# Noniterative computation of infimum in $H_\infty$ optimisation for plants with invariant zeros on the $j\omega$-axis

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**Abstract:** A simple and noniterative procedure for the computation of the exact value of the infimum in the singular $H_\infty$-optimisation problem is presented, as a continuation of our earlier work. Our problem formulation is general and we do not place any restrictions in the finite and infinite zero structures of the system, and the direct feedthrough terms between the control input and the controlled output variables, and between the disturbance input and the measurement output variables. Our method is applicable to a class of singular $H_\infty$-optimisation problems for which the transfer functions from the control input to the controlled output and from the disturbance input to the measurement output satisfy certain geometric conditions. In particular, the paper extends the result of earlier work by allowing these two transfer functions to have invariant zeros on the $j\omega$-axis.

## Notation

$A^T$ = transpose of $A$
$A^H$ = complex conjugate transpose of $A$
$I$ = identity matrix
$\mathbb{R}$ = set of real numbers
$\mathbb{C}$ = whole complex plane
$\mathbb{C}^-$ = open left-half complex plane
$\mathbb{C}^+$ = open right-half complex plane
$\mathbb{C}^o$ = imaginary axis $j\omega$
$\sigma_{max}(A)$ = maximum singular value of $A$
$\lambda(A)$ = set of eigenvalues of $A$
$\lambda_{max}(A)$ = maximum eigenvalue of $A$ where $\lambda(A) \subset \mathbb{R}$
$\rho(A)$ = spectral radius of $A$
Ker $(V)$ = kernel of $V$
Im $(V)$ = image of $V$

We define the following subspaces:

$\mathcal{V}^g(A, B, C, D)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A + BF)$-invariant and contained in $\operatorname{Ker}(C + DF)$ such that the eigenvalues of $(A + BF)|\mathcal{V}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some $F$.

$\mathcal{S}^g(A, B, C, D)$ is the minimal $(A + KC)$-invariant subspace of $\mathbb{R}^n$ containing $\operatorname{Im}(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}^g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some $K$.

For the cases $\mathbb{C}_g = \mathbb{C}$, $\mathbb{C}_g = \mathbb{C}^-$ and $\mathbb{C}_g = \mathbb{C}^o \cup \mathbb{C}^+$ we replace the index $g$ in $\mathcal{V}^g$ and $\mathcal{S}^g$ by $*$, $-$ and $+$, respectively.

## 1 Introduction

This paper is a continuation of our earlier work [1–3] in noniterative computation of the infimum in $H_\infty$-optimisation problem. In our most recent work [3], a noniterative algorithm to compute the infimum (hereafter denoted by $\gamma_o^*$) for a class of $H_\infty$-optimisation for which the transfer functions from the control input to the controlled output and from disturbance input to the measurement output have no invariant zeros of the $j\omega$-axis and also satisfy certain geometric conditions. In this paper, we extend our previous work in Reference 3 by removing the contraints on the invariant zeros of these transfer functions. A similar attempt has been made in a very recent paper [4], however, the results reported are basically restricted to the case of $H_\infty$-optimisation problem using state feedback. This work complements the one in Reference 4 by considering the general case of $H_\infty$-optimisation via measurement output feedback. We show that the infimum $\gamma_o^*$ is equal to the square root of the maximum eigenvalues of a constant matrix, which can be easily obtained from the data of the given $H_\infty$-optimisation problem. Our algorithm for the computation of $\gamma_o^*$ has been implemented efficiently in a Matlab-software environment for numerical solutions [5].

## 2 Problem formulation

Consider the linear system

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1 x + D_1 w \\ z = C_2 x + D_2 u \end{cases} \tag{1}$$

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where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $w \in \mathbb{R}^p$ is the disturbance, $y \in \mathbb{R}^r$ is the measurement output and $z \in \mathbb{R}^q$ is the controlled output. Without loss of generality, assume that both $[C_2, D_2]$ and $[E^T, D_1^T]$ are of full rank. Let $T_{zw}(s)$ denote the transfer function matrix from the disturbance $w$ to the controlled output $z$. The standard $H_\infty$-optimal control problem is concerned with the construction of stabilising feedback control laws that minimise the $H_\infty$-norm of $T_{zw}(s)$. Define

$\gamma_o^* := \inf \{\| T_{zw}(s) \|_\infty$ where $u(s) = F_o(s)y(s)$ for any proper transfer function matrix $F_o(s)$ which internally stabilises the system of eqn. 1$\}$

For the case that $y = x$, i.e. $H_\infty$-optimisation via state feedback, relabel $\gamma_o^*$ as $\gamma_s^*$ to signify that the infimum is taken over all stabilising state feedback laws. We give a simple and noniterative procedure for determining $\gamma_o^*$. The method is applicable to the general system of eqn. 1 satisfying the following assumptions:

A1: $(A, B)$ is stabilisable

A2: $\mathrm{Im}\,(E) \subseteq \mathscr{V}^-(A, B, C_2, D_2) + \mathscr{S}^-(A, B, C_2, D_2)$

B1: $(A, C_1)$ is detectable

B2: $\mathrm{Ker}\,(C_2) \supseteq \mathscr{V}^-(A, E, C_1 D_1) \cap \mathscr{S}^-(A, E, C_1, D_1)$

Assumptions A1 and B1 are necessary for any total problems, hence assumptions A2 and B2 are basically the main conditions in this paper. If $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ are, respectively, right- and left-invertible, then assumptions A2 and B2 are automatically satisfied.

*Remark 2.1:* It might be helpful to interpret our conditions A2 and B2 in the context of 'block characterisation' of the $H_\infty$ optimal control problem, which stems from the frequency-domain approach in early 1980s. This block characterisation in the frequency-domain approach was considered to be an indicator of the degree of the 'complexity' of the problem, although in our opinion, such a block characterisation is dependent on proof technique and cannot be used as a true measure of the complexity of the problem. At any rate, first recall the definition of this block characterisation. We denote $P_1(s)$ and $P_2(s)$ as the Rosenbrock system matrices of the systems $(A, B, C_2, D_2)$ and $(A, E, C_1, D_1)$ respectively, namely,

$$P_1(s) = \begin{bmatrix} sI - A & B \\ C_2 & D_2 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} sI - A & E \\ C_1 & D_1 \end{bmatrix}$$

The $H_\infty$ optimum control problem is said to be

(*a*) general one block if both $P_1(s)$ and $P_2^T(s)$ have maximal row normal rank

(*b*) general two block if precisely one of the matrices $P_1(s)$ and $P_2^T(s)$ has maximal row normal rank

(*c*) general four block if none of the matrices $P_1(s)$ and $P_2^T(s)$ has maximal row normal rank.

Finally, the definition of so-called one, two and four-block Nehari $H_\infty$ control problem is the same as the preceding definitions with the exception that no zeros in $\mathbb{C}^0 \cup \{\infty\}$ in the systems $(A, B, C_2, D_2)$ and $(A^T, C_1^T, E^T, D_1^T)$ are allowed. Now it is easy to verify that the class of $H_\infty$ optimal control problems considered here, namely the class of problems that satisfy conditions A2 and B2 are in fact a subset of the general four-block problem. Moreover, they subsume as special cases the one block Nehari problem and the general one block problem.

## 3 Special co-ordinate basis

We recall the definition of the special co-ordinate basis (SCB) for a linear time-invariant nonstrictly proper system [6]. Such a co-ordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structures of a given system as well as other system geometric properties. It is instrumental in the derivation of the method described in Section 4.

In what follows, we recapitulate the main results in a theorem and some properties of the special co-ordinate basis while leaving detailed derivation and proofs to be found in References 6 and 7. Consider the system described by

$$\begin{cases} \dot{x} = Ax + Bu + Ew \\ z = Cx + Du \end{cases} \tag{2}$$

It can be easily shown that using singular value decomposition one can always find an orthogonal transformation $U$ and a nonsingular matrix $V$ that put the direct feedthrough matrix $D$ into the following form

$$\bar{D} = UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

where $m_0$ is the rank of $D$. Without loss of generality one can assume that the matrix $D$ in eqn. 2 has the form as shown in eqn. 3. Thus the system in eqn. 2 can be rewritten as

$$\dot{x} = Ax + [B_0 \quad B_1] \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + Ew$$

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \tag{4}$$

where $B_0$, $B_1$, $C_0$ and $C_1$ are the matrices of appropriate dimensions. The inputs $u_0$ and $u_1$ and the outputs $z_0$ and $z_1$ are those of the transformed system, namely

$$u = V \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = Uz$$

The $H_\infty$-norm of the system transfer function $T_{sw}(s)$ is unchanged when an orthogonal transformation is applied on the output $z$, and under any nonsingular transformations on the states and control inputs. We have the following main theorem:

*Theorem 3.1 (SCB):* There exist nonsingular transformations $\Gamma_a$, $\Gamma_o$ and $\Gamma_i$ such that

$$x = \Gamma_s[(x_a^+)^T, x_b^T, (x_a^o)^T, (x_a^-)^T, x_c^T, x_f^T]^T$$

$$[z_0^T, z_1^T]^T = \Gamma_o[z_0^T, z_f^T, z_b^T]^T$$

$$[u_0^T, u_1^T]^T = \Gamma_i[u_0^T, u_f^T, u_c^T]^T$$

and

$$\bar{A} = \Gamma_s^{-1}(A - B_0 C_0)\Gamma_s = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b & 0 & 0 & 0 & L_{af}^+ C_f \\ 0 & A_{bb} & 0 & 0 & 0 & L_{bf} C_f \\ 0 & L_{ab}^o C_b & A_{aa}^o & 0 & 0 & L_{af}^o C_f \\ 0 & L_{ab}^- C_b & 0 & A_{aa}^- & 0 & L_{af}^- C_f \\ B_c E_{ca}^+ & L_{cb} C_b & B_c E_{ca}^o & B_c E_{ca}^- & A_{cc} & L_{cf} C_f \\ B_f E_{fa}^+ & B_f E_{fb} & B_f E_{fa}^o & B_f E_{fa}^- & B_f E_{fc} & A_{ff} \end{bmatrix} \tag{5}$$

$$\bar{B} = \Gamma_s^{-1}[B_0 \quad B_1]\Gamma_i = \begin{bmatrix} B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0a}^o & 0 & 0 \\ B_{0a}^- & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix} \tag{6}$$

$$\bar{C} = \Gamma_o^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^+ & C_{0b} & C_{0a}^o & C_{0a}^- & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & 0 & C_f \\ 0 & C_b & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

and

$$\bar{D} = \Gamma_0^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

where $\lambda(A_{aa}^-) \in \mathbb{C}^-$, $\lambda(A_{aa}^o) \in \mathbb{C}^o$, $\lambda(A_{aa}^+) \in \mathbb{C}^+$, $(A_{cc}, B_c)$ is controllable, $(A_{bb}, C_b)$ is observable and the subsystem $(A_{ff}, B_f, C_f)$ is invertible with no invariant zeros.

The proof of this theorem can be found in References 6 and 7. In what follows, we state some important properties of the SCB which are pertinent to our present work. For further details regarding SCB and its properties, see Reference 8.

*Property 3.1:* The given system $(A, B, C, D)$ is right-invertible if and only if $x_b$ and hence $z_b$ are nonexistent, left-invertible if and only if $x_c$ and hence $u_c$ are nonexistent, invertible if and only if both $x_c$ and $x_b$ are nonexistent.

*Property 3.2:* $\lambda(A_{aa}^-) \in \mathbb{C}^-$, $\lambda(A_{aa}^o) \in \mathbb{C}^o$ and $\lambda(A_{aa}^+) \in \mathbb{C}^+$ are respectively the stable, $j\omega$ and unstable invariant zeros of $(A, B, C, D)$.

*Property 3.3:* The pair $(A, B)$ is stabilisable if and only if $(A_{con}, B_{con})$ is stabilisable where

$$A_{con} = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b & 0 \\ 0 & A_{bb} & 0 \\ 0 & L_{ab}^o C_b & A_{aa}^o \end{bmatrix} \quad B_{con} = \begin{bmatrix} B_{0a}^+ & L_{af}^+ \\ B_{0b} & L_{bf} \\ B_{0a}^o & L_{af}^o \end{bmatrix} \tag{9}$$

There are interconnections between the SCB and various invariant and almost invariant geometric subspaces. In the following we list the geometrical interpretations of some state vector components of SCB.

*Property 3.4:*

$x_a^- \oplus x_a^o \oplus x_a^+ \oplus x_c$ spans $\mathscr{V}^*(A, B, C, D)$

$x_a^- \oplus x_c$ spans $\mathscr{V}^-(A, B, C, D)$

$x_a^o \oplus x_a^+ \oplus x_c$ spans $\mathscr{V}^+(A, B, C, D)$

$x_c \oplus x_f$ spans $\mathscr{S}^*(A, B, C, D)$

$x_a^- \oplus x_c \oplus x_f$ spans $\mathscr{S}^+(A, B, C, D)$

$x_a^o \oplus x_a^+ \oplus x_c \oplus x_f$ spans $\mathscr{S}^-(A, B, C, D)$

## 4 Computational algorithm for $\gamma_o^*$

In this Section, we present our main result, namely, the noniterative algorithm for computational of the infimum in $H_\infty$-optimisation for plants with invariant zeros on the $j\omega$-axis. First, we denote $\Sigma_P$ and $\Sigma_Q$, respectively, as the subsystems $(A, B, C_2, D_2)$ and $(A^T, C_1^T, E^T, D_1^T)$ to conform with the notations in our previous work and the work of Stoorvogel [9, 10], which plays a significant role in the development of our results in References 1–3. In what follows, we introduce a step-by-step procedure to compute $\gamma_o^*$.

*Step 1:* Transform the system $(A, B, C_2, D_2)$ into the special co-ordinate basis (SCB) described in Section 3. To all submatrices and transformations in the SCB of $\Sigma_P$, we append the subscript $_P$ to signify their relation to the system $\Sigma_P$. Next we compute

$$\Gamma_{sP}^{-1} E = [(E_{aP}^{-1})^T \ (E_{bP})^T \ (E_{aP}^o)^T \ (E_{aP}^-)^T \ (E_{cP})^T \ (E_{fP})^T]^T \tag{10}$$

It is simple to verify from the properties of SCB that assumption A2 implies $E_{bP} = 0$. Then define the matrices

$$A_{xP} = \begin{bmatrix} A_{aaP}^+ & L_{abP}^+ C_{bP} & 0 \\ 0 & A_{bbP} & 0 \\ 0 & L_{abP}^o C_{bP} & A_{aaP}^o \end{bmatrix}$$

$$B_{xP} = \begin{bmatrix} B_{0aP}^+ & L_{0aP}^+ \\ B_{0bP} & L_{bfP} \\ B_{0aP}^o & L_{afP}^o \end{bmatrix}$$

$$E_{xP} = \begin{bmatrix} E_{aP}^+ \\ E_{bP} \\ E_{aP}^o \end{bmatrix} \tag{11}$$

and

$$C_{xP} = \Gamma_{oP}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_{bP} & 0 \end{bmatrix}$$

$$D_{xP} = \Gamma_{oP}\begin{bmatrix} I_{m0P} & 0 \\ 0 & C_{fP}C_{fP}^T \\ 0 & 0 \end{bmatrix} \tag{12}$$

By some simple algebra, it is straightforward to show that

$$C_{xP}^T[I - D_{xP}(D_{xP}^T D_{xP})^{-1} D_{xP}^T]C_{xP} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{C}_{bP}^T \tilde{C}_{bP} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

for some full row rank $\tilde{C}_{bP}$,

$$A_{xP} - B_{xP}(D_{xP}^T D_{xP})^{-1} D_{xP}^T C_{xP} = \begin{bmatrix} A_{aaP}^+ & \tilde{L}_{abP}^+ \tilde{C}_{bP} & 0 \\ 0 & \tilde{A}_{bbP} & 0 \\ 0 & \tilde{L}_{abP}^o \tilde{C}_{bP} & A_{aaP}^o \end{bmatrix} \tag{14}$$

and

$$B_{xP}(D_{xP}^T D_{xP})^{-1} B_{xP}^T = \begin{bmatrix} B_{0aP}^+ & \tilde{L}_{afP}^+ \\ B_{0bP} & \tilde{L}_{bfP} \\ B_{0aP}^o & \tilde{L}_{afP}^o \end{bmatrix}\begin{bmatrix} B_{0aP}^+ & \tilde{L}_{afP}^+ \\ B_{0bP} & \tilde{L}_{bfP} \\ B_{0aP}^o & \tilde{L}_{afP}^o \end{bmatrix}^T \tag{15}$$

for some appropriate $\tilde{L}_{abP}$, $\tilde{L}_{abP}^o$, $\tilde{L}_{afP}^+$, $\tilde{L}_{bfP}$ and $\tilde{L}_{afP}^o$. It can easily be verified that the pair $(\tilde{A}_{bbP}, \tilde{C}_{bP})$ is observable provided that $(A_{bbP}, C_{bP})$ is observable.

*Step 2: Define*

$$A_P = \begin{bmatrix} A_{aaP}^+ & \tilde{L}_{abP}^+ \tilde{C}_{bP} \\ 0 & \tilde{A}_{bbP} \end{bmatrix}$$

$$B_P = \begin{bmatrix} B_{0bP}^+ & \tilde{L}_{abP}^+ \\ B_{0bP} & \tilde{L}_{bfP} \end{bmatrix}$$

and

$$C_P = [0 \quad \tilde{C}_{bP}] \tag{16}$$

Then solve for the unique positive definite solution $S_P$ of the algebraic matrix Riccati equation,

$$A_P S_P + S_P A_P^T - B_P B_P^T + S_P C_P^T C_P S_P = 0 \tag{17}$$

together with the matrix $T_P$ defined by

$$T_P = \begin{bmatrix} T_{aaP} & 0 \\ 0 & 0 \end{bmatrix}$$

where $T_{aaP}$ is the unique solution of the algebraic matrix Lyapunov equation,

$$A_{aaP}^+ T_{aaP} + T_{aaP}(A_{aaP}^+)^T = E_{aP}^+(E_{aP}^+)^T \tag{18}$$

It is simple to verify from the properties of SCB that under assumption A1, $(A_P, B_P)$ is stabilisable and $(-A_P, C_P)$ is detectable since $\lambda(A_{aa}^+) \in \mathbb{C}^+$ and $(\tilde{A}_{bbP}, \tilde{C}_{bP})$ is observable. Hence the existence and uniqueness of $S_P$ and $T_{aaP}$ follow from results of Reference 11. Next, solve the unique solution $Y_P$ of the following Sylvester equation,

$$(A_P + S_P C_P^T C_P)Y_P + Y_P(A_{aaP}^o)^T + S_P C_P^T(\tilde{L}_{abP}^o)^T - B_P[B_{0aP}^o \quad \tilde{L}_{afP}^o]^T = 0 \tag{19}$$

Denote the set of eigenvalues of $A_{aaP}^o$ with nonnegative imaginary part as $\{j\omega_1, \ldots, j\omega_{k_P}\}$ and for $i = 1, \ldots, k_P$, choose complex matrices $V_{iP}$, whose columns form a basis of the eigenspace $\{x \in \mathbb{C}^{n_{aP}^o} \mid x^H(j\omega_i I - A_{aaP}^o) = 0\}$ where $n_{aP}^o$ is the dimension of $A_{aaP}^o$. Then define

$$F_{iP} = V_{iP}^H([B_{0aP}^o \quad \tilde{L}_{afP}^o][B_{0aP}^o \quad \tilde{L}_{afP}^o]^T + \tilde{L}_{abP}^o(\tilde{L}_{abP}^o)^T - [(\tilde{L}_{abP}^o)^T + C_P Y_P]^T[(\tilde{L}_{abP}^o)^T + C_P Y_P])V_{iP} \tag{20}$$

for $i = 1, \ldots, k_P$, and

$$F_P = \text{blockdiag}\{F_{1P}, \ldots, F_{k_P P}\} \tag{21}$$

It is shown in Reference 12 that $F_P > 0$. Also, define

$$G_P = \text{blockdiag}\{[V_{1P}^H E_{aP}^o(E_{aP}^o)^T V_{1P}], \ldots, [V_{k_P P}^H E_{aP}^o(E_{aP}^o)^T V_{k_P P}]\} \tag{22}$$

*Step 3:* Transform the system $(A^T, C_1^T, E^T, D_1^T)$ into the special co-ordinate basis (SCB) described in Section 3. Here we add the subscript $_Q$ to all submatrices and transformations in the SCB of the system $\Sigma_Q$. Next compute

$$\Gamma_{sQ}^{-1} C_2^T = [(E_{aQ}^+)^T \ (E_{bQ})^T \ (E_{aQ}^o)^T \ (E_{aQ}^-)^T \ (E_{cQ})^T \ (E_{fQ})]^T \tag{23}$$

It is simple to show from the properties of SCB that assumption B2 implies $E_{bQ} = 0$. Then define the matrices

$$A_{xQ} = \begin{bmatrix} A_{aaQ}^+ & L_{abQ}^+ C_{bQ} & 0 \\ 0 & A_{bbQ} & 0 \\ 0 & L_{abQ}^o C_{bQ} & A_{aaQ}^o \end{bmatrix}$$

$$B_{xQ} = \begin{bmatrix} B_{0aQ}^+ & L_{afQ}^+ \\ B_{0bQ} & L_{bfQ} \\ B_{0aQ}^o & L_{afQ}^o \end{bmatrix}$$

$$E_{xQ} = \begin{bmatrix} E_{aQ}^+ \\ E_{bQ} \\ E_{aQ}^o \end{bmatrix} \tag{24}$$

and

$$C_{xQ} = \Gamma_{oQ}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_{bQ} & 0 \end{bmatrix}$$

$$D_{xQ} = \Gamma_{oQ}\begin{bmatrix} I_{m0Q} & 0 \\ 0 & C_{fQ}C_{fQ}^T \\ 0 & 0 \end{bmatrix} \tag{25}$$

By some simple algebra, it is straightforward to show that

$$C_{xQ}^T[I - D_{xQ}(D_{xQ}^T D_{xQ})^{-1} D_{xQ}^T]C_{xQ} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{C}_{bQ}^T \tilde{C}_{bQ} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

for some full row rank $\tilde{C}_{bQ}$,

$$A_{xQ} - B_{xQ}(D_{xQ}^T D_{xQ})^{-1} D_{xQ}^T C_{xQ} = \begin{bmatrix} A_{aaQ}^+ & \tilde{L}_{abQ}^+ \tilde{C}_{bQ} & 0 \\ 0 & \tilde{A}_{bbQ} & 0 \\ 0 & \tilde{L}_{abQ}^o \tilde{C}_{bQ} & A_{aaQ}^o \end{bmatrix} \tag{27}$$

and

$$B_{xQ}(D_{xQ}^T D_{xQ})^{-1} B_{xQ}^T = \begin{bmatrix} B_{0aQ}^+ & \tilde{L}_{afQ}^+ \\ B_{0bQ} & \tilde{L}_{bfQ} \\ B_{0aQ}^o & \tilde{L}_{afQ}^o \end{bmatrix} \begin{bmatrix} B_{0aQ}^+ & \tilde{L}_{afQ}^+ \\ B_{0bQ} & \tilde{L}_{bfQ} \\ B_{0aQ}^o & \tilde{L}_{afQ}^o \end{bmatrix}^T \tag{28}$$

for some appropriate $\tilde{L}_{abQ}$, $\tilde{L}_{abQ}^o$, $\tilde{L}_{afQ}^+$, $\tilde{L}_{bfQ}$ and $\tilde{L}_{afQ}^o$. It can easily be verified that the pair $(\tilde{A}_{bbQ}, \tilde{C}_{bQ})$ is observable provided that $(A_{bbQ}, C_{bQ})$ is observable.

*Step 4:* Define

$$A_Q = \begin{bmatrix} A_{aaQ}^+ & \tilde{L}_{abQ}^+ \tilde{C}_{bQ} \\ 0 & \tilde{A}_{bbQ} \end{bmatrix}$$

$$B_Q = \begin{bmatrix} B_{0aQ}^+ & \tilde{L}_{afQ}^+ \\ B_{0bQ} & \tilde{L}_{bfQ} \end{bmatrix}$$

and

$$C_Q = [0 \quad \tilde{C}_{bQ}] \tag{29}$$

then solve for the unique positive definite solution $S_Q$ of the algebraic matrix Riccati equation,

$$A_Q S_Q + S_Q A_Q^T - B_Q B_Q^T + S_Q C_Q^T C_Q S_Q = 0 \tag{30}$$

together with the matrix $T_Q$ defined by

$$T_Q = \begin{bmatrix} T_{aaQ} & 0 \\ 0 & 0 \end{bmatrix}$$

where $T_{aaQ}$ is the unique solution of the algebraic matrix Lyapunov equation,

$$A_{aaQ}^+ T_{aaQ} + T_{aaQ}(A_{aaQ}^+)^T = E_{aQ}^+(E_{aQ}^+)^T \tag{31}$$

Again, the existence and uniqueness of $S_Q$ and $T_{aaQ}$ follow from assumption B2 and the properties of SCB. Next, solve the unique solution $Y_Q$ of the following Sylvester equation,

$$(A_Q + S_Q C_Q^T C_Q) Y_Q + Y_Q (A_{aaQ}^o)^T + S_Q C_Q^T (\tilde{L}_{abQ}^o)^T - B_Q [B_{0aQ}^o \quad \tilde{L}_{afQ}^o]^T = 0 \tag{32}$$

Denote the set of eigenvalues of $A_{aaQ}^o$ with non-negative imaginary part as $\{j\omega_1, \ldots, j\omega_{k_Q}\}$ and for $i = 1, \ldots, k_Q$, choose complex matrices $V_{iQ}$, whose columns form a basis of the eigenspace $\{x \in \mathbb{C}^{n_{aQ}^o} \mid x^H(j\omega_i I - A_{aaQ}^o) = 0\}$ where $n_{aQ}^o$ is the dimension of $A_{aaQ}^o$. Then define

$$F_{iQ} = V_{iQ}^H([B_{0aQ}^o \quad \tilde{L}_{afQ}^o][B_{0aQ}^o \quad \tilde{L}_{afQ}^o]^T + \tilde{L}_{abQ}^o(\tilde{L}_{abQ}^o)^T - [(\tilde{L}_{abQ}^o)^T + C_Q Y_Q]^T[(\tilde{L}_{abQ}^o)^T + C_Q Y_Q]) V_{iQ} \tag{33}$$

for $i = 1, \ldots, k_Q$, and

$$F_Q = \text{blockdiag}\{F_{1Q}, \ldots, F_{k_Q Q}\} \tag{34}$$

Again, it can be shown that $F_Q > 0$. Also, define

$$G_Q = \text{blockdiag}\{[V_{1Q}^H E_{aQ}^o (E_{aQ}^o)^T V_{1Q}], \ldots, [V_{k_Q Q}^H E_{aQ}^o (E_{aQ}^o)^T V_{k_Q Q}]\} \tag{35}$$

*Step 5:* Define

$$n_P = \dim\{\mathbb{R}^n / \mathscr{S}^+(A, B, C_2, D_2)\} - n_{aP}^o$$

and

$$n_Q = \dim\{\mathscr{V}^+(A, E, C_1, D_1)\} - n_{aQ}^o$$

We introduce a matric $\Gamma$ of dimension $n_P \times n_Q$ that satisfies the following

$$\Gamma_{sP}^{-1}(\Gamma_{sQ}^{-1})^T = \begin{bmatrix} \Gamma & \star \\ \star & \star \end{bmatrix} \tag{36}$$

and define a constant matrix

$$M = \begin{bmatrix} G_P F_P^{-1} & 0 & 0 & 0 \\ 0 & T_P S_P^{-1} + \Gamma S_Q^{-1} S_P^{-1} & -\Gamma S_Q^{-1} & 0 \\ 0 & -T_Q S_Q^{-1} \Gamma^T S_P^{-1} & T_Q S_Q^{-1} & 0 \\ 0 & 0 & 0 & G_Q F_Q^{-1} \end{bmatrix} \tag{37}$$

We have the following main theorem.

*Theorem 4.1:* Consider the system $\Sigma$ given by eqn. 1. Then under assumptions A1, A2, B1 and B2, the infimum of $H_\infty$-optimisation for $\Sigma$ is

$$\gamma_o^* = \sqrt{[\lambda_{max}(M)]} \tag{38}$$

*Proof:* Following the results of Scherer [4] (e.g. Theorem 6), it can be shown that

$$\gamma > \gamma_P^* = \max\{\sqrt{[\lambda_{max}(T_P S_P^{-1})]}, \sqrt{[\lambda_{max}(G_P F_P^{-1})]}\} \tag{39}$$

if and only if the following algebraic Riccati inequality,

$$\begin{aligned} &[A_{xP} - B_{xP}(D_{xP}^T D_{xP})^{-1} D_{xP} C_{xP}] X \\ &\quad + X[A_{xP} - B_{xP}(D_{xP}^T D_{xP})^{-1} D_{xP} C_{xP}]^T \\ &\quad + \gamma^{-2} E_{xP} E_{xP}^T + X C_{xP}[I - D_{xP}(D_{xP}^T D_{xP})^{-1} D_{xP}^T] \\ &\quad \times C_{xP} X - B_{xP}(D_{xP}^T D_{xP})^{-1} B_{xP}^T < 0 \end{aligned}$$

has a positive definite solution. Then it follows from the results of References 4 and 12 and some simple algebraic manipulations that for $\gamma > \gamma_P^*$, the positive semidefinite matrix $P(\gamma)$ given by

$$P(\gamma) = (\Gamma_{sP}^{-1})^T \begin{bmatrix} (S_P - \gamma^{-2} T_P)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{sP}^{-1} \tag{40}$$

is the lower limit point of the set

$$\{P > 0 \mid \exists F: (A + BF)^T P + P(A + BF) + \gamma^{-2} P E E^T P + (C_2 + D_2 F)^T (C_2 + D_2 F) < 0\}$$

Moreover, such a $P(\gamma)$ does not exist when $\gamma < \gamma_P^*$. By dual reasoning, one can shown that

$$\gamma > \gamma_Q^* = \max\{\sqrt{[\lambda_{max}(T_Q S_Q^{-1})]}, \sqrt{[\lambda_{max}(G_Q F_Q^{-1})]}\} \tag{41}$$

if and only if the following algebraic Riccati inequality,

$$\begin{aligned} &[A_{xQ} - B_{xQ}(D_{xQ}^T D_{xQ})^{-1} D_{xQ} C_{xQ}] Z \\ &\quad + Z[A_{xQ} - B_{xQ}(D_{xQ}^T D_{xQ})^{-1} D_{xQ} C_{xQ}]^T \\ &\quad + \gamma^{-2} E_{xQ} E_{xQ}^T + Z C_{xQ}[I - D_{xQ}(D_{xQ}^T D_{xQ})^{-1} D_{xQ}^T] \\ &\quad \times C_{xQ} Z - B_{xQ}(D_{xQ}^T D_{xQ})^{-1} B_{xQ}^T < 0 \end{aligned}$$

has a positive definite solution. And for $\gamma > \gamma_Q^*$, the positive semidefinite matrix $Q(\gamma)$ given by

$$Q(\gamma) = (\Gamma_{sQ}^{-1})^T \begin{bmatrix} (S_Q - \gamma^{-2} T_Q)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \Gamma_{sQ}^{-1} \tag{42}$$

is the lower limit point of the set

$$\{Q > 0 \mid \exists K: (A + KC_1)Q + Q(A + KC_1)^T + \gamma^{-2} Q C_2^T C_2 Q + (E + KD_1)(E + KD_1)^T < 0\}$$

Again, such a $Q(\gamma)$ does not exist when $\gamma < \gamma_Q^*$. Now define

$$\gamma_{PQ} = \max\{\sqrt{[\lambda_{max}(T_P S_P^{-1})]}, \sqrt{[\lambda_{max}(T_Q S_Q^{-1})]}\} \tag{43}$$

and

$$\gamma_{cou} = \sup\{\gamma \in (\gamma_{PQ}, \infty) \mid \rho[P(\gamma) Q(\gamma)] < \gamma^2\} \tag{44}$$

where $P(\gamma)$ and $Q(\gamma)$ are as given in eqns. 40 and 42, respectively. Then following the results of Scherer [4], it can easily be shown that

$$\gamma_o^* = \max\{\gamma_{cou}, \sqrt{[\lambda_{max}(G_P F_P^{-1})]}, \sqrt{[\lambda_{max}(G_Q F_Q^{-1})]}\} \tag{45}$$

Also, using the results of Chen, Saberi and Ly [2, 3], it can be shown that

$$\gamma_{cou} = \left\{\lambda_{max}\begin{bmatrix} T_P S_P^{-1} + \Gamma S_Q^{-1}\Gamma^T S_P^{-1} & -\Gamma S_Q^{-1} \\ -T_Q S_Q^{-1}\Gamma^T S_P^{-1} & T_Q S_Q^{-1}\end{bmatrix}\right\}^{1/2} \tag{46}$$

Hence, the result of Theorem 4.1 follows.

## 5 Example

We illustrate our main result in the following example. Consider a given system characterised by

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 11 & -21.876238 & -4.2239 & -2.425699 \\ 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*Step 1:* It is simple to verify that the subsystem $(A, B, C_2, D_2)$ is left-invertible with two invariant zeros at $\pm j$ and assumption A2 is satisfied. Applying SCB transformation to $(A, B, C_2, D_2)$,

$$\Gamma_{sP} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 1.3660254 & 0.3660254 & 0 & 0 & 0 \\ 0.1988066 & 1.9900945 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_P = \begin{bmatrix} -01614784 & 0.2246812 \\ 0.6026457 & -0.8385216 \end{bmatrix}$$

$$B_P = \begin{bmatrix} 0.6040578 & -0.1762197 \\ 0.4723969 & 0.4878984 \end{bmatrix}$$

$$C_P = \begin{bmatrix} 1.3544397 & 0.2665382 \\ 0.2665382 & 2.0058434 \end{bmatrix}$$

$$E_{bP} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_{aaP}^o = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\tilde{L}_{abP}^o = \begin{bmatrix} 0.9489977 & 1.0485243 \\ -0.9489977 & -1.0485243 \end{bmatrix}$$

and

$$[B_{0aP}^o \quad \tilde{L}_{afP}^o] = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad E_{aP}^o = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Following the procedure in Section 4,

$$S_P = \begin{bmatrix} 0.6180716 & -0.2516670 \\ -0.2516670 & 0.7339429 \end{bmatrix}$$

$$T_P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Y_P = \begin{bmatrix} -0.6928337 & -0.0822109 \\ -0.3161228 & 0.3068152 \end{bmatrix}$$

and

$$F_P = 2.3885733 \quad G_P = 3.5$$

*Step 2:* The subsystem $(A, E, C_1, D_1)$ is invertible and of nonminimum phase with invariant zeros at $\{0.078944, \pm j2.302011, -4.095803\}$. Hence, assumption B2 is automatically satisfied. Applying the SCB transformation to $(A^T, C_1^T, E^T, D_1^T)$,

$$\Gamma_{sQ} = \begin{bmatrix} 0.2148444 & 0.0018481 & 0.2169145 & 0.0698280 & 0.2 \\ 0.5503097 & 0.6645646 & -0.6352193 & 0.8023543 & 0.4 \\ -0.7990597 & -0.7456317 & -0.5938518 & -0.5805731 & 0.6 \\ -0.0941402 & -0.0440333 & 0.3437855 & 0.0892284 & 0.4 \\ -0.0603521 & 0.0210926 & -0.2803500 & -0.0795282 & 0.2 \end{bmatrix}$$

$$A_Q = A_{aaQ}^+ = 0.0789442$$

$$B_Q = [2.3596219 \quad -0.1725085]$$

$$C_Q = 0$$

$$E_{aQ}^+ = [0.1593412 \quad 0.0009204 \quad 0.0116587 \quad 0.1593412]$$

and

$$A_{aaQ}^o = \begin{bmatrix} 0.8733954 & -14.3566212 \\ 0.4222493 & -0.8733953 \end{bmatrix}$$

$$[B_{0aQ}^o \quad \tilde{L}_{afQ}^o] = \begin{bmatrix} 13.8502316 & -10.8089077 \\ 0.3251762 & -1.3752299 \end{bmatrix}$$

$$E_{aQ}^o = \begin{bmatrix} -1.9958628 & 6.3511003 & -0.7973732 & -1.9958628 \\ -0.5082606 & 0.0920508 & -0.4908900 & -0.5082606 \end{bmatrix}$$

Following the procedure in Section 4,

$$S_Q = 35.4527292$$

$$T_Q = 0.3224810$$

$$Y_Q = [-5.2529064 \quad 93.6614674]$$

and

$$F_Q = 8.4694885 \quad G_Q = 35.4527292$$

*Step 3:* Evaluate

$$M = \begin{bmatrix} 1.4653098 & 0 & 0 & 0 & 0 \\ 0 & -0.0000103 & -0.0000451 & 0.0003744 & 0 \\ 0 & 0.0000632 & 0.0002763 & -0.0022958 & 0 \\ 0 & -0.0002503 & -0.0010946 & 0.0090961 & 0 \\ 0 & 0 & 0 & 0 & 0.2110284 \end{bmatrix}$$

and obtain

$$\gamma_o^* = \sqrt{[\lambda_{max}(M)]} = 1.2104998$$

## 6 Conclusion

We have extended the results of References 2 and 3 and presented a simple noniterative algorithm for the computation of the infimum for a class of $H_\infty$-optimisation problem. We have shown that this infimum is equal to the square root of the maximum eigenvalue of a constant matrix that can be easily obtained from the system matrices of $\Sigma$. Our results are obtained under the assumptions that the two subsystems $\Sigma_P$ and $\Sigma_Q$ satisfy certain geometric conditions. The proposed algorithm for computing the infimum is applicable to the general case of singular $H_\infty$-optimisation problem where no restrictions have been placed on finite zeros and infinite zeros of $\Sigma_P$ and $\Sigma_Q$ and the direct feedthrough terms in $\Sigma$.

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