CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol. 8, No. 1, pp. 17-35, March, 1992

C91012R
©MITA PRESS

Contributed Paper

EXACT COMPUTATION OF THE INFIMUM IN H_∞-OPTIMIZATION VIA STATE FEEDBACK*

B. M. CHEN,[1] A. SABERI[1] AND U.-L. LY[2]

Abstract. This paper presents a simple and non-iterative procedure for the computation of the exact value of the infimum in the standard H_∞-optimal control with state feedback. The problem formulation is general and does not place any restrictions on the direct feedthrough term between the control input and the controlled output variables. The algorithm involves solutions of two algebraic Lyapunov equations of a subsystem obtained from the transformation of the original system into a special coordinate basis. The method is applicable to systems where the transfer function from the control input to the controlled output is right-invertible and has no invariant zeros on the $j\omega$ axis. Two applications are also considered. The first one provides a necessary and sufficient condition for the solvability of H_∞-almost disturbance decoupling problem via state feedback with internal stability. The second application deals with the computation of the supremum of the complex stability radii which can be achieved by linear state feedback. Several examples are provided to illustrate the numerical algorithm, one of which is the determination of the achievable reduction in H_∞-norm of aircraft responses to turbulence in a disturbance rejection design using optimal state feedback, and another example is the achievable H_∞-performance in control of a flexible mechanical system.

Key Words—H_∞-optimization, robust control.

1. Introduction

Over the past decade one has witnessed a proliferation of literature on H_∞-optimal control since it was first introduced by Zames (1981). The main focus of the work has been and continues to be on the formulation and solution to the robust multivariable control problem. Since the original formulation in Zames (1981), a great deal of work has been done toward solving the H_∞-optimal control problem. Primarily all early research results involved a mixture of time-domain and frequency-domain techniques (Doyle, 1984; Francis, 1987; Glover, 1984). Recently, considerable attention has been focused on purely time-domain methods based on algebraic Riccati equations (ARE) (Doyle et al., 1989; Doyle and Glover, 1988; Khargonekar et al., 1988; Petersen, 1987; 1988; Stoorvogel, 1991; Stoorvogel and Trentelman, 1990; Tadmor, 1988; Zhou and Khargonekar, 1988). Along this line of research, connections are also made

---

* Received by the editors January 28, 1991 and in revised form June 29, 1991.
*[1] The work of B.M. Chen and A. Saberi is supported in part by Boeing Commercial Airplane Group.
*[2] The work of U.-L. Ly is supported in part by NASA Ames Dryden Flight Research Facility under grant contract NAG-2-629.

[1] School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.
[2] Department of Aeronautics and Astronautics, FS-10, University of Washington, Seattle, WA 98195, U.S.A.

between H_∞-optimal control and differential games (Papavassilopoulos and Safonov, 1989; Rhee and Speyer, 1989). Typically in the ARE approach to H_∞-optimal control problem, the achieved design is suboptimal in the sense that the H_∞-norm of the closed-loop system transfer function from the disturbance to the controlled output is less than a prescribed value. For the regular case†, the existence of the suboptimal state feedback law is formulated in terms of the existence of a stabilizing positive semi-definite solution for an "indefinite" ARE (see Doyle et al., 1989). In the singular case (i.e., not a regular case), the existence of suboptimal state feedback laws is equivalent to the existence of an $\varepsilon>0$ for which a certain ARE has a positive definite solution (Zhou and Khargonekar, 1988). A recent paper by Stoorvogel and Trentelman (1990) has shown that conditions for the existence of suboptimal state feedback laws can be expressed in terms of the existence of a solution to a quadratic matrix inequality. The solution of this inequality must also satisfy two rank conditions. Their conditions are very intuitive and reminiscent of the dissipation inequality in singular linear quadratic optimal control. Similar results are also obtained by Stoorvogel (1991) for suboptimal output-feedback laws.

The state-space approach using algebraic Riccati solutions provides basically an iterative scheme of approximating the infimum (denoted here by γ_s^*) of the norm of the closed-loop transfer function under state feedback laws. For example for the regular case, the computation proceeds as follows: one starts with a value of γ and then determines whether $\gamma>\gamma_s^*$ by solving an indefinite algebraic Riccati equation and looking for the stabilizing positive semi-definiteness of the solution. In the case where such a solution exists then we have $\gamma>\gamma_s^*$, and the procedure is then repeated with a smaller value of γ. In principle, one can approximate the infimum γ_s^* to within any degree of accuracy in this manner. However, this search procedure is exhaustive and can be very costly. More significantly due to the high-gain occurrence as γ gets close to γ_s^*, numerical solutions for algebraic Riccati equation can become highly sensitive and ill-conditioned. In fact, this difficulty becomes more severe in the singular case. So the iterative procedure based on algebraic Riccati solutions is not reliable and should not be used to determine the infimum γ_s^*. This paper presents a simple, accurate and non-iterative method of computing the exact value of γ_s^*. Our method is applicable to the class of systems for which their transfer function from the control input to the controlled output is right invertible and has no invariant zeros on the $j\omega$ axis.

In the non-iterative computation of γ_s^* our problem formulation does not place any restrictions on the direct feedthrough matrix between the control input and the controlled output variables, removing the limitations imposed by Petersen (1988) that require the existence of a nonsingular feedthrough term. Our problem formulation also differs from Scherer (1990) which appeared after the appearance of the preliminary conference version of this paper (Chen et al., 1990). In Scherer (1990), for the system that has no invariant zeros on the $j\omega$ axis and also has no infinite zeros, an iterative algorithm for computing γ_s^* is presented. Moreover, under an additional condition that the H_∞ norm of a certain transfer matrix is zero, his method becomes non-iterative. The major difference between our work and that of Scherer is that we have made no

† Regular case refers to a system where the feedthrough matrix from the input to the controlled output is injective.

restriction on the infinite zero structure of the system, however, in Scherer (1990) no infinite zero in the system is allowed. Such a relaxation of the constraints on infinite zero structure of a system is highly significant in H_∞ theory as has been pointed out in detail by Stoorvogel (1991).

Our method is very simple and avoids the well known complex computational problem associated with time/frequency domain approach. One of the key components of our method is to transform the problem using a special coordinate basis (s.c.b) transformation introduced in Sannuti and Saberi (1987) and Saberi and Sannuti (1990), which exhibits clearly the finite- and infinite-zero structures of the system among other system geometric properties. The other component utilizes the results of Stoorvogel and Trentelman (1990). The algorithm for computing γ_s^* has been implemented in a Matlab-software environment. Numerous examples are given in Sec. 6 to illustrate the computation of γ_s^* for aircraft control applications and control of a flexible mechanical system.

The outline of this paper is as follows. In Sec. 2 we introduce the problem statement. In Sec. 3 we recall the special coordinate basis (s.c.b) and its properties for non-strictly proper systems. This s.c.b transformation is instrumental in the derivation of the main results given in Sec. 4 for the exact computation of γ_s^*. Section 5 gives some applications of the results developed in Sec. 4 such as the problem of almost disturbance decoupling with internal stability, and the computation of the supremum of the complex stability radii which can be achieved by linear state feedback. Section 6 contains numerous illustrative examples and the conclusion is given in Sec. 7.

Throughout this paper we shall adopt the following conventions and notations:

A': transpose of A.
I: an identity matrix of appropriate dimension.
$\mathscr{R}$: the set of real numbers.
C: whole complex plane.
C^-: open left-half complex plane.
C^+: open right-half complex plane.
C^o: imaginary axis $j\omega$.
$\sigma_{\max}(A)$: maximum singular value of A.
$\lambda(A)$: the set of eigenvalues of A.
$\lambda_{\max}(A)$: maximum eigenvalue of A where $\lambda(A)\subset\mathscr{R}$.
$\mathrm{Ker}(V)$: kernal of V.
$\mathrm{Im}(V)$: image of V.

We also refer to the linear dynamical system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{1.1}$$

as the system (A, B, C, D). We also refer to $T_{yu}(s)=C(sI-A)^{-1}B+D$ as the transfer function matrix of the system (A, B, C, D) between the input u and the output y. For any real rational matrix $T(s)$,

$$\|T(s)\|_\infty \triangleq \sup\{\sigma_{\max}[T(j\omega)]:\ \omega\in\mathscr{R}\}, \tag{1.2}$$

then $\|T(s)\|_\infty$ coincides with the L_∞-norm of $T(s)$ if $T(s)$ is proper and has no poles in C^o, and with the H_∞-norm of $T(s)$ if it is proper and stable.

2. Problem Formulation

Let us consider the following linear system:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = x, \\ z = Cx + Du, \end{cases} \tag{2.1}$$

where $x \in \mathscr{R}^n$ is the state, $u \in \mathscr{R}^m$ is the input, $w \in \mathscr{R}^p$ is the disturbance, $y \in \mathscr{R}^n$ is the measured output for feedback control (here we consider state feedback) and $z \in \mathscr{R}^q$ is the controlled output. Let $T_{zw}(s)$ denote the transfer function matrix from the disturbance w to the controlled output z.

The standard H_∞-optimal control problem with state feedback is concerned with the construction of an internally stabilizing state feedback control-law $u = Fx$ that minimizes the H_∞-norm of $T_{zw}(s)$, where

$$T_{zw}(s) = (C+DF)(sI-A-BF)^{-1}E.$$

We define

$$\gamma_s^* \triangleq \inf\{\, \|T_{zw}(s)\|_\infty \quad \text{where} \quad u = Fx \quad \text{and} \quad A + BF \quad \text{is a stability matrix}\}$$

to be the infimum of the H_∞-optimization under state feedback laws. Most current state-space H_∞-optimization algorithms cannot determine γ_s^* exactly and can only provide lower and upper bounds to γ_s^*. In contrast, the problem addressed in the paper is the exact computation of the value of the infimum γ_s^* using a non-iterative method. In the next section we shall recall the definition of the special coordinate basis (s.c.b) for a linear time-invariant non-strictly proper system (Saberi and Sannuti, 1990). Such a coordinate basis has a distinct feature of explicitly displaying the infinite-zero and finite-zero structures of a given system as well as other system geometric properties. It is instrumental in the derivation of the numerical algorithm.

3. Special Coordinate Basis

In the following we recapitulate the main results in a theorem and some properties of the special coordinate basis while leaving detailed derivation and proofs to be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). Consider the system described in Eq. (2.1). It can be easily shown that using singular value decomposition one can always find an orthogonal transformation U and a nonsingular matrix V that put the direct feedthrough matrix D into the following form:

$$\bar{D} = UDV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.1}$$

where r is the rank of D. Without loss of generality, one can assume that the matrix D in Eq. (2.1) has the form as shown in Eq. (3.1). Thus the system in Eq.

(2.1) can be rewritten as

$$\dot{x} = Ax + [B_0 \quad B_1]\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + Ew, \tag{3.2}$$

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \tag{3.3}$$

where B_0, B_1, C_0 and C_1 are the matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely,

$$u = V\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = Uz.$$

Note that the H_∞-norm of the system transfer function $T_{zw}(s)$ is unchanged when we apply an orthogonal transformation on the output z, and under any nonsingular transformations on the states and control inputs. We have the following main theorem:

Theorem 3.1. There exist non-singular transformations Γ_s, Γ_o and Γ_i such that

$$x = \Gamma_s[x_a', x_b', x_c', x_f']', \quad x_a = [(x_a^-)', (x_a^+)']',$$

$$[z_0', z_1']' = \Gamma_o[z_0', z_f', z_b']', \quad [u_0', u_1']' = \Gamma_i[u_0', u_f', u_c']'$$

and

$$\Gamma_s^{-1}(A - B_0C_0)\Gamma_s = \begin{bmatrix} A_{aa}^- & A_o & L_{ab}^-C_b & 0 & L_{af}^-C_f \\ 0 & A_{aa}^+ & L_{ab}^+C_b & 0 & L_{af}^+C_f \\ 0 & 0 & A_{bb} & 0 & L_{bf}C_f \\ B_cE_{ca}^- & B_cE_{ca}^+ & B_cE_{cb} & A_{cc} & L_{cf}C_f \\ B_fE_{fa}^- & B_fE_{fa}^+ & B_fE_{fb} & B_fE_{fc} & A_{ff} \end{bmatrix}, \tag{3.4}$$

$$\Gamma_s^{-1}[B_0 \quad B_1]\Gamma_i = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix}, \tag{3.5}$$

$$\Gamma_o^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0f} \\ 0 & 0 & 0 & 0 & C_f \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix} \tag{3.6}$$

and

$$\Gamma_o^{-1}D\Gamma_i = \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.7}$$

where $C_f = [0 \quad I_{m_f}]$. Moreover, the pair (A_{cc}, B_c) is controllable, pair (A_{bb}, C_b) is observable and the subsystem (A_{ff}, B_f, C_f) is invertible with no invariant zeros.

The proof of this theorem can be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). We note that the output transformation Γ_o is of form,

$$\Gamma_o = \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{or} \end{bmatrix}. \tag{3.8}$$

In what follows, we state some important properties of the s.c.b which are pertinent to our present work.

Property 3.1. The given system (A, B, C, D) is right-invertible, if and only if x_b and hence z_b are nonexistent, left-invertible, if and only if x_c and hence u_c are nonexistent, invertible, if and only if both x_c and x_b are nonexistent.

Property 3.2. Invariant zeros of (A, B, C, D) are the eigenvalues of A_{aa}^- and A_{aa}^+. Moreover, the minimum-phase and nonminimum-phase invariant zeros of (A, B, C, D) are the eigenvalues of A_{aa}^- and A_{aa}^+, respectively.

Property 3.3. The pair (A, B) is stabilizable, if and only if (A_{con}, B_{con}) is stabilizable where

$$A_{con} = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{con} = \begin{bmatrix} B_{0a}^+ & L_{af}^+ \\ B_{0b} & L_{bf} \end{bmatrix}. \tag{3.9}$$

Property 3.4. If the system (A, B, C, D) is stabilizable and right-invertible, i.e., x_b is nonexistent, then the pair $(A_{aa}^+, [B_{0a}^+, L_{af}^+])$ is controllable.

There are interconnections between the s.c.b and various invariant and almost invariant geometric subspaces. To establish these interconnections, let us define the following subspaces:

- $V^g(A, B, C, D)$—the maximal subspace of $\mathscr{R}^n$ which is $(A+BF)$—invariant and contained in $\mathrm{Ker}(C+DF)$ such that the eigenvalues of $(A+BF) \mid V^g$ are contained in $C_g \subseteq C$ for some F.
- $S^g(A, B, C, D)$—the minimal $(A+KC)$—invariant subspace of $\mathscr{R}^n$ containing $\mathrm{Im}(B+KD)$ such that the eigenvalues of the map which is induced by $(A+KC)$ on the factor space $\mathscr{R}^n/S^g$ are contained in $C_g \subseteq C$ for some K.

For the cases that $C_g = C$, $C_g = C^-$ and $C_g = C^o \cup C^+$, we replace the index g in V^g and S^g by "*", "−" and "+", respectively. We list in the following the geometrical interpretations of some state vector components of s.c.b.

Property 3.5.
1. $x_a^- \oplus x_a^+ \oplus x_c$ spans $V^*(A, B, C, D)$.
2. $x_a^- \oplus x_c$ spans $V^-(A, B, C, D)$.
3. $x_a^+ \oplus x_c$ spans $V^+(A, B, C, D)$.
4. $x_c \oplus x_f$ spans $S^*(A, B, C, D)$.
5. $x_a^- \oplus x_c \oplus x_f$ spans $S^+(A, B, C, D)$.
6. $x_a^+ \oplus x_c \oplus x_f$ spans $S^-(A, B, C, D)$.

4. Computational Algorithm for γ_s^*

In this section we give a simple non-iterative procedure for determining γ_s^*. The method assumes that the system (A, B, C, D) is stabilizable, right-invertible and has no invariant zeros in C^o. The assumption of right invertibility is, for a nondegenerate case, equivalent to one where the number of control inputs must be greater than or equal to the number of controlled outputs. The other assumption on the invariant zeros is typical in H_∞-literature.

Before we give the proof of our results, let us outline the steps involved in the computation of γ_s^*,

Step 1: Transform the system (A, B, C, D) into the special coordinate basis s.c.b described in Sec. 3 and apply the state transformation matrix Γ_s to the disturbance input distribution matrix E as follows:

$$\Gamma_s^{-1}E = \begin{bmatrix} E_a^- \\ E_a^+ \\ E_c \\ E_f \end{bmatrix}. \tag{4.1}$$

Note that the component associated with x_b is missing since x_b is nonexistent for a right-invertible system (see Property 3.1).

Step 2: If the system (A, B, C, D) is of nonminimum phase then solve the following Lyapunov equations:

$$A_{aa}^+S + S(A_{aa}^+)' = [B_{0a}^+, L_{af}^+\Gamma_{or}^{-1}]\cdot[B_{0a}^+, L_{af}^+\Gamma_{or}^{-1}]', \tag{4.2}$$

$$A_{aa}^+T + T(A_{aa}^+)' = E_a^+(E_a^+)'. \tag{4.3}$$

Existence and uniqueness of the solutions S and T of the above Lyapunov equations follow from the fact that $\lambda(A_{aa}^+)\in C^+$ (i.e., $-A_{aa}^+$ is a stable matrix) since the eigenvalues of A_{aa}^+ are the right-half plane invariant zeros of the system (A, B, C, D). Moreover, from the Property 3.4 of Sec. 3, the pair $(A_{aa}^+, [B_{oa}^+, L_{af}^+\Gamma_{or}^{-1}])$ is controllable when the system (A, B, C, D) is stabilizable and right-invertible. The solution S of Eq. (4.2) is therefore positive definite and hence invertible.

Step 3: The infimum γ_s^* under state feedback control is given by

$$\gamma_s^* = \begin{cases} \sqrt{\lambda_{\max}(TS^{-1})} & \text{if } (A, B, C, D) \text{ is of nonminimum phase,} \\ 0 & \text{if } (A, B, C, D) \text{ is of minimum phase.} \end{cases} \tag{4.4}$$

Here we note that the eigenvalues of (TS^{-1}) are real and non-negative due to the fact that $S>0$ and $T\geq 0$.

We have the following main theorem:

Theorem 4.1. Consider the system of (2.1) and suppose that (A, B, C, D) is stabilizable, right-invertible and possesses no invariant zeros on C^o. Then γ_s^* as given in Eq. (4.4) is the infimum of the H_∞-optimal control under state feedback.

Proof. Let us apply a pre-feedback law,

$$u_0 = -C_0 x + v$$

to the system of (3.2) and (3.3). Then it is trivial to write the new system as,

$$\dot{x} = (A - B_0 C_0)x + [B_0 \quad B_1]\begin{bmatrix} v \\ u_1 \end{bmatrix} + Ew,$$

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ u_1 \end{bmatrix}.$$

It follows from the theorem s.c.b that there exist non-singular transformations, Γ_s and Γ_i such that

$$x = \Gamma_s[(x_a^-)', (x_a^+)', x_c', x_f']', \quad [v', u_1']' = \Gamma_i[v', u_f', u_c']'.$$

For a right-invertible system, the state component x_b is nonexistent and the transformed system is given by

$$\begin{bmatrix} \dot{x}_a^- \\ \dot{x}_a^+ \\ \dot{x}_c \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} A_{aa}^- & A_o & 0 & L_{af}^- C_f \\ 0 & A_{aa}^+ & 0 & L_{af}^+ C_f \\ B_c E_{ca}^- & B_c E_{ca}^+ & A_{cc} & L_{cf} C_f \\ B_f E_{fa}^- & B_f E_{fa}^+ & B_f E_{fc} & A_{ff} \end{bmatrix}\begin{bmatrix} x_a^- \\ x_a^+ \\ x_c \\ x_f \end{bmatrix} + \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0f} & B_f & 0 \end{bmatrix}\begin{bmatrix} v \\ u_f \\ u_c \end{bmatrix} + \begin{bmatrix} E_a^- \\ E_a^+ \\ E_c \\ E_f \end{bmatrix} w, \tag{4.5}$$

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{or} C_f \end{bmatrix}\begin{bmatrix} x_a^- \\ x_a^+ \\ x_c \\ x_f \end{bmatrix} + \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ u_f \\ v_c \end{bmatrix}, \tag{4.6}$$

where $\Gamma_{or} C_f = [0 \quad \Gamma_{or}]$ and Γ_{or} is a non-singular matrix as defined in (3.8). The above transformation of the system with a pre-state feedback law $u_0 = -C_o x + v$ along with the s.c.b state and control input transformations does not change our problem solution since it does not affect the value of γ_s^*.

Now, suppose that $\gamma > \gamma_s^*$. It is easy to verify that

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_a^+ & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0, \tag{4.7}$$

where

$$P_a^+ = \begin{cases} (S - T/\gamma^2)^{-1} > 0 & \text{if } (A, B, C, D) \text{ is of nonminimum phase,} \\ 0 & \text{if } (A, B, C, D) \text{ is of minimum phase,} \end{cases} \tag{4.8}$$

satisfies the quadratic matrix inequality and the rank conditions of Stoorvogel and Trentelman (1990). Hence it follows from Theorem 2.1 of Stoorvogel and Trentelman (1990) that there exists a state feedback law $F \in \mathscr{R}^{m\times n}$ such that $\|T_{zw}(s)\|_\infty < \gamma$ and $\lambda(A+BF) \in C^-$. If the system (A, B, C, D) is of minimum phase, $\gamma_s^* = 0$ and hence the converse of the theorem is trivial. We need to introduce the following lemmas in order to prove the converse part of this theorem when (A, B, C, D) is of nonminimum phase.

Lemma 4.1. Given the system of (2.1) along with the assumptions stated in the main theorem and $\gamma > 0$. Then there exists an $F \in \mathscr{R}^{m\times n}$ such that $\|T_{zw}(s)\|_\infty < \gamma$ and $\lambda(A+BF) \subset C^-$, if and only if there exists a real symmetric solution $P_a \geq 0$ to the algebraic Riccati equation

$$\begin{aligned} &P_a A_{aa} + A_{aa}' P_a + P_a E_a E_a' P_a / \gamma^2 \\ &\quad - P_a [B_{0a}, L_{af} \Gamma_{or}^{-1}][B_{0a}, L_{af} \Gamma_{or}^{-1}]' P_a = 0, \end{aligned} \tag{4.9}$$

such that

$$\lambda(A_{aa} + E_a E_a' P_a / \gamma^2 - [B_{0a}, L_{af}\Gamma_{or}^{-1}][B_{0a}, L_{af}\Gamma_{or}^{-1}]' P_a) \subset C^-, \tag{4.10}$$

where

$$\left.\begin{aligned} A_{aa} &= \begin{bmatrix} A_{aa}^- & A_o \\ 0 & A_{aa}^+ \end{bmatrix}, \quad B_{0a} = \begin{bmatrix} B_{0a}^- \\ B_{0a}^+ \end{bmatrix} \\ E_a &= \begin{bmatrix} E_a^- \\ E_a^+ \end{bmatrix}, \quad L_{af} = \begin{bmatrix} L_{af}^- \\ L_{af}^+ \end{bmatrix} \end{aligned}\right\}. \tag{4.11}$$

Proof. Without loss of generality, we assume that the given system has been transformed into the form of (4.5) and (4.6). Now let us define the new state variables,

$$x_a = \begin{bmatrix} x_a^- \\ x_a^+ \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_c \\ x_f \end{bmatrix},$$

where x_3 contains only the last m_f states of x_f which are directly associated with the controlled output z_1 while x_2 contains x_c and the remaining states of x_f. Hence, the dynamics of the transformed system can be partitioned as follows:

$$\left.\begin{aligned} \dot{x}_a &= A_{aa} x_a + [B_{0a} \quad L_{af}] \begin{bmatrix} v \\ x_3 \end{bmatrix} + E_a w \\ \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{22} \\ B_{32} \end{bmatrix} u_1 \\ &\quad + \begin{bmatrix} B_{21} & A_{21} \\ B_{31} & A_{31} \end{bmatrix} \begin{bmatrix} v \\ x_a \end{bmatrix} + \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} w \\ \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} x_a + \begin{bmatrix} I_r & 0 \\ 0 & \Gamma_{or} \end{bmatrix} \begin{bmatrix} v \\ x_3 \end{bmatrix} \end{aligned}\right\}, \tag{4.12}$$

where $A_{22}, A_{23}, \cdots, E_3$ are the matrices with appropriate dimensions. It is now straightforward to verify that the new system given above satisfies all the properties of Stoorvogel and Trentelman (1990) decomposition. Then the result follows from Corollary 5.2 and Theorem 6.2 of Stoorvogel and Trentelman (1990).

Lemma 4.2. Suppose that (A, B, C, D) is of nonminimum phase. Then the Riccati equation of (4.9) has a solution $P_a \geq 0$ such that condition (4.10) is satisfied, if and only if $S > T/\gamma^2$.

Proof. Suppose that $S > T/\gamma^2$ and define the positive definite matrix $X \triangleq S - T/\gamma^2$. It follows from (4.2) and (4.3) that

$$A_{aa}^+ X + X(A_{aa}^+)' - [B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}][B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}]' + E_a^+ (E_a^+)'/\gamma^2 = 0. \tag{4.13}$$

Now, let us pre- and post-multiply (4.13) by $P_a^+ \triangleq X^{-1}$, we obtain

$$P_a^+ A_{aa}^+ + (A_{aa}^+)' P_a^+ - P_a^+ [B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}][B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}]' P_a^+ + P_a^+ E_a^+ (E_a^+)' P_a^+ /\gamma^2 = 0.$$

From the above Riccati equation, we conclude

$$P_a^+ [A_{aa}^+ - [B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}][B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}]' P_a^+ + E_a^+ (E_a^+)' P_a^+ /\gamma^2] (P_a^+)^{-1} = -(A_{aa}^+)'.$$

Thus, the matrix $A_{aa}^+ - [B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}][B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}]' P_a^+ + E_a^+ (E_a^+)' P_a^+ /\gamma^2$ is stable. We now let

$$P_a \triangleq \begin{bmatrix} 0 & 0 \\ 0 & P_a^+ \end{bmatrix} \geq 0.$$

It can be verified by substitution that P_a is a solution to (4.9). Furthermore,

$$A_{aa} - [B_{0a}, L_{af} \Gamma_{or}^{-1}][B_{0a}, L_{af} \Gamma_{or}^{-1}]' P_a + E_a E_a' P_a /\gamma^2 = \begin{bmatrix} A_{aa}^- & A_o - [B_{0a}^-, L_{af}^- \Gamma_{or}^{-1}][B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}]' P_a^+ + E_a^- (E_a^+)' P_a^+ /\gamma^2 \\ 0 & A_{aa}^+ - [B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}][B_{0a}^+, L_{af}^+ \Gamma_{or}^{-1}]' P_a^+ + E_a^+ (E_a^+)' P_a^+ /\gamma^2 \end{bmatrix}, \tag{4.14}$$

is a stability matrix. Hence, P_a is a solution to (4.9) and satisfies condition (4.10).

Conversely, suppose that (4.9) has a solution $P_a \geq 0$ that satisfies (4.10). And let $N(P_a)$ denote the null space of matrix P_a. If $\mathrm{x} \in N(P_a)$, then it follows from (4.9) that

$$P_a A_{aa} \mathrm{x} = 0. \tag{4.15}$$

That is, $A_{aa} \mathrm{x} \in N(P_a)$. Hence, $N(P_a)$ is an A_{aa}-invariant subspace. Therefore, there exists an orthogonal transformation Q such that

$$\widetilde{P}_a = Q'P_aQ = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{P}_a^+ \end{bmatrix},$$

$$\widetilde{A}_{aa} = Q'A_{aa}Q = \begin{bmatrix} \widetilde{A}_{aa}^- & \widetilde{A}_o \\ 0 & \widetilde{A}_{aa}^+ \end{bmatrix},$$

$$\widetilde{E}_a = Q'E_a = \begin{bmatrix} \widetilde{E}_a^- \\ \widetilde{E}_a^+ \end{bmatrix}$$

and

$$[\widetilde{B}_{0a},\ \widetilde{L}_{af}\Gamma_{or}^{-1}] = Q'[B_{0a},\ L_{af}\Gamma_{or}^{-1}] = \begin{bmatrix} \widetilde{B}_{0a}^- & \widetilde{L}_{af}^-\Gamma_{or}^{-1} \\ \widetilde{B}_{0a}^+ & \widetilde{L}_{af}^+\Gamma_{or}^{-1} \end{bmatrix}. \tag{4.16}$$

Condition (4.10) implies that the matrix

$$\begin{aligned} &\widetilde{A}_{aa} - [\widetilde{B}_{0a},\ \widetilde{L}_{af}\Gamma_{or}^{-1}][\widetilde{B}_{0a},\ \widetilde{L}_{af}\Gamma_{or}^{-1}]'\widetilde{P}_a + \widetilde{E}_a\widetilde{E}_a'\widetilde{P}_a/\gamma^2 \\ &= \begin{bmatrix} \widetilde{A}_{aa}^- & \widetilde{A}_o - [\widetilde{B}_{0a}^-,\ \widetilde{L}_{af}^-\Gamma_{or}^{-1}][\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}]'\widetilde{P}_a^+ + \widetilde{E}_a^-(\widetilde{E}_a^+)'\widetilde{P}_a^+/\gamma^2 \\ 0 & \widetilde{A}_{aa}^+ - [\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}][\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}]'\widetilde{P}_a^+ + \widetilde{E}_a^+(\widetilde{E}_a^+)'\widetilde{P}_a^+/\gamma^2 \end{bmatrix}, \end{aligned} \tag{4.17}$$

is stable and hence the submatrices $\widetilde{A}_{aa}^+ - [\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}][\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}]'\widetilde{P}_a^+ + \widetilde{E}_a^+(\widetilde{E}_a^+)'\widetilde{P}_a^+/\gamma^2$ and $\widetilde{A}_{aa}^-$ must also be stable.

When we substitute (4.16) into (4.9), we obtain the following matrix Riccati equation:

$$\begin{aligned} &\widetilde{P}_a^+\widetilde{A}_{aa}^+ + (\widetilde{A}_{aa}^+)'\widetilde{P}_a^+ - \widetilde{P}_a^+[\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}][\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}]'\widetilde{P}_a^+ \\ &\quad + \widetilde{P}_a^+\widetilde{E}_a^+(\widetilde{E}_a^+)'\widetilde{P}_a^+/\gamma^2 = 0 \end{aligned} \tag{4.18}$$

or

$$\begin{aligned} &\widetilde{P}_a^+(\widetilde{A}_{aa}^+ - [\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}][\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}]'\widetilde{P}_a^+ + \widetilde{E}_a^+(\widetilde{E}_a^+)'\widetilde{P}_a^+/\gamma^2)(\widetilde{P}_a^+)^{-1} \\ &= -(\widetilde{A}_{aa}^+)'. \end{aligned} \tag{4.19}$$

Hence $\lambda(\widetilde{A}_{aa}^+)$ must be contained in C^+. Thus, we conclude that $\lambda(\widetilde{A}_{aa}^-) = \lambda(A_{aa}^-)$ and $\lambda(\widetilde{A}_{aa}^+) = \lambda(A_{aa}^+)$. Returning to the Riccati equation (4.18) and letting $X \trianglelefteq (\widetilde{P}_a^+)^{-1} > 0$, we have

$$\begin{aligned} &\widetilde{A}_{aa}^+X + X(\widetilde{A}_{aa}^+)' - [\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}][\widetilde{B}_{0a}^+,\ \widetilde{L}_{af}^+\Gamma_{or}^{-1}]' \\ &\quad + \widetilde{E}_a^+(\widetilde{E}_a^+)'/\gamma^2 = 0. \end{aligned} \tag{4.20}$$

Moreover, let $\widetilde{T}$ be the solution to the Lyapunov equation

$$\widetilde{A}_{aa}^+\widetilde{T} + \widetilde{T}(\widetilde{A}_{aa}^+)' = \widetilde{E}_a^+(\widetilde{E}_a^+)'. \tag{4.21}$$

Defining $\widetilde{S} = \widetilde{T}/\gamma^2 + X > \widetilde{T}/\gamma^2$, it can be shown that $\widetilde{S}$ satisfies the Lyapunov equation

$$\tilde{A}_{aa}^{+}\tilde{S} + \tilde{S}(\tilde{A}_{aa}^{+})' = [\tilde{B}_{0a}^{+}, \tilde{L}_{af}^{+}\Gamma_{or}^{-1}][\tilde{B}_{0a}^{+}, \tilde{L}_{af}^{+}\Gamma_{or}^{-1}]'. \tag{4.22}$$

Thus, the condition $\tilde{S}>\tilde{T}/\gamma^2$ is satisfied. Noting that the state transformation Γ_s in the special coordinate basis is in general non-unique, and the transformation Q can be exhausted into the state transformation Γ_s. Thus one can redefine the original state transformation to include the transformation Q and therefore reducing $Q=I$ in (4.16). Hence we can conclude that $S>T/\gamma^2$ also holds. This completes our proof of Lemma 4.2.

The converse part of our main theorem follows immediately from Lemmas 4.1 and 4.2 since the condition $\gamma>\sqrt{\lambda_{\max}(TS^{-1})}$ is equivalent to $S>T/\gamma^2$. This completes our proof of Theorem 4.1.

Remark 4.1: Under the condition that the feedthrough matrix D is nonsingular, i.e., the system (A, B, C, D) has no infinite zeros, it is simple to verify that matrix

$$P = (\Gamma_s^{-1})'\begin{bmatrix} 0 & 0 \\ 0 & (S-T/\gamma^2)^{-1} \end{bmatrix}\Gamma_s^{-1}$$

satisfies the well-known Riccati equation of the state feedback H_∞ control given in Doyle et al. (1989). Also, we would like to remark that for this special case, a similar result had been obtained by Petersen (1988) and the work of Kawatani and Kimura (1989) contains the similar reasoning.

Remark 4.2: The right invertibility condition of the system (A, B, C, D) in Theorem 4.1 can be weakened and replaced by the assumption,

$$\mathrm{Im}(E) \subseteq S^{+}(A, B, C, D) \cup S^{-}(A, B, C, D).$$

In this case, the algorithm for the computation of γ_s^* should be slightly modified. For the sake of brevity we have omitted the detailed discussion of this case in this paper.

5. Some Applications

Results developed in Sec. 4 can be used to examine the solvability condition of almost disturbance decoupling problem with internal stability via state feedback and can also be applied to a certain robustness problem, namely, the computation of the supremum of the complex stability radii which can be achieved under linear state feedback. These issues are considered in the following subsections:

5.1 Almost disturbance decoupling with stability The problem of almost disturbance decoupling was first introduced by Willems (see Weiland and Willems, 1989, for recent results and related references). The basic problem is the design of a linear state feedback such that the controlled output z is approximately decoupled from the disturbance input w. The more precise definition of this problem is given below:

Definition 5.1. Consider the system of (2.1). Then we say that the H_∞-Almost Disturbance Decoupling Problem with internal Stability $(\mathrm{ADDPS})_{H_\infty}$ is solvable if for all $\varepsilon>0$ there exists a state feedback law $u=F_x$ for the system defined above such that the closed-loop system is internally stable and the H_∞-norm of the transfer function between the disturbance input w and the controlled output z is less than ε.

From the above formulation, it is obvious that solvability condition for $(\mathrm{ADDPS})_{H_\infty}$ is exactly the condition under which $\gamma_s^*=0$. Solvability condition for $(\mathrm{ADDPS})_{H_\infty}$ with $D=0$ is well-known (see Weiland and Willems, 1989). In the following theorem, we extend this result to the general case where $D\neq 0$.

Lemma 5.1. Consider the system Σ as given by (2.1). Assume that (A, B, C, D) is right-invertible and has no invariant zeros on $j\omega$ axis. Then $(\mathrm{ADDPS})_{H_\infty}$ is solvable, i.e., $\gamma_s^*=0$, if and only if $\mathrm{Im}(E)\subseteq S^+(A, B, C, D)$.

Proof. If the system (A, B, C, D) is of minimum phase, $\gamma_s^*=0$ and for minimum-phase system $\mathrm{Im}(E)$ is always contained in $S^+(A, B, C, D)$. In what follows, we proceed to prove the case when the system (A, B, C, D) is of nonminimum phase. It is straightforward to verify that $\mathrm{Im}(E)\subseteq S^+(A, B, C, D)$ implies that $E_a^+=0$. Then from (4.3), we have $T=0$ and hence $\gamma_s^*=0$. Conversely, if $\mathrm{Im}(E)\not\subseteq S^+(A, B, C, D)$, then $E_a^+\neq 0$. Again it is simple to see from (4.3) that $T\neq 0$ and hence $\gamma_s^*\neq 0$. This completes our proof of the corollary.

5.2 Maximizing the complex stability radius In this subsection, we consider an uncertain linear system,

$$\dot{x} = Ax + Bu + D\Delta Ex, \tag{5.1}$$

where $A\in\mathscr{R}^{n\times n}$, $B\in\mathscr{R}^{n\times m}$, $D\in\mathscr{R}^{n\times l}$ and $E\in\mathscr{R}^{p\times n}$ are given constant matrices while Δ expresses the uncertainty which is structured by the matrices D and E. Moreover, we assume that (A, B) is stabilizable. For any stabilizing state feedback law $u=Fx$, $F\in\mathscr{R}^{n\times m}$ (i.e., $\lambda(A+BF)\subset C^-$), the complex stability radius is defined as (Hinrichsen and Pritchard, 1989)

$$r_c(A, B, D, E, F) \triangleq \inf\{\|\Delta\|:\ \Delta\in C^{l\times p}\quad \text{such that}\quad A+BF+D\Delta E\quad \text{is unstable}\}.$$

The supremum of the complex stability radii that can be achieved by stabilizing linear state feedback law is defined as

$$\bar{r}_c(A, B, D, E) = \sup\{r_c(A, B, D, E, F):\ F\in\mathscr{R}^{m\times n}\quad \text{and}\quad A+BF\quad \text{is stable}\}.$$

At a first glance, it seems that complex perturbation is not natural and should not play a role in robustness analysis. However, it turned out that complex stability radius is important for two good reasons: First of all, it provides a lower bound for the real stability radius (defined as the complex stability radius but with the restriction that Δ be a real matrix), and there are important special cases where the real and complex stability radii coincide. Moreover, there are elegant results for the complex stability radii while that is not the case for the real stability radii. Secondly, it turned out that the complex stability radii are equivalent to real dynamic stability radii, i.e., Δ is a real dynamic perturbation (for further details and a recent survey of literature, see Hinrichsen and Pritchard, 1989).

Our results in this paper provide a simple non-iterative way of computing $\bar{r}_c(A, B, D, E)$. We assume that (E, A, B) is right-invertible and has no invariant zeros on the $j\omega$ axis. The algorithm for computing $\bar{r}_c(A, B, D, E)$ is given below:

Step 1: Using the result of Sec. 4, find the infimum of the H_∞-optimization for the system

$$\Sigma_e: \begin{cases} \dot{x} = Ax + Bu + Dw, \\ y = x, \\ z = Ex. \end{cases} \tag{5.2}$$

Let this infimum be denoted by γ^*_{se}.

Step 2:

$$\bar{r}_c(A, B, D, E) = \frac{1}{\gamma^*_{se}}. \tag{5.3}$$

We have the following lemma and corollary:

Lemma 5.2. Assume that (A, B, E) is right-invertible and has no invariant zeros on the $j\omega$ axis. Then $\bar{r}_c(A, B, D, E) = 1/\gamma^*_{se}$.

Proof. The proof follows from the fact (Hinrichsen and Pritchard, 1989) that

$$r_c(A, B, D, E, F) = \|G_F\|_\infty^{-1},$$

where G_F denotes the transfer function matrix of $(A+BF, D, E, 0)$.

Corollary 5.1. Assume that (A, B, E) is right-invertible and has no invariant zeros on the $j\omega$ axis. Then $\bar{r}_c(A, B, D, E) = \infty$, if and only if $\mathrm{Im}(D) \subseteq S^+(A, B, E, 0)$.

Proof. It follows from Lemmas 5.1 and 5.2.

6. Numerical Examples

We will demonstrate the procedure of the computation of γ^*_s with two examples in aircraft control, one in control of a flexible mechanical system and one illustrating the result of $(\mathrm{ADDPS})_{H_\infty}$.

Example 1. The first example involves the minimization of the H_∞-norm of the aircraft normal acceleration z response to longitudinal turbulence w for a B767 longitudinal aircraft model using elevator control u. State model of this system is given in the following for a flight condition of mach 0.80, altitude 35,000 [ft] and center of gravity at 0.18MAC (mean aerodynamic chord):

$$\dot{x} = \begin{bmatrix} -0.01675 & 0.11214 & 0.00028 & -0.56083 & 0.01675 & -0.02432 \\ -0.0164 & -0.77705 & 0.99453 & 0.00147 & 0.0164 & -0.06339 \\ -0.04167 & -3.6595 & -0.95443 & 0 & 0.04167 & -3.6942 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4447 & 0 \\ 0 & 0 & 0 & 0 & 0 & -15 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 15 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.9431 \\ 0 \end{bmatrix} w,$$

$$z = [0.00694 \quad 0.32795 \quad 0.00231 \quad 0 \quad -0.00694 \quad 0.02679]x.$$

The system $(A, B, C, 0)$ is invertible with invariant zeros at: $\{-6.7743, 6.13546, -8.1557\times10^{-3}, -4.3926\times10^{-4}, -0.4447\}$. Note that the system has a nonminimum phase zero at 6.13546 [rad/s]. Following the procedure developed in Sec. 4, we obtain

$$A^+_{aa} = 6.13546002, \quad B^+_{0a} = 0, \quad L^+_{af} = -169.9647,$$

$$\Gamma_{or} = 1 \quad \text{and} \quad E^+_a = -0.14448956,$$

which yield

$$S = 2354.18357133, \quad T = 0.00170136$$

and

$$\gamma^*_s = \sqrt{TS^{-1}} = 8.50115113 \times 10^{-4}.$$

Example 2. The second example is the control of a flexible mechanical system consisting of four discs connected by flexible rods and given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w,$$

$$z = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]x.$$

The system $(A, B, C, 0)$ is invertible and has no invariant zeros; hence it is of minimum-phase. The infimum of H_∞-optimization with state feedback is therefore equal to zero, i.e., $\gamma_s{}^*=0$. Again it is difficult to determine this solution using an ARE-based method. Usually the Riccati-solver tends to break down even when γ is not close to γ^*_s.

Example 3. This example is a disturbance rejection design for a fighter aircraft. The model is for an AFTI-F16 longitudinal aircraft flying at mach 0.90 and altitude of 20,000 [ft] and is given by

$$\dot{x} = \begin{bmatrix} -0.011669 & 0.024753 & -0.5271 & -0.5601 & 0.001496 & 0.011669 & -0.024753 & 0 \\ -0.044669 & -1.437 & 16.14 & -0.018286 & -1.065\times10^{-7} & 0.044669 & 1.437 & 0 \\ -0.088168 & -0.080495 & -0.7046 & 0 & 7.2141\times10^{-7} & 0.088168 & 0.080495 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.5303 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5303 & 0.005303 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.005303 & -0.5303 \end{bmatrix} x$$

$$+ \begin{bmatrix} 0.017112 & 0 \\ -2.304 & 0 \\ -21.7 & 0 \\ 0 & 0 \\ 0 & 99.06 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.03 & 0 \\ 0 & 1.261 \\ 0 & -53.32 \end{bmatrix} w,$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0.00073647 & 0.044067 & -0.003463 & -1.298\times10^{-6} & 8.6435\times10^{-9} & -0.00073647 & -0.044067 & 0 \end{bmatrix} x$$

$$+ \begin{bmatrix} 0 & 0 \\ -0.044389 & 0 \end{bmatrix} u.$$

The system (A, B, C, D) is invertible, unstable and nonminimum phase with invariant zeros at $\{-1.3692 \pm 18.637i,\ 1.1338\times10^{-3},\ -0.5303,\ -0.5303 \pm 5.3030\times10^{-3} i\}$. Again, following the procedure developed in Sec. 4, we obtain

$$A_{aa}^{+} = 0.00113375, \quad B_{0a}^{+} = -750.07999321, \quad L_{af}^{+} = -0.13170630$$

and

$$\Gamma_{or} = 1, \quad E_{a}^{+} = [0.0000016557,\ -0.03608666].$$

Then solving two Lyapunov equations (4.2) and (4.3), we have

$$S = 2.48122601\times 10^{8}, \quad T = 0.57430758$$

and

$$\gamma_s^* = \sqrt{TS^{-1}} = 4.81104160\times 10^{-5}.$$

Example 4. Consider a system characterized by

$$\dot{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} w,$$

$$z = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u.$$

One can easily check that system (A, B, C, D) is square and invertible with two nonminimum phase invariant zeros at $\{1, 2\}$. Furthermore, this system is already in the form of s.c.b and $E_a^+ = [0 \quad 0]'$. Hence, it follows from Lemma 5.1 that $\gamma_s^* = 0$.

7. Conclusions

In this paper we have presented a simple and non-iterative algorithm for the computation of the infimum in the standard H_∞-optimization problem via state feedback. The results are applicable to systems where the transfer function from the control input to the controlled output is right-invertible and has no invariant zeros on the $j\omega$ axis. The procedure involves solutions of two Lyapunov equations usually of low dimensionality determined from the number of nonminimum phase finite invariant zeros. Two applications of our results have also been considered. In the first application we gave the solvability condition for H_∞-almost disturbance decoupling problem via state feedback with internal stability, while in the second application we provide the computation of the supremum of the complex stability radii which can be achieved under linear state feedback. The algorithm for the computation of the infimum in H_∞-optimization using output feedback is the subject of Chen et al. (1992).

References

Chen, B.M., A. Saberi and U. Ly (1990). Exact computation of the infimum in H_∞-optimization via state feedback. *Proc. of the 28th Annual Allerton Conference on Communication, Control and Computing*, Monticello, IL, 745–754.

Chen, B.M., A. Saberi and U. Ly (1992). Exact computation of the infimum in H_∞-optimization via output feedback. *IEEE Trans. Automatic Control*, **AC-37**, 1, 70–78.

Doyle, J.C. (1984). *Lecture Notes in Advances in Multivariable Control.* ONR/Honeywell Workshop.

Doyle, J.C., K. Glover, P.P. Khargonekar and B.A. Francis (1989). State space solutions to standard H_2 and H_∞ control problems. *IEEE Trans. Automatic Control*, **AC-34**, 8, 831–847.

Doyle, J.C. and K. Glover (1988). State space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity. *Systems & Control Letters*, **11**, 167–172.

Francis, B.A. (1987). *A Course in H_∞ Control Theory*. Lecture Notes in Control and Information Sciences, **88**, Springer-Verlag, Berlin.

Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds. *Int. J. Control*, **39**, 6, 1115–1193.

Hinrichsen, D. and A.J. Pritchard (1989). Real and complex stability radii: A survey. *Proc. of an International Workshop*, Bremen, West Germany, 119–162.

Kawatani, R. and H. Kimura (1989). Synthesis of reduced-order H_∞ controllers based on conjugation. *Int. J. Control*, **50**, 2, 525–541.

Khargonekar, P., I.R. Petersen and M.A. Rotea (1988). H_∞-optimal control with state feedback. *IEEE Trans. Automatic Control*, **AC-33**, 8, 786–788.

Papavassilopoulos, G.P. and M.G. Safonov (1989). Robust control design via game theoretic methods. *Proc. of IEEE Conf. on Decision and Control*, Tampa, FL, 382–387.

Petersen, I.R. (1987). Disturbance attenuation and H_∞-optimization: A design method based on the algebraic Riccati equation. *IEEE Trans. Automatic Control*, **AC-32**, 5, 427–429.

Petersen, I.R. (1988). Complete results for a class of state feedback disturbance attenuation problems. *Proc. of IEEE Conf. on Decision and Control*, Austin, TX, 1349–1353.

Rhee, I. and J.L. Speyer (1989). A game theoretic controller and its relationship to H_∞ and linear exponential Gaussian synthesis. *Proc. of IEEE Conf. on Decision and Control*, Tampa, FL, 909–915.

Saberi, A. and P. Sannuti (1990). Squaring down of non-strictly proper systems. *Int. J. Control*, **51**, 3, 621–629.

Sannuti, P. and A. Saberi (1987). Special coordinate basis for multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling. *Int. J. Control*, **45**, 5, 1655–1704.

Scherer, C. (1990). H_∞ control by state feedback and fast algorithm for the computation of optimal H_∞ norms. *IEEE Trans. Automatic Control*, **35**, 10, 1090–1099.

Stoorvogel, A.A. (1991). The singular H_∞-control problem with dynamic measurement feedback. *SIAM J. Control and Optimization*, **29**, 1, 160–184.

Stoorvogel, A.A. and H.L. Trentelman (1990). The quadratic matrix inequality in singular H_∞-control with state feedback. *SIAM J. Control and Optimization*, **28**, 5, 1190–1208.

Tadmor, G. (1988). H_∞ in the time domain: The standard four blocks problem. Preprint.

Weiland, S. and J.C. Willems (1989). Almost disturbance decoupling with internal stability. *IEEE Trans. Automatic Control*, **AC-34**, 3, 277–286.

Zames, G. (1981). Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Automatic Control*, **AC-26**, 2, 301–320.

Zhou, K. and P. Khargonekar (1988). An algebraic Riccati equation approach to H_∞-optimization. *Systems & Control Letters*, **11**, 85–91.

Ben M. Chen was born in Fuqing, Fujian, China, on November 25, 1963. He received the B.S. in mathematics and computer science from Amoy University, Xiamen, China, in 1983, the M.S. in electrical engineering from Gonzaga University, Spokane, Washington, in 1988, and the Ph.D. in electrical and computer engineering from Washington State University, Pullman, Washington, in 1991.

From 1983 to 1986 he worked as a software engineer in the South-China Computer Corporation, China. From 1986 to 1988 he held a Presidential scholarship at Gonzaga University and a Cardinal Yu-Pin scholarship from the Sino-American Amity Fund, Inc., New York. He is currently a post-doctoral research associate in the School of Electrical Engineering and Computer Science, Washington State University, Pullman. His current research interests are in robust and optimal control theory, linear system theory and computer aided design for control systems.

Ali Saberi is working in the School of Electrical Engineering & Computer Science, Washington State University, Pullman, Washington, U.S.A.

Uy-Loi Ly received the B.Sc. degree in aeronautics and mathematics from the University of California, Davis in 1975 and the M.Sc. degree in applied mechanics and the Engineer degree in aeronautics from the California Institute of Technology, Pasadena, in 1976 and 1977, respectively. From 1980 to 1982 he returned to complete the Ph.D. degree at Stanford University under the guidance of Professors A.E. Bryson and R.H. Cannon in the Department of Aeronautics and Astronautics. From 1977 to 1988 he was with the Boeing Company as a Principal Engineer in the Flight Controls Research organization responsible for the development of practical design methods for the synthesis and analysis of multivariable control systems. Since 1988, he has been on the faculty of the Department of Aeronautics and Astronautics at the University of Washington. His current interests are in the research and practical application of multivariable control theory to aerospace related problems.