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Brief Paper

Further results on almost disturbance decoupling with global asymptotic stability for nonlinear systems[1]

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Abstract

As a complement to some new breakthroughs on global almost disturbance decoupling problem with stability for nonlinear systems, in a recent note, we identified a class of unstable zero dynamics that are allowed to be affected by disturbances. The class of the unstable zero dynamics identified in that note is linear and have all the poles at the origin. In this paper, we enlarge such a class of zero dynamics to include any linear system with all its poles in the closed left-half plane. The condition on the way the disturbance affects this part of zero dynamics is also identified. This enlargement is due to a new scaling technique that views each pair of jw axis zeros as a "generalized integrator" and transforms the zero dynamics into a number of chains of "generalized integrators". © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction and preliminaries

We revisit the problem of global almost disturbance decoupling with stability for nonlinear systems of the form,

$$\dot{x} = f(x) + g(x)u + p(x)w, \quad y = h(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $w \in \mathbb{R}$ is the disturbance, $y \in \mathbb{R}$ is the regulated output, f, g, and p are smooth vector fields with $f(0) = 0$, and h is a smooth function with $h(0) = 0$. The problem of almost disturbance decoupling with stability was originally formulated and solved for linear systems by Willems (1981). Since then various generalizations to nonlinear systems have been made (see, for example, Saberi and Sannuti, 1988; Marino et al., 1989, 1994; Isidori, 1996a, b; Lin, 1998a and the references therein). The problem we are to consider in this note is the following one (see, for example, Isidori, 1995).

Definition 1 (*L_2 almost disturbance decoupling with global asymptotic stability*). The problem of L_2 almost disturbance decoupling with global asymptotic stability is said to be solvable for system (1) if, for any given $\gamma > 0$, there is a smooth feedback law $u = u(x; \gamma)$ with $u(0; \gamma) = 0$, such that the corresponding closed-loop system

(a) has a globally asymptotically stable equilibrium at $x = 0$;
(b) has an L_2 gain, from the disturbance input w to the regulated output y, that is less than or equal to γ, i.e.,

$$\int_0^\infty y^2(t)\,\mathrm{d}t \le \gamma^2 \int_0^\infty w^2(t)\,\mathrm{d}t,$$

$$\forall w \in L_2 \quad \text{and} \quad \text{for } x(0) = 0 \tag{2}$$

and all the states of the closed-loop system, and hence the control $u = u(x;\gamma)$ are bounded.

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In this paper, we consider the problem of L_2 almost disturbance decoupling with global asymptotic stability for system (1) in the following special form;

$$\begin{aligned} \dot{z} &= f_0(z, \xi_1) + p_0(z, \xi_1) w, \\ \dot{\xi}_i &= \xi_{i+1} + p_i(z, \xi_1, \xi_2, \dots, \xi_i) w, \quad i = 1, 2, \dots, r-1, \\ \dot{\xi}_r &= u + p_r(z, \xi_1, \xi_2, \dots, \xi_{r-1}, \xi_r) w, \\ y &= \xi_1 \end{aligned} \tag{3}$$

in which it is assumed that $f_0(0, 0) = 0$ and the dynamics

$$\dot{z} = f_0(z, 0) \tag{4}$$

is referred to as the zero dynamics of system (3) (Isidori, 1995). Throughout this note, we will also, by somewhat abuse of terminology, refer to the first equation of Eq. (3) as the zero dynamics equation.

Our result represents a generalization of our recent note (Lin, 1998a) and a further complement to the recent series of results (Marino et al., 1994; Isidori, 1995, 1996a, b). More specifically, it is shown in Marino et al. (1994) that the problem of L_2 almost disturbance decoupling with global asymptotic stability is solvable if

(i) the equilibrium $z = 0$ of the zero dynamics (4) is globally asymptotically stable; and
(ii) $p_0(z, 0) = 0$,

and, under these conditions, feedback laws of high gain type that solve the problem are also explicitly constructed. This result of Marino et al. (1994) was recently generalized in Isidori (1996a) in the sense that Condition (ii) is replaced by a weaker one. The requirement that the zero dynamics be globally asymptotically stable (Condition (i)), however, remains unrelaxed.

More recently, the results of Marino et al. (1994) and Isidori (1996a) are further generalized in Isidori (1996b) to allow part of the zero dynamics to be unstable as long as it satisfies certain stabilizability condition and its corresponding zero dynamics equation *is unaffected* by the disturbance w. The zero dynamics equation (recall that the first equation of Eq. (1) is referred to as the zero dynamics equation) considered in Isidori (1996b) takes the following cascade-connected form with two subsystems:

$$\begin{aligned} \dot{z}_a &= f_a(z_a, z_c, \xi_1) + p_a(z_a, z_c, \xi_1) w, \\ \dot{z}_c &= f_c(z_c, \xi_1) \end{aligned} \tag{5}$$

where the first one characterizes a "stable part" of the zero dynamics (more precisely, $z_a = 0$ is a globally asymptotically stable equilibrium of $\dot{z}_a = f_a(z_a, 0, 0)$), and the second one characterizes a possibly unstable but stabilizable and *disturbance unaffected* part of the zero dynamics. The conditions needed on both subsystems for solving the problem of L_2 almost disturbance decoupling with globally asymptotic stability can be made more precise by recalling the following result from Isidori (1996b). We note here that the bounded state property as required by Definition 1, although not stated explicitly in the previous formulation, is ensured by all the recent designs (Marino et al., 1994; Isidori, 1995, 1996a, b; Lin, 1998a).

Theorem 1. *Suppose that*

(i) *there exists a smooth real-valued function $V_a(z_a)$, which is positive definite and proper, such that*

$$\frac{\partial V_a}{\partial z_a}[f_a(z_a, z_c, \xi_1) + p_a(z_a, z_c, \xi_1) w] \le -\alpha_a(|z_a|) + \gamma_0^2 |w|^2 + \gamma_0^2 |z_c|^2 + \gamma_0^2 |\xi_1|^2 \tag{6}$$

for some $\mathscr{K}_\infty$ function α_a and some positive real number γ_0 (here $|\cdot|$ *denotes the Euclidean norm and the class $\mathscr{K}_\infty$ consists of all functions α: $\mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ which are continuous, strictly increasing, and satisfy $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$*), *and*
(ii) *there exists a smooth real-valued function $v_c(z_c)$, with $v_c(0) = 0$, and a smooth real-valued function $V_c(z_c)$, which is positive definite and proper, such that*

$$\frac{\partial V_c}{\partial z_c} f_c(z_c, v_c(z_c)) + |v_c(z_c)|^2 \le -\alpha_c(|z_c|) \tag{7}$$

for some $\mathscr{K}_\infty$ function α_c.

Then, the problem of L_2 almost disturbance decoupling with global asymptotic stability is solvable for system (3) *with its first equation in the form of Eq.* (5).

For use in the proof of our main result, we also recall the following observation from Isidori (1996b).

Observation 1. *We note here the emphasis on that Eq.* (6) *be true for some positive real number γ_0. Indeed what is needed in the proof is that for the arbitrary positive number γ there exist V_a and α_a such that Eq.* (6) *hold. One observes that the former is sufficient to guarantee the latter, for one can multiply both sides of Eq.* (6) *by γ^2/γ_0^2 and redefine V_1 and α_a accordingly.*

As a complement to the above-mentioned results where unstable zero dynamics is either not allowed to be present or to be affected by disturbance, we observed in a recent note (Lin, 1998a) that a certain class of unstable zero dynamics is actually allowed to be affected by the disturbance in solving the problem of L_2 almost disturbance decoupling with global asymptotic stability. The class of disturbance affected unstable zero dynamics we consider is linear and contains a chain of integrators of arbitrary length with every integrator except the last one affected by the disturbance. The key to arriving at this result is to use low gain feedback to stabilize this disturbance affected unstable part of the zero dynamics.

In this paper, we generalize the results of our recent note (Lin, 1998a) by enlarging the class of disturbance affected unstable zero dynamics to include any linear system with all its poles in the closed left-half plane. This enlargement is due to a new scaling technique that views each pair of jw-axis zeros as a "generalized integrator" and transfers the zero dynamics into a number of chains of "generalized integrators". After the main results section, we will also provide some brief discussions on the impossibility of allowing the exponentially unstable zero dynamics to be affected by disturbance by invoking relevant results of linear systems.

2. Main results

We consider system (3) with its first equation in the form of

$$\begin{aligned}\dot{z}_a &= f_a(z_a, z_b, \xi_1) + p_a(z_a, z_b, \xi_1)w,\\ \dot{z}_b &= Az_b + B\xi_1 + p_b(z_a, z_b, \xi_1)w,\end{aligned} \tag{8}$$

where $z_b = [z_{b1}, z_{b2}, \dots, z_{bq}]'$, (A, B) is stabilizable, and all the eigenvalues of A are in the closed left-half plane. Without loss of generality, assume that all the eigenvalues of A are on the jw-axis. For if there are some eigenvalues of A that are in the open left-half plane, their corresponding dynamics can be viewed as part of z_a. Further, we assume that the pair (A, B) is in the following canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_q & -a_{q-1} & -a_{q-2} & \cdots & -a_1 \end{bmatrix};$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

We also made the following assumption on the disturbance vector field $p_b(z_a, z_b, \xi_1)$.

Assumption 1. (i) *There exists a constant number $\delta \geq 0$ such that*

$$|p_b(z_a, z_b, \xi_1)| \leq \delta. \tag{10}$$

(ii) *For any z_a, z_b, ξ_1,*

$$p_b(z_a, z_b, \xi_1) \in \mathcal{S}(A), \tag{11}$$

where $\mathcal{S}(A) = \bigcap_{\omega \in \lambda(A)} \operatorname{Im}\{\omega I - A\}$.

Our main result is presented in a theorem as follows.

Theorem 2. *Consider system* (3) *with its first equation in the form of* Eq. (8). *If*
(i) *there exists a smooth real-valued function $V_a(z_a)$, which is positive definite and proper, such that*

$$\frac{\partial V_a}{\partial z_a}[f_a(z_a, z_b, \xi_1) + p_a(z_a, z_b, \xi_1)w] \leq -\alpha_a(|z_a|) + \gamma_0^2|w|^2 + \gamma_0^2|z_b|^2 + \gamma_0^2|\xi_1|^2 \tag{12}$$

for some $\mathcal{K}_\infty$ function α_a and some *positive real number γ_0, and*
(ii) *Assumption* 1 *is satisfied*;

then, the problem of L_2 almost disturbance decoupling with global asymptotic stability is solvable.

Remark 1. We note here that in the case that all the eigenvalues of A are at the origin the last row of A is zero and hence the zero dynamics (8) reduces to the one considered in Lin, (1996a). Also, Assumption 1 reduces to the fact that the first $q-1$ elements of $p_b(z_a, z_b, \xi_1)$ is bounded by a constant while the last element of $p_b(z_a, z_b, \xi_1)$ is identically zero. Consequently, the main result Theorem 2, includes that of Lin (1998a) as a special case.

In the case that all functions in Eq. (3) are linear, Assumption 1(ii) is a necessary condition for the solution of the problem (Scherer, 1992; Schwartz et al., 1996). It means that all states except those corresponding to the last row [two row] of any [real] Jordan block of A can be directly affected by disturbance. More specifically, let $T \in \mathbb{R}^n$ and an integer k be such that

$$T^{-1}AT = \operatorname{blkdiag}\{J_1, J_2, \dots, J_k\},$$

where each block J_i, $i = 1, 2, \dots, k$, has the following real Jordan canonical form: if $\lambda_i \in \lambda(A)$ is real,

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix},$$

or if $\lambda_i = \mu_i + \mathrm{j}\omega_i \in \lambda(A)$ and $\bar{\lambda}_i = \mu_i - \mathrm{j}\omega_i \in \lambda(A)$ with $\omega_i \neq 0$,

$$J_i = \begin{bmatrix} \Lambda_i & I_2 & & \\ & \ddots & \ddots & \\ & & \Lambda_i & I_2 \\ & & & \Lambda_i \end{bmatrix}, \qquad \Lambda_i = \begin{bmatrix} \mu_i & \omega_i \\ -\omega_i & \mu_i \end{bmatrix}.$$

Correspondingly, let

$$T^{-1}p_b(z_a, z_b, \xi_1) = \begin{bmatrix} \check{p}_{b1}(z_a, z_b, \xi_1) \\ \check{p}_{b2}(z_a, z_b, \xi_1) \\ \vdots \\ \check{p}_{bk}(z_a, z_b, \xi_1) \end{bmatrix},$$

with $\check{p}_{bi}(z_a, z_b, \xi_1)$, $i = 1, 2, \ldots, k$, being further partitioned in conformity with J_i as

$$\check{p}_{bi}(z_a, z_b, \xi_1) = \begin{bmatrix} \star \\ \vdots \\ \star \\ \check{p}_{bi\star}(z_a, z_b, \xi_1) \end{bmatrix}.$$

Then Assumption 1(ii) implies $p_{bi\star}(z_a, z_b, \xi_1) \equiv 0$, $i = 1, 2, \ldots, k$.

To prove the theorem, we need first to establish the following four lemmas.

Lemma 1. *Given a matrix pair* (A, B) *in the form of Eq.* (9) *with all eigenvalues of* A *on the imaginary axis. Let* $F(\varepsilon) \in \mathbb{R}^{1\times q}$ *be the unique matrix such that* $\lambda(A - BF(\varepsilon)) = -\varepsilon + \lambda(A)$, $\varepsilon \in (0, 1]$. *Then, there exists a nonsingular transformation matrix* $Q(\varepsilon) \in \mathbb{R}^{q\times q}$ *such that*

$$Q^{-1}(\varepsilon)(A - BF(\varepsilon))Q(\varepsilon) = J(\varepsilon)$$
$$= \mathrm{blkdiag}\{J_0(\varepsilon), J_1(\varepsilon), \ldots, J_l(\varepsilon)\}, \tag{13}$$

where

$$J_0(\varepsilon) = \begin{bmatrix} -\varepsilon & 1 & & \\ & \ddots & \ddots & \\ & & -\varepsilon & 1 \\ & & & -\varepsilon \end{bmatrix}_{q_0\times q_0} \tag{14}$$

and for each $i = 1$ *to* l,

$$J_i(\varepsilon) = \begin{bmatrix} J_i^\star & I_2 & & \\ & \ddots & \ddots & \\ & & J_i^\star(\varepsilon) & I_2 \\ & & & J_i^\star(\varepsilon) \end{bmatrix}_{2q_i\times 2q_i},$$

$$J_i^\star(\varepsilon) = \begin{bmatrix} -\varepsilon & \beta_i \\ -\beta_1 & -\varepsilon \end{bmatrix} \tag{15}$$

with $\beta_i > 0$ *for all* $i = 1$ *to* l *and* $\beta_i \neq \beta_j$ *for* $i \neq j$.

Remark 2. We note here that $F(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. Feedback with such a feedback gain $F(\varepsilon)$ is hence referred to as low gain feedback.

For each $i = 1$ to l, $J_i(0)$ can be viewed as a chain of "generalized integrators" of length q_i, with each "generalized integrator corresponding to a pair of jw-axis poles. As it will become clear shortly, such a notion of chains of generalized integrators allows us to develop a certain scaling technique (see the proof of Theorem 2).

Proof of Lemma 1. Let

$$\det(sI - A + BF(\varepsilon)) = (s + \varepsilon)^{q_0} \prod_{i=1}^{l} (s + \varepsilon - \mathrm{j}\beta_i)^{q_i}$$
$$\times (s + \varepsilon + \mathrm{j}\beta_i)^{q_i}. \tag{16}$$

Then, the q_0 generalized eigenvectors $A - BF(\varepsilon)$ corresponding to the eigenvalue $\lambda_0(\varepsilon) = -\varepsilon$ are (Kailath, 1980)

$$Q_{01}(\varepsilon) = \begin{bmatrix} 1 \\ \lambda_0(\varepsilon) \\ \lambda_0^2(\varepsilon) \\ \vdots \\ \lambda_0^{q-2}(\varepsilon) \\ \lambda_0^{q-1}(\varepsilon) \end{bmatrix}, \quad Q_{02}(\varepsilon) = \begin{bmatrix} 0 \\ 1 \\ 2\lambda_0(\varepsilon) \\ 3\lambda_0^2(\varepsilon) \\ \vdots \\ (q-1)\lambda_0^{q-2}(\varepsilon) \end{bmatrix}, \ldots,$$

$$Q_{0q_0}(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ C_{q-2}^{q_0-1}\lambda_0^{q-q_0-1}(\varepsilon) \\ C_{q-1}^{q_0-1}\lambda_0^{q-q_0}(\varepsilon) \end{bmatrix}. \tag{17}$$

Similarly, for $i = 1$ to l, the q_i generalized eigenvectors of $A - BF(\varepsilon)$ corresponding to eigenvalues of $\lambda_i(\varepsilon) = -\varepsilon + \mathrm{j}\beta_i$ and $\bar{\lambda}_i(\varepsilon) = -\varepsilon - \mathrm{j}\beta_i$ are given, respectively, by

$$Q_{i1}(\varepsilon) = \begin{bmatrix} 1 \\ \lambda_i(\varepsilon) \\ \lambda_i^2(\varepsilon) \\ \vdots \\ \lambda_i^{q-2}(\varepsilon) \\ \lambda_i^{q-1}(\varepsilon) \end{bmatrix}, \quad Q_{i2}(\varepsilon) = \begin{bmatrix} 0 \\ 1 \\ 2\lambda_i(\varepsilon) \\ 3\lambda_i^2(\varepsilon) \\ \vdots \\ (q-1)\lambda_i^{q-2}(\varepsilon) \end{bmatrix}, \ldots,$$

$$Q_{iq_i}(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ C_{q-2}^{q_i-1}\lambda_i^{q-q_i-1}(\varepsilon) \\ C_{q-1}^{q_i-1}\lambda_i^{q-q_i}(\varepsilon) \end{bmatrix} \tag{18}$$

and their complex conjugates $\bar{Q}_{ij}(\varepsilon)$, $j = 1$ to q_i.

We then form the following real nonsingular transformation matrix;

$$Q(\varepsilon) = [Q_0(\varepsilon) \quad Q_1(\varepsilon) \quad Q_2(\varepsilon) \quad \cdots \quad Q_l(\varepsilon)], \tag{19}$$

where

$$Q_0(\varepsilon) = [Q_{01}(\varepsilon) \quad Q_{02}(\varepsilon) \quad \cdots \quad Q_{0q_0}(\varepsilon)]$$

and for $i = 1$ to l,

$$Q_i(\varepsilon) = \left[\frac{Q_{i1} + \bar{Q}_{i1}}{2} \quad \frac{Q_{i1} - \bar{Q}_{i1}}{2j} \quad \frac{Q_{i2} + \bar{Q}_{i2}}{2} \quad \frac{Q_{i2} - \bar{Q}_{i2}}{2j} \quad \cdots \quad \frac{Q_{iq_i} + \bar{Q}_{iq_i}}{2j} \quad \frac{Q_{iq_i} - \bar{Q}_{iq_i}}{2j}\right].$$

It can now be readily verified that

$$Q^{-1}(\varepsilon)(A - BF(\varepsilon))Q(\varepsilon) = J(\varepsilon), \tag{20}$$

where $J(\varepsilon)$ is as defined in Eq. (13). This completes the proof of Lemma 1. □

Lemma 2. *Let*

$$\tilde{J}(\varepsilon) = \mathrm{blkdiag}\{\tilde{J}_0, \tilde{J}_1(\varepsilon), \ldots, \tilde{J}_l(\varepsilon)\}, \tag{21}$$

where

$$\tilde{J}_0 = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}_{q_0 \times q_0} \tag{22}$$

and for each $i = 1$ to l,

$$\tilde{J}_i(\varepsilon) = \begin{bmatrix} \tilde{J}_i^\star(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & \tilde{J}_i^\star(\varepsilon) & I_2 \\ & & & \tilde{J}_i^\star(\varepsilon) \end{bmatrix}_{2q_i \times 2q_i}$$

$$J_i^\star(\varepsilon) = \begin{bmatrix} -1 & \beta_i/\varepsilon \\ -\beta_i/\varepsilon & -1 \end{bmatrix} \tag{23}$$

with $\beta_i > 0$ for all $i = 1$ to l and $\beta_i \neq \beta_j$ for $i \neq j$. Then the unique positive-definite solution $\tilde{P}$ to the Lyapunov equation

$$\tilde{J}(\varepsilon)^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{J}(\varepsilon) = -I \tag{24}$$

is independent of ε.

Proof of Lemma 2. We observe that the solution $\tilde{P}$ to the Lyapunov equation (24) is of block diagonal form

$$\tilde{P} = \mathrm{blkdiag}\{\tilde{P}_0, \tilde{P}_1(\varepsilon), \tilde{P}_2(\varepsilon), \ldots, \tilde{P}_l(\varepsilon)\}, \tag{25}$$

where $\tilde{P}_0$ is the unique positive-definite solution to the Lyapunov equation

$$\tilde{J}_0^{\mathrm{T}}\tilde{P}_0 + \tilde{P}_0\tilde{J}_0 = -I \tag{26}$$

and, for $i = 1$ to l, $\tilde{P}_i(\varepsilon)$ is the positive-definite solution to the Lyapunov equation

$$\tilde{J}_i^{\mathrm{T}}(\varepsilon)\tilde{P}_i + \tilde{P}_i\tilde{J}_i(\varepsilon) = -I. \tag{27}$$

Clearly, $\tilde{P}_0$ is independent of ε. It remains to show that for each $i = 1$ to l, $\tilde{P}_i(\varepsilon)$ is also independent of ε. To this end, we notice that

$$T_i^{-1}\tilde{J}_i(\varepsilon)T_i = \mathrm{blkdiag}\{J_i^+(\varepsilon), J_i^-(\varepsilon)\}, \tag{28}$$

where

$$J_i^+(\varepsilon) = \begin{bmatrix} -1 + \mathrm{j}\frac{\beta_i}{\varepsilon} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -1 + \mathrm{j}\frac{\beta_i}{\varepsilon} \end{bmatrix}, \tag{29}$$

$$J_i^-(\varepsilon) = \bar{J}_i^+(\varepsilon) = \begin{bmatrix} -1 - \mathrm{j}\frac{\beta_i}{\varepsilon} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -1 - \mathrm{j}\frac{\beta_i}{\varepsilon} \end{bmatrix} \tag{30}$$

and the nonsingular transformation matrix T_i is given by

$$T_i = \begin{bmatrix} 1 & & & 1 & & \\ \mathrm{j} & & & -\mathrm{j} & & \\ & 1 & & & 1 & \\ & \mathrm{j} & & & -\mathrm{j} & \\ & & \ddots & & & \ddots \\ & & 1 & & & 1 \\ & & \mathrm{j} & & & -\mathrm{j} \end{bmatrix}_{2q_i \times 2q_i} \tag{31}$$

Noting that

$$\mathrm{e}^{J_i^+(\varepsilon)t} = \mathrm{e}^{-t + j\beta_i/\varepsilon}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots \\ & 1 & t & \cdots \\ & & \ddots & \ddots \\ & & & 1 \end{bmatrix},$$

$$\mathrm{e}^{(J_i^+(\varepsilon))^* t} = \mathrm{e}^{-t - j\beta_i/\varepsilon}\begin{bmatrix} 1 & & & & \\ t & 1 & & & \\ \frac{t^2}{2} & t & 1 & & \\ \vdots & \vdots & \vdots & \ddots & 1 \end{bmatrix}, \tag{32}$$

we have

$$\mathrm{e}^{(J_i^+(\varepsilon))^* t}\mathrm{e}^{J_i^+(\varepsilon)t} = \mathrm{e}^{-2t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots \\ t & 1 + t^2 & \cdots & \cdots \\ \frac{t^2}{2!} & \vdots & \vdots & \ddots \end{bmatrix} \tag{33}$$

is independent of ε. Similarly,

$$e^{(J_i^-(\varepsilon))^* t} e^{J_i^-(\varepsilon)t} = e^{-2t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots \\ t & 1+t^2 & \cdots & \cdots \\ \frac{t^2}{2!} & \vdots & \vdots & \ddots \end{bmatrix} \tag{34}$$

is also independent of ε.

Hence, using the fact that $T_i^* T_i = 2I_{2q_i}$, we have

$$\begin{aligned} \tilde{P}_i(\varepsilon) &= \int_0^\infty e^{\tilde{J}_i^{\mathrm{T}}(\varepsilon)t} e^{\tilde{J}_i(\varepsilon)t}\, dt \\ &= 2\int_0^\infty (T_i^{-1})^* \operatorname{blkdiag}\{e^{(J_i^+(\varepsilon))^* t} e^{J_i^+(\varepsilon)t}, \\ &\qquad e^{(J_i^-(\varepsilon))^* t} e^{J_i^-(\varepsilon)t}\} T_i^{-1} dt \end{aligned} \tag{35}$$

and is independent of ε. □

Lemma 3. *Let $p_b(z_a, z_b, \xi_1)$ satisfy Assumption 1. Let $Q(\varepsilon)$ be as given in the proof of Lemma 1 and B be as given in Eq. (8). Also let*

$$Q^{-1}(\varepsilon) p_b(z_a, z_b, \xi_1) = \bar{p}_b(z_a, z_b, \xi_1; \varepsilon),\ Q^{-1}(\varepsilon) B = \bar{B}(\varepsilon). \tag{36}$$

Then, there exists a $\bar{\delta} \geq 0$ and a $\bar{b} \geq 0$, both independent of ε, such that for sufficiently small $\varepsilon > 0$,

$$|\bar{p}_b(z_a, z_b, \xi_1; \varepsilon)| \leq \bar{\delta},\ |\bar{B}(\varepsilon)| \leq \bar{b},\ \forall z_a, z_b, \xi_1. \tag{37}$$

Moreover, if we partition $\bar{p}_b(z_a, z_b, \xi_1; \varepsilon)$ according to that of $J(\varepsilon)$ in Eq. (13), as

$$\bar{p}_b(z_a, z_b, \xi_1; \varepsilon) = \begin{bmatrix} \bar{p}_{b0}(z_a, z_b, \xi_1; \varepsilon) \\ \bar{p}_{b1}(z_a, z_b, \xi_1; \varepsilon) \\ \vdots \\ \bar{p}_{pl}(z_a, z_b, \xi_1; \varepsilon) \end{bmatrix},$$

$$\bar{p}_{b0}(\cdot) = \begin{bmatrix} \bar{p}_{b01}(\cdot) \\ \bar{p}_{b02}(\cdot) \\ \vdots \\ \bar{p}_{b0q_0}(\cdot) \end{bmatrix}_{q_0 \times 1}, \qquad \bar{p}_{bi}(\cdot) = \begin{bmatrix} \bar{p}_{bi1}(\cdot) \\ \bar{p}_{bi2}(\cdot) \\ \vdots \\ \bar{p}_{biq_i}(\cdot) \end{bmatrix}_{2q_i \times 1}, \tag{38}$$

then there exists a $\bar{\delta}_0 \geq 0$, independent of ε, such that, for sufficiently small $\varepsilon > 0$,

$$|\bar{p}_{biq_i}(z_a, z_b, \xi_1; \varepsilon)| \leq \bar{\delta}_0 \varepsilon,\ \forall z_a, z_b, \xi_1 \quad \text{and} \quad \forall i = 0, 1, \ldots, l. \tag{39}$$

Proof of Lemma 3. The existence of a $\bar{\delta} \geq 0$ and a $\bar{b}$ that satisfy Eq. (37) follows readily from Item (i) of Assumption 1 and the fact that $Q(\varepsilon)$ is a polynomial matrix in ε and $Q(0)$, being a transformation matrix that takes A into its real Jordan canonical form, is nonsingular (and hence $Q^{-1}(\varepsilon)$ is continuously differentiable in ε).

To show the existence of $\bar{\delta}_0 \geq 0$ that satisfies Eq. (39), we note that Item (ii) of Assumption 1 implies that, for each $i = 0$ to l,

$$|\bar{p}_{biq_i}(z_a, z_b, \xi_1; 0)| = 0,\ \forall z_a, z_b, \xi_1. \tag{40}$$

The existence of such a $\bar{\delta}_0$ now follows trivially from the continuous differentiability of $Q^{-1}(\varepsilon)$. □

Lemma 4. *Let $A, B, F(\varepsilon), Q(\varepsilon), l, q_i$ for $i = 0$ to l, be as defined in Lemma 1 and its proof. Define a scaling matrix $S(\varepsilon)$ as*

$$S(\varepsilon) = \operatorname{blkdiag}\{S_0(\varepsilon), S_1(\varepsilon), S_2(\varepsilon), \cdots, S_l(\varepsilon)\}. \tag{41}$$

where $S_0(\varepsilon) = \operatorname{diag}\{\varepsilon^{q_0-1}, \varepsilon^{q_0-2}, \ldots, \varepsilon, 1\}$ and for $i = 1$ to l, $S_i(\varepsilon) = \operatorname{blkdiag}\{\varepsilon^{q_i-1} I_2, \varepsilon^{q_i-2} I_2, \ldots, \varepsilon I_2, I_2\}$.

Then, there exists a $\kappa \geq 0$ independent of ε such that

$$|F(\varepsilon) Q(\varepsilon) S^{-1}(\varepsilon)| \leq \kappa\varepsilon. \tag{42}$$

Proof of Lemma 4. Observe that

$$F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon) = [F(\varepsilon)Q_0 S_0^{-1}(\varepsilon) \quad F(\varepsilon)Q_1 S_1^{-1}(\varepsilon) \quad \cdots \quad F(\varepsilon)Q_l(\varepsilon) S_l^{-1}(\varepsilon)], \tag{43}$$

where $Q_0(\varepsilon)$ and $Q_i(\varepsilon)$ are defined in Eq. (19). We next recall from Lin (1998b, equation (2.2.17)) that for each $i = 0$ to l and for each $j = 1$ to q_i, there exists a $\kappa_{ij} \geq 0$, independent of ε, such that,

$$|F(\varepsilon) Q_{ij}(\varepsilon)| \leq \kappa_{ij} \varepsilon^{q_i - j + 1}, \quad \forall \varepsilon \in (0, 1]. \tag{44}$$

It is now clear that there exists a $\kappa_0 \geq 0$ such that

$$|F(\varepsilon) Q_0(\varepsilon) S_0^{-1}(\varepsilon)| \leq \kappa_0 \varepsilon, \quad \forall \varepsilon \in (0, 1]. \tag{45}$$

For each $i = 1$ to l, noting the definition of $Q_i(\varepsilon)$, it is also straightforward to verify that there exists a $\kappa_i \geq 0$, independent of ε, such that

$$|F(\varepsilon) Q_i(\varepsilon) S_i^{-1}(\varepsilon)| \leq \kappa_i \varepsilon,\ \forall \varepsilon \in (0, 1]. \tag{46}$$

The results of the Lemma now follows readily. □

We are now ready to prove our main result Theorem 2.

Proof of Theorem 2. We begin by defining

$$u_b(z_b) = -F(\varepsilon) z_b,\ \varepsilon \in (0, 1], \tag{47}$$

where $F(\varepsilon)$ is such that $\lambda(A - BF(\varepsilon)) = -\varepsilon + \lambda(A)$. Such an $F(\varepsilon)$ exists and is unique since the pair (A, B) is controllable and is of single input.

We next rename the output of the system as

$$\tilde{y} = \xi_1 - u_b(z_b) = \xi_1 + F(\varepsilon) z_b. \tag{48}$$

With this new output $\tilde{y}$, we define a new set of state variables for the system, $\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_r$, as

$$\begin{aligned} \tilde{z}_a &= z_a, \\ \tilde{z}_b &= S(\varepsilon) Q^{-1}(\varepsilon) z_b, \\ \tilde{\xi}_1 &= \tilde{y} = \xi_1 + F(\varepsilon) z_b, \\ \tilde{\xi}_2 &= \xi_2 + F(\varepsilon) A z_b + F(\varepsilon) B \xi_1, \\ &\ \vdots \\ \tilde{\xi}_r &= \xi_r + F(\varepsilon) A^{r-1} z_b + F(\varepsilon) A^{r-2} B \xi_1 \\ &\quad + F(\varepsilon) A^{r-3} B \xi_2 + \cdots + F(\varepsilon) B \xi_{r-1}, \end{aligned} \tag{49}$$

where $Q(\varepsilon)$ is as defined in the proof of Lemma 1 and $S(\varepsilon)$ is as defined in Lemma 4.

We also choose a pre-feedback law as

$$u = -F(\varepsilon)A^r z_b - F(\varepsilon)A^{r-1}B\xi_1 - F(\varepsilon)A^{r-2}B\xi_2 - \cdots - F(\varepsilon)B\xi_r + \tilde{u}. \tag{50}$$

Under this pre-feedback law, it follows from Lemma 1 that the closed-loop system in the new state variables can be rewritten as follows:

$$\begin{aligned}
\dot{\tilde{z}}_a &= f_a(\tilde{z}_a, Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b, \tilde{\xi}_1 + u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b)) \\
&\quad + p_a(\tilde{z}_a, Q(\varepsilon)S^{-1}\tilde{z}_b, \tilde{\xi}_1 + u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b))w, \\
\dot{\tilde{z}}_b &= \varepsilon\tilde{J}(\varepsilon)\tilde{z}_b + \tilde{B}(\varepsilon)\tilde{\xi}_1 + \varepsilon\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1; \varepsilon)w, \\
\dot{\tilde{\xi}}_1 &= \tilde{\xi}_2 + \tilde{p}_1(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1)w, \\
\dot{\tilde{\xi}}_2 &= \tilde{\xi}_3 + \tilde{p}_2(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1, \tilde{\xi}_2)w, \\
&\ \vdots \\
\dot{\tilde{\xi}}_{r-1} &= \tilde{\xi}_q + \tilde{p}_{r-1}(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_{r-1})w, \\
\dot{\tilde{\xi}}_r &= \tilde{u} + \tilde{p}_r(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_r)w,
\end{aligned} \tag{51}$$

where $\tilde{J}(\varepsilon)$ is as defined in Lemma 2,

$$\tilde{B}(\varepsilon) = S(\varepsilon)Q^{-1}(\varepsilon)B,$$

$$\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1; \varepsilon) = S(\varepsilon)Q^{-1}(\varepsilon)p_b(\tilde{z}_a, Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b, \tilde{\xi}_1 + u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b))/\varepsilon$$

and $\tilde{p}_i(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_i)$ for $i = 1$ to r, are defined in an obvious way. For later use, we note that Lemma 3 implies that, for sufficiently small $\varepsilon > 0$,

$$|\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1; \varepsilon)| \le \tilde{\delta}, \forall \tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1. \tag{52}$$

We now observe that system (51) is in the form of Eq. (3) and (5) with the first equation of Eq. (5) corresponding to the dynamics of $\tilde{z}_a$ and $\tilde{z}_b$ and the second equation of Eq. (5) non-existent. We hence can apply Theorem 1 to system (5). Condition (ii) of Theorem 1 is automatically satisfied. To verify Condition (i) of Theorem 1, we will show that there exists an $\varepsilon^* \in (0, 1]$ such that for each $\varepsilon \in (0, \varepsilon^*]$, there exists a $V_{ab}(\tilde{z}_a, \tilde{z}_b)$ and α_{ab}, and the following inequality corresponding to (6) holds

$$\begin{aligned}
&\frac{\partial V_{ab}}{\partial \tilde{z}_a}[f_a(\tilde{z}_a, Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b, \tilde{\xi}_1 + u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b)) \\
&\quad + p_a(\tilde{z}_a, Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b, \tilde{\xi}_1 + u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b))w] \\
&\quad + \frac{\partial V_{ab}}{\partial \tilde{z}_b}[\varepsilon\tilde{J}(\varepsilon)\tilde{z}_b + \tilde{B}(\varepsilon)\tilde{\xi}_1 + \varepsilon\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1)w] \\
&\le -\alpha_{ab}(|[\tilde{z}_a^{\mathrm{T}}, \tilde{z}_b^{\mathrm{T}}]^{\mathrm{T}}|) + \varepsilon^2|w|^2 + \varepsilon^2|\tilde{\xi}_1|^2.
\end{aligned} \tag{53}$$

Let us choose

$$V_{ab}(\tilde{z}_a, \tilde{z}_b) = \varepsilon^{2\bar{q}+6}V_a(\tilde{z}_a) + \varepsilon^5\tilde{z}_b^{\mathrm{T}}\tilde{P}\tilde{z}_b, \quad \bar{q} = \max_{i=0,1,\ldots,l} q_i, \tag{54}$$

where the function V_a is as given by Condition (i) of the theorem, $\tilde{P}$ is the positive-definite solution of the following Lyapunov function:

$$\tilde{J}(\varepsilon)^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{J}(\varepsilon) = -I \tag{55}$$

and, by Lemma 2, is independent of ε. Noting that $u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b) = -F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b$, it follows from Eqs. (12), (55), (52), and Lemma 4 that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&\frac{\partial V_{ab}}{\partial \tilde{z}_a}[f_a(\tilde{z}_a, Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b, \tilde{\xi}_1 + u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b)) \\
&\quad + p_a(\tilde{z}_a, Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b, \tilde{\xi}_1 + u_b(Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b))w] \\
&\quad + \frac{\partial V_{ab}}{\partial \tilde{z}_b}[\varepsilon\tilde{J}(\varepsilon)\tilde{z}_b + \tilde{B}(\varepsilon)\tilde{\xi}_1 + \varepsilon\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1)w] \\
&\le -\varepsilon^{2\bar{q}+6}\alpha_a(|\tilde{z}_a|) + \varepsilon^{2\bar{q}+6}\gamma_0^2|w|^2 \\
&\quad + \varepsilon^{2\bar{q}+6}\gamma_0^2|Q(\varepsilon)|^2|S^{-1}(\varepsilon)|^2|\tilde{z}_b|^2 \\
&\quad + \varepsilon^{2\bar{q}+6}\gamma_0^2[2|\tilde{\xi}_1|^2 + 2|F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)|^2|\tilde{z}_b|^2] \\
&\quad - \varepsilon^6|\tilde{z}_b|^2 + 2\varepsilon^5\tilde{z}_b^{\mathrm{T}}\tilde{P}\tilde{B}(\varepsilon)\tilde{\xi}_1 + 2\varepsilon^6\tilde{z}_b^{\mathrm{T}}\tilde{P}\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1)w \\
&\le -\varepsilon^{2\bar{q}+6}\alpha_a(|\tilde{z}_a|) - [\varepsilon^6 - \varepsilon^7 - \varepsilon^8 - \varepsilon^8\gamma_0^2|Q(\varepsilon)|^2 \\
&\quad - 2\varepsilon^{2\bar{q}+8}\kappa^2\gamma_0^2]|\tilde{z}_b|^2 \\
&\quad + [\varepsilon^{2\bar{q}+6}\gamma_0^2 + \varepsilon^4\tilde{\delta}^2|\tilde{P}|^2]|w|^2 + [2\varepsilon^{2\bar{q}+6}\gamma_0^2 \\
&\quad + \varepsilon^3|\tilde{P}|^2\bar{b}^2]|\tilde{\xi}_1|^2.
\end{aligned} \tag{56}$$

It is straightforward to verify that there exists an $\varepsilon^* \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon^*]$,

$$\begin{aligned}
&\varepsilon^6 - \varepsilon^7 - \varepsilon^8 - \varepsilon^8\gamma_0^2|Q(\varepsilon)|^2 - 2\varepsilon^{2\bar{q}+8}\kappa^2\gamma_0^2 \ge \varepsilon^6/2, \\
&\varepsilon^{2\bar{q}+6}\gamma_0^2 + \varepsilon^4\tilde{\delta}^2|\tilde{P}|^2 \le \varepsilon^2, \\
&2\varepsilon^{2\bar{q}+6}\gamma_0^2 + \varepsilon^3|\tilde{P}|^2\bar{b}^2 \le \varepsilon^2.
\end{aligned} \tag{57}$$

Also note that, for every $\varepsilon \in (0, \varepsilon^*]$, the function $W(\tilde{z}_a, \tilde{z}_b) = \varepsilon^{2\bar{q}+6}\alpha_a(|z_a|) + 1/2\varepsilon^6|\tilde{z}_b|^2$ is continuous positive definite and is radially unbounded. It follows from Khalil (1996, Lemma 3.5) that there exists a $\mathscr{K}_\infty$ function α_{ab} such that $W(\tilde{z}_a, \tilde{z}_b) \ge \alpha_{ab}(|[\tilde{z}_a^{\mathrm{T}}, \tilde{z}_b^{\mathrm{T}}]^{\mathrm{T}}|)$. Thus, with this choice of α_{ab}, Eq. (53) is satisfied for every $\varepsilon \in (0, \varepsilon^*]$.

We now apply Theorem 1 to system (51) and obtain that, for every $\varepsilon \in (0, \varepsilon^*]$, there exists a smooth state feedback $\tilde{u}(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_r; \varepsilon)$ such that the closed-loop system consisting of system (51) and this feedback law

(a) has a globally asymptotically stable equilibrium at the origin;
(b) has an L_2 gain, from the disturbance w to the renamed output $\tilde{y} = \tilde{\xi}_1$, that is less than or equal to ε, i.e.,

$$\int_0^\infty \tilde{y}^2(t)\,\mathrm{d}t = \int_0^\infty \tilde{\xi}_1^2(t)\,\mathrm{d}t \le \varepsilon^2\int_0^\infty w^2(t)\,\mathrm{d}t. \tag{58}$$

Moreover, the state $(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_r)$, and hence the control $\tilde{u}$, are bounded.

To obtain the L_2 gain from the disturbance w to the regulated output y, we examine the second equation of Eq. (51) with $\tilde{\xi}_1$ viewed as a disturbance. For the Lyapunov function $V_b(\tilde{z}_b) = \tilde{z}_b^{\mathrm{T}}\tilde{P}\tilde{z}_b$, we have

$$\frac{\partial V_b}{\partial \tilde{z}_b}[\varepsilon\tilde{J}(\varepsilon)\tilde{z}_b + \tilde{B}(\varepsilon)\tilde{\xi}_1 + \varepsilon\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1)w]$$
$$\leq -\varepsilon|\tilde{z}_b|^2 + 2\tilde{z}_b^{\mathrm{T}}\tilde{P}\tilde{B}(\varepsilon)\tilde{\xi}_1 + 2\varepsilon\tilde{z}_b^{\mathrm{T}}\tilde{P}\tilde{p}_b(\tilde{z}_a, \tilde{z}_b, \tilde{\xi}_1)w$$
$$\leq -\frac{\varepsilon}{2}|\tilde{z}_b|^2 + \frac{4}{\varepsilon}\bar{b}^2|\tilde{P}|^2|\tilde{\xi}_1|^2 + 4\varepsilon\tilde{\delta}^2|\tilde{P}|^2w^2. \tag{59}$$

Integrating both sides of the above inequality and using $V_b(0) = 0$ and Eq. (58), we obtain that

$$\int_0^\infty |\tilde{z}_b(t)|^2\,\mathrm{d}t \leq 8[\bar{b}^2 + \tilde{\delta}^2]|\tilde{P}|^2\int_0^\infty w^2(t)\,\mathrm{d}t. \tag{60}$$

Recalling that $y = \xi_1 = \tilde{\xi}_1 - F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b$ and the fact that $|F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)\tilde{z}_b| \leq \kappa\varepsilon$ (see Lemma 4), it follows from Eqs. (58) and (60) that

$$\int_0^\infty y^2(t)\,\mathrm{d}t \leq \int_0^\infty (2\tilde{\xi}_1^2(t) + 2\varepsilon^2\kappa^2|\tilde{z}_b|^2(t))\,\mathrm{d}t.$$
$$\leq [2 + 16\kappa^2(\bar{b}^2 + \tilde{\delta}^2)|\tilde{P}|^2]\varepsilon^2\int_0^\infty w^2(t)\,\mathrm{d}t, \tag{61}$$

Finally, for any given $\gamma > 0$, let $\varepsilon \in (0, \varepsilon^*]$ be such that

$$[2 + 16\kappa^2(\bar{b}^2 + \tilde{\delta}^2)|\tilde{P}|^2]\varepsilon^2 \leq \gamma^2 \tag{62}$$

to complete the proof.

3. Discussions and conclusions

We have generalized a recent result on almost disturbance decoupling with global asymptotic stability for nonlinear systems (Lin, 1998a) by allowing a larger class of disturbance affected unstable zero dynamics. In comparison with Lin (1998a) where the disturbance affected unstable zero dynamics has all its poles at the origin, the current paper requires only that its poles be in the closed-left plane, thus allowing the unstable zeros on the jw-axis. A natural question is whether it is still possible to allow the disturbance affected unstable zero dynamics has poles in the open right half plane. The answer turns out to be negative. To see this, we consider the linear counterpart of system (3) and (8),

$$\begin{aligned}
\dot{z}_a &= A^- z_a + A^{\mp} z_b + B^- \xi_1 + E^- w,\\
\dot{z}_b &= A^+ z_b + B^+ \xi_1 + E^+ w,\\
\dot{\xi}_i &= \xi_{i+1} + E_i w,\ i = 1, 2, \dots, r,\\
\dot{\xi}_r &= u + E_r w,
\end{aligned} \tag{63}$$

where A^- is asymptotically stable and A^+ is exponentially unstable. It follows from Scherer (1992) or Schwartz et al. (1996) that a necessary condition for the almost disturbance decoupling problem for the above system to be solvable is that $E^+ \equiv 0$, i.e., the exponentially unstable zero dynamics should not be affected by any disturbances.

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References

Isidori, A. (1995). *Nonlinear control systems* (3rd ed.). Berlin: Springer.

Isidori, A. (1996a). A note on almost disturbance decoupling for nonlinear minimum phase systems. *Systems Control Lett.*, *27*, 191–194.

Isidori, A. (1996b). Global almost disturbance decoupling with stability for non minimum-phase single-input single-output nonlinear systems. *Systems Control Lett.*, *28*, 115–122.

Kailath, T. (1980). *Linear systems*. Englewood Cliffs, NJ: Prentice-Hall.

Khalil, H. K. (1996). *Nonlinear systems*. (2nd ed.). Upper Saddle River, NJ: Prentice-Hall.

Lin, Z. (1998a). Almost disturbance decoupling with global asymptotic stability for nonlinear systems with disturbance affected unstable zero dynamics. *Systems Control Lett.*, *33*, 163–169.

Lin, Z. (1998b). *Low gain feedback*, Lecture Notes in Control and Information Sciences, (Vol. 240), London: Springer.

Marino, R., Respondek, W., & van der Schaft, A. J. (1989). Almost disturbance decoupling for single-input single-output nonlinear systems. *IEEE Trans. Automat. Control*, *34*, 1013–1017.

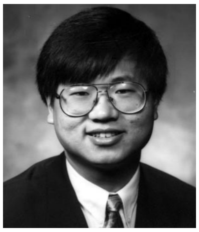
Marino, R., Respondek, W., van der Schaft, A. J., & Tomei, P. (1994). Nonlinear H_∞ almost disturbance decoupling. *Systems Control Lett.*, *23*, 159–168.

Saberi, A., & Sannuti, P. (1988). Global stabilization with almost disturbance decoupling of a class of uncertain non-linear systems. *Int. J. Control*, *47*, 717–727.

Scherer, C. (1992). H_∞-optimization without assumptions on finite and infinite zeros. *SIAM J. Control Optim.*, *30*, 143–166.

Schwartz, B., Isidori, A., & Tarn, T. J. (1996). L_2 disturbance attenuation and performance bounds for linear non-minimum phase square invertible systems, personal communications, short version in *Proc. 35th IEEE Conf. on Decision and Control* (pp. 227–228).

Willems, J. C. (1981). Almost invariant subspaces: an approach to high gain feedback design. Part I: Almost controlled invariant subspaces. *IEEE Trans. Automat. Control*, *26*, 235–252.



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