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Brief paper

Explicit construction of H_∞ control law for a class of nonminimum phase nonlinear systems ☆

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Abstract

This paper addresses a nonlinear H_∞ control problem for a class of nonminimum phase nonlinear systems. The given system is first transformed into a special coordinate basis, in which the system zero dynamics is divided into a stable part and an unstable part. A sufficient solvability condition is then established for solving the nonlinear H_∞ control problem. Moreover, based on the sufficient solvability condition, an upper bound of the best achievable L_2 gain from the system disturbance to the system controlled output is estimated for the nonlinear H_∞ control problem. The proofs of our results yield explicit algorithms for constructing required control laws for solving the nonlinear H_∞ control problem. In particular, the solution to the nonlinear H_∞ control problem does not require solving any Hamilton–Jacobi equations. Finally, the obtained results are utilized to solve a benchmark problem on a rotational/translational actuator (RTAC) system.

Keywords: Disturbance attenuation; Nonlinear systems; L_2 gain; Nonminimum phase

1. Introduction

The nonlinear H_∞ control problem is to design a feedback control law for a nonlinear system such that the closed-loop system is internally stable, and has an L_2-gain, from its disturbance input to its system output, less than or equal to a prescribed value $\gamma > 0$. This problem has attracted much research effort since the works of Van der Schaft (1991, 1992), and many interesting results are available in the literature (see, for example, Battilotti, 1996; Isidori, Schwartz, & Tarn, 1999; Jiang & Hill, 1998; Van der Schaft, 2000 and references therein). If $\gamma > 0$ is arbitrary, the problem is also known as the problem of almost disturbance decoupling with internal stability. It was shown that the almost disturbance decoupling problem is solvable if the disturbance input does not affect the unstable part of zero dynamics of the system, see, for example, Isidori (1996a, 1996b), Marino, Respondek, van der Schaft, and Tomei (1994). When the unstable zero dynamics of the system is affected by disturbance input, the almost disturbance decoupling problem is also solvable for a special class of nonlinear systems whose zero dynamics contains a chain of integrators affected by disturbance (Lin, 1998). However, for more general situations, disturbance decoupling is generally not feasible. One has to seek to design a controller that achieves a pre-specified L_2-gain $\gamma > \gamma^*$, where γ^* is the best achievable performance for the problem, i.e., the problem is solvable for $\gamma > \gamma^*$ and not for $\gamma < \gamma^*$.

For linear systems, the optimal value γ^* can be perfectly calculated by solving two Lyapunov equations if the system is single-input and single-output (SISO) (see, for example, Chen, 2000; Chen, Saberi, & Ly, 1992a, 1992b; Peterson, 1987; Scherer, 1992). It is shown in Chen (2000) and Chen et al. (1992b) that the optimal value is only related to the unstable zero dynamics of the given system even for the singular problem. For nonlinear systems, the problem of estimating the optimal γ^* is investigated in Isidori et al. (1999) and Ji and Gao (1995). An estimation of optimal H_∞-gain for nonlinear H_∞ control problem is obtained in Ji and Gao (1995). In Isidori et al. (1999), the authors computed an upper bound of the

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optimal value γ^* for a class of nonlinear systems with second order zero dynamics. In general, if the optimal value γ^* is unknown, we need to solve Hamilton–Jacobi equations recursively to construct an optimal control law. However, for most nonlinear systems, it is very difficult to solve the Hamilton–Jacobi equation analytically. To solve the nonlinear H_∞ control problem practically, some approximation methods have been developed for solving the Hamilton–Jacobi equation numerically, for example, Taylor series approximation (Huang & Lin, 1995), and neural networks approximation (Abu-Khalaf & Lewis, 2005; Abu-Khalaf, Lewis, & Huang, 2006). Some properties of viscosity solutions of Hamilton–Jacobi equations are investigated in Crandall, Evans, and Lions (1984).

In this paper, we aim to construct an H_∞ control law without solving any Hamilton–Jacobi equations. Consider a class of nonlinear systems in the so-called output feedback form (Marino & Tomei, 1991, 1993, 1995) characterized by

$$\dot{\xi} = A\xi + Bu + \Psi(y) + \mathscr{G}(\xi)w, \quad y = C\xi, \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^s$ the disturbance input, $u \in \mathbb{R}$ the control input, and $y \in \mathbb{R}$ the system output or the controlled output, and $\Psi(y)=[\phi_i(y)]$, $\mathscr{G}(\xi)=[g_{ij}(\xi)]$ where $\psi_i(y) : \mathbb{R} \to \mathbb{R}$, $i=1,\dots,n$, $g_{ij}(\xi) : \mathbb{R}^n \to \mathbb{R}$, $i=1,\dots,n$, $j=1,\dots,s$ are some smooth nonlinear functions. Using special coordinate basis (SCB) technique, it follows from Chen, Lin, and Shamash (2004) and Sannuti and Saberi (1987) that there exists a nonsingular matrix $\Gamma_s \in \mathbb{R}^{n\times n}$ such that the state transformation $\xi=\Gamma_s x$ transforms the system (1) into the following SCB form:

$$\begin{aligned}
\dot{x}_a^- &= A_a^- x_a^- + L_a^- y + \phi_a^-(y) + \mathscr{H}_a^-(x)w,\\
\dot{x}_a^0 &= A_a^0 x_a^0 + L_a^0 y + \phi_a^0(y) + \mathscr{H}_a^0(x)w,\\
\dot{x}_a^+ &= A_a^+ x_a^+ + L_a^+ y + \phi_a^+(y) + \mathscr{H}_a^+(x)w,\\
\dot{x}_1 &= x_2 + \phi_1(y) + \mathscr{H}_1(x)w, \quad y = x_1,\\
\dot{x}_i &= x_{i+1} + \phi_i(y) + \mathscr{H}_i(x)w, \quad i = 2,\dots,r-1,\\
\dot{x}_r &= Ex + bu + \phi_r(y) + \mathscr{H}_r(x)w,
\end{aligned} \tag{2}$$

where b is a nonzero scalar, $A_a^- \in \mathbb{R}^{n_a^-\times n_a^-}$ has all its eigenvalues strictly in the left-half plane, $A_a^0 \in \mathbb{R}^{n_a^0\times n_a^0}$ has all its eigenvalues on the imaginary axis, and all those of $A_a^+ \in \mathbb{R}^{n_a^+\times n_a^+}$ are strictly in the right-half plane. Moreover, $n_a^+ + n_a^0 + n_a^- + r = n$.

In the past two decades, various control problems have been investigated for the nonlinear system in the output feedback form, such as global stabilization (Marino & Tomei, 1993), adaptive control (Khalil, 1999), nonlinear output regulation (Ding, 2003b; Serrani, Isidori, & Marconi, 2001), and disturbance rejection (Ding, 2003a, 2006; Lan, Chen, & Ding, 2006; Lin, Qian, & Huang, 2003; Marino & Tomei, 2005), just to name a few. However, most of these works are based on an assumption that the given system is of minimum phase, that is, $n_a^+ + n_a^0 = 0$. In particular, the nonlinear H_∞ control problem has been studied in Ezal, Kokotovic, Teel, and Basar (2001) and Isidori (1999) for a class of nonlinear systems of the form (2) with $n_a^+ + n_a^0 + n_a^- = 0$. In Ezal et al. (2001), an output feedback control law is obtained to achieve local near-optimality and semi-global inverse H_∞ optimality. The output feedback control law is constructed based on the solutions of two Riccati equations. In Isidori (1999), the existence of the nonlinear smooth state feedback control law that solves the global H_∞ control problem is investigated under the assumption that there exists a pair $\{K, X\}$ solves a Riccati equation. In fact, it is shown that if $\{K, X\}$ solves the Riccati equation, there exists $\{u(x), V(x)\}$ such that it solves the Hamilton–Jacobi equation corresponding to the nonlinear H_∞ control problem. In this paper, we consider the nonlinear H_∞ control problem for the nonminimum phase systems of the form (2), that is, $n_a^+ + n_a^0 > 0$. It should be noted that the systematic design of the global stabilizer for the nonminimum phase systems is limited to one-dimensional unstable zero dynamics (see, e.g., Ding, 2001; Robertsson & Johansson, 1999). In Lan and Chen (2006), we developed a global stabilization technique for the nonminimum phase nonlinear systems with higher order zero dynamics. The stabilizing control law is constructed explicitly by combining a robust stabilization technique (Khargonekar, Petersen, & Zhou, 1990; Petersen, 1988) and a backstepping-like strategy proposed by Tsinias (1991). We will extend this technique to solve the following nonlinear H_∞ control problem which gives an explicit construction of the H_∞ control law without solving any Hamilton–Jacobi equations. Though only the state feedback control is considered in this paper, it is possible to extend our results to the output feedback control by using the available observer techniques (Arcak & Kokotovic, 2001a, 2001b; Krstic, Kanellakopoulos, & Kokotovic, 1995).

Nonlinear H_∞ Control Problem by Linear Feedback. Given $\gamma > 0$, find, if possible, a linear state feedback control law of the form:

$$u = K\xi \tag{3}$$

such that the equilibrium at $\xi = 0$ of the closed-loop system consisting of (1) and (3) is globally asymptotically stable, and has an L_2 gain, from the exogenous disturbance input w to the output y, that is less than or equal to γ, i.e.,

$$\int_0^T \|y(t)\|^2\,\mathrm{d}t \leqslant \gamma^2 \int_0^T \|w(t)\|^2\,\mathrm{d}t \tag{4}$$

for all $T \geqslant 0$ and zero initial state $\xi(0) = 0$.

The following assumptions on the given system are made:

A1: (A, B) is stabilizable;

A2: $\psi_i(0) = 0$, $i = 1,\dots,n$, and, there exist n positive real numbers l_i, $i = 1,\dots,n$, such that

$$|\psi_i(y)| \leqslant l_i|y|, \quad \forall y \in \mathbb{R}. \tag{5}$$

and lastly,

A3: There exist positive real numbers k_{ij}, $i = 1,\dots,n$, $j = 1,\dots,s$ such that for all $\xi \in \mathbb{R}^n$

$$|g_{ij}(\xi)| \leqslant k_{ij}.$$

Remark 1.1. Under Assumption A2, and noting that $\phi_a^+(0)=0$ and $\phi_a^0(0) = 0$, there exist constant matrices $D_a^+ \in \mathbb{R}^{n_a^+\times p}$, $D_a^0 \in \mathbb{R}^{n_a^0\times p}$ and a Lebesgue measurable matrix function

$G(y):\mathbb{R}\to\mathbb{R}^p$, $i=1,\ldots,p$, where p is an appropriate positive integer, such that

$$\begin{bmatrix}\phi_a^0(y)\\ \phi_a^+(y)\end{bmatrix}=\begin{bmatrix}D_a^0\\ D_a^+\end{bmatrix}G(y)y := D_a^s G(y)y, \tag{6}$$

where $(G(y))^{\mathrm{T}}G(y)\leqslant 1$ for all $y\in\mathbb{R}$. Moreover, under Assumption A3, it is clear that there exist constant matrices $H_a^+\in\mathbb{R}^{n_a^+\times s}$, $H_a^0\in\mathbb{R}^{n_a^0\times s}$, such that

$$\begin{bmatrix}\mathscr{H}_a^+(x)\\ \mathscr{H}_a^0(x)\end{bmatrix}\begin{bmatrix}\mathscr{H}_a^+(x)\\ \mathscr{H}_a^0(x)\end{bmatrix}^{\mathrm{T}}\leqslant\begin{bmatrix}H_a^+\\ H_a^0\end{bmatrix}\begin{bmatrix}H_a^+\\ H_a^0\end{bmatrix}^{\mathrm{T}} := H_a^s(H_a^s)^{\mathrm{T}}.$$

2. Solution to nonlinear H_∞ control problem

In this section, we will consider the solvability of the nonlinear H_∞ control problem by linear feedback for the system (1). The sufficient conditions are described by the following theorem, and a nonlinear H_∞ control law is constructed explicitly in its proof.

Theorem 2.1. *Under Assumptions* A1–A3, *let* $P_L>0$, $P_D\geqslant 0$ *and* $P_h\geqslant 0$ *be the unique solutions of* (7)–(9).

$$A_a^+P_L+P_L(A_a^+)^{\mathrm{T}}=L_a^+(L_a^+)^{\mathrm{T}}, \tag{7}$$

$$A_a^+P_D+P_D(A_a^+)^{\mathrm{T}}=D_a^+(D_a^+)^{\mathrm{T}}, \tag{8}$$

$$A_a^+P_h+P_h(A_a^+)^{\mathrm{T}}=H_a^+(H_a^+)^{\mathrm{T}}. \tag{9}$$

If there exists a real $0<c<1$ *such that*

$$P_c=P_L-\frac{1}{c}P_D>0 \tag{10}$$

and

$$x^\star\left(\frac{1}{c}D_a^0(D_a^0)^{\mathrm{T}}-L_a^0(L_a^0)^{\mathrm{T}}\right)x<0 \tag{11}$$

for any eigenvector x *of* $-(A_a^0)^{\mathrm{T}}$, *then the nonlinear* H_∞ *control problem is solvable by a linear state feedback for given* $\gamma>\hat\gamma:=\max\{\hat\gamma_+,\hat\gamma_0\}$, *where*

$$\hat\gamma_+=\sqrt{\lambda_{\max}(P_c^{-1}P_h)/(1-c)}$$

and

$$\hat\gamma_0=\sqrt{\max_{\|x\|=1}\left\{\frac{x^\star H_a^0(H_a^0)^{\mathrm{T}}x}{(1-c)x^\star\left(L_a^0(L_a^0)^{\mathrm{T}}-\frac{1}{c}D_a^0(D_a^0)^{\mathrm{T}}\right)x}\right\}}$$

for any eigenvector x *of* $-(A_a^0)^{\mathrm{T}}$.

Proof. Let us ignore the dynamics of x_a^-, and denote

$$x_a^s=\begin{bmatrix}x_a^0\\ x_a^+\end{bmatrix},\quad A_a^s=\begin{bmatrix}A_a^0 & 0\\ 0 & A_a^+\end{bmatrix},\quad L_a^s=\begin{bmatrix}L_a^0\\ L_a^+\end{bmatrix},$$
$$\phi_a^s(y)=\begin{bmatrix}\phi_a^0(y)\\ \phi_a^+(y)\end{bmatrix},\quad \mathscr{H}_a^s(x)=\begin{bmatrix}\mathscr{H}_a^0(x)\\ \mathscr{H}_a^+(x)\end{bmatrix}.$$

Since $\gamma>\hat\gamma:=\max\{\hat\gamma_+,\hat\gamma_0\}$, we have

$$x^\star\left(\frac{1}{(1-c)\gamma^2}H_a^0(H_a^0)^{\mathrm{T}}+\frac{1}{c}D_a^0(D_a^0)^{\mathrm{T}}-L_a^0(L_a^0)^{\mathrm{T}}\right)x<0$$

for any eigenvector x of $-(A_a^0)^{\mathrm{T}}$. Then, by Theorem 4 of Scherer (1992), for any Z_0, there exists a solution Z of the Lyapunov inequality

$$A_a^0Z+Z(A_a^0)^{\mathrm{T}}+\frac{1}{(1-c)\gamma^2}H_a^0(H_a^0)^{\mathrm{T}} + \frac{1}{c}D_a^0(D_a^0)^{\mathrm{T}}-L_a^0(L_a^0)^{\mathrm{T}}<0 \tag{12}$$

such that $Z>Z_0$. Let

$$P=\begin{bmatrix}Z & Y^{\mathrm{T}}\\ Y & X\end{bmatrix}^{-1}, \tag{13}$$

where

$$X=P_L-\frac{1}{c}P_D-\frac{1}{(1-c)\gamma^2}H_a^+(H_a^+)^{\mathrm{T}} \tag{14}$$

and Y is the unique solution of

$$A_a^+Y+Y(A_a^0)^{\mathrm{T}}+\frac{1}{(1-c)\gamma^2}H_a^+(H_a^0)^{\mathrm{T}} + \frac{1}{c}D_a^+(D_a^0)^{\mathrm{T}}-L_a^+(L_a^0)^{\mathrm{T}}=0 \tag{15}$$

and Z is a solution of (12) such that $P>0$. Then we have

$$(A_a^s)^{\mathrm{T}}P+PA_a^s+P\left(\frac{1}{(1-c)\gamma^2}H_a^s(H_a^s)^{\mathrm{T}} + \frac{1}{c}D_a^s(D_a^s)^{\mathrm{T}}-L_a^s(L_a^s)^{\mathrm{T}}\right)P\leqslant 0.$$

Now let

$$V_0(x_a^s)=\frac{1}{(1-c)}(x_a^s)^{\mathrm{T}}Px_a^s;\quad \alpha_0(x_a^s)=-(L_a^s)^{\mathrm{T}}Px_a^s.$$

It is not difficult to show that

$$\frac{\partial V_0(x_a^s)}{\partial x_a^s}(A_a^sx_a^s+L_a^s\alpha_0(x_a^s)+\phi_a^s(\alpha_0(x_a^s))+\mathscr{H}_a^s(x)w) \leqslant -\varepsilon_0\|x_a^s\|^2+\alpha_0^2(x_a^s)+\gamma^2\|w\|^2 \tag{16}$$

for some $\varepsilon_0>0$.

Then, based on (16), a linear state feedback control law for solving the nonlinear H_∞ control problem of the system (1) can be constructed by using the inductive procedure developed in Tsinias (1991) for solving a global stabilization problem for a class of minimum phase nonlinear systems.

*Step H.*1: Consider the system

$$\dot x_a^s=A_a^sx_a^s+L_a^sx_1+\phi_a^s(x_1)+\mathscr{H}_a^s(x)w,$$

$$\dot x_1=x_2+\phi_1(x_1)+\mathscr{H}_1(x)w,$$

$$y=x_1$$

viewing x_2 as control input and let

$$V_1(x_a^s,x_1)=V_0(x_a^s)+\tfrac{1}{2}\sigma_1(x_1-\alpha_0(x_a^s))^2 \tag{17}$$

and

$$\alpha_1(x_a^s, x_1) = -\frac{1}{\sigma_1}\frac{\partial V_0(x_a^s)}{\partial x_a^s}L_a^s - \frac{m_1}{\sigma_1}(x_1 - \alpha_0(x_a^s)) - \frac{\sigma_1}{4\delta_1^2}(x_1 - \alpha_0(x_a^s)) \times \left(H_1^{\mathrm{T}}H_1 + \frac{\partial\alpha_0(x_a^s)}{\partial x_a^s}H_a^s(H_a^s)^{\mathrm{T}}\left(\frac{\partial\alpha_0(x_a^s)}{\partial x_a^s}\right)^{\mathrm{T}}\right) + \frac{\partial\alpha_0(x_a^s)}{\partial x_a^s}(A_a^s x_a^s + L_a^s x_1) - \frac{1}{\sigma_1}(x_1 + \alpha_0(x_a^s)), \tag{18}$$

where σ_1 and m_1 are design parameters to be determined later, and $\delta_1 > 0$ is an arbitrary small constant real. Then $x_2 = \alpha(x_a^s, x_1)$ yields

$$\dot V_1(x_a^s, x_1) \leqslant -\varepsilon_0\|x_a^s\|^2 - m_1(x_1 - \alpha_0(x_a^s))^2 - y^2 - (\gamma^2 + \delta_1^2)w^2 + \frac{\partial V_0(x_a^s)}{\partial x_a^s}D_a^+G(x_1)(x_1 - \alpha_0(x_a^s)) - \sigma_1(x_1 - \alpha_0(x_a^s))\frac{\partial\alpha_0(x_a^s)}{\partial x_a^s}\phi_a^+(x_1) + \sigma_1(x_1 - \alpha_0(x_a^s))\phi_1(x_1).$$

It is clear that there exist two positive constants $0 < r_{01} < \varepsilon_0$ and $r_{02} > 0$ such that

$$\left|\frac{\partial V_0(x_a^s)}{\partial x_a^s}D_a^sG(x_1)(x_1 - \alpha_0(x_a^s))\right| \leqslant r_{01}\|x_a^s\|^2 + r_{02}(x_1 - \alpha_0(x_a^s))^2 \tag{19}$$

for all $(x_a^s, x_1) \in \mathbb{R}^{n_a^0+n_a^++1}$. Moreover, it was shown in Tsinias (1991) that there exist two positive constants $r_{11} > 0$ and $r_{12} > 0$ such that

$$\left|(x_1 - \alpha_0(x_a^s))\frac{\partial\alpha_0(x_a^s)}{\partial x_a^s}\phi_a^s(x_1)\right| \leqslant r_{11}(\|x_a^s\|^2 + (x_1 - \alpha_0(x_a^s))^2) \tag{20}$$

and

$$|(x_1 - \alpha_0(x_a^s))\phi_1(x_1)| \leqslant r_{12}(\|x_a^s\|^2 + (x_1 - \alpha_0(x_a^s))^2) \tag{21}$$

for all $(x_a^s, x_1) \in \mathbb{R}^{n_a^0+n_a^++1}$. By (19), (20) and (21), we have

$$\dot V_1(x_a^s, x_1) \leqslant -(\varepsilon_0 - r_{01} - \sigma_1(r_{11} + r_{12}))\|x_a^+\|^2 - (m_1 - r_{02})(x_1 - \alpha_0(x_a^s))^2 + \sigma_1(r_{11} + r_{12})(x_1 - \alpha_0(x_a^s))^2 - y^2 + (\gamma^2 + \delta_1^2)w^2. \tag{22}$$

Select σ_1 and m_1 such that

$$0 < \sigma_1 < \frac{\varepsilon_0 - r_{01}}{r_{11} + r_{12}}, \quad r_{02} + \sigma_1(r_{11} + r_{12}) < m_1 \tag{23}$$

and note that $\alpha_0(x_a^s)$ is linear, it is clear that there exists a positive constant $\varepsilon_1 > 0$ such that

$$\dot V_1(x_a^s, x_1) \leqslant -\varepsilon_1\left\|\begin{matrix} x_a^s \\ x_1 \end{matrix}\right\|^2 - y^2 + (\gamma^2 + \delta_1^2)w^2. \tag{24}$$

Step H.i: ($i \geqslant 2$) Denote

$$X_i = \begin{bmatrix} x_a^s \\ x_1 \\ \vdots \\ x_i \end{bmatrix}, \quad \Phi_i(y) = \begin{bmatrix} \phi_a^s(y) \\ \phi_1(y) \\ \vdots \\ \phi_i(y) \end{bmatrix}, \quad \mathbf{H_i} = \begin{bmatrix} H_a^s \\ H_1 \\ \vdots \\ H_i \end{bmatrix}$$

$$\bar{\mathscr{H}}_{\mathbf{i}}(x) = \begin{bmatrix} \mathscr{H}_a^s(x) \\ \mathscr{H}_1(x) \\ \vdots \\ \mathscr{H}_i(x) \end{bmatrix}, \quad F_i(X_i, x_{i+1}) = \begin{bmatrix} A_a^+x_a^+ + L_a^+x_1 \\ x_2 \\ \vdots \\ x_{i+1} \end{bmatrix}$$

and consider the system

$$\dot X_i = F_i(X_i, x_{i+1}) + \Phi_i(y) + \bar{\mathscr{H}}_{\mathbf{i}}(x)w,$$

$$y = x_1$$

with x_{i+1} as control input. Define

$$V_i(X_i) = V_{i-1}(X_{i-1}) + \tfrac{1}{2}\sigma_i(x_i - \alpha_{i-1}(X_{i-1}))^2, \tag{25}$$

$$\alpha_i(X_i) = -\frac{1}{\sigma_i}\frac{\partial V_{i-1}(X_{i-1})}{\partial x_{i-1}} + \frac{\partial\alpha_{i-1}(X_{i-1})}{\partial X_{i-1}}F_{i-1}(X_{i-1}, x_i) - \frac{m_i}{\sigma_i}(x_i - \alpha_{i-1}(X_{i-1})) - \frac{\sigma_i}{4\delta_i^2}(x_i - \alpha_{i-1}(X_{i-1}))\left(\frac{\partial\alpha_{i-1}(X_{i-1})}{\partial X_{i-1}} \times \mathbf{H_{i-1}H_{i-1}^T}\left(\frac{\partial\alpha_{i-1}(X_{i-1})}{\partial X_{i-1}}\right)^{\mathrm{T}} + H_i^{\mathrm{T}}H_i\right) \tag{26}$$

recursively where $\sigma_i > 0$ and $m_i > 0$ are design parameters to be defined later, and $\delta_i > 0$ is any arbitrary small constant real. Assume there exists a scalar $\varepsilon_{i-1} > 0$ such that

$$\frac{\partial V_{i-1}(X_{i-1})}{\partial X_{i-1}}[F_{i-1}(X_{i-1}, \alpha_{i-1}(X_{i-1})) + \Phi_{i-1}(x_1) + \bar{\mathscr{H}}_{\mathbf{i-1}}(x)w] \leqslant -\varepsilon_{i-1}\|X_{i-1}\|^2 - y^2 + (\gamma^2 + \delta_1^2 + \cdots + \delta_{i-1}^2)w^2$$

then let $x_{i+1} = \alpha_i(X_i)$ we have

$$\dot V_i(X_i) \leqslant -(\varepsilon_{i-1} - \sigma_i(r_{i1} + r_{i2}))\|X_{i-1}\|^2 - (m_i - \sigma_i(r_{i1} + r_{i2}))(x_i - \alpha_{i-1}(X_{i-1}))^2 - y^2 + (\gamma^2 + \delta_1^2 + \cdots + \delta_i^2)w^2,$$

where $r_{i1} > 0$ and $r_{i2} > 0$ are positive constants such that

$$\left|(x_i - \alpha_{i-1}(X_{i-1}))\frac{\partial\alpha_{i-1}(X_{i-1})}{\partial X_{i-1}}\Phi_{i-1}(x_1)\right| \leqslant r_{i1}((x_i - \alpha_{i-1}(X_{i-1}))^2 + \|X_{i-1}\|^2)$$

and

$$|(x_i - \alpha_{i-1}(X_{i-1}))\phi_i(x_1)| \leqslant r_{i2}((x_i - \alpha_{i-1}(X_{i-1}))^2 + \|X_{i-1}\|^2).$$

Selecting σ_i and m_i such that

$$0 < \sigma_i < \frac{\varepsilon_{i-1}}{r_{i1} + r_{i2}}, \quad \sigma_i(r_{i1} + r_{i2}) < m_i$$

it is clear that there exists a positive constant $\varepsilon_i > 0$ such that

$$\dot{V}_i(X_i) \leqslant -\varepsilon_i \|X_i\|^2 - y^2 + (\gamma^2 + \delta_1^2 + \cdots + \delta_i^2) w^2. \tag{27}$$

Step H.r: Let

$$V_r(X_r) = V_{r-1}(X_{r-1}) + \tfrac{1}{2}\sigma_r(x_r - \alpha_{i-1}(X_{r-1}))^2,$$

$$\begin{aligned}\alpha_r(X_r) = &-\frac{1}{\sigma_r}\frac{\partial V_{r-1}(X_{r-1})}{\partial x_{r-1}} \\ &+ \frac{\partial \alpha_{r-1}(X_{r-1})}{\partial X_{r-1}} F_{r-1}(X_{r-1}, x_r) \\ &- \frac{m_r}{\sigma_r}(x_r - \alpha_{r-1}(X_{r-1})) \\ &- \frac{\sigma_r}{4\delta_r^2}(x_r - \alpha_{r-1}(X_{r-1}))\left(\frac{\partial \alpha_{r-1}(X_{r-1})}{\partial X_{r-1}}\right. \\ &\left.\times \mathbf{H_{r-1}H_{r-1}^T}\left(\frac{\partial \alpha_{r-1}(X_{r-1})}{\partial X_{r-1}}\right)^{\mathrm{T}}\right) \\ &- \frac{\sigma_r}{4\delta_r^2} H_r^{\mathrm{T}} H_r (x_r - \alpha_{r-1}(X_{r-1})).\end{aligned} \tag{28}$$

By induction, there is a positive constant $\varepsilon_r > 0$ such that

$$\begin{aligned}&\frac{\partial V_r(X_r)}{\partial X_r}(F_r(X_r, \alpha_r(X_r)) + \Phi_r(x_1) + \mathscr{H}_{\mathbf{r}}(x) w) \\ &\quad \leqslant -\varepsilon_r \|X_r\|^2 - y^2 + (\gamma^2 + \delta_1^2 + \cdots + \delta_r^2) w^2.\end{aligned} \tag{29}$$

Therefore, the linear state feedback control law

$$u = Kx := \frac{1}{b}(\alpha_r(X_r) - Ex) \tag{30}$$

solves the H_∞ control problem of the system (2). Noting that A_a^- is stable, (30) completes the proof of Theorem 2.1. □

Remark 2.1. By Theorem 2.1, the achievable L_2 gain can be estimated by solving the following minimization problem on c

$$\hat{\gamma}^* = \min_{\substack{0<c<1 \\ P_L - P_D/c > 0 \\ x^\star(\frac{1}{c}D_a^0(D_a^0)^{\mathrm{T}} - L_a^0(L_a^0)^{\mathrm{T}})x < 0}} \max\{\hat{\gamma}_+, \hat{\gamma}_0\} \tag{31}$$

where x is the eigenvector of $-(A_a^0)^{\mathrm{T}}$.

3. H_∞ control of the RTAC system

The model of the RTAC system is shown in Fig. 1, and the normalized motion equation of the RTAC system is given by Bupp, Bernstein, and Coppola (1998a, 1998b) and Huang (2004)

$$\ddot{\varsigma} + \varsigma = \varepsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + 0.1w, \tag{32}$$

$$\ddot{\theta} = -\varepsilon\ddot{\varsigma}\cos\theta + \upsilon, \tag{33}$$

where ς is the normalized displacement of the cart, θ the angular position of the eccentric mass, w the normalized disturbance,

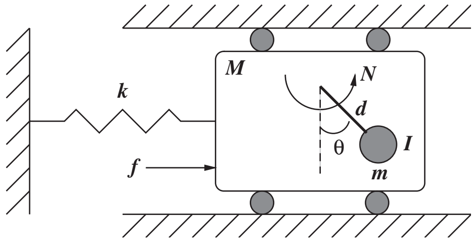

Fig. 1. Model of the RTAC system.

υ the normalized control input. ε is the coupling ratio between the translational and rotational motion. Let $y = \theta$, $\xi_1 = \varsigma + \varepsilon\sin\theta$, $\xi_2 = \dot{\varsigma} + \varepsilon\dot{\theta}\cos\theta$, $\xi_3 = \theta$, $\xi_4 = \dot{\theta}$. Moreover, define $x_a^0 = [\xi_1 \ \ \xi_2]^{\mathrm{T}}$, $x_1 = \xi_3$, and $x_2 = \xi_4$. Then, with a pre-state feedback

$$\upsilon = \varepsilon\cos\xi_3(\xi_1 - (1 + \xi_4^2)\varepsilon\sin\xi_3) - (1 - \varepsilon^2\cos^2\xi_3)u$$

the state space representation of (32)–(33) is given in the SCB form with $n_a^+ = n_a^- = 0$, $n_a^0 = 2$ and $r = 2$, that is,

$$\dot{x}_a^0 = A_a^0 x_a^0 + L_a^0 y + \phi_a^0(y) + H_a^0 w, \tag{34}$$

$$\dot{x}_1 = x_2, \quad y = x_1, \tag{35}$$

$$\dot{x}_2 = u - (0.1\varepsilon\cos y)w/(1 - \varepsilon^2\cos^2 y), \tag{36}$$

where

$$A_a^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad L_a^0 = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}, \quad H_a^0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$\phi_a^0(y) = \begin{bmatrix} 0 \\ \varepsilon(\sin(y) - y) \end{bmatrix}.$$

It is clear that the system (34)–(36) satisfies Assumption A1–A3. Since all the invariant zeros $\pm 1j$ are on the imaginary axis, the achievable L_2 gain can be estimated by solving the optimal problem (31) which gives $\hat{\gamma}^* = 0.7854$ under $c = 0.3634$. Let $\gamma = 0.9$, and $Z > 0$ be a solution of

$$\begin{aligned}&Z(A_a^0)^{\mathrm{T}} + A_a^0 Z + \frac{1}{(1-c)\gamma^2} H_a^0 (H_a^0)^{\mathrm{T}} \\ &\quad + \frac{1}{c} D_a^0 (D_a^0)^{\mathrm{T}} - L_a^0 (L_a^0)^{\mathrm{T}} < 0.\end{aligned} \tag{37}$$

Let $P = Z^{-1}$ and define

$$V_0(x_a^0) = \frac{1}{1-c}(x_a^0)^{\mathrm{T}} P x_a^0, \quad \alpha_0(x_a^0) = -(L_a^0) P x_a^0$$

then, (16) is satisfied with $\varepsilon_0 = 0.1$. By Step H.1, we have

$$V_1(x_a^0, x_1) = V_0(x_a^0) + \tfrac{1}{2}\sigma_1(x_1 - \alpha_0(x_a^0))^2$$

$$\begin{aligned}\alpha_1(x_a^0, x_1) = &-\frac{2}{\sigma_1}(L_a^0)^{\mathrm{T}} P x_a^0 - (L_a^0)^{\mathrm{T}} P (A_a^0 x_a^0 + L_a^0 x_1) \\ &- \frac{1}{\sigma_1}(x_1 + \alpha_0(x_a^0)) - \frac{m_1}{\sigma_1}(x_1 - \alpha_0(x_a^0)) \\ &- \frac{\sigma_1}{4\delta_1^2}((L_a^0)^{\mathrm{T}} P H_a^0)^2 (x_1 - \alpha_0(x_a^0)),\end{aligned}$$

where $\sigma_1 = 0.1$, $m_1 = 15$ and $\delta_1 = 0.05$. Then by Step H.2, let $X_1 = [(x_a^0)^{\mathrm{T}} \;\; x_1]^{\mathrm{T}}$, and

$$\mathbf{H}_1 = \begin{bmatrix} H_a^0 \\ 0 \end{bmatrix}, \quad F_1(X_1, x_2) = \begin{bmatrix} A_a^0 x_a^0 + L_a^0 x_1 \\ x_2 \end{bmatrix}$$

the H_∞ control law is given by

$$\begin{aligned} u = & -\frac{1}{\sigma_2}\frac{\partial V_1(X_1)}{\partial x_1} + \frac{\partial \alpha_1(X_1)}{\partial X_1} F_1(X_1, x_2) \\ & - \frac{\sigma_2}{4\delta_2^2}\left(H_2^2 + \left(\frac{\partial \alpha_1(X_1)}{\partial X_1}\mathbf{H}_1\right)^2\right)(x_2 - \alpha_1(X_1)) \\ & - \frac{m_2}{\sigma_2}(x_2 - \alpha_1(X_1)), \end{aligned} \tag{38}$$

where $H_2 = -0.1\varepsilon/(1-\varepsilon^2)$, $\sigma_2 = \delta_2 = 0.05$, and $m_2 = 0.16$.

4. Conclusions

The nonlinear H_∞ control problem is investigated for a class of nonminimum phase nonlinear system. The nonlinearity of the system relies the measurable output only, and satisfies some linear growth conditions. After transforming the system into the SCB form, the sufficient condition of the nonlinear H_∞ control problem is related to the solvability of three Lyapunov equations on the unstable zero dynamics. And the achievable L_2 gain estimation can be calculated based on the solutions of these Lyapunov equations.

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