for some appropriate dimensional matrices L_{af}^{m}, L_{ab}^{m}, E_b^{m}, and E_f^{m}. The state equations in (B.1) is now in the form of SCB. Hence, it follows from the properties of SCB that Σ_m and Σ have the same infinite zero structure and that Σ_m is left invertible. Furthermore, the invariant zeros of Σ_m are given by

$$\lambda\begin{bmatrix} A_{aa}^{+} - K_a^{+}E_a^{+} & -K_a^{+}E_a^{-} \\ 0 & A_{aa}^{-} \end{bmatrix}. \tag{B.2}$$

Evidently, these eigenvalues lie in $\mathscr{C}^{-}$. Hence, Σ_m is of minimum phase. Moreover, it is straightforward to verify that the left state and input zero directions associated with the minimum phase invariant zeros of Σ remain unchanged in Σ_m.

Next, we proceed to show that $V(s)V^{H}(s) = I$. From the well known Woodbury or Schur formula, and (2.3) and (2.4), we have

$$\begin{aligned} V^{-1}(s) &= \Gamma_3\left[I + E_a^{+}(sI - A_{aa}^{+})^{-1}K_a^{+}\right]\Gamma_3^{-1} \\ &= I + \Gamma_3E_a^{+}(sI - A_{aa}^{+})^{-1}P^{-1}E_a^{+\prime}\Gamma_3' \\ &= I + \Gamma_3E_a^{+}(sP - PA_{aa}^{+})^{-1}E_a^{+\prime}\Gamma_3' \\ &= I + \Gamma_3E_a^{+}(sP + A_{aa}^{+\prime}P - E_a^{+\prime}\Gamma_3'\Gamma_3E_a^{+})^{-1}E_a^{+\prime}\Gamma_3' \\ &= I - \Gamma_3E_a^{+}P^{-1}(-sI - A_{aa}^{+\prime} + E_a^{+\prime}K_a^{+\prime})^{-1}E_a^{+\prime}\Gamma_3' \\ &= I - (\Gamma_3')^{-1}K_a^{+\prime}\left[(sI - A_{aa}^{+} + K_a^{+}E_a^{+})^{-1}\right]^{H}E_a^{+\prime}\Gamma_3' \\ &= V^{H}(s). \end{aligned}$$

Here, we note that the poles of $V(s)$ are the eigenvalues of the stable matrix $-A_{aa}^{+}$, and the poles of $V^{-1}(s)$ are the nonminimum phase invariant zeros of Σ, namely $\lambda(A_{aa}^{+})$. Finally, we are ready to show that $G(s) = G_m(s)V(s)$. Let us define

$$\tilde{\Phi} = (sI - \tilde{A})^{-1} = \begin{bmatrix} sI - A_{aa}^{+} & 0 & -L_{ab}^{+}C_b & -L_{af}^{+}C_f \\ 0 & sI - A_{aa}^{-} & -L_{ab}^{-}C_b & -L_{af}^{-}C_f \\ 0 & 0 & sI - A_{bb} & -L_{bf}C_f \\ -B_fE_a^{+} & -B_fE_a^{-} & -B_fE_b & sI - A_{ff} \end{bmatrix}^{-1},$$

$$K = \begin{bmatrix} K_a^{+} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$E = \begin{bmatrix} E_a^{+} & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\Phi} = \begin{bmatrix} sI - A_{aa}^{+} & 0 & -L_{ab}^{+}C_b & -L_{af}^{+}C_f \\ 0 & sI - A_{aa}^{-} & -L_{ab}^{-}C_b & -L_{af}^{-}C_f \\ 0 & 0 & sI - A_{bb} & -L_{bf}C_f \\ 0 & -B_fE_a^{-} & -B_fE_b & sI - A_{ff} \end{bmatrix}^{-1}.$$

Then it is straightforward to verify that

$$\tilde{B}_m = \tilde{B} + K$$
$$\tilde{\Phi} = (\hat{\Phi}^{-1} - \tilde{B}E)^{-1}$$
$$\tilde{C}\hat{\Phi}K = 0$$
$$E\hat{\Phi}K = E_a^{+}(sI - A_{aa}^{+})^{-1}K_a^{+}.$$

Hence

$$\begin{aligned} G(s)V^{-1}(s) &= \Gamma_2\tilde{C}\tilde{\Phi}\tilde{B}\Gamma_3^{-1}\Gamma_3\left[I + E_a^{+}(sI - A_{aa}^{+})^{-1}K_a^{+}\right]\Gamma_3^{-1} \\ &= \Gamma_2\left[\tilde{C}\tilde{\Phi}\tilde{B} + \tilde{C}(\hat{\Phi}^{-1} - \tilde{B}E)^{-1}\tilde{B}E\hat{\Phi}K\right]\Gamma_3^{-1} \\ &= \Gamma_2\left[\tilde{C}\tilde{\Phi}\tilde{B} + \tilde{C}\left[(\hat{\Phi}^{-1} - \tilde{B}E)^{-1}\hat{\Phi}^{-1} - I\right]\hat{\Phi}K\right]\Gamma_3^{-1} \\ &= \Gamma_2\left[\tilde{C}\tilde{\Phi}\tilde{B} + \tilde{C}\tilde{\Phi}K - \tilde{C}\hat{\Phi}K\right]\Gamma_3^{-1} \\ &= \Gamma_2\tilde{C}\tilde{\Phi}(\tilde{B} + K)\Gamma_3^{-1} \\ &= G_m(s). \end{aligned} \tag{B.3}$$

This completes the Proof of Theorem 2.1. ■

Appendix C
Proof of Theorem 3.1

Without loss of generality, we assume that Σ is in the form of SCB of Appendix A. Then following the procedures of Appendix B, one can show that there exists a nonsingular state transformation such that Σ_M can be transformed into the form of SCB as

$$\dot{\bar{x}}_a^M = \left(A_{aa}^+ - K_a^+E_a^+\right)\bar{x}_a^M - K_a^+E_a^+x_a^- + L_{af}^M y_f + L_{ab}^M y_b$$
$$\dot{x}_a^- = A_{aa}^- x_a^- + L_{af}^- y_f + L_{ab}^- y_b$$
$$\dot{x}_b = A_{bb} x_b + L_{bf} y_f, \; y_b = C_b x_b$$
$$\dot{x}_f = A_f x_f + L_f y_f + B_f\left[u + E_a^+\bar{x}_a^M + E_a^- x_a^- + E_b^M x_b + E_f^M x_f\right], \; y_f = C_f x_f \tag{C.1}$$

for some appropriate dimensional matrices L_{af}^M, L_{ab}^M, E_b^M, and E_f^M. Hence, it follows from the properties of SCB that Σ_M and Σ have the same infinite zero structure and that Σ_M is left invertible. Furthermore, the invariant zeros of Σ_M are given by

$$\lambda\begin{bmatrix} A_{aa}^+ - K_a^+E_a^+ & -K_a^+E_a^- \\ 0 & A_{aa}^- \end{bmatrix} \subset \mathscr{C}^-. \tag{C.2}$$

Hence, Σ_M is of minimum phase. Moreover, the left state and input zero directions associated with the invariant zeros $\lambda(A_{aa}^-)$ of Σ remain unchanged in Σ_M. Next, it is simple to verify that the equality of $G(s) = G_M(s)U(s)$ follows directly from (B.3) and the property of $U(s)U^H(s) \to I$ as $|s| \to \infty$ follows from the fact that $U(s) \to I$ as $|s| \to \infty$. This completes the Proof of Theorem 3.1. ■

References

[1] B. M. Chen, "Software manual for the special coordinate basis of multivariable linear systems, Tech. Rep. ECE0094, Washington State Univ., Nov. 1988.

[2] Y. Halevi and Z. J. Palmor, "Extended limiting forms of optimum observers and LQG regulators," *Int. J. Contr.*, vol. 43, pp. 193-212, 1986.

[3] Z. L. Lin, "The implementation of special coordinate basis for linear multivariable systems in Matlab," Tech. Rep. ECE0100, Washington State Univ., Nov. 1989.

[4] A. G. J. MacFarlane and N. Karcanias, "Poles and zeros of linear multivariable systems: A survey of the algebraic, geometric, and complex variable theory," *Int. J. Contr.*, vol. 24, pp. 33-74, 1976.

[5] B. C. Moore, "On the flexibility offered by state feedback in multivariable systems beyond closed-loop eigenvalue assignment," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 689-692, 1976.

[6] A. Saberi, B. M. Chen, and P. Sannuti, "Theory of LTR for nonminimum phase systems, recoverable target loops, recovery in a subspace—Part I: Analysis," *Int. J. Contr.*, vol. 53, no. 5, pp. 1067-1160, May 1991.

[7] P. Sannuti and A. Saberi, "A special coordinate basis of multivariable linear systems—Finite and infinite zero structure, squaring down, and decoupling," *Int. J. Contr.*, vol. 45, no. 5, pp. 1655-1704, 1987.

[8] U. Shaked, "An explicit expression for the minimum-phase image of transfer function matrices," *IEEE Trans. Automat. Contr.*, vol. 34, no. 12, pp. 1290-1293, 1989.

[9] ——, "Nearly singular filtering for uniform and nonuniform rank linear continuous systems," *Int. J. Contr.*, vol. 38, pp. 811-830, 1983.

[10] ——, "The all-pass property of optimal open loop tracking systems," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 465-467, 1984.

[11] U. Shaked and E. Soroka, "Explicit solution to the unstable stationary filtering problem," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 185-189, 1986.

[12] E. Soroka and U. Shaked, "The properties of reduced-order minimum-variance filters for systems with partially perfect measurements," *IEEE Trans. Automat. Contr.*, vol. 33, no. 11, pp. 1022-1034, 1988.

[13] M. G. Strintzis, "A Solution to the matrix factorization problem," *IEEE Trans. Informat. Theory*, vol. IT-18, pp. 225-232, 1972.

[14] W. G. Tuel, Jr., "Computer algorithm for spectral factorization of rational matrices," *IBM J.*, pp. 163-170, Mar. 1968.

[15] Z. Zhang and J. S. Freudenberg, "Loop transfer recovery for nonminimum phase plants," *IEEE Trans. Automat. Contr.*, vol. 35, no. 5, May 1990.

Inversion of Polynomial Matrices by Interpolation

A. Schuster and P. Hippe

***Abstract*—Generalizing known polynomial interpolation methods to polynomial matrices new algorithms for the computation of the inverse of such matrices are developed.**

I. Introduction

The inversion of polynomial matrices is a problem arising in various fields of control system synthesis [10], [9], [16], [17], [6]. Using Cramer's rule the inversion can be carried out by computing determinants [4], [7], [11], [15]. Another possibility is the direct computation of the adjoint matrix which can, e.g., be obtained by polynomial operations [10], [14]. Such polynomial operations, however, are known to cause numerical problems. Therefore numerous methods have been developed which are based on the manipulation of constant matrices. Some of these methods require restrictive assumptions as, e.g., the algorithm by Emre *et al.* [5] where the determinant must be known at the outset or the method proposed by Inouye [8] which only works for row or column proper polynomial matrices. Chang *et al.* [3] and Zhang [18] present algorithms which yield the inverse already in irreducible form, however, at the expense of increased computational effort. Buslowicz [1] develops a recursive algorithm for the adjoint matrix and for the determinant starting from the coefficient matrices. This algorithm is especially elegant for matrices of low dimensions.

Here, we present interpolation methods for the computation of the adjoint matrix which constitute a generalization of known polynomial interpolation approaches. One of these seems especially suitable for computer applications.

II. Interpolation of Polynomial Matrices

In what follows adj $A(s)$ denotes the adjoint matrix, det $A(s)$ the determinantal polynomial, degree $[A(s)]$ the highest degree of all elements of $A(s)$, $\delta_{ci}[A(s)]$ the ith column degree, and $\delta_{rj}[A(s)]$ the jth row degree of a $p \times p$ nonsingular polynomial matrix $A(s)$, and $\otimes$ denotes the Kronecker product.

It is well known that there is one and only one polynomial of degree $\varrho \leq n$ which assumes the values $f(x_0), f(x_1), \cdots, f(x_n)$ at distinct base points $x_0, x_1, \cdots, x_n$. This polynomial is called the ϱth degree interpolation polynomial. Three important interpolation methods are [2]

i) the direct approach using Vandermonde's matrix

Manuscript received July 20, 1990; revised January 4, 1991.
The authors are with the Institut für Regelungstechnik, Universität Erlangen-Nürnberg, D-8520 Erlangen, West Germany.
IEEE Log Number 9104393.

0018-9286/92$03.00