CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol.6, No.4, pp.573–594, December, 1990
C90019R

Contributed Paper

LOOP TRANSFER RECOVERY FOR NON-STRICTLY PROPER PLANTS*

B. M. CHEN,[1] A. SABERI,[1] S. BINGULAC[2] AND P. SANNUTI[3]

Abstract. Observer based controllers for loop transfer recovery of non-strictly proper systems which are left invertible and of minimum phase are considered. A complete analysis of loop transfer recovery problem using either full or reduced order observer based controller is provided.

Key Words—Robust control, loop transfer recovery.

1. Introduction and Problem Statement

In multi-input and multi-output feedback control system design, performance specifications such as command following, disturbance rejection, closed-loop band-width, stability robustness with respect to unstructured dynamic uncertainties etc., are naturally posed in frequency domain in terms of sensitivity and complementary sensitivity functions (Doyle and Stein, 1981). These sensitivity and complementary sensitivity functions are related to the loop transfer matrices evaluated by breaking the control loop at critical points, commonly either the input or output point of the given plant. Thus typically, one is interested in designing a closed-loop control system to arrive at a specified loop transfer function. In this paper, we concentrate on a case when the uncertainties are modeled at the input point of a nominal plant model and hence the required loop transfer function is specified at the plant input point. However, our results can be dualized for the case when the required loop transfer function is specified at the output point. In recent years, a design procedure called LQG/LTR, originally proposed by Doyle and Stein (1979) has gained some prominences. Essentially, LQG/LTR is a two step design procedure. In the first step of design, a standard state feedback design is done so that the resulting loop transfer function at the plant input point, here after called as a target loop transfer function, meets the given specifications. In the second step of design, one first assumes a closed-loop configuration as in Fig. 1 where $C(s)$ and $P(s)$ are respectively the transfer functions of a controller and the given plant. Given $P(s)$ and the target loop transfer function $L(s)$, one seeks to design a $C(s)$ such

* Received by the editors February 13, 1990 and in revised form July 16, 1990.
The work of B.M. Chen and A. Saberi is supported in part by National Science Foundation under grant No. ECS 8618953 and in part by Boeing Commercial Airplane Group.

[1] Department of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.

[2] Bradley Department of Electrical Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.

[3] Department of Electrical and Computer Engineering, P.O. Box 909, Rutgers University, Piscataway, NJ 08855-0909, U.S.A.

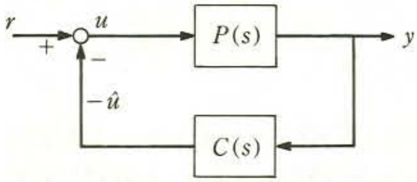


Fig. 1. Closed-loop configuration—plant with controller.

that $C(j\omega)P(j\omega)$ is either exactly or approximately equal to $L(j\omega)$ in the frequency region of interest. This second step of design is termed as LTR design.

Ever since the seminal work of Doyle and Stein (1979), there have been many papers on LTR using either full or reduced order observer based controllers. All these papers, however, assume that the given plant is strictly proper. So far there is no method what so ever in the literature to deal with LTR design for non-strictly proper systems. In this paper, we focus our attention on the loop transfer recovery design for non-strictly proper plants. Let us consider a left invertible and minimum phase plant Σ,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1.1}$$

where the state vector $x \in R^n$, output vector $y \in R^p$ and input vector $u \in R^m$. Let us assume that Σ is stabilizable and detectable. Without loss of generality, we also assume that $[B', D']'$ and $[C, D]$ are of maximal rank. Let F be a stabilizing full state feedback gain matrix such that (a) the closed-loop system is asymptotically stable, i.e., eigenvalues of $A - BF$ lie in the open left half s-plane, and (b) the open-loop transfer function when the loop is broken at the input point of the plant meets the given frequency dependent specifications. The state feedback control law is

$$u = -Fx, \tag{1.2}$$

and the loop transfer function evaluated when the loop is broken at the input point of the plant, the so called target loop transfer function, is

$$L(s) = F\Phi B, \tag{1.3}$$

where $\Phi = (sI - A)^{-1}$. Instead of using the state feedback control law of (1.2), if one uses output feedback controller $C(s)$ as in Fig. 1, then the achieved loop transfer function evaluated when the loop is broken at the input of the plant is

$$\hat{L}(s) = C(s)P(s), \quad P(s) = C\Phi B + D, \tag{1.4}$$

and thus our goal is to design $C(s)$ such that the mismatch function $E(j\omega)$ with $E(s)$ defined as

$$E(s) = L(s) - \hat{L}(s) \tag{1.5}$$

is either exactly zero or in some sense approximately zero over the frequency range of interest. More precisely, we say exact LTR (ELTR) is achieved if

$$C(s)P(s) = L(s) \quad \text{for all } s.$$

Achieving ELTR is in general not possible. In an attempt to achieve "approximate" LTR, one normally parameterizes $C(s)$ as a function of a scalar or a vector parameter σ and thus obtains a family of controllers $C(s, \sigma)$. We say asymptotic LTR (ALTR) is achieved if

$$C(s, \sigma)P(s) \to L(s) \quad \text{pointwise in } s$$

as the tuning parameter $\sigma \to \infty$, or equivalently $E(s, \sigma) \to 0$ pointwise in s as $\sigma \to \infty$. Achievability of ALTR enables the designer to choose a member of the family of controllers that corresponds to a particular value of σ that achieves a desired level of recovery.

Regarding the structure for controller $C(s)$ or $C(s, \sigma)$, one has complete freedom to choose any appropriate structure for it. All the existing LTR literature for strictly proper systems, except for Chen, Saberi and Sannuti (1990) assumes a full or reduced order observer based controller. In this paper, we will study only the traditional observer based controllers, either full or reduced order type. A study similar to this paper but using appropriate compensator structures will be a subject of our future research.

The paper is organized as follows. In Sec. 2, we develop the full order observer based controller which achieves either ELTR or ALTR. And in Sec. 3, we consider the LTR design by using the reduced order observer based controller. Two numerical examples are given in Sec. 4 to illustrate the developed theory of loop transfer recovery for non-strictly proper systems.

Throughout this paper, A' denotes the transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. $\lambda(A)$ denotes the set of eigenvalues of A. Similarly, $\sigma_{\max}[A]$ and $\sigma_{\min}[A]$ respectively denote the maximum and minimum singular values of A. The open left and closed right half s-plane are respectively denoted by C^- and C^+.

2. Full Order Observer Based Controller

In this section, we consider the loop transfer recovery design via the full order observer based controllers. Without loss of generality but for simplicity of presentation, we will assume that D matrix in (1.1) has been transformed in the form of

$$D = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where r is the rank of D and $r \le m$ (left invertibility of Σ implies that $m \le p$). Let us partition the state equation of (1.1) as

$$\left.\begin{aligned} \dot{x} &= Ax + B_0 u_0 + B_1 u_1 \\ y_0 &= C_0 x + u_0, \quad y_1 = C_1 x \end{aligned}\right\}. \tag{2.2}$$

Here the control u is partitioned as $u=[u_0', \ u_1']'$ where u_0 and u_1 are of dimension r and $m-r$ respectively. And the output y is partitioned as $y=[y_0', y_1']'$ where y_0 and y_1 are respectively of dimension r and $p-r$. Hence, we can rewrite (2.2) as a new system characterized by the triple (A_1, B_1, C_1),

$$\left. \begin{aligned} \dot{x} = A_1 x + B_1 u_1 + B_0 y_0, \quad y_0 &= C_0 x + u_0 \\ y_1 &= C_1 x \end{aligned} \right\}, \tag{2.3}$$

where $A_1=A-B_0C_0$ and y_0 is considered as a known signal. We have the following results:

Remark 2.1: Note that if D is of full rank (i.e., $r=m$), then $B_1=0$.

Lemma 2.1. Consider the system Σ_1 characterized by the triple (A_1, B_1, C_1) with $B_1 \neq 0$. Then, we have

1. Σ_1 is left invertible and of minimum phase if and only if Σ is left invertible and of minimum phase.
2. Invariant zeros of Σ_1 are the same as those of the given plant Σ.

Proof. It is trivial to verify this lemma by following the properties of a special coordinate basis (s.c.b) for non-strictly proper systems (Saberi and Sannuti, 1990; Also, see the details of s.c.b in Appendix A).

Now consider a full order observer based controller configuration as given in Fig. 2. Note that in Fig. 2 we have partitioned the state feedback gain matrix F in conformity with u_0 and u_1 as

$$F = \begin{bmatrix} F_{u_0} \\ F_{u_1} \end{bmatrix}.$$

The dynamics of the observer-based controller in Fig. 2 can be expressed as

$$\dot{\hat{x}} = A_1 \hat{x} + B_0 y_0 + B_1 u_1 + K(y_1 - C_1 \hat{x}) \tag{2.4}$$

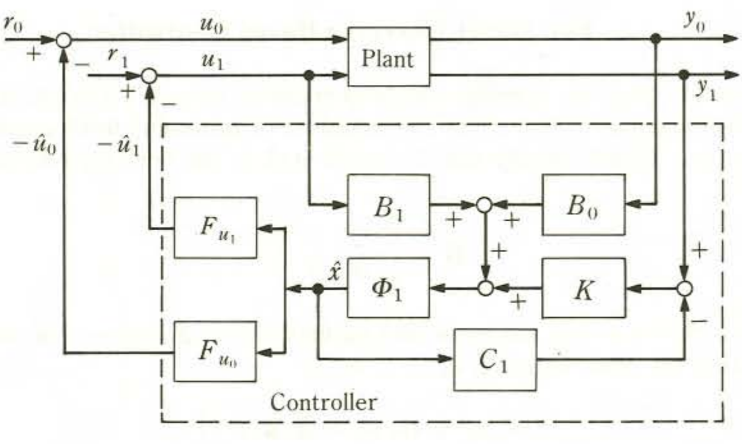


Fig. 2. Plant with the full order observer based controller.

and

$$u_0 = \hat{u}_0 = -F_{u_0}\hat{x}, \quad u_1 = \hat{u}_1 = -F_{u_1}\hat{x}. \tag{2.5}$$

By examining (2.3) and (2.4), we note that (2.4) is nothing more than a conventional observer. Thus, it is simple to verify that the well-known separation principle holds and the stability of the closed-loop system is guaranteed whenever $A-BF$ and A_1-KC_1 are stable. The transfer function of the observer based controller (i.e., the transfer function from y to $-\hat{u}$, when u and $\hat{u}$ are considered as two separate variables) can be easily obtained,

$$C(s) = F(sI-A_1+B_1F_{u_1}+KC_1)^{-1}[B_0,\ K], \tag{2.6}$$

where K is the only unknown matrix which is considered as a free design parameter. In what follows, we examine the necessary and sufficient conditions for loop transfer recovery of non-strictly proper systems.

Lemma 2.2. Consider the closed-loop system comprising of the given plant Σ along with the observer based controller (2.4) and (2.5) as given in Fig. 2. Then the mismatch function as defined in (1.6) can be expressed as

$$E(s) = M(s)[I_m+M(s)]^{-1}(I_m+F\Phi B), \tag{2.7}$$

where

$$M(s) = [M_0(s),\ M_1(s)], \tag{2.8}$$

and where $m\times r$ and $m\times(m-r)$ dimensional matrices $M_0(s)$ and $M_1(s)$ are given by

$$M_0(s) \equiv 0, \quad M_1(s) = F(\Phi_1^{-1}+KC_1)^{-1}B_1 \tag{2.9}$$

with $\Phi_1=(sI-A_1)^{-1}=(sI-A+B_0C_0)^{-1}$.

Proof. See Appendix B.

Lemma 2.3.

$$E(j\omega) = 0 \quad \text{if and only if } M_1(j\omega) = 0 \quad \text{for all } \omega \in \Omega,$$

where Ω is the set of all $0\le\omega<\infty$ for which $L(j\omega)$ and $\hat{L}(j\omega)$ are well defined (i.e., all required inverses exist).

Proof. It is obvious.

Thus Eq. (2.7) presents a clear perspective to study the basic mechanism by which both exact and approximate LTR occurs for non-strictly proper systems. It is clear that ELTR is achievable if $M_1(j\omega)=0$ exactly and on the other hand ALTR is achievable if $\sigma_{\max}[M_1(j\omega)]$ can be made arbitrarily small for all ω. We have the following theorem for ELTR.

Theorem 2.1. Consider the closed-loop system comprising of the given plant Σ and the full order observer based controller (2.4) and (2.5) as given in Fig. 2.

Then both the asymptotic stability of the closed-loop system and ELTR can be achieved under the following conditions:
1. $FB_1=0$.
2. C_1B_1 is of maximal rank (i.e., the plant has no infinite zero of order higher than one).
3. The given plant is left invertible and is of minimum phase.

Proof. Under the conditions given in the theorem, a constructive method of obtaining the observer gain K to achieve both closed-loop stability and ELTR is given in Appendix C.

Corollary 2.1. For a left invertible and minimum phase plant as in (1.1), if D matrix is of full rank, i.e., $r=m$, then ELTR is achievable.

Proof. $r=m$ implies that $B_1=0$ and hence $M(s)\equiv M_0(s)\equiv 0$.

Remark 2.2: ELTR is achievable for any single-input and single-output non-strictly proper system if it is of minimum phase.

Since $FB_1=0$ severely restricts the class of loop transfer functions that are exactly achievable, we now focus our attention on ALTR for non-strictly proper systems. In ALTR, the gain K is parameterized in terms of a tuning parameter σ. Thus we rewrite $M_1(s)$ as

$$M_1(s,\ \sigma) = F(\Phi_1^{-1}+K(\sigma)C_1)^{-1}B_1. \tag{2.10}$$

Most of the existing literature focuses attention on how to design the observer gain $K(\sigma)$ such that

$$(\Phi_1^{-1}+K(\sigma)C_1)^{-1}B_1\rightarrow 0 \quad \text{pointwise in } s \quad \text{as } \sigma\rightarrow\infty, \tag{2.11}$$

which is known as Doyle-Stein condition. (2.11) is a sufficient condition to render $\sigma_{\max}[M_1(j\omega,\ \sigma)]$ and hence $\sigma_{\max}[M(j\omega,\ \sigma)]$ arbitrarily small for all ω. Doyle and Stein (1979) gave another sufficient condition under which (2.11) is true. Their condition is as follows: Let $K(\sigma)$ be chosen such that as $\sigma\rightarrow\infty$, $K(\sigma)/\sigma\rightarrow B_1W$ for some nonsingular matrix W. Then, (2.11) is true and consequently ALTR is achieved as $\sigma\rightarrow\infty$. Also, there were several attempts later on to weaken Doyle-Stein condition (Madiwale and Williams, 1985; Matson and Maybeck, 1987; Saberi and Sannuti, 1990). It is well known that in order to satisfy (2.11), one needs that the triple $(A_1,\ B_1,\ C_1)$, namely Σ_1, is left invertible and of minimum phase. However, from Lemma 2.1, it follows that Σ_1 is left invertible and of minimum phase if and only if the given plant is left invertible and minimum phase. Thus existing design methods can be used to find the gain $K(\sigma)$. In comparison with the sufficient conditions for ELTR as stated in Theorem 2.1, one finds a drastic relaxation of the required conditions for ALTR.

Theorem 2.2. Consider the closed-loop system comprising of the given plant Σ and the full order observer based controller (2.4) and (2.5) as given in Fig. 2. Let the given non-strictly proper plant be left invertible and be of minimum phase. Then a gain $K(\sigma)$ can be designed such that both asymptotic stability of the closed-loop system and ALTR can be achieved.

Proof. The proof is obvious in view of (2.11) and the well known results for strictly proper systems.

As discussed above, most often one opts for ALTR design as it requires less stringent conditions than ELTR design. In ALTR, the level of recovery depends on $\sigma_{\max}[M(j\omega,\ \sigma)]$. However in order to render $\sigma_{\max}[M(j\omega,\ \sigma)]$ small, one needs to increase the tuning parameter σ which itself increases the gain $K(\sigma)$. Thus as discussed by Sogaard-Andersen and Niemann (1989), there is a fundamental trade-off between the level of recovery and the size of gain. This trade-off can be visualized in a natural way in terms of the trade-off between the singular values of sensitivity and complementary sensitivity functions and singular values of $M(j\omega,\ \sigma)$. We can extend their results to the case when the given plant is non-strictly proper. Let $S_0(s,\ \sigma)$ and $T_0(s,\ \sigma)$ be the achieved sensitivity and complementary sensitivity functions in the configuration of Fig. 2 when the loop is broken at the input point of the plant,

$$S_0(s,\ \sigma) = [I_m + C(s,\ \sigma)P(s)]^{-1}$$

and

$$T_0(s,\ \sigma) = I_m - S_0(s,\ \sigma) = [I_m + C(s,\ \sigma)P(s)]^{-1}C(s,\ \sigma)P(s).$$

Let $S_F(s)$ and $T_F(s)$ be the sensitivity and complementary sensitivity functions corresponding to the target loop-shape. Then, we have an identical lemma as we have for the case when the given plant is strictly proper:

Lemma 2.4. Consider the configuration of Fig. 2. Let the given non-strictly proper plant be left invertible and of minimum phase. Then, there exists a gain $K(\sigma)$ such that as $\sigma \to \infty$,

$$S_0(j\omega,\ \sigma) \to S_F(j\omega)$$

and

$$T_0(j\omega,\ \sigma) \to T_F(j\omega).$$

Moreover, for any given value of σ, we have following bounds on all singular values $i=1$ to m of $S_0(j\omega,\ \sigma)$ and $T_0(j\omega,\ \sigma)$:

$$\frac{|\sigma_i[S_0(j\omega,\ \sigma)] - \sigma_i[S_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \leq \sigma_{\max}[M_1(j\omega,\ \sigma)]$$

and

$$\frac{|\sigma_i[T_0(j\omega,\ \sigma)] - \sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \leq \sigma_{\max}[M_1(j\omega,\ \sigma)].$$

Proof. It follows Sogaard-Andersen and Niemann (1989). Also, see Chen, Saberi and Sannuti (1990).

As in the case when the given plant is strictly proper, the expressions given above can be used to analyze the inevitable trade-off between good recovery as indicated by $\sigma_{\max}[M(j\omega,\ \sigma)]$ and robustness and performance as reflected in the sensitivity and complementary sensitivity functions. To do this, one can apply the so-called recovery diagrams developed by Sogaard-Andersen and Niemann (1989).

3. Reduced Order Observer Based Controller

Now, let us consider a reduced order observer based controller. In this section, without loss of generality but for simplicity of presentation, we will assume that the given system Σ is in the form of a special coordinate basis (Saberi and Sannuti, 1990; See also Appendix A). Then some of the state variables correspond exactly to the given output y_1 and hence need not be estimated. Let us first partition the state equations (2.3) of Σ as

$$\left.\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_{11}u_{11} + \widetilde{B}_{01}y_0 \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_{12}u_{12} + B_{02}y_0 \end{aligned}\right\}, \tag{3.1}$$

$$y_0 = C_0 x + u_0, \quad y_1 = x_1. \tag{3.2}$$

Here x_1 consists of all the output y_1 and is of dimension $n_1=p-r$ while x_2 consists of the rest of the state variables and is of dimension $n_2=n-p+r$. The control u_1 is partitioned as $u_1=[u_{11}', u_{12}']'$ where u_{11} does not directly control the state variable x_2 and is of dimension m_1 while u_{12} controls directly x_2 and is of dimension $m_2=m-m_1-r$. Similarly, let us also partition the output y_1 as $y_1=[y_{11}', y_{12}']'$. Here y_{11} is directly controlled by u_{11} and y_{12} consists of the rest output which is not directly influenced by u_{11}. Moreover, y_{11} and y_{12} are respectively of dimension m_1 and $p_2=p-r-m_1$. In view of the special coordinate basis given in Appendix A, one can eaesily rewrite (3.1) as

$$\left.\begin{aligned} \dot{y}_{11} &= D_{11}y_1 + D_{12}x_2 + B_{00}y_0 + G_{11}u_{11} \\ \dot{y}_{12} &= C_{11}y_1 + C_{12}x_2 + B_{01}y_0 \\ \dot{x}_2 &= A_{22}x_2 + B_{12}u_{12} + A_{21}y_1 + B_{02}y_0 \end{aligned}\right\} \tag{3.3}$$

for some appropriate matrices $D_{11}, D_{12}, \cdots, B_{02}$, and moreover, G_{11} is non-singular. Since $x_1=y_1$, we need to estimate only x_2. For this purpose, we consider the following reduced order system Σ_r characterized by the triple (A_{22}, B_{12}, C_{12}):

$$\dot{x}_2 = A_{22}x_2 + B_{12}u_{12} + A_{21}y_1 + B_{02}y_0, \tag{3.4}$$

$$w = C_{12}x_2 = \dot{y}_{12} - C_{11}y_1 - B_{01}y_0. \tag{3.5}$$

In (3.4) and (3.5), y_0 and y_1 are considered as known signals and w is the output by which x_2 is to be estimated. We have the following lemma.

Lemma 3.1. Consider the system Σ_r characterized by the triple (A_{22}, B_{12}, C_{12}) with $B_{12}\neq 0$. Then, we have

1. Σ_r is left invertible and of minimum phase if and only if Σ is left invertible and of minimum phase.
2. Invariant zeros of Σ_r are the same as those of the given plant Σ.

Proof. It follows from the special coordinate basis of Σ as given in Appendix A and the partition of dynamic equations of Σ as in (3.3).

Now we can design a full order observer for the reduced order system Σ_r. Consider

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + B_{12}u_{12} + A_{21}y_1 + B_{02}y_0 + K_r(w - C_{12}\hat{x}_2), \tag{3.6}$$

where K_r is the observer gain for the reduced order system Σ_r. As in the case of full order observers, K_r is the only unknown matrix here and hence it is considered as a free design parameter. The estimate $\hat{x}_2$ given by (3.6) requires $w = \dot{y}_{12} - C_{11}y_1 - B_{01}y_0$. This implies that we need a differentiator to obtain $\hat{x}_2$. However, following Kwakernaak and Sivan (1972), one can rewrite (3.6) in another variable z. Let

$$\hat{x}_2 = z + K_r y_{12}.$$

Then

$$\dot{z} = A_r(z + K_r y_{12}) + B_{12}u_{12} + (A_{21} - K_r C_{11})y_1 + (B_{02} - K_r B_{01})y_0, \tag{3.7}$$

where A_r is the dynamic matrix of the reduced order observer,

$$A_r = A_{22} - K_r C_{12}.$$

Thus, by implementing (3.7), $\hat{x}_2$ can be obtained without generating $\dot{y}_{12}$.

In Fig. 3, the state feedback gain matrix F has been partitioned as

$$F = \begin{bmatrix} F_{01} & F_{02} \\ F_{11} & F_{12} \end{bmatrix},$$

where F_{01}, F_{02}, F_{11} and F_{12} are of dimension $r \times n_1$, $r \times n_2$, $(m-r) \times n_1$ and $(m-r) \times n_2$, respectively. Now in order to bring the theory of full and reduced order observers to the same frame work and to understand the conditions for

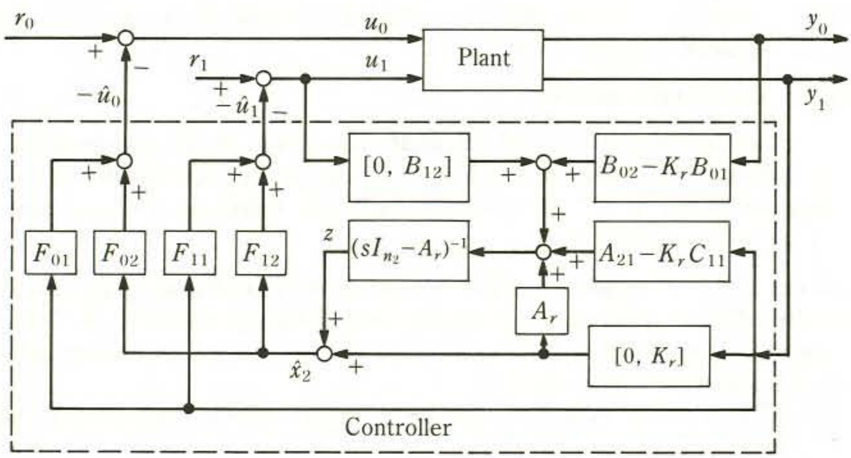


Fig. 3. Plant with the reduced order observer based controller.

either ELTR or ALTR clearly, we present the following results which are analogous to Lemmas 2.2 and 2.3.

Lemma 3.2. Let us partition the state feedback gain F to correspond with the dynamic system given in (3.1),

$$F = [F_1, F_2],$$

where F_1 and F_2 are the feedback gain associated with $x_1 = y_1$ and x_2 respectively. Then the mismatch function $E_r(s)$, the error between the target loop transfer function $L(s)$ and that achieved by the reduced order observer based controller is given by

$$E_r(s) = M_r(s)[I_m + M_r(s)]^{-1}(I_m + F\Phi B), \tag{3.8}$$

where

$$M_r(s) = [M_{r0}(s), M_{r1}(s), M_{r2}(s)] \tag{3.9}$$

and where $m \times r$, $m \times m_1$ and $m \times m_2$ dimensional matrices $M_{r0}(s)$, $M_{r1}(s)$ and $M_{r2}(s)$ are given by

$$M_{r0}(s) \equiv 0, \quad M_{r1}(s) \equiv 0, \quad M_{r2}(s) = F_2(\Phi_{22}^{-1} + K_r C_{12})^{-1} B_{12} \tag{3.10}$$

with $\Phi_{22} = (sI_{n_2} - A_{22})^{-1}$.

Proof. See Appendix D.

Remark 3.1: The expression for $E_r(s)$ is identical to the corresponding one when full order observer based controller is used, see (2.7), except that now $M_r(s)$ takes the place of $M(s)$.

Lemma 3.3.

$$E_r(j\omega) = 0, \quad \text{if and only if } M_{r2}(j\omega) = 0 \quad \text{for all } \omega \in \Omega_r,$$

where Ω_r is the set of all $0 \le \omega < \infty$ for which $\hat{L}_r(j\omega)$ and $L(j\omega)$ are well defined (i.e., all required inverses exist).

Proof. The proof is obvious.

It is clear that $M_r(s) \equiv 0$ if and only if $M_{r2}(s) \equiv 0$ and $M_{r2}(s)$ is dependent only on the reduced order system Σ_r which is strictly proper and left invertible with no invariant zeros in C^+. Hence, we have the following theorem which is analogous to Theorem 2.1.

Theorem 3.1. Consider the closed-loop system comprising of the given plant Σ and the reduced order observer based controller as given in Fig. 3. Then both the asymptotic stability of the closed-loop system and ELTR can be achieved under the following conditions:

1. $C_1 B_1$ is of maximal rank (i.e., the plant has no infinite zero of order higher than one).
2. The given plant is left invertible and is of minimum phase.

Proof. Condition 1 implies that $m_2=0$ and $M_{r2}(s)$ is nonexistent. Hence $M_r(s)\equiv 0$ and thus ELTR takes place. It is straightforward to verify through the properties of the special coordinate basis that the stability of the reduced order observer is guaranteed by the condition 2.

Remark 3.2: When reduced order observer based controllers are used, the condition $FB_1=0$ is not necessary for ELTR.

Since in general $M_r(s)$ or equivalently $M_{r2}(s)$ cannot exactly be made zero, one focuses attention on ALTR. That is, one needs

$$M_{r2}(s,\ \sigma) = F_2(\Phi_{22}^{-1}+K_r(\sigma)C_{12})^{-1}B_{12} \to 0 \quad \text{pointwise in } s \quad \text{as } \sigma \to \infty,$$

where the gain $K_r(\sigma)$ is now parameterized in terms of a tuning parameter σ. However, as in the previous section, in order to have the state feedback and observer design to be independent of one another, one needs to require that

$$(\Phi_{22}^{-1}+K_r(\sigma)C_{12})^{-1}B_{12} \to 0 \quad \text{pointwise in } s \quad \text{as } \sigma \to \infty, \tag{3.11}$$

which is the Doyle-Stein condition. Again it is well known that (3.11) can be satisfied if $(A_{22},\ B_{12},\ C_{12})$, namely Σ_r, is left invertible and of minimum phase. Furthermore, from Lemma 3.1, it follows that Σ_r is left invertible and of minimum phase if and only if the given plant is left invertible and of minimum phase. Hence, we have the following result which is analogous to Theorem 2.2.

Theorem 3.2. Consider the closed-loop system comprising of the given plant Σ and the reduced order observer based controller as given in Fig. 3. Let the given plant be left invertible and be of minimum phase. Then a gain $K_r(\sigma)$ can be designed using the triple $(A_{22},\ B_{12},\ C_{12})$ such that both asymptotic stability of the closed-loop system and ALTR can be achieved.

Proof. The proof is obvious in view of (3.11).

Now as in Lemma 2.4, we would like to develop bounds on the sensitivity and complementary sensitivity functions generated by the use of reduced order observer based controllers. Let $S_{0r}(s,\ \sigma)$ and $T_{0r}(s,\ \sigma)$ be the generated sensitivity and complementary sensitivity functions in the configuration of Fig. 3 when the loop is broken at the input points of the plant,

$$S_{0r}(s,\ \sigma) = [I_m+C_r(s,\ \sigma)P(s)]^{-1}$$

and

$$T_{0r}(s,\ \sigma) = I_m - S_{0r}(s,\ \sigma) = [I_m+C_r(s,\ \sigma)P(s)]^{-1}C_r(s,\ \sigma)P(s),$$

where $C_r(s,\ \sigma)$ is the transfer function of the reduced order observer based controller. We have the following results analogous to Lemma 2.4.

Lemma 3.4. Consider the configuration of Fig. 3. Let the given non-strictly proper plant be left invertible and of minimum phase. Then there exists a gain $K_r(\sigma)$ such that as $\sigma\to\infty$,

$$S_{0r}(j\omega,\ \sigma) \to S_F(j\omega)$$

and

$$T_{0r}(j\omega,\ \sigma) \to T_F(j\omega).$$

Moreover, for any given value of σ, we have the following bounds on all singular values $i=1$ to m of $S_{0r}(j\omega,\ \sigma)$ and $T_{0r}(j\omega,\ \sigma)$:

$$\frac{|\sigma_i[S_{0r}(j\omega,\ \sigma)]-\sigma_i[S_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \sigma_{\max}[M_{r2}(j\omega,\ \sigma)]$$

and

$$\frac{|\sigma_i[T_{0r}(j\omega,\ \sigma)]-\sigma_i[T_F(j\omega)]|}{\sigma_{\max}[S_F(j\omega)]} \le \sigma_{\max}[M_{r2}(j\omega,\ \sigma)].$$

Proof. It follows Chen, Saberi and Sannuti (1990).

4. Examples

Two examples are presented in this section to illustrate the theory of loop transfer recovery for non-strictly proper systems developed in the previous sections. As in the existing LTR literature, we give the maximum and minimum singular value graphs of the target loop and achieved loop transfer matrices for each example over a given range of ω. We also include the plots of the maximum singular values of the mismatch function between the target loop and achieved loop transfer matrices in the case of ALTR, since it is the best way to check whether true recovery has taken place or not (Chen, Saberi and Sannuti, 1990).

Example 1. Consider a non-strictly proper plant described by

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 1 & 1 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & -1 & 0 & 0 & -2 & 1 \\ 2 & -2 & 0 & -4 & 2 & 0 & -1 \\ 0 & 2 & 3 & 0 & -2 & 1 & -1 \\ 1 & 0 & 2 & -3 & 2 & 2 & 0 \\ -1 & -1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is square and invertible with two invariant zeros at $z_1=-2$ and $z_2=-4$. The state feedback gain matrix is given as

$$F = \left[\begin{matrix} 19.0075 & -15.2595 & 6.8147 \\ -13.5241 & 26.8185 & -2.7557 \\ 7.1266 & -10.0804 & -2.7934 \end{matrix} \right.$$
$$\left. \begin{matrix} -0.2864 & 0.2532 & 1.3794 & 3.1722 \\ 0.0581 & -0.1105 & -0.6243 & -6.2823 \\ -2.2915 & 1.4206 & 7.4193 & 0.3031 \end{matrix} \right].$$

We design ALTR via full order observer based controllers by using Kalman filter formalism. The observer gain is obtained as

$$K(\sigma) = PC_1',$$

where P is the solution of the algebraic Riccati equation,

$$A_1P + PA_1' - PC_1'C_1P + \sigma^2 B_1B_1' = 0$$

with A_1, B_1 and C_1 as defined in (2.3). The plots of singular values in Figs. 4(a) and 4(b) clearly show that ALTR takes place as σ increases.

Example 2. Consider the given plant,

$$A = \begin{bmatrix} 5 & 0 & 9 & -3 & 3 \\ 4 & -9 & 2 & 9 & 7 \\ 9 & 9 & 1 & 6 & 3 \\ 6 & -1 & 6 & -8 & 9 \\ 7 & -1 & 8 & -6 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & 9 \\ -3 & 3 & 5 \\ -5 & 2 & 2 \\ 2 & -2 & 2 \\ 4 & 8 & 6 \end{bmatrix},$$

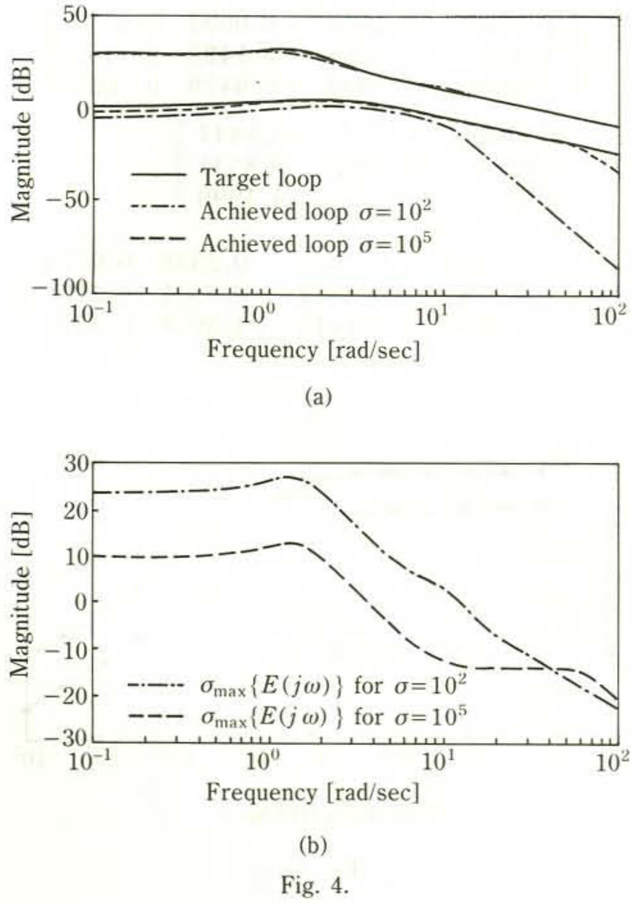


Fig. 4.

$$C = \begin{bmatrix} 5 & 0 & 9 & -3 & 3 \\ 5 & -2 & 3 & 4 & 8 \\ 1 & -2 & 1 & -6 & 0 \\ 1 & 5 & -9 & 3 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 9 & 9 & 1 \\ 6 & 4 & 0 \\ 3 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix},$$

which is left invertible with no invariant zeros. The state feedback gain matrix is designed as

$$F = \begin{bmatrix} 0.1238 & -0.0853 & -1.9687 & 1.0131 & 1.2659 \\ 1.3416 & -0.3102 & 2.1914 & -0.0171 & 1.8150 \\ 3.3664 & 0.6067 & 3.7863 & 0.7023 & 2.2218 \end{bmatrix}.$$

It is straightforward to verify that ELTR can be achieved by using a reduced order observer based controller. The following reduced order observer based controller, which achieves ELTR as shown in Fig. 5, is obtained by ATEA method (asymptotic time-scale and eigenstructure assignment method developed by Saberi and Sannuti (1990)):

$$\dot{z}_2 = \begin{bmatrix} -5.5331 & 4.4470 & 14.1642 \\ -6.2615 & -12.1601 & 2.8991 \\ -2.5236 & -3.0757 & -1.4197 \end{bmatrix} z_2 + \begin{bmatrix} 0.5631 & -0.1482 & -0.6093 & 0.8944 \\ 1.4034 & -0.5858 & -3.1420 & 0.3465 \\ 1.9158 & -1.3809 & -0.9479 & 0.7657 \end{bmatrix} y,$$

$$-\hat{u} = \begin{bmatrix} 0.4362 & -0.3675 & -0.5447 \\ -1.3304 & 0.3803 & 2.8331 \\ -3.1625 & 0.6465 & 4.1990 \end{bmatrix} z_2 + \begin{bmatrix} -0.2612 & 0.2885 & 0.2338 & 0.0274 \\ 0.4462 & -0.2353 & -0.6571 & 0.2109 \\ 0.9222 & -0.4446 & -1.3998 & 0.4776 \end{bmatrix} y.$$

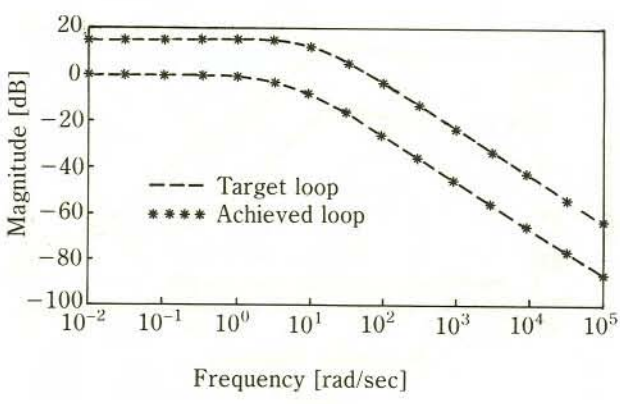


Fig. 5.

5. Conclusions

The loop transfer recovery (LTR) methods for non-strictly proper systems, using both full order and reduced order observers are developed in this paper. We have converted the problem of loop transfer recovery design for a non-strictly proper system into the one for a corresponding strictly proper system. Hence, all the existing results of LTR can be directly applied. Two numerical examples are presented to demonstrate the results.

References

Athans, M. (1986). A tutorial on LQG/LTR methods. *Proceedings of ACC*, Seattle, Washington, 1289–1296.

Chen, B.M., S. Bingulac and A. Saberi (1989). A C-A-D package for LTR design via time-scale structure and eigenvalue assignment. *Proceedings of 27th annual Allerton Conference on Communications, Control and Computing*, Monticello, Illinois, 439–440.

Chen, B.M., A. Saberi and P. Sannuti (1990). A new stable compensator design for exact and approximate loop transfer recovery. *Proceedings of ACC*, San Diego, CA, 812–817; Also to appear in *Automatica*.

Dowdle, J.R., G. Stein and N.R. Sandell (1982). Minimal order robust observer-based compensator. *Proceedings of CDC*, Orlando, FL, 894–896.

Doyle, J.C. and G. Stein (1979). Robustness with observers. *IEEE Trans. Automatic Control*, **AC-24**, 4, 607–611.

Doyle, J.C. and G. Stein (1981). Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Trans. Automatic Control*, **AC-26**, 1, 4–16.

Goodman, G.C. (1984). The LQG/LTR method and discrete-time control systems. Report No. LIDS-TH-1392, MIT, MA.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. John Wiley, N.Y.

Madiwale, A.N. and D.E. Williams (1985). Some extensions of loop transfer recovery. *Proceedings of ACC*, Boston, MA, 790–795.

Matson, C.L. and P.S. Maybeck (1987). On an assumed convergence result in the LQG/LTR technique. *Proceedings of CDC*, Los Angeles, CA, 951–952.

Ridgely, D.B. and S.S. Banda (1986). Introduction to robust multivariable control. Report No. AFWAL-TR-3120, Flight Dynamics Laboratories, Air Force Systems Command, Wright-Patterson Air Force Base, OH.

Saberi, A. and P. Sannuti (1990 a). Observer design for loop transfer recovery and for uncertain dynamical systems. *IEEE Trans. Automatic Control*, **AC-35**, 8, 878–897.

Saberi, A. and P. Sannuti (1990 b). Squaring down of non-strictly proper systems. *Int. J. Control*, **51**, 3, 621–629.

Sannuti, P. and A. Saberi (1987). A special coordinate basis of multivariable linear system—Finite and infinite zero structure, squaring down and decoupling. *Int. J. Control*, **45**, 5, 1655–1704.

Sogaard-Andersen, P. (1987). Loop transfer recovery with minimal order observers. *Proceedings of CDC*, Los Angeles, CA, 933–938.

Sogaard-Andersen, P. and H.H. Niemann (1989). Trade-offs in LTR-based feedback design. *Proceedings of ACC*, Pittsburgh, PA, 922–928.

Stein, G. and M. Athans (1987). The LQG/LTR procedure for multivariable feedback control design. *IEEE Trans. Automatic Control*, **AC-32**, 2, 105–114.

Appendix A: A Special Coordinate Basis (s.c.b) for Non-strictly Proper Multivariable Linear Systems and Its Properties

Consider a linear system Σ,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

where x, y and u are respectively n-, p- and m-dimensional state, output and input vectors. Without loss of generality, we assume that $[B', D']'$ and $[C, D]$ are of full rank.

Theorem of s.c.b There exist nonsingular transformations Γ_1, Γ_2 and Γ_3, an integer $KI \leq n$, and integer indexes q_i and r_i, $i=0$ to KI, with $r_0=0$, such that

$$x = \Gamma_1 \tilde{x}, \quad y = \Gamma_2[\tilde{y}_f', \tilde{y}_b']', \quad u = \Gamma_3[\tilde{u}', \tilde{v}']', \quad \tilde{x} = [\tilde{x}_a', \tilde{x}_b', \tilde{x}_c', \tilde{x}_f']',$$

$$\tilde{x}_b = [\tilde{x}'_{b1}, \tilde{x}'_{b2}, \cdots, \tilde{x}'_{bKI}]', \quad \tilde{x}_{bi} = [\tilde{x}'_{b1i-1}, \tilde{x}'_{b2i-2}, \cdots, \tilde{x}'_{bi0}]',$$

$$\tilde{x}_f = [\tilde{x}'_{1f}, \tilde{x}'_{2f}, \cdots, \tilde{x}'_{KIf}]', \quad \tilde{x}_{if} = [\tilde{x}'_{1i-1}, \tilde{x}'_{2i-2}, \cdots, \tilde{x}'_{i0}]',$$

$$\tilde{y}_b = [\tilde{y}'_{1b}, \tilde{y}'_{2b}, \cdots, \tilde{y}'_{KIb}]', \quad \tilde{y}_f = [\tilde{y}'_{0f}, \tilde{y}'_{1f}, \tilde{y}'_{2f}, \cdots, \tilde{y}'_{KIf}]',$$

$$\tilde{u} = [\tilde{u}'_0, \tilde{u}'_1, \cdots, \tilde{u}'_{KI}]', \quad \dot{\tilde{x}}_a = A_{aa}\tilde{x}_a + L_{af}\tilde{y}_f + L_{ab}\tilde{y}_b,$$

$$\dot{\tilde{x}}_b = A_{bb}\tilde{x}_b + L_{bf}\tilde{y}_f, \quad \tilde{y}_{ib} = C_{ib}\tilde{x}_{bi} \equiv \tilde{x}_{b1i-1}, \quad \tilde{y}_b = C_b\tilde{x}_b,$$

$$\dot{\tilde{x}}_c = A_{cc}\tilde{x}_c + A_{ca}\tilde{x}_a + L_{cf}\tilde{y}_f + L_{cb}\tilde{y}_b + B_c\tilde{v},$$

$$\tilde{y}_{0f} = E_{0a}\tilde{x}_a + E_{0b}\tilde{x}_b + E_{0c}\tilde{x}_c + \sum_{j=1}^{KI} E_{0j}\tilde{x}_{jf} + \tilde{u}_0.$$

For each $i=1$ to KI,

$$\dot{\tilde{x}}_{if} = A_{if}\tilde{x}_{if} + L_{if}\tilde{y}_f + B_{if}\left[\tilde{u}_i + E_{ia}\tilde{x}_a + E_{ib}\tilde{x}_b + E_{ic}\tilde{x}_c + \sum_{j=1}^{KI} E_{ij}\tilde{x}_{jf}\right],$$

$$\tilde{y}_{if} = C_{if}\tilde{x}_{if} \equiv \tilde{x}_{1i-1}, \quad \tilde{y}_f = C_f\tilde{x}_f.$$

Here, the last rows of L_{if} have zero elements and that $A_{1f} \equiv 0$,

$$A_{if} = \begin{bmatrix} 0 & I_{(i-1)q_i} \\ 0 & 0 \end{bmatrix}, \quad B_{if} = \begin{bmatrix} 0 \\ I_{q_i} \end{bmatrix}.$$

Furthermore, the pair (A_{cc}, B_c) is controllable and (A_{bb}, C_b) is observable.

For clarity, the dimension of each variable is given now: $\tilde{x}_a$, $\tilde{x}_b$, $\tilde{x}_{bi}$, $\tilde{x}_{bij}$, $\tilde{x}_c$, $\tilde{x}_f$, $\tilde{x}_{if}$, $\tilde{x}_{ij}$, $\tilde{u}$, $\tilde{u}_i$, $\tilde{v}$, $\tilde{y}_f$, $\tilde{y}_{if}$, $\tilde{y}_b$ and $\tilde{y}_{ib}$ are respectively of dimension n_a, n_b, ir_i, r_{i+j}, n_c, n_f, n_{if}, q_{i+j}, m_u, q_i, m_v, p_f, q_i, p_b and r_i, where

$$p = \sum_{i=0}^{KI}(q_i + r_i), \quad n = n_a + n_b + n_c + n_f,$$

$$m_u = p_f = \sum_{i=0}^{KI} q_i, \quad m_v = m - m_u, \quad p_b = p - p_f = \sum_{i=1}^{KI} r_i,$$

$$n_{if} = iq_i, \quad n_f = \sum_{i=1}^{KI} n_{if}, \quad n_b = \sum_{i=1}^{KI} ir_i.$$

Property 1. The given system Σ is right invertible if and only if $\tilde{x}_b$ and hence $\tilde{y}_b$ are nonexistent, left invertible if and only if $\tilde{x}_c$ and hence $\tilde{v}$ are nonexistent, invertible, if and only if both $\tilde{x}_b$ and $\tilde{x}_c$ are nonexistent.

Property 2. Invariant zeros of Σ are the eigenvalues of A_{aa}.

Property 3. Σ is left invertible and of minimum phase implies that Σ is detectable.

Appendix B: Proof of Lemma 2.2

Rewrite the transfer function of the given plant as

$$P(s) = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} [\Phi B_0, \ \Phi B_1] + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{B.1}$$

Then, from (1.4), (2.6) and (B.1),

$$\begin{aligned}
\hat{L}(s) &= C(s)P(s) \\
&= F(\Phi_1^{-1}+KC_1+B_1F_{u_1})^{-1} \\
&\quad \times [(\Phi^{-1}+B_0C_0+KC_1)\Phi B_0, \ (B_0C_0+KC_1)\Phi B_1] \\
&= F[I_n+(\Phi_1^{-1}+KC_1)^{-1}B_1F_{u_1}]^{-1}(\Phi_1^{-1}+KC_1)^{-1} \\
&\quad \times [(\Phi_1^{-1}+KC_1)\Phi B_0, \ (B_0C_0+KC_1)\Phi B_1] \\
&= F[I_n+(\Phi_1^{-1}+KC_1)^{-1}B_1F_{u_1}]^{-1} \\
&\quad \times [\Phi B_0, \ (\Phi^{-1}+B_0C_0+KC_1)^{-1}(B_0C_0+KC_1)\Phi B_1] && \text{(B.2)} \\
&= F[I_n+(\Phi_1^{-1}+KC_1)^{-1}[0, \ B_1]F]^{-1} \\
&\quad \times [\Phi B_0, \ \Phi B_1-(\Phi_1^{-1}+KC_1)^{-1}B_1] && \text{(B.3)} \\
&= [I_m+F(\Phi_1^{-1}+KC_1)^{-1}[0, \ B_1]]^{-1} \\
&\quad \times [F\Phi B_0, \ F\Phi B_1-F(\Phi_1^{-1}+KC_1)^{-1}B_1] && \text{(B.4)} \\
&= [I_m+M(s)]^{-1}[F\Phi B-M(s)].
\end{aligned}$$

Note that we use $\Phi_1^{-1}=(sI_n-A+B_0C_0)=\Phi^{-1}+B_0C_0$ in reducing some of the above algebra. Also, to go from (B.2) to (B.3), we use matrix identity,

$$(\Phi^{-1}+B_0C_0+KC_1)^{-1}(B_0C_0+KC_1) = I_n - (\Phi^{-1}+B_0C_0+KC_1)^{-1}\Phi^{-1}.$$

Moreover, to go from (B.3) to (B.4), we use the identity

$$X[I_n+YX]^{-1} = [I_m+XY]^{-1}X$$

for $X \in R^{m\times n}$ and $Y \in R^{n\times m}$. Thus, we have

$$
\begin{aligned}
E(s) &= L(s) - \hat{L}(s) \\
&= F\Phi B - [I_m + M(s)]^{-1}[F\Phi B - M(s)] \\
&= [I_m + M(s)]^{-1}[(I_m + M(s))F\Phi B - F\Phi B + M(s)] \\
&= [I_m + M(s)]^{-1}M(s)(I_m + F\Phi B) \\
&= M(s)[I_m + M(s)]^{-1}(I_m + F\Phi B).
\end{aligned}
$$

Appendix C: Proof of Theorem 2.1

In view of conditions (2) and (3) of Theorem 2.1, it follows theorem of s.c.b in Appendix A and Saberi and Sannuti (1990 b) that there exist nonsingular transformations Γ_1, Γ_2 and Γ_3 such that

$$x = \Gamma_1 \tilde{x}, \quad y = \Gamma_2 [\tilde{y}'_{0f}, \tilde{y}'_{1f}, \tilde{y}'_b]',$$

$$u = \Gamma_3 [\tilde{u}'_0, \tilde{u}'_1]', \quad \tilde{x} = [\tilde{x}'_a, \tilde{x}'_b, \tilde{x}'_f]',$$

$$\dot{\tilde{x}}_a = A_{aa}\tilde{x}_a + L_{ab}\tilde{y}_b + L_{a1f}\tilde{y}_{1f} + L_{a0f}\tilde{y}_{0f},$$

$$\dot{\tilde{x}}_b = A_{bb}\tilde{x}_b + L_{b1f}\tilde{y}_{1f} + L_{b0f}\tilde{y}_{0f}, \; \tilde{y}_b = C_b\tilde{x}_b,$$

$$\dot{\tilde{x}}_f = E_a\tilde{x}_a + E_b\tilde{x}_b + E_{1f}\tilde{x}_f + E_{0f}\tilde{y}_{0f} + \tilde{u}_1,$$

$$\tilde{y}_{1f} = \tilde{x}_f, \quad \tilde{y}_{0f} = C_{0f}\tilde{x} + \tilde{u}_0.$$

Here, the pair (A_{bb}, C_b) is observable. Furthermore, $\lambda(A_{aa})$ are the invariant zeros of the given plant and hence in view of condition (2) of Theorem 2.1, they are in C^-.

Since (A_{bb}, C_b) is observable, one can select a gain K_{bb} such that $\lambda(A_{bb} - K_{bb}C_b)$ are in the desired locations in C^-. Also, one can always choose a gain K_{f1f} such that $\lambda(E_{1f} - K_{f1f})$ are in the desired locations in C^-. Now choose a gain $\tilde{K}$ as

$$\tilde{K} = \begin{bmatrix} 0 & L_{a1f} & L_{ab} \\ 0 & L_{b1f} & K_{bb} \\ 0 & K_{f1f} & K_{fb} \end{bmatrix},$$

where K_{fb} is an arbitrary matrix with appropriate dimensions. Finally, let

$$[K_0, K] = \Gamma_1 \tilde{K} \Gamma_2^{-1},$$

where K_0 and K are of dimension $n \times r$ and $n \times (p-r)$, respectively. It is now straightforward to verify that $A_1 - KC_1$ has eigenvalues in C^- and that

$$F(\Phi_1^{-1} + KC_1)^{-1}B_1 \equiv 0$$

provided $FB_1 \equiv 0$. Hence in view of Lemmas 2.2 and 2.3, ELTR is achieved.

Appendix D: Proof of Lemma 3.2

In the reduced order observer based feedback control system of Fig. 3, at first we want to evaluate the loop transfer function $\hat{L}_r(s)$ when the loop is broken at the input point of the plant. For this purpose, consider the plant input u and the controller output $\hat{u}$ as two separate variables. Then, from (3.1),

$$B_{12}u_{12} = \dot{x}_2 - A_{22}x_2 - A_{21}x_1 - B_{02}y_0.$$

This implies

$$B_{12}u_{12}(s) = \Phi_{22}^{-1}x_2(s) - A_{21}x_1(s) - B_{02}y_0(s).$$

From (3.3), one has

$$K_r\dot{y}_{12}(s) = K_r[C_{11}x_1(s)+C_{12}x_2(s)+B_{01}y_0(s)].$$

Hence, we obtain

$$\begin{aligned} B_{12}u_{12}(s) = {} & (\Phi_{22}^{-1}+K_rC_{12})x_2(s) - (A_{21}-K_rC_{11})x_1(s) \\ & - (B_{02}-K_rB_{01})y_0(s) - K_r\dot{y}_{12}(s). \end{aligned} \tag{D.1}$$

Then, from (3.7),

$$\begin{aligned} \dot{\hat{x}}_2 &= K_r\dot{y}_{12} + \dot{z} \\ &= (A_{22}-K_rC_{12})\hat{x}_2 + B_{12}u_{12} + (A_{21}-K_rC_{11})x_1 \\ &\quad + (B_{02}-K_rB_{01})y_0 + K_r\dot{y}_{12}. \end{aligned}$$

Hence,

$$\begin{aligned} \hat{x}_2(s) = {} & (\Phi_{22}^{-1}+K_rC_{12})^{-1}[(A_{21}-K_rC_{11})x_1(s) \\ & + (B_{02}-K_rB_{01})y_0(s) + K_r\dot{y}_{12}(s) + B_{12}u_{12}(s)] \end{aligned}$$

and

$$\begin{aligned} -\hat{u}(s) &= F_1x_1(s) + F_2\hat{x}_2(s) \\ &= F_1x_1(s) + F_2(\Phi_{22}^{-1}+K_rC_{12})^{-1} \\ &\quad \times [(A_{21}-K_rC_{11})x_1(s) + (B_{02}-K_rB_{01})y_0(s) \\ &\quad + K_r\dot{y}_{12}(s) + B_{12}u_{12}(s)]. \end{aligned}$$

Thus

$$\begin{aligned} & (I_m+F_2(\Phi_{22}^{-1}+K_rA_{12})^{-1}[0,\ 0,\ B_{12}])[-\hat{u}(s)] \\ & \quad = F_1x_1(s) + F_2(\Phi_{22}^{-1}+K_rC_{12})^{-1}[(A_{21}-K_r)x_1(s) \\ & \quad + (B_{02}-K_rB_{01})y_0(s) + K_r\dot{y}_{12}(s)] \end{aligned}$$

and therefore

$$\begin{aligned}
-\hat{u}(s) &= [I_m+M_r(s)]^{-1}[F_1x_1(s) + F_2(\Phi_{22}^{-1}+K_rC_{12})^{-1} \\
&\quad \times [(A_{21}-K_r)x_1(s) + (B_{02}-K_rB_{01})y_0(s) + K_r\dot{y}_{12}(s)] \\
&= [I_m+M_r(s)]^{-1}(F_1x_1(s) + F_2x_2(s) - F_2(\Phi_{22}^{-1}+K_rC_{12})^{-1} \\
&\quad \times [(\Phi_{22}^{-1}+K_rC_{12})x_2(s) - (A_{21}-K_r)x_1(s) \\
&\quad - (B_{02}-K_rB_{01})y_0(s)-K_r\dot{y}_{12}(s)]),
\end{aligned} \tag{D.2}$$

where

$$M_r(s) = F_2(\Phi_{22}^{-1}+K_rC_{12})^{-1}[0,\ 0,\ B_{12}].$$

Note that

$$F_1x_1(s) + F_2x_2(s) = F\Phi Bu(s). \tag{D.3}$$

Thus, (D.1) to (D.3) imply that

$$\begin{aligned}
-\hat{u}(s) &= [I_m+M_r(s)]^{-1}[F\Phi Bu(s) - F_2(\Phi_{22}^{-1}+K_rC_{12})^{-1}B_{12}u_{12}(s)] \\
&= [I_m+M_r(s)]^{-1}[F\Phi B-M_r(s)]u(s).
\end{aligned}$$

Hence,

$$\hat{L}_r(s) = [I_m+M_r(s)]^{-1}[F\Phi B-M_r(s)].$$

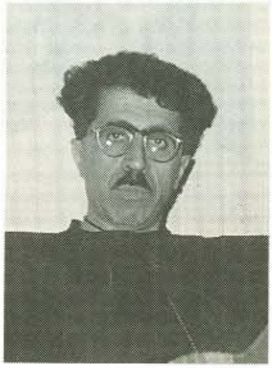
Then, we have

$$\begin{aligned}
E_r(s) &= L(s) - \hat{L}_r(s) \\
&= [I_m+M_r(s)]^{-1}[(I_m+M_r(s))F\Phi B - F\Phi B + M_r(s)] \\
&= [I_m+M_r(s)]^{-1}M_r(s)(I_m+F\Phi B) \\
&= M_r(s)[I_m+M_r(s)]^{-1}(I_m+F\Phi B).
\end{aligned}$$

Ben M. Chen was born in Fujian, China, on November 25, 1963. He received the B.S. degree in mathematics and computer science from Amoy University, Xiamen, China, in July, 1983, the M.S. degree in electrical engineering from Gonzaga University, Spokane, Washington, in May, 1988.

From July, 1983 to March, 1986, he worked as a Software Engineer in South-China Computer Corporation, Guangzhou, China. From September, 1986 to May, 1988, he held a Presidential scholarship at Gonzaga University and a Cardinal Yu-Pin scholarship from Sino-American Amity

Fund, Inc., New York. He is currently working toward the Ph.D. degree in the Department of Electrical and Computer Engineering at Washington State University, Pullman. His current research interests are in robust control theory.

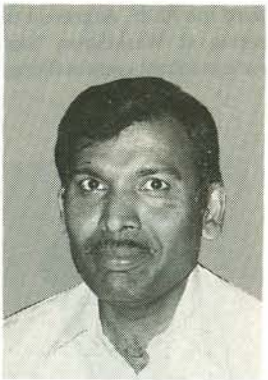


Ali Saberi received his Ph.D. degree in electrical engineering from Michigan State University, East Lansing, in 1983.

He is currently an Associate Professor in the Department of Electrical and Computer Engineering, Washington State University, Pullman, Washington.

Stanoje Bingulac received the B.S. and Ph.D. degrees in electrical engineering from the University of Belgrade, Belgrade, Yugoslavia, in 1959 and 1964, respectively. He is currently a professor of electrical engineering at Virginia Polytechnic Institute and State University, Blacksburg, Virginia. He has conducted research in control theory and computational methods in control. He has developed the interactive conversational computer aided design package L-A-S which has been used in more than 20 universities and research organizations in the U.S. and abroad. He has more than 20 years experience in teaching at all levels and is author or coauthor of more than 80 technical papers published in technical journals and presented at technical conferences. His current research interests are in identification and control of multivariable control systems, with emphasis on computer application and algorithm development.

He is a member of Sigma Xi and a senior member of IEEE.



Peddapullaiah Sannuti obtained his Ph.D. degree in electrical engineering from the University of Illinois at Urbana-Champaign. He is presently a professor in the department of Electrical and Computer Engineering, Rutgers, the State University of New Jersey, U.S.A.