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Brief paper

Further results on structural assignment of linear systems via sensor selection☆

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Abstract

The problem of assigning structural properties of a linear system through sensor selection is, for a given pair (A, B), to find an output pair (C, D) such that the resulting system (A, B, C, D) has the pre-specified structural properties, such as the finite and infinite zero structures and the invertibility properties. In this paper, by introducing the notion of infinite zero assignable sets for the pair (A, B), we establish necessary and sufficient conditions for the assignability of a given set of infinite zeros and a set of structural properties which includes the left invertibility property. In establishing these conditions, we develop a numerical algorithm for the construction of the required (C, D).

Keywords: Finite zeros; Infinite zeros; Linear systems; Structural assignment; Sensor selection

1. Introduction

Structural properties of linear systems, such as the finite and infinite zero structures and the invertibility properties, have played a very important role in many linear systems and control areas, including robust and H_∞ control (see, e.g., Chen, 2000; Lin, 1998), H_2 optimal control (e.g., Saberi, Sannuti, & Chen, 1995), and control with saturation (e.g., Lin, 1998). One of the major obstacles to successful applications of multivariable control synthesis techniques to practical control problems is the lack of adequate understanding of the linkage between achievable control performances and hardware implementation such as the selection and location of sensors and actuators. Indeed, this linkage provides a foundation upon which trade-offs can be incorporated in the preliminary design stage of an engineering system. For example, it is well understood in the literature that nonminimum-phase zeros are troublesome to deal with. However, simple examples show that such zeros can be removed by properly adding, removing or relocating sensors and actuators. This is exactly what motivated the interest in the problem of structural assignment. This problem is, for a linear system, $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, to find an output $y = Cx + Du$, such that the resulting system (A, B, C, D) has the pre-specified structural properties, such as the finite and infinite zero structures and the invertibility properties.

Most results on structural assignment in the literature pertain to the assignment of finite zero (invariant zero or transmission zero) structures (see, e.g., Emami-Naeini & Dooren, 1982; Karcanias & Giannakopoulos, 1989; Karcanias, Laios, & Giannakopoulos, 1988; Kouvariatkis & MacFarlane, 1976; Patel, 1978; Patel, Geniele, & Khorasani, 1994; Rosenbrock, 1970; Smagina, 2002; Sorokin, 1998; Syrmos, 1993; Syrmos & Lewis, 1993; Vardulakis, 1980). Chen and Zheng (1995) proposed a technique which is capable of simultaneously assigning finite and infinite zero structures. Recently, we successfully attempted to deal with the assignment of complete system structures, including finite and infinite zero structures and invertibility structures in Liu, Chen, and Lin (2003). In particular, in Liu et al. (2003), we identified a set of sufficient conditions, and under these conditions, an algorithm that leads to the assignment of a set of complete structural properties is developed.

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By using the similar technique of Rosenbrock (1970) and Amparan, Marcaida, and Zaballa (2004) presented the necessary and sufficient conditions under which an infinite zero structure can be assigned. Other structural properties, such as finite zero structure and invertibility properties were not considered. Moreover, the tool they used to establish these necessary and sufficient conditions is the rational function matrix, which, though mathematically elegant, does not lead to computational algorithms to construct the required (C, D).

In this paper, we will first introduce the notion of infinite zero assignable sets. With this notion, we establish necessary and sufficient conditions for the assignability of a given set of infinite zeros and a set of structural properties which includes left invertibility property. These conditions indicate the conservativeness of the existing conditions. In establishing these conditions, we develop a numerical algorithm for the construction of (C, D).

Throughout the paper, $\star$ denotes a submatrix of less interest in the context. For an integer k, $\varrho_k = [1 \;\; 0] \in \mathbb{R}^{1\times k}$, $\vartheta_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^k$, $\aleph_k = \begin{bmatrix} 0 & I_{k-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}$.

2. Background materials

Consider a linear system Σ:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Without loss of generality, we assume that both $[B' \;\; D']$ and $[C \;\; D]$ are of full row rank. In what follows, we give a compact form of the special coordinate basis, which was introduced in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). A toolkit (Lin, Chen, & Liu, 2004) in the Matlab environment is available online at http://linearsystemskit.net. This canonical form, implemented in the toolkit, is based on a numerically stable algorithm recently reported in Chu, Liu, and Tan (2002), together with an enhanced procedure reported in Chen, Lin, and Shamash (2004).

Theorem 2.1. *Given* (1), *there exist state, output and input transformations* Γ_{S}, Γ_{O} *and* Γ_{I}, *such that*

$$\begin{aligned}\tilde{A} &= \Gamma_{\mathrm{S}}^{-1} A \Gamma_{\mathrm{S}} \\ &= A_{\mathrm{s}} + B_0 C_0 \\ &= \begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ab}}C_{\mathrm{b}} & 0 & L_{\mathrm{ad}}C_{\mathrm{d}} \\ 0 & A_{\mathrm{bb}} & 0 & L_{\mathrm{bd}}C_{\mathrm{d}} \\ B_{\mathrm{c}}E_{\mathrm{ca}} & L_{\mathrm{cb}}C_{\mathrm{b}} & A_{\mathrm{cc}} & L_{\mathrm{cd}}C_{\mathrm{d}} \\ B_{\mathrm{d}}E_{\mathrm{da}} & B_{\mathrm{d}}E_{\mathrm{db}} & B_{\mathrm{d}}E_{\mathrm{dc}} & A_{\mathrm{dd}} \end{bmatrix} \\ &\quad + \begin{bmatrix} B_{0\mathrm{a}} \\ B_{0\mathrm{b}} \\ B_{0\mathrm{c}} \\ B_{0\mathrm{d}} \end{bmatrix} [C_{0\mathrm{a}} \;\; C_{0\mathrm{b}} \;\; C_{0\mathrm{c}} \;\; C_{0\mathrm{d}}],\end{aligned} \tag{2}$$

$$\tilde{B} = \Gamma_{\mathrm{S}}^{-1} B \Gamma_{\mathrm{I}} = [B_0 \;\; B_{\mathrm{s}}] = \begin{bmatrix} B_{0\mathrm{a}} & 0 & 0 \\ B_{0\mathrm{b}} & 0 & 0 \\ B_{0\mathrm{c}} & 0 & B_{\mathrm{c}} \\ B_{0\mathrm{d}} & B_{\mathrm{d}} & 0 \end{bmatrix}, \tag{3}$$

$$\tilde{C} = \Gamma_{\mathrm{O}}^{-1} C \Gamma_{\mathrm{S}} = \begin{bmatrix} C_0 \\ C_{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ 0 & 0 & 0 & C_{\mathrm{d}} \\ 0 & C_{\mathrm{b}} & 0 & 0 \end{bmatrix}, \tag{4}$$

$$\tilde{D} = \Gamma_{\mathrm{O}}^{-1} D \Gamma_{\mathrm{I}} = D_{\mathrm{s}} = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

where $(A_{\mathrm{cc}}, B_{\mathrm{c}})$ *is controllable*, $(A_{\mathrm{bb}}, C_{\mathrm{b}})$ *is observable and* $A_{\mathrm{dd}} = A_{\mathrm{dd}}^* + B_{\mathrm{d}}E_{\mathrm{dd}} + L_{\mathrm{dd}}C_{\mathrm{d}}$ *with* $A_{\mathrm{dd}}^* = \mathrm{blkdiag}\{\aleph_{q_1}, \aleph_{q_2}, \ldots, \aleph_{q_{m_{\mathrm{d}}}}\}$, $B_{\mathrm{d}} = \mathrm{blkdiag}\{\vartheta_{q_1}, \vartheta_{q_2}, \ldots, \vartheta_{q_{m_{\mathrm{d}}}}\}$, *and* $C_{\mathrm{d}} = \mathrm{blkdiag}\{\varrho_{q_1}, \varrho_{q_2}, \ldots, \varrho_{q_{m_{\mathrm{d}}}}\}$.

Proposition 1. *The structural decomposition of* (2)–(5) *shows explicitly the finite zero and infinite zero structures, as well as left and right invertibility structures.*

1. *The finite zero structure of* Σ *is characterized by the eigenstructure of* A_{aa}.
2. *Left invertibility structure* $S_{\mathrm{L}}^\star(\Sigma)$ *is the observability indices of* $(A_{\mathrm{bb}}, C_{\mathrm{b}})$, *and right invertibility structure* $S_{\mathrm{R}}^\star(\Sigma)$ *is the controllability indices of* $(A_{\mathrm{cc}}, B_{\mathrm{c}})$.
3. Σ *has* $m_0 = \mathrm{rank}(D)$ *infinite zeros of order* 0. *The infinite zeros (of order greater than* 0*) of* Σ *is given by* $S_\infty^\star(\Sigma) = \{q_1, q_2, \ldots, q_{m_{\mathrm{d}}}\}$. *That is, each* q_i *corresponds to an infinite zero of* Σ *of order* q_i.
4. *The finite zero structure*, $S_{\mathrm{R}}^\star$, $S_{\mathrm{L}}^\star$ *and* $S_\infty^\star$ *correspond to Morse index lists* $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$ *and* $\mathscr{I}_4$ (Morse, 1973), *respectively. Also,* Σ *is left invertible if* $S_{\mathrm{R}}^\star$ *is empty, right invertible if* $S_{\mathrm{L}}^\star$ *is empty, invertible if both* $S_{\mathrm{R}}^\star$ *and* $S_{\mathrm{L}}^\star$ *are empty, and degenerate if both* $S_{\mathrm{R}}^\star$ *and* $S_{\mathrm{L}}^\star$ *are present.*

Lemma 2.1 (Chen et al., 2004). *The pair* (A, B) *is controllable if and only if* $(A_{\mathrm{con}}, B_{\mathrm{con}})$ *is controllable, where*

$$A_{\mathrm{con}} = \begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ab}}C_{\mathrm{b}} \\ 0 & A_{\mathrm{bb}} \end{bmatrix}, \quad B_{\mathrm{con}} = \begin{bmatrix} B_{0\mathrm{a}} & L_{\mathrm{ad}} \\ B_{0\mathrm{b}} & L_{\mathrm{bd}} \end{bmatrix}.$$

If (A, B) *is uncontrollable, its uncontrollable eigenvalues are included in* $\lambda(A_{\mathrm{con}})$.

3. Preliminary results

Consider a pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and a vector of positive integers $\eta = (\eta_1, \eta_2, \ldots, \eta_\varpi)$. Let b_k be the kth column of B. Define

$$\Theta(A, B, \eta) = \{b_1 A b_1 \cdots A^{\eta_1 - 1} b_1 | b_2 A b_2 \cdots A^{\eta_2 - 1} b_2 | \cdots | b_\varpi A b_\varpi \cdots A^{\eta_\varpi - 1} b_\varpi\}.$$

Definition 3.1. A set of positive integers $\eta = (\eta_1, \eta_2, \ldots, \eta_\varpi)$ is called an infinite zero assignable set of (A, B), if there exist a state feedback K and an input transformation T_I such that $\Theta(A - BT_\mathrm{I}K, BT_\mathrm{I}, \eta)$ is of full column rank.

Lemma 3.1. *For $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and a set of positive integers $\eta = (\eta_1, \eta_2, \ldots, \eta_\varpi)$ with $\Theta(A, B, \eta)$ being of full column rank, there exists a $T_\mathrm{S} \in \mathbb{R}^{n\times n}$ such that $(T_\mathrm{S}^{-1}AT_\mathrm{S}, T_\mathrm{S}^{-1}B)$ is as follows*:

$$\left(\left[\begin{array}{cc|cc|c|cc} A_{11} & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \star & 0 & \cdots & \star & 0 \\ \hline \star & \star & \star & I_{\eta_1-1} & \cdots & \star & 0 \\ \star & \star & \star & 0 & \cdots & \star & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \star & \star & \star & 0 & \cdots & \star & I_{\eta_\varpi-1} \\ \star & \star & \star & 0 & \cdots & \star & 0 \end{array}\right], \left[\begin{array}{c} 0 \cdots 0 \quad 0 \\ 0 \cdots 0 \quad B_{22} \\ \hline 0 \cdots 0 \quad \star \\ 1 \cdots 0 \quad \star \\ \hline \vdots \ddots \vdots \quad \vdots \\ \hline 0 \cdots 0 \quad \star \\ 0 \cdots 1 \quad \star \end{array}\right]\right),$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}$ contains the uncontrollable eigenvalues of A, $A_{22} \in \mathbb{R}^{n_2\times n_2}$ and $B_{22} \in \mathbb{R}^{n_2\times(m-\varpi)}$ with $n_2 = n - n_1 - \sum_{j=1}^{\varpi}\eta_j$.

Proof. Let $T_{\mathrm{S}1}$ such that

$$A_* = T_{\mathrm{S}1}^{-1}AT_{\mathrm{S}1} = \begin{bmatrix} A_{11} & 0 \\ \star & A_{*2} \end{bmatrix}, \quad B_* = T_{\mathrm{S}1}^{-1}B = \begin{bmatrix} 0 \\ B_{*2} \end{bmatrix},$$

where (A_{*2}, B_{*2}) is controllable. Thus, $\Theta(A_*, B_*, \eta) = T_{\mathrm{S}1}^{-1}\Theta(A, B, \eta)$. Since the first n_1 rows of $\Theta(A_*, B_*, \eta)$ are zeros, there exists a $T_0 = \mathrm{blkdiag}\{I_{n_1}, T_{22}\} \in \mathbb{R}^{n\times(n_1+n_2)}$ such that $[T_0 \;\; \Theta(A_*, B_*, \eta)]$ is nonsingular. Let $T_\mathrm{S} := [T_{\mathrm{S}1}T_0 \;\; \Theta(A, B, \eta)]$, we have

$$\begin{aligned} B &= [T_\mathrm{S}e_{g_2} \;\; T_\mathrm{S}e_{g_3} \cdots T_\mathrm{S}e_n | b_{\varpi+1} \cdots b_m] \\ &= T_\mathrm{S}[e_{g_2} \;\; e_{g_3} \cdots e_n | T_\mathrm{S}^{-1}b_{\varpi+1} \cdots T_\mathrm{S}^{-1}b_m], \end{aligned} \tag{6}$$

$$\begin{aligned} AT_\mathrm{S} &= [AT_0 | A^{\eta_1}b_1T_\mathrm{S}e_{g_1+1} \cdots T_\mathrm{S}e_{g_2} | \\ &\qquad \cdots | A^{\eta_\varpi}b_\varpi T_\mathrm{S}e_{g_\varpi+1} \cdots T_\mathrm{S}e_{n-1}] \\ &= T_\mathrm{S}[T_\mathrm{S}^{-1}AT_0 | T_\mathrm{S}^{-1}A^{\eta_1}b_1e_{g_1+1} \cdots e_{g_2} | \\ &\qquad \cdots | T_\mathrm{S}^{-1}A^{\eta_\varpi}b_\varpi e_{g_\varpi+1} \cdots e_{n-1}], \end{aligned} \tag{7}$$

where $g_j = n_1 + n_2 + \sum_{i=1}^{j-1}\eta_i$, $j = 1, 2, \ldots, \varpi$, and e_i is the ith column of I_n. Multiplying both sides of (6) and (7) from the left by T_S^{-1}, we obtain the result of the lemma. □

Lemma 3.2. *Consider a triple (A, B, C) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$,*

$$A = \begin{bmatrix} A_0 & \gamma_1 & 0 & \cdots & \gamma_m & 0 \\ \Delta_1 & \star & I_{\tau_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_m & \star & 0 & \cdots & \star & I_{\tau_m-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where $A_0 \in \mathbb{R}^{n_o\times n_o}$ and $\gamma_i \in \mathbb{R}^{n_o}$, $\Delta_i \in \mathbb{R}^{(\tau_i-1)\times n_o}$, $i = 1, 2, \ldots, m$, with $n_o = n - \sum_{i=1}^{m}\tau_i$. There exists a T_S such that

$$B_1 = T_\mathrm{S}^{-1}B = B, \quad C_1 = CT_\mathrm{S} = C, \tag{8}$$

$$A_1 = T_\mathrm{S}^{-1}AT_\mathrm{S} = \begin{bmatrix} A_0 & \gamma_1 & 0 & \cdots & \gamma_m & 0 \\ 0 & \star & I_{\tau_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{\tau_m-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \tag{9}$$

which reveals that the system (A, B, C) is invertible with finite zeros $\lambda(A_0)$ and infinite zeros $\{\tau_1, \tau_2, \ldots, \tau_m\}$.

Proof. Let

$$x = \begin{pmatrix} x_0 \\ x_\mathrm{d} \end{pmatrix}, \quad x_\mathrm{d} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad x_i = \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,\tau_i} \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

Then the system (A, B, C) can be written as

$$\dot{x}_0 = A_0x_0 + \sum_{k=1}^{m}\gamma_k x_{i,1},$$

$$\dot{x}_{i,j} = A_{i,j}x_0 + x_{i,j+1} + \sum_{k=1}^{m}a_{i,j,k}x_{i,1},$$

$$\dot{x}_{i,\tau_i} = A_{i,\tau_i} x_0 + \sum_{k=1}^{m} a_{i,\tau_i,k} x_{i,1} + u_i,$$

$$j = 1, 2, \ldots, \tau_i, \quad i = 1, 2, \ldots, m.$$

Define $x_{i,2}^1 = A_{i,1} x_0 + x_{i,2}$, then,

$$\dot{x}_{i,1} = x_{i,2}^1 + \sum_{k=1}^{m} a_{i,1,k} x_{i,1},$$

$$\begin{aligned}\dot{x}_{i,2}^1 &= A_{i,1} \dot{x}_0 + \dot{x}_{i,2} \\ &= (A_{i,2} + A_{i,1} A_0) x_0 + x_{i,3} + \sum_{k=1}^{m} (a_{i,2,k} + A_{i,1} \gamma_k) x_{i,1} \\ &:= A_{i,2}^1 x_0 + x_{i,3} + \sum_{k=1}^{m} a_{i,2,k}^1 x_{i,1}.\end{aligned}$$

Similarly, defining $x_{i,3}^1 = A_{i,2}^1 x_0 + x_{i,3}$, we have

$$\dot{x}_{i,2} = x_{i,3}^1 + \sum_{k=1}^{m} a_{i,2,k} x_{i,1},$$

$$\dot{x}_{i,3}^1 = A_{i,2}^1 \dot{x}_0 + \dot{x}_{i,3} := A_{i,3}^1 x_0 + x_{i,4} + \sum_{k=1}^{m} a_{i,3,k}^1 x_{i,1}.$$

Proceeding recursively, we finally obtain

$$\dot{x}_{i,1} = x_{i,2}^1 + \sum_{k=1}^{m} a_{i,1,k} x_{i,1},$$

$$\dot{x}_{i,j}^1 = x_{i,j+1}^1 + \sum_{k=1}^{m} a_{i,j,k}^1 x_{i,1},$$

$$\dot{x}_{i,\tau_i}^1 = A_{i,\tau_i}^1 x_0 + \sum_{k=1}^{m} a_{i,\tau_i,k}^1 x_{i,1} + u_i,$$

$$j = 2, 3, \ldots, \tau_i, \quad i = 1, 2, \ldots, m.$$

Thus, there exists a T_S,

$$T_\mathrm{S} = \begin{bmatrix} I_{n_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \Phi_1 & 0 & I_{\tau_1 - 1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & \cdots & 1 & 0 \\ \Phi_m & 0 & 0 & \cdots & 0 & I_{\tau_m - 1} \end{bmatrix},$$

with some appropriate matrices $\Phi_i \in \mathbb{R}^{(\tau_i - 1)\times n_0}$, $i = 1, 2, \ldots, m$, such that $A_1 = T_\mathrm{S}^{-1} A T_\mathrm{S}$, $B_1 = T_\mathrm{S}^{-1} B = B$ and $C_1 = C T_\mathrm{S} = C$ are in the form of (8)–(9), which in turn is in the form of (2)–(5). □

The following is a simple algorithm that, for a given A, assigns B such that (A, B) has the prescribed controllability indices.

Lemma 3.3. *Given $A \in \mathbb{R}^{n\times n}$ with its eigenvalues Δ having unity geometric multiplicities. Let $k = \{k_1, k_2, \ldots, k_m\}$ be a set of nonnegative integers, and Δ_1 be a set of n_1 complex scalars. Then, there exists a $B \in \mathbb{R}^{n\times m}$ such that the pair (A, B) has controllability indices k and uncontrollable eigenvalues Δ_1 if and only if $\Delta_1 \subset \Delta$ is self-conjugated and $n_1 + \sum_{i=1}^{m} k_i = n$.*

Proof. Necessity: It is obvious since there exists a T such that $T^{-1}AT = \left[\begin{smallmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{smallmatrix}\right]$, $T^{-1}B = \left[\begin{smallmatrix} 0 \\ B_{22} \end{smallmatrix}\right]$, where (A_{22}, B_{22}) is controllable.

Sufficiency: Without loss generality, we assume that the matrix A is already in Jordan form, and $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_m$. There exists a T such that $T^{-1}AT = \left[\begin{smallmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{smallmatrix}\right]$, where $\lambda(A_{11}) = \Delta_1$, and $\lambda(A_{22})$ have unity geometric multiplicities. Hence, there exists $b_2 \in \mathbb{R}^{n-n_1}$ such that (A_{22}, b_2) is controllable. Let $B = [b \;\; A^{w_2} b \;\; \cdots \;\; A^{w_m} b]$, $b = [0 \;\; b_2']'$, where $w_j = \sum_{i=1}^{j-1} k_i$, $j = 2, 3, \ldots, m$. It can be verified that (A, B) has the prescribed controllability indices. □

4. Main results

We first give necessary and sufficient conditions for the assignability of a set of infinite zeros.

Lemma 4.1. *Consider a pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let $\Lambda_4 = \{q_1, q_2, \ldots, q_{m_\mathrm{d}}\}$ be a set of positive integers. Then, there exist the matrices C and D such that the infinite zeros of order greater than 0 of the resulting system (A, B, C, D) are given by Λ_4 if and only if Λ_4 is an infinite zero assignable set of the pair (A, B).*

Proof. Necessity: Let $T_\mathrm{S} \in \mathbb{R}^{n\times n}$ and $T_\mathrm{I} \in \mathbb{R}^{m\times m}$ be such that $T_\mathrm{S}^{-1} A T_\mathrm{S}$ and $T_\mathrm{S}^{-1} B T_\mathrm{I}$ are in the forms of (2)–(3). Define

$$K = T_\mathrm{I} \begin{bmatrix} C_{0\mathrm{a}} & C_{0\mathrm{b}} & C_{0\mathrm{c}} & C_{0\mathrm{d}} \\ 0 & 0 & 0 & E_{\mathrm{dd}} \\ 0 & 0 & 0 & 0 \end{bmatrix} T_\mathrm{S}^{-1},$$

$$T_{\mathrm{I}1} = \begin{bmatrix} 0 & I_{m_0} & 0 \\ I_{m_\mathrm{d}} & 0 & 0 \\ 0 & 0 & I_{m_\mathrm{c}} \end{bmatrix}.$$

Then,

$$A - BK = T_\mathrm{S} \begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ab}} C_\mathrm{b} & 0 & L_{\mathrm{ad}} C_\mathrm{d} \\ 0 & A_{\mathrm{bb}} & 0 & L_{\mathrm{bd}} C_\mathrm{d} \\ B_\mathrm{c} E_{\mathrm{ca}} & L_{\mathrm{cb}} C_\mathrm{b} & A_{\mathrm{cc}} & L_{\mathrm{cd}} C_\mathrm{d} \\ B_\mathrm{d} E_{\mathrm{da}} & B_\mathrm{d} E_{\mathrm{db}} & B_\mathrm{d} E_{\mathrm{dc}} & A_{\mathrm{dd}}^* + L_{\mathrm{dd}} C_\mathrm{d} \end{bmatrix} T_\mathrm{S}^{-1}.$$

Therefore, $\Theta(A - BK, BT_\mathrm{I} T_{\mathrm{I}1}, \Lambda_4) = T_\mathrm{S}[0 \;\; \Delta_\mathrm{d}']'$, where $\Delta_\mathrm{d} = \mathrm{blkdiag}\,\{\delta_{q_1}, \delta_{q_2}, \ldots, \delta_{q_{m_\mathrm{d}}}\}$, $\delta_k \in \mathbb{R}^{k\times k}$ with the elements in the inverse diagonal being 1s, and all the other elements being 0s. Thus, Λ_4 is an infinite zero assignable set of (A, B).

Sufficiency: We will give a constructive proof that would yield the desired (C, D). By Lemma 3.1, there exist $T_{\mathrm{S1}} \in \mathbb{R}^{n\times n}$, $T_{\mathrm{I}} \in \mathbb{R}^{m\times m}$, and $K_1 \in \mathbb{R}^{m\times n}$ such that

$$A_1 = T_{\mathrm{S1}}^{-1}(A - BT_{\mathrm{I}}K_1)T_{\mathrm{S1}} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \alpha_1 & 0 & \cdots & \alpha_{m_{\mathrm{d}}} & 0 \\ \varDelta_{11} & \varDelta_{12} & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varDelta_{m_{\mathrm{d}}1} & \varDelta_{m_{\mathrm{d}}2} & \star & 0 & \cdots & \star & I_{q_{m_{\mathrm{d}}}-1} \\ \star & \star & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \tag{10}$$

$$B_1 = T_{\mathrm{S1}}^{-1}BT_{\mathrm{I}} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & B_{22} \\ 0 & \cdots & 0 & \star \\ 1 & \cdots & 0 & \star \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \star \\ 0 & \cdots & 1 & \star \end{bmatrix}, \tag{11}$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}$ contains uncontrollable eigenvalues of A, $A_{22} \in \mathbb{R}^{n_2\times n_2}$ and $B_{22} \in \mathbb{R}^{n_2\times m_0}$ with $m_0 = m - m_{\mathrm{d}}$ and $n_2 = n - n_1 - \sum_{i=1}^{m_{\mathrm{d}}} q_i$. Thus, by Lemma 3.2, there exists a T_{S2} such that

$$\tilde{A}_2 = T_{\mathrm{S2}}^{-1}A_1T_{\mathrm{S2}} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \alpha_1 & 0 & \cdots & \alpha_{m_{\mathrm{d}}} & 0 \\ 0 & 0 & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \star & 0 & \cdots & \star & I_{q_{m_{\mathrm{d}}}-1} \\ \star & \star & \star & 0 & \cdots & \star & 0 \end{bmatrix}, \tag{12}$$

$$\tilde{B}_2 = T_{\mathrm{S2}}^{-1}B_1 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & B_{22} \\ 0 & \cdots & 0 & \star \\ 1 & \cdots & 0 & \star \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \star \\ 0 & \cdots & 1 & \star \end{bmatrix}. \tag{13}$$

Clearly, $A = T_{\mathrm{S1}}T_{\mathrm{S2}}\tilde{A}_2T_{\mathrm{S2}}^{-1}T_{\mathrm{S1}}^{-1} + T_{\mathrm{S1}}T_{\mathrm{S2}}\tilde{B}_2K_1$, $B = T_{\mathrm{S1}}T_{\mathrm{S2}}\tilde{B}_2T_{\mathrm{I}}^{-1}$. Let us define

$$\tilde{C}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0_{m_0\times n_1} & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \tilde{D}_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{m_0} \end{bmatrix},$$

which is in conformity with the structures of $\tilde{A}_2$ and $\tilde{B}_2$. The system $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_2, \tilde{D}_2)$ is invertible with m_0 infinite zeros of order 0 and infinite zeros (of order greater than 0) $\varLambda_4$. Assign $C = \tilde{C}_2T_{\mathrm{S2}}^{-1}T_{\mathrm{S1}}^{-1} + \tilde{D}_2K_1$, $D = \tilde{D}_2T_{\mathrm{I}}^{-1}$. The systems (A, B, C, D) and $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_2, \tilde{D}_2)$ are equivalent under state and input transformations and state feedback. Thus, they have the same Morse index lists. □

Remark 4.1. Following the proof of Lemma 4.1, a set of necessary conditions can be established under which a complete structure can be assigned: consider a pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let n_{a} be a nonnegative integer, $\varLambda_2 = \{\ell_1, \ell_2, \ldots, \ell_{m_{\mathrm{c}}}\}$, $\varLambda_3 = \{\mu_1, \mu_2, \ldots, \mu_{p_{\mathrm{b}}}\}$ and $\varLambda_4 = \{q_1, q_2, \ldots, q_{m_{\mathrm{d}}}\}$ be three sets of positive integers. If there exist $C \in \mathbb{R}^{(m+p_{\mathrm{b}}-m_{\mathrm{c}})\times n}$ and $D \in \mathbb{R}^{(m+p_{\mathrm{b}}-m_{\mathrm{c}})\times m}$ such that the system (A, B, C, D) has n_{a} finite zeros, $m - m_{\mathrm{d}}$ infinite zeros of order 0, and the Morse index lists $\mathscr{I}_2 = \varLambda_2$, $\mathscr{I}_3 = \varLambda_3$ and $\mathscr{I}_4 = \varLambda_4$, then n_{a}, $\varLambda_2$, $\varLambda_3$ and $\varLambda_4$ must satisfy

(1) $\{\ell_1, \ell_2, \ldots, \ell_{m_{\mathrm{c}}}, q_1, q_2, \ldots, q_{m_{\mathrm{d}}}\}$ is an infinite zero assignable set of the pair (A, B);
(2) $n_{\mathrm{a}} + \sum_{i=1}^{p_{\mathrm{b}}}\mu_i + \sum_{i=1}^{m_{\mathrm{c}}}\ell_i + \sum_{i=1}^{m_{\mathrm{d}}}q_i = n$.

In what follows, we present the necessary and sufficiency conditions for the assignability of a set of structural properties which includes the left invertibility property. In the statement of the theorem, repeated uncontrollable eigenvalues are counted repeatedly.

Theorem 4.1. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that A has n_1 uncontrollable eigenvalues $\varDelta$, all of which have unity geometric multiplicities. Let $\varLambda_1$ be a set of n_{a} self-conjugated complex scalars, and $\varLambda_3 = \{\mu_1, \mu_2, \ldots, \mu_{p_{\mathrm{b}}}\}$ and $\varLambda_4 = \{q_1, q_2, \ldots, q_{m_{\mathrm{d}}}\}$ be two sets of positive integers. Then, there exist C and D such that the resulting system (A, B, C, D) is left invertible ($\mathscr{I}_2 = \emptyset$), and has finite zeros $\varLambda_1$, $m - m_{\mathrm{d}}$ infinite zeros of order 0, and the Morse index lists $\mathscr{I}_3 = \varLambda_3$ and $\mathscr{I}_4 = \varLambda_4$ if and only if*

(1) *$\varLambda_1 = \varTheta_1 \cup \varDelta_1$, where $\varDelta_1 \subset \varDelta$, and $\varTheta_1$ is a set of n_{e} self-conjugated complex scalars, $n_{\mathrm{e}} \leqslant n - n_1 - \sum_{i=1}^{m_{\mathrm{d}}}q_i$;*
(2) *$\varLambda_4$ is an infinite zero assignable set of (A, B);*
(3) *$n_{\mathrm{a}} + \sum_{i=1}^{p_{\mathrm{b}}}\mu_i + \sum_{i=1}^{m_{\mathrm{d}}}q_i = n$.*

Proof. Necessity: Condition (2) follows from Lemma 4.1. Condition (3) is obvious since A_{cc} is an empty matrix. By Lemma 2.1, $\lambda(A_{\mathrm{aa}})$ contains $n_{\mathrm{a}} - n_{\mathrm{e}}$ self-conjugated uncontrollable eigenvalues, and $\lambda(A_{\mathrm{bb}})$ contains $\sum_{i=1}^{p_{\mathrm{b}}}\mu_i - n_*$ self-conjugated uncontrollable eigenvalues, where $n_* \geqslant 0$. Thus, $n_{\mathrm{a}} - n_{\mathrm{e}} + \sum_{i=1}^{p_{\mathrm{b}}}\mu_i - n_* = n_1$. Hence, we have Condition (1).

Sufficiency: We will give a constructive proof. Following the proof of Lemma 4.1, there exist $T_{\mathrm{S1}}, T_{\mathrm{S2}} \in \mathbb{R}^{n\times n}$, $T_{\mathrm{I}} \in \mathbb{R}^{m\times m}$ and $K_1 \in \mathbb{R}^{m\times n}$, such that $(A_1, B_1) := (T_{\mathrm{S1}}^{-1}(A - BT_{\mathrm{I}}K_1)T_{\mathrm{S1}}, T_{\mathrm{S1}}^{-1}BT_{\mathrm{I}})$ is in the form of (10)–(11) and $(\tilde{A}_2, \tilde{B}_2) := (T_{\mathrm{S2}}^{-1}A_1T_{\mathrm{S2}}, T_{\mathrm{S2}}^{-1}B_1)$ is in the form of

(12)–(13). Obviously, there exists $K_\star \in \mathbb{R}^{m\times n}$ such that

$$\tilde{A}_2 - \tilde{B}_2 K_\star = \begin{bmatrix} A_{11} & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \alpha_1 & 0 & \cdots & \alpha_{m_d} & 0 \\ 0 & 0 & \star & I_{q_1-1} & \cdots & \star & 0 \\ 0 & 0 & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \star & 0 & \cdots & \star & I_{q_{m_d}-1} \\ 0 & 0 & \star & 0 & \cdots & \star & 0 \end{bmatrix}.$$

Let $L_\alpha = [\alpha_1 \ \alpha_2 \ \cdots \ \alpha_{m_d} \ B_{22}]$. By the PBH test, $\mathrm{rank}[A_{22} - sI \ \ L_\alpha] = n_2$, $s \in \mathbb{C}$, thus (A_{22}, L_α) is controllable. Define $A_\gamma = A_{22} - L_\alpha K_\beta = A_{22} - B_{22}K_{22} - \sum_{i=1}^{m_d} \alpha_i \beta_i$, where $K_\beta = [\beta_1' \ \beta_2' \ \cdots \ \beta_{m_d}' \ K_{22}']'$ with $\beta_1, \beta_2, \ldots, \beta_{m_d} \in \mathbb{R}^{1\times n_2}$ and $K_{22} \in \mathbb{R}^{m_0 \times n_2}$. $\lambda(A_\gamma)$ can be freely relocated by K_β. We select K_β such that $\lambda(A_\gamma)$ includes Θ_1 and some other distinct eigenvalues. Consider

$$A_{*1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} - B_{22}K_{21} - \sum_{i=1}^{m_d} \alpha_i \delta_i & A_\gamma \end{bmatrix},$$

where $\delta_1, \delta_2, \ldots, \delta_{m_d} \in \mathbb{R}^{1\times n_1}$ and $K_{21} \in \mathbb{R}^{m_0\times n_1}$. Note that $\delta_1, \delta_2, \ldots, \delta_{m_d}$ and K_{21} will not change $\lambda(A_{*1})$, but can change the Jordan form of A_{*1}. Let T_{*2} be such that

$$A_{*2} = T_{*2}^{-1} A_{*1} T_{*2} = \begin{bmatrix} A_{aa} & M_{ab} \\ 0 & A_{bb} \end{bmatrix},$$

where $\lambda(A_{aa})$ are given by Λ_1, and $\lambda(A_{bb})$ have unity geometric multiplicities. Thus, by the proof of Lemma 3.3, we can construct a C_b such that (A_{bb}, C_b) is observable and has observability indices Λ_3. And thus, there exists an L_b such that $\lambda(A_{bb} - L_b C_b) \cap \lambda(A_{aa}) = \emptyset$. Consequently, the Sylvester equation $-A_{aa}N + N(A_{bb} - L_b C_b) = M_{ab}$ has a unique solution $N \in \mathbb{R}^{n_1 \times n_2}$. Let $C_{*1} = [C_{21} \ \ C_{22}] := [0 \ \ C_b] T_{*2}^{-1}$, $C_{21} \in \mathbb{R}^{p_b \times n_1}$ and

$$T_* = T_{*2} \begin{bmatrix} I & N \\ 0 & I \end{bmatrix}.$$

We have

$$T_*^{-1} A_{*1} T_* = \begin{bmatrix} A_{aa} & N L_b C_b \\ 0 & A_{bb} \end{bmatrix}, \quad C_{*1} T_* = [0 \ \ C_b].$$

Denote

$$K_2 = \begin{bmatrix} 0 & 0 & 0 \\ K_{21} & K_{22} & 0 \end{bmatrix} \in \mathbb{R}^{m\times n},$$

and let $B_2 = B_1$ and

$$A_2 = A_1 - B_1 K_2 = \begin{bmatrix} A_{11} & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{21} - B_{22}K_{21} & A_{22} - B_{22}K_{22} & \alpha_1 & 0 & \cdots & \alpha_{m_d} & 0 \\ \Delta_{11} & \Delta_{12} & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{m_d1} & \Delta_{m_d2} & \star & 0 & \cdots & \star & I_{q_{m_d}-1} \\ \star & \star & \star & 0 & \cdots & \star & 0 \end{bmatrix}.$$

We assign (C_2, D_2) as follows:

$$C_2 = \begin{bmatrix} \delta_1 & \beta_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{m_d} & \beta_{m_d} & 0 & 0 & \cdots & 1 & 0 \\ C_{21} & C_{22} & 0 & 0 & \cdots & 0 & 0 \\ 0_{m_0\times n_1} & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{m_0} \end{bmatrix},$$

which are in conformity with the structures of A_2 and B_2. It can be verified that $A = T_{S1} A_2 T_{S1}^{-1} + T_{S1} B_2 (K_1 + K_2 T_{S1}^{-1})$, $B = T_{S1} B_2 T_I^{-1}$. Let

$$C = C_2 T_{S1}^{-1} + D_2(K_1 + K_2 T_{S1}^{-1}), \quad D = D_2 T_I^{-1}.$$

In what follows, we will show that the system (A, B, C, D) has the desired structural properties. It is obvious that the systems (A, B, C, D) and (A_2, B_2, C_2, D_2) are equivalent under state and input transformations and state feedback. We further define T_{S3} as

$$T_{S3} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & \cdots & 0 & 0 \\ -\delta_1 & -\beta_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{q_1-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\delta_{m_d} & -\beta_{m_d} & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I_{q_{m_d}-1} \end{bmatrix}.$$

Consequently,

$$A_3 = T_{S3} A_2 T_{S3}^{-1} = \begin{bmatrix} A_{*1} & \star & 0 & \cdots & \star & 0 \\ \Delta_1 & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{m_d} & \star & 0 & \cdots & \star & I_{q_{m_d}-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix},$$

$$C_3 = T_{S3} C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ C_{21} & C_{22} & 0 & 0 & \cdots & 0 & 0 \\ 0_{m_0\times n_1} & 0_{m_0\times n_2} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$B_3 = T_{S3} B_2 = B_2, \quad D_3 = D_2.$$

By Lemma 3.2, we can find a T_{S4} such that

$$A_4 = T_{S4} A_3 T_{S4}^{-1} = \begin{bmatrix} A_{*1} & \star & 0 & \cdots & \star & 0 \\ 0 & \star & I_{q_1-1} & \cdots & \star & 0 \\ \star & \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \star & 0 & \cdots & \star & I_{q_{m_d}-1} \\ \star & \star & 0 & \cdots & \star & 0 \end{bmatrix},$$

$$C_4 = T_{S4} C_3 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ C_{*1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$B_4 = T_{S4} B_3 = B_3, \quad D_4 = D_3.$$

Define $T_{S5} = \operatorname{blkdiag}\{T_*, I\} \in \mathbb{R}^{n\times n}$. The quadruple $(T_{S5}^{-1} A_4 T_{S5}, T_{S5}^{-1} B_4, C_4 T_{S5}, D_4)$ is in the form of (2)–(5), having finite zeros Λ_1, $m - m_d$ infinite zeros of order 0, and the Morse index lists $\mathscr{I}_2 = \emptyset$, $\mathscr{I}_3 = \Lambda_3$ and $\mathscr{I}_4 = \Lambda_4$. And so are the systems (A_2, B_2, C_2, D_2) and (A, B, C, D). In conclusion, we have obtained a set of the desired (C, D) as given by $\Omega = \{(\Gamma_O C, \Gamma_O D) | \Gamma_O \in \mathbb{R}^{(p_b+m)\times(p_b+m)} \text{ is nonsingular}\}$. □

Remark 4.2. For the given (A, B), if the uncontrollable eigenvalues are not of unity geometric multiplicities, then the assignment of $\mathscr{I}_3$ will be subject to more constraints and thus will be slightly more complicated. We also note that the selection of Θ_1 is free as long as it satisfies conditions in Theorem 4.1, but the eigenstructure of finite zeros corresponding to Θ_1 are not necessarily freely assignable.

Remark 4.3. In our earlier algorithm (Liu et al., 2003), in order to be assignable, each desired order of infinite zeros must be equal to or less than a corresponding element in the controllability indices of (A, B). In our current algorithm, no such a constraint is imposed. We, however, note that according to Commault and Dion (1982) and Amparan et al. (2004), the majorization relation between the controllability indices and the assignable infinite zero orders still need to be satisfied.

The following corollary deals with the assignment of structural properties of invertible systems.

Corollary 4.1. *Consider the pair* (A, B) *with* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ *and uncontrollable eigenvalues* Δ. *Let* Λ_1 *be a set of* n_a *complex scalars, and* $\Lambda_4 = \{q_1, q_2, \ldots, q_{m_d}\}$ *be a set of positive integers. Then, there exist* $C \in \mathbb{R}^{m\times n}$ *and* $D \in \mathbb{R}^{m\times m}$ *such that* (A, B, C, D) *is invertible, and has finite zeros* Λ_1, $m - m_d$ *infinite zeros of order* 0, *and infinite zeros* (*of order greater than* 0) Λ_4 *if and only if*

(1) $\Lambda_1 = \Theta_1 \cup \Delta$, *where* Θ_1 *is a set of self-conjugated complex scalars*;
(2) Λ_4 *is an infinite zero assignable set of* (A, B);
(3) $n_a + \sum_{i=1}^{m_d} q_i = n$.

5. An example

We consider a benchmark problem for robust control of a flexible mechanical system in Wie and Bernstein (1990). The problem is to control the displacement of the second mass by applying a force to the first mass as shown in Fig. 1, where x_1 and x_2 are, respectively, the positions of Mass 1 ($m_1 = 1$) and Mass 2 ($m_2 = 1$), $k = 1$ is the spring constant, u is the input force, and w_1 and w_2 are the frictions (disturbances). The output to be controlled is $z = x_2$, the dynamic model is given by

$$\dot{x} = Ax + Bu + Ew$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$
$$z = C_2 x = [0 \;\; 0 \;\; 1 \;\; 0] x.$$

It is simple to verify that the subsystem (A, B, C_2) is of minimum-phase and invertible. Hence, the disturbance w can be totally decoupled from z under the full state feedback. Our objective is to identify sets of measurement output or the locations of sensors such that an output feedback could yield the same performance as the state feedback. It follows from Chen (2000) that this can be made possible by choosing a measurement $y = C_1 x$, such that (A, E, C_1) is left invertible and of minimum-phase. It can be verified that (A, E) is in the controllability canonical form, with controllability index $\{2, 2\}$. Following the algorithm given in the previous section, we obtain the measurement matrices,

$$\mathbf{\Omega}_1 = \left\{ \Gamma_O \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right\},$$

where $\Gamma_O \in \mathbb{R}^{2\times 2}$, $\det(\Gamma_O) \neq 0$, such that for any $C_1 \in \mathbf{\Omega}_1$, (A, E, C_1) is square invertible with two infinite zeros of order 2 and no finite zeros. Similarly, we assign

$$\mathbf{\Omega}_2 = \left\{ \Gamma_O \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\},$$

such that for any $C_1 \in \mathbf{\Omega}_2$, (A, E, C_1) is square invertible with two infinite zeros of order 1 and two finite zeros at -1. We can also assign

$$\mathbf{\Omega}_3 = \left\{ \Gamma_O \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\},$$

such that for any $C_1 \in \mathbf{\Omega}_3$, (A, E, C_1) is square invertible with two infinite zeros $\{1, 3\}$ and no finite zeros. For the case of

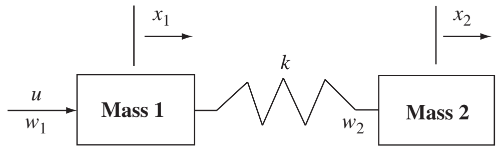

Fig. 1. A two-mass-spring flexible mechanical system.

$D_1 \neq 0$, we assign

$$\mathbf{\Omega}_4 = \left\{ \left(\Gamma_{\mathrm{O}} \begin{bmatrix} 0 & 1 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \Gamma_{\mathrm{O}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) \right\},$$

such that for $(C_1, D_1) \in \mathbf{\Omega}_4$, (A, E, C_1, D_1) is square invertible with two infinite zeros $\{0, 1\}$ and three finite zeros at -2, $-1 + j$ and $-1 - j$.

For any $C_1 \in \mathbf{\Omega}_1$, (A, E, C_1) is of minimum-phase, but has higher order infinite zeros. It is well known that higher orders of infinite zeros would yield higher controller gains, which is in general not desirable. Thus, the measurement output $C_1 \in \mathbf{\Omega}_2$ is more desirable. It is straightforward to verify that the H_∞ almost disturbance decoupling is achievable by measurement feedback for any $C_1 \in \mathbf{\Omega}_1$ or $C_1 \in \mathbf{\Omega}_2$.

6. Conclusions

In this paper, we have revisited the problem of structural assignment for linear systems. By introducing the notion of infinite zero assignable set for a matrix pair, we established necessary and sufficient conditions for the assignability of a set of structural properties which includes the left invertibility property. These results significantly improve the existing results on the topic.

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