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Solutions to general H_∞ almost disturbance decoupling problem with measurement feedback and internal stability for discrete-time systems[☆]

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State feedback and full order/reduced order measurement feedback controllers are explicitly constructed that solve the H_∞ almost disturbance decoupling problems for general linear discrete-time systems. Keys to the explicit construction of these controllers are structural decompositions of the systems and a low gain feedback design technique.

Abstract

In this paper, we construct feedback controllers, explicitly parameterized in a single parameter ε, that solve the well-known H_∞ almost disturbance decoupling problem with measurement feedback and with internal stability (H_∞-ADDPMS) for discrete-time linear systems. In particular, we explicitly construct parameterized solutions for the following three cases: the full state feedback, the full information feedback and the general measurement output feedback. The first two cases have static solutions while the last one has only dynamic solutions. Both the full order and the reduced order measurement feedback controllers are presented for the latter case. The problem considered in this paper is general and complete. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

We consider the problem of H_∞ almost disturbance decoupling with measurement feedback and internal stability for discrete-time linear systems. The problem of almost disturbance decoupling has a vast history behind it, occupying a central part of classical as well as modern control theory. Several important problems, such as robust control, decentralized control, non-interactive control, model reference or tracking control, H_2 and H_∞ suboptimal control problems can all be recast into an almost disturbance decoupling problem. Roughly speaking, the basic almost disturbance decoupling problem is to find an output feedback control law such that in the closed-loop system the disturbances are quenched, say in an L_p sense, up to any pre-specified degree of accuracy while maintaining internal stability. Such a problem was originally formulated by Willems (1981, 1982) for continuous-time systems and labeled as ADDPMS (the almost disturbance decoupling problem with measurement feedback and internal stability). The prefix H_∞ in the acronym H_∞-ADDPMS is used to specify that the degree of accuracy in disturbance quenching is measured in L_2 gain.

There is extensive literature on the almost disturbance decoupling problem for continuous-time systems (see, for example, Ozcetin, Saberi & Sannuti, 1992; Scherer, 1992; Trentlman, 1986; Weiland & Willems, 1989, and the references therein). Recently, we have proposed solutions, which are explicitly parameterized in a single parameter ε, to this well-known problem for continuous-time systems (Chen, Lin & Hang, 1998). The problem considered in Chen et al. (1998) is general and complete in that the system is allowed to have invariant zeros on the imaginary axis. In contrast, the problem of almost disturbance decoupling for general discrete-time systems is less

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studied. Only very recently has the necessary and sufficient conditions under which the H_∞-ADDPMS for general discrete-time systems is solvable been derived by Chen, He and Chen (1999). As in Chen et al. (1998), the problem considered in Chen et al. (1999) is general in that the system is allowed to have invariant zeros on the unit circle. Under the solvability conditions of Chen et al. (1999), the problem of constructing feedback laws that solve the H_∞-ADDPMS for discrete-time linear systems, however, remains unattempted. The objective of this paper is to present algorithms for the explicit construction of feedback laws that solve the H_∞-ADDPMS for general discrete-time systems whose subsystems are allowed to have invariant zeros on the unit circle.

More specifically, we consider the H_∞-ADDPMS for the following discrete-time linear system:

$$\Sigma: \quad \begin{aligned} \delta x &= Ax + Bu + Ew, \\ y &= C_1 x + D_1 w, \\ h &= C_2 x + D_2 u + D_{22} w, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measurement, $w \in \mathbb{R}^q$ is the disturbance and $h \in \mathbb{R}^p$ is the output to be controlled, $A, B, E, C_1, C_2, D_1, D_2$, and D_{22} are constant matrices of appropriate dimensions, and finally, here and elsewhere in this paper, we suppress the running index k in $x(k)$ and use δx to denote $x(k+1)$ of the left-hand side of a difference equation.

For easy reference in future development, throughout this paper, we define Σ_P to be the subsystem characterized by the matrix quadruple (A, B, C_2, D_2) and Σ_Q to be the subsystem characterized by the matrix quadruple (A, E, C_1, D_1). Dynamic feedback control laws of the following form are investigated:

$$\Sigma_c: \quad \begin{aligned} \delta x_c &= A_c x_c + B_c y, \\ u &= C_c x_c + D_c y. \end{aligned} \tag{2}$$

The controller Σ_c of (2) is said to be internally stabilizing when applied to the system Σ, if the following matrix is asymptotically stable:

$$A_{cl} := \begin{bmatrix} A + BD_cC_1 & BC_c \\ B_cC_1 & A_c \end{bmatrix}, \tag{3}$$

i.e., all its eigenvalues lie on the open unit disc. Denote by T_{hw} the corresponding closed-loop transfer matrix from the disturbance w to be controlled output h, i.e.,

$$T_{hw}(z) = [C_2 + D_2D_cC_1 \quad D_2C_c] \left(zI - \begin{bmatrix} A + BD_cC_1 & BC_c \\ B_cC_1 & A_c \end{bmatrix} \right)^{-1} \begin{bmatrix} E + BD_cD_1 \\ B_cD_1 \end{bmatrix} + D_2D_cD_1 + D_{22}. \tag{4}$$

The H_∞ norm of the transfer matrix T_{hw} is given by

$$\|T_{hw}\|_\infty := \sup_{\omega \in [0,\pi]} \sigma_{\max}[T_{hw}(e^{j\omega})], \tag{5}$$

where $\sigma_{\max}[\cdot]$ denotes the maximal singular value. Then the general H_∞-ADDPMS for the given discrete-time system Σ of (1) can be formally defined as follows.

Definition 1.1. The general H_∞ almost disturbance decoupling problem with measurement feedback and with internal stability (the general H_∞-ADDPMS) for (1) is said to be solvable if, for any given scalar $\gamma > 0$, there exists at least one controller of the form (2) such that,

1. in the absence of disturbance, the closed-loop system comprising system (1) and the controller (2) is asymptotically stable, i.e., the matrix A_{cl} as given by (3) is asymptotically stable;
2. the closed-loop system has an l_2-gain, from the disturbance w to the controlled output h, that is less than or equal to γ, i.e.,

$$\|h\|_{l_2} \le \gamma \|w\|_{l_2}, \quad \forall w \in l_2 \text{ and for } (x(0), x_c(0)) = (0,0). \tag{6}$$

Equivalently, the H_∞-norm of the closed-loop transfer matrix from w to h, T_{hw}, is less than or equal to γ, i.e., $\|T_{hw}\|_\infty \le \gamma$.

We referred to such a problem as the general H_∞-ADDPMS since our solution does not require the subsystems of (1) to have no invariant zeros in the unit circle.

The main objective of this paper is to explicitly construct feedback control laws that solve the general H_∞-ADDPMS for discrete-time systems. The outline of this paper is as follows: Section 2 recalls the background materials on the solvability conditions of the general H_∞-ADDPMS for discrete-time systems and the special coordinate basis of linear systems. Section 3 deals with the design of feedback control laws for the special case that full state or full information is measured for feedback. Section 4 deals with the construction of both full and reduced order measurement feedback controllers. Concluding remarks are made in Section 5.

Throughout this paper, the following notation will also be used: X' denotes the transpose of matrix X; $|X|$ denotes the 2-norm of matrix X; I denotes an identity matrix with appropriate dimensions; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$ is the set of all complex numbers; $\mathbb{C}^{\odot}$, $\mathbb{C}^{\circ}$ and $\mathbb{C}^{\otimes}$ are, respectively, the open unit disc, the unit circle and the set of complex numbers outside the unit circle; $\mathrm{Ker}(X)$ is the kernel of X; $\mathrm{Im}(X)$ is the image of X; $\lambda(X)$ is the set of eigenvalues of a real square matrix X; $X^\dagger$ is the generalized inverse of X; and $\sigma_{\max}(X)$ denotes the maximal singular value of matrix X.

2. Background materials and preliminary results

In this section, we recall the necessary and sufficient conditions of Chen et al. (1999) under which the general H_∞-ADDPMS for a discrete-time system is solvable and the special coordinate basis (SCB) of Sannuti and Saberi (1987) and Saberi and Sannuti (1990). The latter serves as a basic tool in our development of algorithms for constructing control laws that solve the general H_∞-ADDPMS.

2.1. Solvability conditions for discrete-time general H_∞-ADDPMS

In order to state the necessary and sufficient conditions for solving the general H_∞-ADDPMS for discrete-time systems, we need the following geometric subspaces.

Definition 2.1. Consider a linear time-invariant system Σ_* characterized by a matrix quadruple (A_*, B_*, C_*, D_*). The weakly unobservable subspaces of Σ_*, $\mathscr{V}^{\odot}$, and the strongly controllable subspaces of Σ_*, $\mathscr{S}^{\odot}$, are defined as follows:

1. $\mathscr{V}^{\odot}(\Sigma_*)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A_* + B_*F_*)$-invariant and contained in $\mathrm{Ker}(C_* + D_*F_*)$ such that the eigenvalues of $(A_* + B_*F_*)|\mathscr{V}^{\odot}$ are contained in $\mathbb{C}^{\odot} \cup \mathbb{C}^{\circ}$ for some constant matrix F_*.
2. $\mathscr{S}^{\odot}(\Sigma_*)$ is the minimal $(A_* + K_*C_*)$-invariant subspace of $\mathbb{R}^n$ containing $\mathrm{Im}(B_* + K_*D_*)$ such that the eigenvalues of the map which is induced by $(A_* + K_*C_*)$ on the factor space $\mathbb{R}^n/\mathscr{S}^{\odot}$ are contained in $\mathbb{C}^{\odot} \cup \mathbb{C}^{\circ}$ for some constant matrix K_*.

Definition 2.2. Consider a linear system Σ_* characterized by a quadruple (A_*, B_*, C_*, D_*). For any $\lambda \in \mathbb{C}$, we define

$$\mathscr{S}_\lambda(\Sigma_*) := \left\{ x \in \mathbb{C}^n \,\middle|\, \exists u \in \mathbb{C}^{n+m}: \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} u \right\} \tag{7}$$

and

$$\mathscr{V}_\lambda(\Sigma_*) := \left\{ x \in \mathbb{C}^n \,\middle|\, \exists u \in \mathbb{C}^m: 0 = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\}. \tag{8}$$

$\mathscr{V}_\lambda(\Sigma_*)$ and $\mathscr{S}_\lambda(\Sigma_*)$ are associated with the so-called state zero directions of Σ_* if λ is an invariant zero of Σ_*.

The following results are mainly due to Chen et al. (1999).

Theorem 2.1. *Consider the discrete-time linear system Σ as given by* (1) *with the measurement output being*

$$y = \begin{pmatrix} x \\ w \end{pmatrix}, \quad or \quad C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 \\ I \end{pmatrix}, \tag{9}$$

i.e., all state variables and disturbances (*full information*) *are measured and are available for feedback. The general H_∞-ADDPMS is solvable if and only if the following conditions are satisfied*:

(a) (A, B) *is stabilizable.*
(b) $\mathrm{Im}(D_{22}) \subset \mathrm{Im}(D_2)$, *i.e.*, $D_{22} + D_2 S = 0$, *where* $S = -(D_2'D_2)^{\dagger} D_2' D_{22}$.
(c) $\mathrm{Im}(E + BS) \subset \{\mathscr{V}^{\odot}(\Sigma_{\mathrm{P}}) + B\,\mathrm{Ker}(D_2)\} \cap \{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_{\mathrm{P}})\}$, *where* $S = -(D_2'D_2)^{\dagger} D_2' D_{22}$.

The result for the general measurement feedback case is given in the next.

Theorem 2.2. *Consider the discrete-time linear system Σ as given by* (1). *The H_∞-ADDPMS for* (1) *is solvable by the control law of* (2) *if and only if the following conditions are satisfied*:

(a) (A, B) *is stabilizable.*
(b) (A, C_1) *is detectable.*
(c) $D_{22} + D_2 S D_1 = 0$, *where* $S = -(D_2'D_2)^{\dagger} D_2' D_{22} D_1' (D_1 D_1')^{\dagger}$.
(d) $\mathrm{Im}(E + BSD_1) \subset \{\mathscr{V}^{\odot}(\Sigma_{\mathrm{P}}) + B\,\mathrm{Ker}(D_2)\} \cap \{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_{\mathrm{P}})\}$, *where* $S = -(D_2'D_2)^{\dagger} D_2' D_{22} D_1' (D_1 D_1')^{\dagger}$.
(e) $\mathrm{Ker}(C_2 + D_2 S C_1) \supset \{\mathscr{S}^{\odot}(\Sigma_{\mathrm{Q}}) \cap C_1^{-1}\{\mathrm{Im}(D_1)\}\} \cup \{\bigcup_{|\lambda|=1} \mathscr{V}_\lambda(\Sigma_{\mathrm{Q}})\}$, *where* $S = -(D_2'D_2)^{\dagger} D_2' D_{22} D_1' (D_1 D_1')^{\dagger}$.
(f) $\mathscr{S}^{\odot}(\Sigma_{\mathrm{Q}}) \subset \mathscr{V}^{\odot}(\Sigma_{\mathrm{P}})$.

The following result deals with the case when only strictly proper measurement feedback laws are used.

Theorem 2.3. *Consider the discrete-time linear system Σ as given by* (1). *The H_∞ almost disturbance decoupling problem with internally stability and with a strictly proper measurement feedback law, i.e., the control law of the form* (2) *with $D_c = 0$, for* (1) *is solvable if and only if the following conditions are satisfied*:

(a) (A, B) *is stabilizable.*
(b) (A, C_1) *is detectable.*
(c) $D_{22} = 0$.
(d) $\mathrm{Im}(E) \subset \mathscr{V}^{\odot}(\Sigma_{\mathrm{P}}) \cap \{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_{\mathrm{P}})\}$.
(e) $\mathrm{Ker}(C_2) \supset \mathscr{S}^{\odot}(\Sigma_{\mathrm{Q}}) \cup \{\bigcup_{|\lambda|=1} \mathscr{V}_\lambda(\Sigma_{\mathrm{Q}})\}$.
(f) $\mathscr{S}^{\odot}(\Sigma_{\mathrm{Q}}) \subset \mathscr{V}^{\odot}(\Sigma_{\mathrm{P}})$.
(g) $A\mathscr{S}^{\odot}(\Sigma_{\mathrm{Q}}) \subset \mathscr{V}^{\odot}(\Sigma_{\mathrm{P}})$.

The following remark concerns the full state feedback case.

Remark 2.1. For special case when all the states of the system (1) are measured and available for feedback, i.e., $C_1 = I$ and $D_1 = 0$, it can be easily derived from Theorem 2.2 that the H_∞-ADDPMS is solvable if and only if the following conditions are satisfied: (a) (A, B) is stabilizable, (b) $D_{22} = 0$, and (c) $\mathrm{Im}(E) \subset \mathscr{V}^{\odot}(\Sigma_{\mathbf{P}}) \cap \{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_{\mathbf{P}})\}$.

2.2. Special coordinate basis of linear systems

Consider a discrete-time linear time-invariant system Σ_* characterized by the quadruple (A_*, B_*, C_*, D_*) or in the state space form

$$\delta x = A_* x + B_* u,\qquad y = C_* x + D_* u, \tag{10}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output of Σ_*. It is simple to verify that there exist non-singular transformations U and V such that

$$UD_* V = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

where m_0 is the rank of matrix D_*. In fact, U can be chosen as an orthogonal matrix. Hence hereafter, without loss of generality, it is assumed that the matrix D_* has the form given on the right-hand side of (11). One can now rewrite the system of (10) as

$$\delta x = A_* x + [B_{*0} \quad B_{*1}]\begin{pmatrix} u_0 \\ u_1 \end{pmatrix},\qquad \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} C_{*0} \\ C_{*1} \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \tag{12}$$

where the matrices B_{*0}, B_{*1}, C_{*0} and C_{*1} have appropriate dimensions. We have the following theorem.

Theorem 2.4 (SCB). *Given the linear system Σ_* of (10), there exist*

1. *coordinate-free non-negative integers n_a^-, n_a^0, n_a^+, n_b, n_c, n_d, $m_d \le m - m_0$ and q_i, $i = 1, \ldots, m_d$, and*
2. *non-singular state, output and input transformations Γ_s, Γ_o and Γ_i which take the given Σ_* into a special coordinate basis that explicitly displays various properties of Σ_*.*

The special coordinate basis is described by the following set of equations:

$$x = \Gamma_\mathrm{s} \mathbf{x},\quad y = \Gamma_\mathrm{o} \mathbf{y},\quad u = \Gamma_\mathrm{i} \mathbf{u}, \tag{13}$$

$$\mathbf{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix},\quad x_a = \begin{pmatrix} x_a^- \\ x_a^0 \\ x_a^+ \end{pmatrix},\quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix}, \tag{14}$$

$$\mathbf{y} = \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix},\quad y_d = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_d} \end{pmatrix},\quad \mathbf{u} = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix},\quad u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix} \tag{15}$$

and

$$\delta x_a^- = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{ad}^- y_d + L_{ab}^- y_b, \tag{16}$$

$$\delta x_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 y_0 + L_{ad}^0 y_d + L_{ab}^0 y_b, \tag{17}$$

$$\delta x_a^+ = A_{aa}^+ x_a^+ + B_{0a}^+ y_0 + L_{ad}^+ y_d + L_{ab}^+ y_b, \tag{18}$$

$$\delta x_b = A_{bb} x_b + B_{0b} y_0 + L_{bd} y_d,\quad y_b = C_b x_b, \tag{19}$$

$$\delta x_c = A_{cc} x_c + B_{0c} y_0 + L_{cb} y_b + L_{cd} y_d + B_c[E_{ca}^- x_a^- + E_{ca}^0 x^0 + E_{ca}^+ x_a^+ + u_c], \tag{20}$$

$$y_0 = C_{0c} x_c + C_{0a}^- x_a^- + C_{0a}^+ x_a^0 + C_{0a}^+ x_a^+ + C_{0d} x_d + C_{0b} x_b + u_0 \tag{21}$$

and for each $i = 1, \ldots, m_d$,

$$\delta x_i = A_{q_i} x_i + L_{i0} y_0 + L_{id} y_d + B_{q_i}\left[u_i + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} x_j\right], \tag{22}$$

$$y_i = C_{q_i} x_i,\quad y_d = C_d x_d. \tag{23}$$

Here the states x_a^-, x_a^0, x_a^+, x_b, x_c and x_d are, respectively, of dimensions n_a^-, n_a^0, n_a^+, n_b, n_c and $n_d = \sum_{i=1}^{m_d} q_i$, while x_i is of dimension q_i for each $i = 1, \ldots, m_d$. The control vectors u_0, u_d and u_c are, respectively, of dimensions m_0, m_d and $m_c = m - m_0 - m_d$ while the output vectors y_0, y_d and y_b are, respectively, of dimensions $p_0 = m_0$, $p_d = m_d$ and $p_b = p - p_0 - p_d$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i - 1} \\ 0 & 0 \end{bmatrix},\quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad C_{q_i} = [1, 0, \ldots, 0]. \tag{24}$$

Assuming that x_i, $i = 1, 2, \ldots, m_d$, are arranged such that $q_i \le q_{i+1}$, the matrix L_{id} has the particular form

$$L_{id} = [L_{i1} \quad L_{i2} \quad \cdots \quad L_{ii-1} \quad 0 \quad \cdots \quad 0]. \tag{25}$$

Also, the last row of each L_{id} is identically zero. Moreover, we have $\lambda(A_{aa}^-) \subset \mathbb{C}^{\odot}$, $\lambda(A_{aa}^0) \subset \mathbb{C}^{\circ}$, $\lambda(A_{aa}^+) \subset \mathbb{C}^{\otimes}$. Also, the pair (A_{cc}, B_c) is controllable and the pair (A_{bb}, C_b) is observable.

Proof. See Sannuti and Saberi (1987) and Saberi and Sannuti (1990). The software realizations of the above decomposition in MATLAB can be found in Lin and Chen (1998). □

In what follows, we state some important properties of the SCB which are pertinent to our present work. The rigorous proofs of these properties can be found in Chen (1998).

Property 2.1. The given system Σ_* is observable (detectable) if and only if the pair $(A_{\rm obs}, C_{\rm obs})$ is observable (detectable), where

$$A_{\rm obs} := \begin{bmatrix} A_{aa} & 0 & L_{ad}C_d \\ B_cE_{ca} & A_{cc} & L_{cd}C_d \\ B_dE_{da} & B_dE_{dc} & A_{dd} \end{bmatrix},$$

$$C_{\rm obs} := \begin{bmatrix} C_{0a} & C_{0c} & C_{0d} \\ 0 & 0 & C_d \end{bmatrix} \tag{26}$$

and where

$$A_{aa} := \begin{bmatrix} A_{aa}^- & 0 & 0 \\ 0 & A_{aa}^0 & 0 \\ 0 & 0 & A_{aa}^+ \end{bmatrix}, \qquad L_{ad} := \begin{bmatrix} L_{ad}^- \\ L_{ad}^0 \\ L_{ad}^+ \end{bmatrix}, \tag{27}$$

$$C_{0a} := [C_{0a}^- \quad C_{0a}^0 \quad C_{0a}^+], \quad E_{da} := [E_{da}^- \quad E_{da}^0 \quad E_{da}^+],$$

$$E_{ca} := [E_{ca}^- \quad E_{ca}^0 \quad E_{ca}^+]. \tag{28}$$

Also, define

$$B_{0a} := \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \end{bmatrix}, \qquad L_{ab} := \begin{bmatrix} L_{ab}^- \\ L_{ab}^0 \\ L_{ab}^+ \end{bmatrix}, \tag{29}$$

and

$$A_{\rm con} := \begin{bmatrix} A_{aa} & L_{ab}C_b & L_{ad}C_d \\ 0 & A_{bb} & L_{bd}C_d \\ B_dE_{da} & B_dE_{db} & A_{dd} \end{bmatrix},$$

$$B_{\rm con} := \begin{bmatrix} B_{0a} & 0 \\ B_{0b} & 0 \\ B_{0d} & B_d \end{bmatrix}. \tag{30}$$

Similarly, Σ_* is controllable (stabilizable) if and only if the pair $(A_{\rm con}, B_{\rm con})$ is controllable (stabilizable).

Property 2.2. Invariant zeros of Σ_* are the eigenvalues of A_{aa}, which are the unions of the eigenvalues of A_{aa}^-, A_{aa}^0 and A_{aa}^+.

Clearly, the SCB decomposes the state-space $\mathscr{X}$ into the following several distinct parts:

$$\mathscr{X} = \mathscr{X}_a^- \oplus \mathscr{X}_a^0 \oplus \mathscr{X}_a^+ \oplus \mathscr{X}_b \oplus \mathscr{X}_c \oplus \mathscr{X}_d. \tag{31}$$

The following property shows interconnections between the special coordinate basis and various invariant geometric subspaces.

Property 2.3.

$$\mathscr{V}^{\odot}(\Sigma_*) = \mathrm{Im}\left\{\Gamma_{\rm s}\begin{bmatrix} I_{n_a^-} & 0 & 0 \\ 0 & I_{n_a^0} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_c} \\ 0 & 0 & 0 \end{bmatrix}\right\},$$

$$\mathscr{S}^{\odot}(\Sigma_*) = \mathrm{Im}\left\{\Gamma_{\rm s}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_{n_a^+} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & I_{n_d} \end{bmatrix}\right\}. \tag{32}$$

Next,

$$\mathscr{S}_\lambda(\Sigma_*) = \mathrm{Im}\left\{\Gamma_{\rm s}\begin{bmatrix} \lambda I - A_{aa} & 0 & 0 & 0 \\ 0 & Y_{b\lambda} & 0 & 0 \\ 0 & 0 & I_{n_c} & 0 \\ 0 & 0 & 0 & I_{n_d} \end{bmatrix}\right\}, \tag{33}$$

where

$$\mathrm{Im}\{Y_{b\lambda}\} = \mathrm{Ker}[C_b(A_{bb} + K_bC_b - \lambda I)^{-1}] \tag{34}$$

and where K_b is any matrix such that $A_{bb} + K_bC_b$ has no eigenvalues at λ. Such a K_b always exists as (A_{bb}, C_b) is completely observable.

$$\mathscr{V}_\lambda(\Sigma_*) = \mathrm{Im}\left\{\Gamma_{\rm s}\begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix}\right\}, \tag{35}$$

where $X_{a\lambda}$ is a matrix whose columns form a basis for the subspace

$$\{\zeta_a \in \mathbb{C}^{n_a} \,|\, (\lambda I - A_{aa})\zeta_a = 0\} \tag{36}$$

and

$$X_{c\lambda} := (A_{cc} + B_cF_c - \lambda I)^{-1}B_c \tag{37}$$

with F_c being any matrix such that $A_{cc} + B_cF_c$ has no eigenvalues at λ. Again, the existence of such an F_c is guaranteed by the controllability of (A_{cc}, B_c).

3. The state and full information feedback cases

In this section, we consider feedback control law design for the general H_∞-ADDPMS for the case that either full state or full information feedback is measured for feedback.

We first consider the case that full state is measured for feedback. We will present a design procedure that constructs a family of parameterized static state feedback control laws

$$u = F(\varepsilon)x, \tag{38}$$

that solves the general H_∞-ADDPMS for the following system:

$$\delta x = Ax + Bu + Ew,$$

$$y = x \tag{39}$$

$$\mathrm{h} = C_2 x + D_2 u + D_{22} w.$$

That is, under this family of state feedback control laws, the resulting closed-loop system is asymptotically stable for sufficiently small ε and the H_∞-norm of the closed-loop transfer matrix from w to h, $T_{hw}(z,\varepsilon)$, tends to zero as ε tends to zero, where

$$T_{hw}(z,\varepsilon) = [C_2 + D_2 F(\varepsilon)][zI - A - BF(\varepsilon)]^{-1}E + D_{22}. \tag{40}$$

Our algorithm for obtaining this $F(\varepsilon)$ utilizes the asymptotic time scale and eigenstructure assignment (ATEA) procedure. The ATEA design procedure was originally conceived in Saberi and Sannuti (1989) and was used to solve many control problems (see for example, Chen, 1998; Saberi, Sannuti & Chen, 1995, to name a few). It was further developed in Lin (1998) to include slow time-scale assignment via low gain feedback. As will be clear shortly, the low gain component is critical in handling the situation when the zero dynamics corresponding to unit circle invariant zeros is affected by disturbances. In comparison with its continuous-time counterparts (Chen et al., 1998), where high gain feedback is an important component of the feedback laws, here in discrete-time setting, we do not have high gain feedback. It is because of this lack of high gain feedback in discrete time that the solvability conditions for the H_∞-ADDPMS exclude the presence of disturbance in the part of the dynamics associated with the infinite zero structure of the system whenever the disturbance is not available for feedback.

Step S.1 (*Decomposition of* Σ_P): Transform the subsystem Σ_P, i.e., the matrix quadruple (A, B, C_2, D_2), into the SCB as given by Theorem 2.4. Denote the state, output and input transformation matrices as Γ_{sP}, Γ_{oP} and Γ_{iP}, respectively.

Step S.2 (*Gain matrix for the subsystem associated with* $\mathscr{X}_c$): Let F_c be any matrix such that

$$A^c_{cc} = A_{cc} + B_c F_c \tag{41}$$

is an asymptotically stable matrix. The existence of such an F_c is guaranteed by the property of the SCB, i.e., (A_{cc}, B_c) is controllable.

Step S.3 (*Gain matrix for the subsystem associated with* $\mathscr{X}_a^+$, $\mathscr{X}_b$ *and* $\mathscr{X}_d$): Let

$$F_{abd} := \begin{bmatrix} 0 & 0 & F^+_{a0} & F_{b0} & F_{d0} \\ E^-_{da} & E^0_{da} & F^+_{ad} & F_{bd} & F_{dd} \end{bmatrix}, \tag{42}$$

where

$$F^+_{abd} := \begin{bmatrix} F^+_{a0} & F_{b0} & F_{d0} \\ F^+_{ad} & F_{bd} & F_{dd} \end{bmatrix} \tag{43}$$

is any matrix such that

$$A^{+c}_{abd} := \begin{bmatrix} A^+_{aa} & L^+_{ab}C_b & L^+_{ad}C_d \\ 0 & A_{bb} & L_{bd}C_d \\ B_d E^+_{da} & B_d E_{db} & A_{dd} \end{bmatrix} + \begin{bmatrix} B^+_{0a} & 0 \\ B_{0b} & 0 \\ B_{0d} & B_d \end{bmatrix} F^+_{abd} \tag{44}$$

is an asymptotically stable matrix. Again, the existence of such an F^+_{abd} is guaranteed by the property of the SCB.

Step S.4 (*Gain matrix for the subsystem associated with* A^0_{aa}): The construction of this gain matrix is carried out in the following four substeps:

Step S.4.1 (*Preliminary coordinate transformation*): Noting that

$$A_{\mathrm{con}} := \begin{bmatrix} A_{aa} & L_{ab}C_b & L_{ad}C_d \\ 0 & A_{bb} & L_{bd}C_d \\ B_d E_{da} & B_d E_{db} & A_{dd} \end{bmatrix}, \quad B_{\mathrm{con}} := \begin{bmatrix} B_{0a} & 0 \\ B_{0b} & 0 \\ B_{0d} & B_d \end{bmatrix},$$

we have

$$A_{\mathrm{con}} + B_{\mathrm{con}}F_{abd} = \begin{bmatrix} A^-_{aa} & 0 & A^-_{abd} \\ 0 & A^0_{aa} & A^0_{abd} \\ 0 & 0 & A^{+c}_{abd} \end{bmatrix},$$

$$B_{\mathrm{con}} = \begin{bmatrix} B^-_{0a} & 0 \\ B^0_{0a} & 0 \\ B^+_{0abd} & B^+_{abd} \end{bmatrix}, \tag{45}$$

where

$$B^+_{0abd} = \begin{bmatrix} B^+_{0a} \\ B_{0b} \\ B_{0d} \end{bmatrix}, \quad B^+_{abd} = \begin{bmatrix} 0 \\ 0 \\ B_d \end{bmatrix}, \tag{46}$$

$$A^0_{abd} = [0 \quad L^0_{ab}C_b \quad L^0_{ad}C_d] + [B^0_{0a} \quad 0]F^+_{abd} \tag{47}$$

and

$$A^-_{abd} = [0 \quad L^-_{ab}C_b \quad L^-_{ad}C_d] + [B^-_{0a} \quad 0]F^+_{abd}. \tag{48}$$

Clearly, the pair $(A_{\mathrm{con}} + B_{\mathrm{con}}F_{abd}, B_{\mathrm{con}})$ remains stabilizable. Construct the following non-singular transformation matrix:

$$\Gamma_{abd} = \begin{bmatrix} I_{n_a^-} & 0 & 0 \\ 0 & 0 & I_{n_a^+ + n_b + n_d} \\ 0 & I_{n_a^0} & T^0_a \end{bmatrix}^{-1}, \tag{49}$$

where T_a^0 is the unique solution to the following Lyapunov equation:

$$A_{aa}^0 T_a^0 - T_a^0 A_{abd}^{+c} = A_{abd}^0. \tag{50}$$

Such a unique solution to the above Lyapunov equation always exists since all the eigenvalues of A_{aa}^0 are on the unit circle and all the eigenvalues of A_{abd}^{+c} are on the open unit disc. It is now easy to verify that

$$\Gamma_{abd}^{-1}(A_{\rm con} + B_{\rm con}F_{abd})\Gamma_{abd} = \begin{bmatrix} A_{aa}^- & A_{abd}^- & 0 \\ 0 & A_{abd}^{+c} & 0 \\ 0 & 0 & A_{aa}^0 \end{bmatrix} \tag{51}$$

and

$$\Gamma_{abd}^{-1}B_{\rm con} = \begin{bmatrix} B_{0a}^- & 0 \\ B_{0abd}^+ & B_{abd}^+ \\ B_{0a}^0 + T_a^0 B_{0abd}^+ & T_a^0 B_{abd}^+ \end{bmatrix}. \tag{52}$$

Hence, the matrix pair (A_{aa}^0, B_a^0) is controllable, where

$$B_a^0 = [B_{0a}^0 + T_a^0 B_{0abd}^+ \ \ T_a^0 B_{abd}^+]. \tag{53}$$

Step S.4.2 (*Further coordinate transformation*): Find the non-singular transformation matrices $\Gamma_{\rm sa}^0$ and $\Gamma_{\rm ia}^0$ such that (A_{aa}^0, B_a^0) can be transformed into the block diagonal controllability canonical form

$$(\Gamma_{\rm sa}^0)^{-1} A_{aa}^0 \Gamma_{\rm sa}^0 = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_l \end{bmatrix} \tag{54}$$

and

$$(\Gamma_{\rm sa}^0)^{-1} B_a^0 \Gamma_{\rm ia}^0 = \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1l} & \star \\ 0 & B_2 & \cdots & B_{2l} & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_l & \star \end{bmatrix}, \tag{55}$$

where l is an integer and for $i = 1,2,\ldots,l$,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{n_i}^i & -a_{n_i-1}^i & -a_{n_i-2}^i & \cdots & -a_1^i \end{bmatrix},$$

$$B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{56}$$

We note that all the eigenvalues of A_i are on the unit circle. Here, the $\star$'s represent submatrices of less interest. The existence of the above canonical form was shown in Wonham (1979) while its software realization can be found in Lin and Chen (1998).

Step S.4.3 (*Subsystem design*): For each (A_i, B_i), let $F_i(\varepsilon) \in \mathbb{R}^{1 \times n_i}$ be the state feedback gain such that

$$\lambda(A_i + B_iF_i(\varepsilon)) = (1-\varepsilon)\lambda(A_i).$$

Clearly, all eigenvalues of $A_i + B_iF_i(\varepsilon)$ are on the open unit disc and $F_i(\varepsilon)$ is unique.

Step S.4.4 (*Composition of gain matrix for subsystem associated with $\mathscr{X}_a^0$*): Let

$$F_a^0(\varepsilon) := \Gamma_{\rm ia}^0 \begin{bmatrix} F_1(\varepsilon) & 0 & \cdots & 0 & 0 \\ 0 & F_2(\varepsilon) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & F_{l-1}(\varepsilon) & 0 \\ 0 & 0 & \cdots & 0 & F_l(\varepsilon) \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} (\Gamma_{\rm sa}^0)^{-1}, \tag{57}$$

where $\varepsilon \in (0,1]$ is a design parameter whose value is to be specified later.

Clearly, we have

$$|F_a^0(\varepsilon)| \le f_a^0 \varepsilon, \quad \varepsilon \in (0,1], \tag{58}$$

for some positive constant f_a^0, independent of ε. For future use, we partition

$$F_a^0(\varepsilon) = \begin{bmatrix} F_{a0}^0(\varepsilon) \\ F_{ad}^0(\varepsilon) \end{bmatrix} \tag{59}$$

and

$$F_a^0(\varepsilon) T_a^0 = \begin{bmatrix} F_{a0+}^0(\varepsilon) & F_{a0b}^0(\varepsilon) & F_{a0d}^0(\varepsilon) \\ F_{ad+}^0(\varepsilon) & F_{adb}^0(\varepsilon) & F_{add}^0(\varepsilon) \end{bmatrix}. \tag{60}$$

Step S.5 (*Composition of parameterized gain matrix $F(\varepsilon)$*): Various gains calculated in Steps S.2–S.4 are now put together to form a composite state feedback gain matrix $F(\varepsilon)$, which is explicitly a polynomial matrix in ε and is given by

$$F(\varepsilon) := \Gamma_{\rm iP}[F_0 + F_\star(\varepsilon)]\Gamma_{\rm sP}^{-1}, \tag{61}$$

where

$$F_0 = -\begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ - F_{a0}^+ & C_{0b} - F_{b0} & C_{0c} & C_{0d} - F_{d0} \\ E_{da}^- & E_{da}^0 & -F_{ad}^+ & -F_{bd} & E_{dc} & -F_{dd} \\ E_{ca}^- & E_{ca}^0 & E_{ca}^+ & 0 & F_c & 0 \end{bmatrix} \tag{62}$$

and

$$F_\star(\varepsilon)=\begin{bmatrix}0 & F^0_{a0}(\varepsilon) & F^0_{a0+}(\varepsilon) & F^0_{a0b}(\varepsilon) & 0 & F^0_{a0d}(\varepsilon)\\ 0 & F^0_{ad}(\varepsilon) & F^0_{ad+}(\varepsilon) & F^0_{adb}(\varepsilon) & 0 & F^0_{add}(\varepsilon)\\ 0&0&0&0&0&0\end{bmatrix}. \quad (63)$$

This completes the construction of the parameterized state feedback gain matrix $F(\varepsilon)$. □

Theorem 3.1. *Consider the given system* (39) *in which all the states are measured and are available for feedback. Assume that the general H_∞-ADDPMS for* (39) *is solvable, i.e., the solvability conditions of Remark* 2.1 *are satisfied. Then, the closed-loop system comprising* (39) *and the full state feedback control law*

$$u=F(\varepsilon)x \quad (64)$$

with $F(\varepsilon)$ given by (61), *has the following properties: for any given $\gamma>0$, there exists a positive scalar $\varepsilon^*>0$ such that for all $0<\varepsilon\le\varepsilon^*$,*

1. *the closed-loop system is asymptotically stable, i.e., $\lambda(A+BF(\varepsilon))$ are on the open unit disc; and,*
2. *the H_∞-norm of the closed-loop transfer matrix from the disturbance w to the controlled output h is less than or equal to γ, i.e., $\|T_{hw}(z,\varepsilon)\|_\infty\le\gamma$.*

Hence, by Definition 1.1, *the family of control laws as given by* (64) *solves the general H_∞-ADDPMS for* (39).

Proof. See Appendix A. □

Next, we proceed to design a family of parameterized full information feedback control laws

$$u=F_x(\varepsilon)x+F_w w, \quad (65)$$

which solves the general H_∞-ADDPMS for the following system:

$$\delta x=Ax+Bu+Ew,$$
$$y=\begin{pmatrix}I\\0\end{pmatrix}x+\begin{pmatrix}0\\I\end{pmatrix}w, \quad (66)$$
$$h=C_2x+D_2u+D_{22}w.$$

That is, under the above full information feedback control laws, the resulting closed-loop system is asymptotically stable for sufficiently small ε and the H_∞-norm of the closed-loop transfer matrix from w to h, $T_{hw}(z,\varepsilon)$, tends to zero as ε tends to zero, where

$$T_{hw}(z,\varepsilon)=[C_2+D_2F_x(\varepsilon)][zI-A-BF_x(\varepsilon)]^{-1}\times(E+BF_w)+(D_{22}+D_2F_w). \quad (67)$$

The following is a step-by-step algorithm for constructing $F_x(\varepsilon)$ and F_w.

Step F.1 (*Computation of S*): Compute

$$S=-(D_2'D_2)^\dagger D_2'D_{22}. \quad (68)$$

Step F.2 (*Computation of $F_x(\varepsilon)$*): Follow Steps S.1–S.5 of the previous algorithm to yield a gain matrix $F(\varepsilon)$. Then, let

$$F_x(\varepsilon)=F(\varepsilon). \quad (69)$$

Also, we need to retain the transformation matrices Γ_{sP} and Γ_{iP}, as well as the submatrix B_d of the SCB of Σ_{P} in order to compute F_w in the next step.

Step F.3 (*Construction of gain matrix F_w*): Let

$$\Gamma_{\mathrm{sP}}^{-1}(E+BS)=[(E_a^-)'\ (E_a^0)'\ (E_a^+)'\ (E_b)'\ (E_c)'\ (E_d)']'. \quad (70)$$

Then, the gain matrix F_w is given by

$$F_w=-\Gamma_{\mathrm{iP}}\begin{bmatrix}0\\(B_d'B_d)^{-1}B_d'E_d\\0\end{bmatrix}+S. \quad (71)$$

It is informative to note that the first portion of matrix F_w is used to cancel the disturbance associated with E_d and in the range space of B_d, while the second portion is used to reject disturbance entering into the system through D_{22}.

Theorem 3.2. *Consider the given system* (66) *in which all the states and disturbances are measured and are available for feedback. Assume that the general H_∞-ADDPMS for* (66) *is solvable, i.e., the solvability conditions of Theorem* 2.1 *are satisfied. Then, the closed-loop system comprising* (66) *and the full information feedback control law*

$$u=F_x(\varepsilon)x+F_w w \quad (72)$$

with $F_x(\varepsilon)$ and F_w being given by (69) *and* (71), *respectively, has the following properties: for any given $\gamma>0$, there exists a positive scalar $\varepsilon^*>0$ such that for all $0<\varepsilon\le\varepsilon^*$,*

1. *the closed-loop system is asymptotically stable, i.e., $\lambda(A+BF_x(\varepsilon))$ are on the open unit disc; and,*
2. *the H_∞-norm of the closed-loop transfer matrix from the disturbance w to the controlled output h is less than or equal to γ, i.e., $\|T_{hw}(z,\varepsilon)\|_\infty\le\gamma$.*

Hence, by Definition 1.1, *the family of control laws as given by* (72) *solves the general H_∞-ADDPMS for* (66).

Proof. See Appendix B. □

Example 3.1. Consider a discrete-time system characterized by (1) with

$$A=\begin{bmatrix}1&1&1&1&0\\0&1&1&1&0\\0&0&0.1&1&0\\0&0&0&0&1\\0.1&0.1&0.1&0.1&0.1\end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \alpha_e & 0 \end{bmatrix}, \tag{73}$$

where α_e is a scalar, and

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_2 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \quad D_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{74}$$

We will consider both the state feedback case and the full information feedback case in this example. Using the toolbox Lin and Chen (1998), we can verify that (A, B) is controllable and Σ_P, i.e., (A, B, C_2, D_2), has two invariant zeros at $z = 1$. Moreover,

$$\mathscr{V}^{\odot}(\Sigma_P) = \mathrm{Im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}, \qquad B\,\mathrm{Ker}(D_2) = \mathrm{Im}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

and

$$\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_P) = \mathrm{Im}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\},$$

$$\mathscr{V}^{\odot}(\Sigma_P) \cap \left\{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_P)\right\} = \mathrm{Im}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

and

$$\{\mathscr{V}^{\odot}(\Sigma_P) + B\,\mathrm{Ker}(D_2)\} \cap \left\{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_P)\right\} = \mathrm{Im}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

It is now clear that

- if the full state is measured for feedback, i.e., $y = x$, then the general H_∞-ADDPMS is solvable if and only if $\alpha_e = 0$; and,
- if the full information is measured for feedback, i.e., $y = [x', w']'$, then the general H_∞-ADDPMS is always solvable for any α_e.

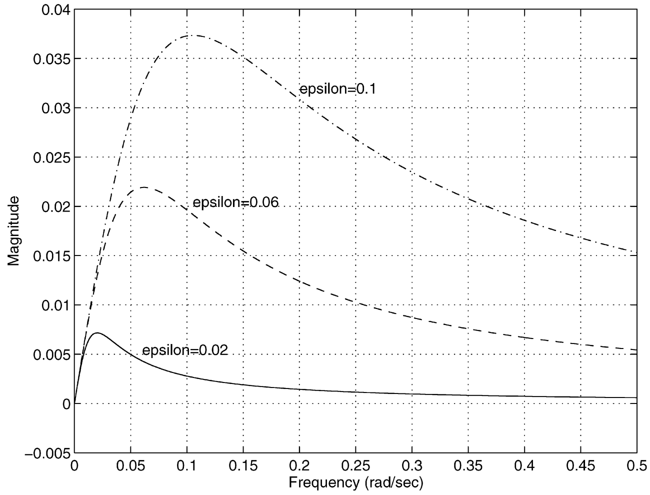

Fig. 1. Max. singular values of T_{hw} — full information case.

Following the algorithms of this section, we obtain the following parameterized gain matrices:

$$F_x(\varepsilon) = \begin{bmatrix} -0.526316(\varepsilon-1)^2 - 1.052632(\varepsilon-1) - 0.626316 \\ -0.775623(\varepsilon-1)^2 - 2.603878(\varepsilon-1) - 1.928255 \\ -0.798061(\varepsilon-1)^2 - 2.763490(\varepsilon-1) - 2.066429 \\ -(\varepsilon-1)^2 - 4.2(\varepsilon-1) - 3.31 \\ -2(\varepsilon-1) - 2.2 \end{bmatrix}', \tag{75}$$

which places the eigenvalues of $A + BF_x(\varepsilon)$ at 0, 0, 0, $1-\varepsilon$ and $1-\varepsilon$, and

$$F_w = [-\alpha_e \quad 0]. \tag{76}$$

The maximum singular value plots of the corresponding closed-loop transfer matrix $T_{hw}(z, \varepsilon)$ in Fig. 1 clearly show that the general H_∞-ADDMPS is indeed solved for both the full information case and the state feedback case (i.e., when $\alpha_e = 0$).

4. The measurement feedback case

4.1. Full order output feedback controller design

In this subsection, we focus on the design of a family of full order proper measurement feedback control laws, which solves the H_∞-ADDPMS for the given system (1) under the solvability conditions of Theorem 2.2. For the case when the given system satisfies the conditions listed in Theorem 2.3, slight modifications on the algorithm below, i.e., letting $S=0$ and $N=0$ in Steps F.C.1 and F.C.2, would yield a strictly proper solution. The following is a step-by-step algorithm for constructing a parameterized full order output feedback controller that solves the H_∞-ADDPMS for (1):

Step F.C.1 (*Computation of S*): Compute

$$S = -(D_2'D_2)^\dagger D_2'D_{22}D_1'(D_1D_1')^\dagger. \tag{77}$$

Step F.C.2 (*Computation of N*): Use the properties of the special coordinate basis to compute two constant matrices X and Y such that $\mathscr{V}^\odot(\Sigma_P)=\mathrm{Ker}(X)$ and $\mathscr{S}^\odot(\Sigma_Q)=\mathrm{Im}(Y)$. Then, compute

$$N = -(B'X'XB + D_2'D_2)^\dagger [B'X' \;\; D_2'] \times \begin{bmatrix} X(A+BSC_1)Y & X(E+BSD_1) \\ (C_2+D_2SC_1)Y & 0 \end{bmatrix} \times \begin{bmatrix} Y'C_1' \\ D_1' \end{bmatrix} (C_1YY'C_1' + D_1D_1')^\dagger. \tag{78}$$

Step F.C.3 (*Construction of the gain matrix $F_P(\varepsilon)$*): Define an auxiliary system

$$\begin{aligned} \delta x &= [A+B(S+N)C_1]x + Bu + [E+B(S+N)D_1]w, \\ y &= x \\ h &= [C_2+D_2(S+N)C_1]x + D_2u + 0w, \end{aligned} \tag{79}$$

and then perform Steps S.1–S.5 of the previous section to the above system to obtain a parameterized gain matrix $F(\varepsilon)$. We let $F_P(\varepsilon)=F(\varepsilon)$.

Step F.C.4 (*Construction of the gain matrix $K_Q(\varepsilon)$*): Define another auxiliary system

$$\begin{aligned} \delta x &= [A+B(S+N)C_1]'x + C_1'u + [C_2+D_2(S+N)C_1]'w, \\ y &= x \\ h &= [E+B(S+N)D_1]'x + D_1'u + 0w \end{aligned} \tag{80}$$

and then perform Steps S.1–S.6 of the previous section to the above system to get the parameterized gain matrix $F(\varepsilon)$. We let $K_Q(\varepsilon)=F(\varepsilon)'$.

Step F.C.5 (*Construction of the full order controller $\Sigma_{FC}(\varepsilon)$*): Finally, the parameterized full order output feedback controller is given by

$$\Sigma_{FC}(\varepsilon): \quad \begin{aligned} \delta x_c &= A_{FC}(\varepsilon)x_c + B_{FC}(\varepsilon)y, \\ u &= C_{FC}(\varepsilon)x_c + D_{FC}(\varepsilon)y, \end{aligned} \tag{81}$$

where

$$\begin{aligned} A_{FC}(\varepsilon) &:= A + B(S+N)C_1 + BF_P(\varepsilon) + K_Q(\varepsilon)C_1, \\ B_{FC}(\varepsilon) &:= -K_Q(\varepsilon), \\ C_{FC}(\varepsilon) &:= F_P(\varepsilon), \\ D_{FC}(\varepsilon) &:= S+N. \end{aligned} \tag{82}$$

Theorem 4.1. *Consider the given system* Σ *of* (1). *Assume that the general* H_∞*-ADDPMS for* (1) *is solvable, i.e., the solvability conditions of Theorem* 2.2 *are satisfied. Then, the closed-loop system comprising* (1) *and the full order measurement feedback controller* (81) *has the following properties: For any given* $\gamma>0$, *there exists a positive scalar* $\varepsilon^*>0$ *such that for all* $0<\varepsilon\le\varepsilon^*$,

1. *the closed-loop system is asymptotically stable; and,*
2. *the* H_∞*-norm of the closed-loop transfer matrix from the disturbance w to the controlled output h is less than or equal to* γ, *i.e.,* $\|T_{hw}(z,\varepsilon)\|_\infty \le \gamma$.

Hence, by Definition 1.1, *the family of control laws as given by* (81) *solves the general* H_∞*-ADDPMS for* (1).

Proof. See Appendix C. □

Example 4.1. We now consider a discrete-time system characterized by (1) with A, B, E, C_2, D_2 and D_{22} being given as in Example 3.1, and

$$C_1 = \begin{bmatrix} 0.5 & 0.1 & 0.5 & 0.2 & 0.1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{83}$$

For simplicity, we let $\alpha_e=1$ in matrix E. Using the toolbox Lin and Chen (1998), one can verify that (A,C_1) is observable and Σ_Q, i.e., (A,E,C_1,D_1), has four invariant zeros at -0.6554, $0.3777\pm j0.6726$, and 1. Moreover,

$$\mathscr{S}^\odot(\Sigma_Q) = \mathrm{Im}\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right\}, \quad C_1^{-1}\{\mathrm{Im}(D_1)\} = \mathrm{Im}\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right\},$$

$$\bigcup_{|\lambda|=1} \mathscr{V}_\lambda(\Sigma_Q) = \mathrm{Im}\left\{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\right\}.$$

Hence,

$$\{\mathscr{S}^\odot(\Sigma_Q)\cap C_1^{-1}\{\mathrm{Im}(D_1)\}\}\cup\left\{\bigcup_{|\lambda|=1}\mathscr{V}_\lambda(\Sigma_Q)\right\} = \mathrm{Im}\left\{\begin{bmatrix}1&0\\0&1\\0&0\\0&0\\0&0\end{bmatrix}\right\}.$$

It is readily seen that all conditions in Theorem 2.2 are satisfied. Hence, the general H_∞-ADDPMS for the given system is solvable. Following the algorithm of this subsection, we obtain a full order output feedback controller of the form (81) with

$$S=[0\quad 0],\quad N=[-1\quad 0.4],$$

$$F_P(\varepsilon)=\begin{bmatrix} -0.526316(\varepsilon-1)^2-1.052632(\varepsilon-1)-0.526316\\ -0.775623(\varepsilon-1)^2-2.603878(\varepsilon-1)-1.828255\\ -0.798061(\varepsilon-1)^2-2.763490(\varepsilon-1)-1.566429\\ -(\varepsilon-1)^2-4.2(\varepsilon-1)-3.11\\ -2(\varepsilon-1)-2.1\end{bmatrix}', \tag{84}$$

which places the eigenvalues of $\tilde{A}+BF_P(\varepsilon)$ at 0, 0, 0, $1-\varepsilon$ and $1-\varepsilon$, and

$$K_Q(\varepsilon)=\begin{bmatrix} -10 & 4\\ -10\varepsilon & 5\varepsilon\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}, \tag{85}$$

which places the eigenvalues of $\tilde{A}+K_Q(\varepsilon)C_1$ at -0.6554, $0.3777\pm j0.6726$, 0 and $1-\varepsilon$. The maximum singular value plots of the corresponding closed-loop transfer matrix $T_{hw}(z,\varepsilon)$ in Fig. 2 show that the general H_∞-ADDPMS is indeed solved.

4.2. Reduced order output feedback controller design

We will follow the procedure of Chen et al. (1998) to design a reduced order output feedback controller, which

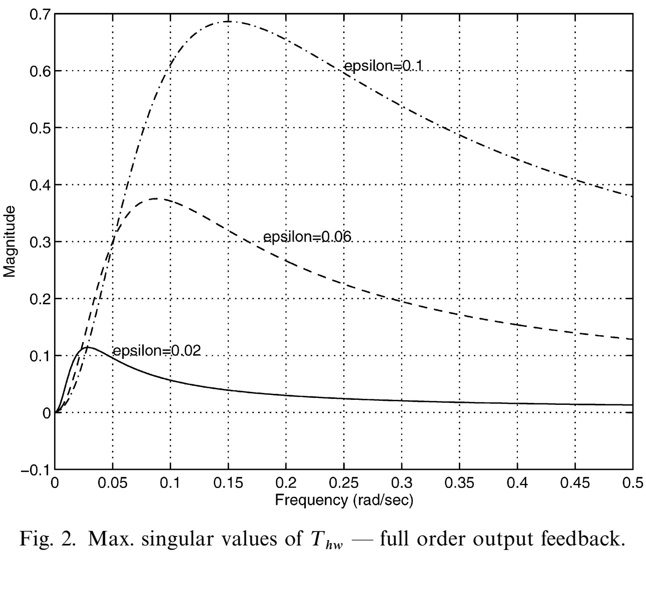

Fig. 2. Max. singular values of T_{hw} — full order output feedback.

also solves the general H_∞-ADDPMS for the discrete-time system (1). First, without loss of generality, we assume that the matrices C_1 and D_1 are already in the form of

$$C_1=\begin{bmatrix}0 & C_{1,02}\\ I_k & 0\end{bmatrix}\quad\text{and}\quad D_1=\begin{bmatrix}D_{1,0}\\ 0\end{bmatrix}, \tag{86}$$

where $k=\ell-\mathrm{rank}(D_1)$ and $D_{1,0}$ is of full rank. Next, we follow Steps F.C.1 and F.C.2 of the previous subsection to compute the constant matrices S and N and partition the following system:

$$\begin{aligned}\delta x&=[A+B(S+N)C_1]x+Bu+[E+B(S+N)D_1]w,\\ y&=C_1x+D_1w,\\ h&=[C_2+D_2(S+N)C_1]x+D_2u+0w,\end{aligned} \tag{87}$$

as follows:

$$\begin{aligned}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix}&=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u+\begin{bmatrix}E_1\\ E_2\end{bmatrix}w,\\ \begin{pmatrix}y_0\\ y_1\end{pmatrix}&=\begin{bmatrix}0 & C_{1,02}\\ I_k & 0\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{bmatrix}D_{1,0}\\ 0\end{bmatrix}w,\\ h&=[C_{2,1}\quad C_{2,2}]\begin{pmatrix}x_1\\ x_2\end{pmatrix}+D_2u+0w,\end{aligned} \tag{88}$$

where the state x of (87) is partitioned into two parts, x_1 and x_2; and y is partitioned into y_0 and y_1 with $y_1\equiv x_1$. Thus, one needs to estimate only the state x_2 in designing the reduced order controller. Define an auxiliary subsystem Σ_{QR} characterized by a matrix quadruple (A_R,E_R,C_R,D_R), where

$$(A_R,E_R,C_R,D_R)=\left(A_{22},E_2,\begin{bmatrix}C_{1,02}\\ A_{12}\end{bmatrix},\begin{bmatrix}D_{1,0}\\ E_1\end{bmatrix}\right). \tag{89}$$

The following is a step-by-step algorithm that constructs the reduced order output feedback controllers for the general H_∞-ADDPMS.

Step R.C.1 (*Construction of the gain matrix* $F_P(\varepsilon)$): Define an auxiliary system

$$\begin{aligned}\delta x&=[A+B(S+N)C_1]x+Bu+[E+B(S+N)D_1]w,\\ y&=x\\ h&=[C_2+D_2(S+N)C_1]x+D_2u+0w,\end{aligned} \tag{90}$$

and then perform Steps S.1–S.5 of the previous section to the above system to obtain a parameterized gain matrix $F(\varepsilon)$. We let $F_P(\varepsilon)=F(\varepsilon)$.

Step R.C.2 (*Construction of the gain matrix* $K_R(\varepsilon)$): Define another auxiliary system

$$\begin{aligned}\delta x&=A_R'x+C_R'u+C_{2,2}'w,\\ y&=x\\ h&=E_R'x+D_R'u+0w,\end{aligned} \tag{91}$$

and then perform Steps S.1–S.5 of the previous section to the above system to obtain a parameterized gain matrix $F(\varepsilon)$. We let $K_{\mathrm{R}}(\varepsilon) = F(\varepsilon)'$.

Step R.C.3 (*Construction of the reduced order controller* $\Sigma_{\mathrm{RC}}(\varepsilon)$): Let us partition $F_{\mathrm{P}}(\varepsilon)$ and $K_{\mathrm{R}}(\varepsilon)$ as

$$F_{\mathrm{P}}(\varepsilon) = [F_{\mathrm{P}1}(\varepsilon) \quad F_{\mathrm{P}2}(\varepsilon)]$$

and

$$K_{\mathrm{R}}(\varepsilon) = [K_{\mathrm{R}0}(\varepsilon) \quad K_{\mathrm{R}1}(\varepsilon)] \tag{92}$$

in conformity with the partition

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},$$

respectively. Then define

$$G_{\mathrm{R}}(\varepsilon) = [-K_{\mathrm{R}0}(\varepsilon), A_{21} + K_{\mathrm{R}1}(\varepsilon)A_{11} - (A_{\mathrm{R}} + K_{\mathrm{R}}(\varepsilon)C_{\mathrm{R}})K_{\mathrm{R}1}(\varepsilon)]. \tag{93}$$

Finally, the parameterized reduced order output feedback controller is given by

$$\Sigma_{\mathrm{RC}}(\varepsilon): \quad \begin{aligned} \delta x_c &= A_{\mathrm{RC}}(\varepsilon)x_c + B_{\mathrm{RC}}(\varepsilon)y, \\ u &= C_{\mathrm{RC}}(\varepsilon)x_c + D_{\mathrm{RC}}(\varepsilon)y, \end{aligned} \tag{94}$$

where

$$A_{\mathrm{RC}}(\varepsilon) := A_{\mathrm{R}} + B_2F_{\mathrm{P}2}(\varepsilon) + K_{\mathrm{R}}(\varepsilon)C_{\mathrm{R}} + K_{\mathrm{R}1}(\varepsilon)B_1F_{\mathrm{P}2}(\varepsilon),$$

$$B_{\mathrm{RC}}(\varepsilon) := G_{\mathrm{R}}(\varepsilon) + [B_2 + K_{\mathrm{R}1}(\varepsilon)B_1][0, F_{\mathrm{P}1}(\varepsilon) - F_{\mathrm{P}2}(\varepsilon)K_{\mathrm{R}1}(\varepsilon)],$$

$$C_{\mathrm{RC}}(\varepsilon) := F_{\mathrm{P}2}(\varepsilon),$$

$$D_{\mathrm{RC}}(\varepsilon) := [0, F_{\mathrm{P}1}(\varepsilon) - F_{\mathrm{P}2}(\varepsilon)K_{\mathrm{R}1}(\varepsilon)] + S + N. \tag{95}$$

Theorem 4.2. *Consider the given system* Σ *of* (1). *Assume that the general* H_∞*-ADDPMS for* (1) *is solvable, i.e., the solvability conditions of Theorem* 2.2 *are satisfied. Then, the closed-loop system comprising* (1) *and the reduced order measurement feedback controller* (94), *which has a dynamic order* $n - \ell + \mathrm{rank}(D_1)$, *has the following properties: for any given* $\gamma > 0$, *there exists a positive scalar* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon \le \varepsilon^*$,

1. *the closed-loop system is asymptotically stable; and,*
2. *the* H_∞*-norm of the closed-loop transfer matrix from the disturbance* w *to the controlled output* h *is less than or equal to* γ, *i.e.*, $\|T_{hw}(z, \varepsilon)\|_\infty \le \gamma$.

Hence, by Definition 1.1, *the family of control laws as given by* (94) *solves the general* H_∞*-ADDPMS for* (1).

Proof. See Appendix D. □

5. Conclusions

We have provided a complete solution to the general H_∞ almost disturbance decoupling problem with measurement feedback and with internal stability for discrete-time linear systems. The problem considered in this paper is general as we allow the subsystems of the given plant to have invariant zeros on the unit circle of the complex plane.

Appendix A — Proof of Theorem 3.1

Under the feedback law $u = F(\varepsilon)x$, the closed-loop system on the SCB can be written as follows:

$$\delta x_a^- = A_{aa}^- x_a^- + B_{0a}^- h_0 + L_{ad}^- h_d + L_{ab}^- h_b + E_a^- w, \tag{A.1}$$

$$\delta x_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 h_0 + L_{ad}^0 h_d + L_{ab}^0 h_b + E_a^0 w, \tag{A.2}$$

$$\delta x_{abd}^+ = A_{abd}^{+c} x_{abd}^+ + [B_{0abd}^+, B_{abd}^+]F_a^0(\varepsilon) \times [x_a^0 + T_a^0 x_{abd}^+] + E_{abd}^+ w, \tag{A.3}$$

$$\delta x_c = A_{cc}^c x_c + B_{0c}h_0 + L_{cb}h_b + L_{cd}h_d + E_c w, \tag{A.4}$$

$$h_0 = [F_{a0}^+, F_{b0}, F_{d0}]x_{abd}^+ + F_{a0}^0(\varepsilon)(x_a^0 + T_a^0 x_{abd}^+), \tag{A.5}$$

$$h_b = [0_{m_b \times n_a^+}, C_b, 0_{m_b \times n_d}]x_{abd}^+, \tag{A.6}$$

$$h_d = [0_{m_b \times n_a^+}, 0_{m_b \times n_b}, C_d]x_{abd}^+, \tag{A.7}$$

where $x_{abd}^+ = [(x_a^+)', x_b', x_d']'$, and B_{0abd}^+ is as defined in Step 4.1 of the state feedback design algorithm. We have also used Condition (b) of Remark 2.1, i.e., $D_{22} = 0$. Matrices E_a^-, E_a^0, E_{abd}^+, E_b and E_c are defined as follows:

$$\Gamma_{\mathrm{sP}}^{-1}E = [(E_a^-)' \quad (E_a^0)'(E_{ab}^+)' \quad E_c' \quad E_d']', \qquad \mathrm{E}_{abd}^+ = [(E_{ab}^+)' \quad E_d']'. \tag{A.8}$$

Condition (c) of Remark 2.1 then implies that $E_{abd}^+ = 0$ and

$$\mathrm{Im}(E_a^0) \subset \mathscr{S}(A_{aa}^0) := \bigcap_{\omega \in \lambda(A_{aa}^0)} \mathrm{Im}\{\omega I - A_{aa}^0\}. \tag{A.9}$$

From the dynamic equations of the closed-loop system (A.1)–(A.7), we observe that the states x_a^- and x_c do not contribute to the controlled output h_0, h_b and h_d and hence are allowed to be afftected by disturbances. The state x_{abd}^+ contributed directly to the control output h_0, h_b and h_d and hence the solvability conditions imply that $E_{abd}^+ = 0$. On the other hand, although the state x_a^0 contributes to the controlled output h_0, its contribution can be reduced arbitrarily by the appropriate choice of the low gain feedback gain matrix $F_{a0}^0(\varepsilon)$.

To complete the proof, we will make two state transformations on the closed-loop system (A.1)–(A.7). The first state transformation is given as follows:

$$\bar{x}_{abd} = \Gamma_{abd}^{-1} x_{abd}, \quad \bar{x}_c = x_c, \tag{A.10}$$

where $x_{abd} = [(x_a^-)', (x_a^0)', (x_{abd}^+)']'$ and $\bar{x}_{abd} = [(\bar{x}_a^-)', (\bar{x}_{abd}^+)', (\bar{x}_a^0)']'$. In the new state variables (A.10), the closed-loop system becomes

$$\delta \bar{x}_a^- = A_{aa}^- \bar{x}_a^- + A_{aabd+}^- \bar{x}_{abd}^+ + B_{0a}^- F_{a0}^0(\varepsilon)\bar{x}_a^0 + E_a^- w, \tag{A.11}$$

$$\delta \bar{x}_{abd}^+ = A_{abd}^{+c} \bar{x}_{abd}^+ + [B_{0abd}^+, B_{abd}^+] F_a^0(\varepsilon)\bar{x}_a^0, \tag{A.12}$$

$$\delta \bar{x}_a^0 = (A_{aa}^0 + B_a^0 F_a^0(\varepsilon))\bar{x}_a^0 + E_a^0 w, \tag{A.13}$$

$$\delta \bar{x}_c = A_{cc}^c \bar{x}_c + A_{cabd+} \bar{x}_{abd}^+ + B_{0c} F_{a0}^0(\varepsilon)\bar{x}_a^0 + E_c w, \tag{A.14}$$

$$h_0 = [F_{a0}^+, F_{b0}, F_{d0}] x_{abd}^+ + F_{a0}^0(\varepsilon)\bar{x}_a^0, \tag{A.15}$$

$$h_b = [0_{m_b \times n_a^+}, C_b, 0_{m_b \times n_d}] x_{abd}^+, \tag{A.16}$$

$$h_d = [0_{m_b \times n_a^+}, 0_{m_b \times n_b}, C_d] x_{abd}^+, \tag{A.17}$$

where

$$A_{aabd+}^- = B_{0a}^-[F_{a0}^+ \quad F_{b0} \quad F_{d0}] + L_{ad}^-[0 \quad 0 \quad C_d] + L_{ab}^-[0 \quad C_b \quad 0]$$

and

$$A_{cabd+} = B_{0c}[F_{a0}^+ \quad F_{b0} \quad F_{d0}] + L_{cb}[0 \quad C_b \quad 0] + L_{cd}[0 \ 0 \ C_d].$$

We now proceed to construct the second transformation. We need to recall the following preliminary results from Lin (1998). It is a summary of Lemmas 2.3.2–2.3.5 of Lin (1998).

Lemma A.1. *Consider a single input pair (A, B) in the form of (56) with all eigenvalues of A on the unit circle. Let $F(\varepsilon) \in \mathbb{R}^{1 \times n}$ be the unique matrix such that $\lambda(A + BF(\varepsilon)) = (1-\varepsilon)\lambda(A)$, $\varepsilon \in (0,1]$. Then, there exists a non-singular transformation matrix $Q(\varepsilon) \in \mathbb{R}^{n \times n}$ such that:*

1. *$Q(\varepsilon)$ transforms $A + BF(\varepsilon)$ into a real Jordan form, i.e.,*

$$Q^{-1}(\varepsilon)(A + BF(\varepsilon))Q(\varepsilon) = J(\varepsilon)$$
$$:= \mathrm{blkdiag}\{J_{-1}(\varepsilon), J_{+1}(\varepsilon), J_1(\varepsilon), \ldots, J_l(\varepsilon)\}, \tag{A.18}$$

where

$$J_{-1}(\varepsilon) = \begin{bmatrix} -(1-\varepsilon) & 1 & & \\ & \ddots & \ddots & \\ & & -(1-\varepsilon) & 1 \\ & & & -(1-\varepsilon) \end{bmatrix}_{n_{-1} \times n_{-1}}, \tag{A.19}$$

$$J_{+1}(\varepsilon) = \begin{bmatrix} 1-\varepsilon & 1 & & \\ & \ddots & \ddots & \\ & & 1-\varepsilon & 1 \\ & & & 1-\varepsilon \end{bmatrix}_{n_{+1} \times n_{+1}}, \tag{A.20}$$

and for each $i = 1$ to l,

$$J_i(\varepsilon) = \begin{bmatrix} J_i^\star(\varepsilon) & I_2 & & \\ & \ddots & \ddots & \\ & & J_i^\star(\varepsilon) & I_2 \\ & & & J_i^\star(\varepsilon) \end{bmatrix}_{2n_i \times 2n_i},$$

$$J_i^\star(\varepsilon) = (1-\varepsilon)\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \tag{A.21}$$

with $\alpha_i^2 + \beta_i^2 = 1$ for all $i = 1$ to l and $\alpha_i \neq \alpha_j$ for $i \neq j$.

2. *Both* $|Q(\varepsilon)|$ and $|Q^{-1}(\varepsilon)|$ *are bounded, i.e.,*

$$|Q(\varepsilon)| \le \theta, \quad |Q^{-1}(\varepsilon)| \le \theta, \quad \varepsilon \in (0,1] \tag{A.22}$$

for some positive constant θ, independent of ε.

3. *Let $E \in \mathbb{R}^{n \times q}$ be such that* $\mathrm{Im}(E) \subset \bigcap_{w \in \lambda(A)} \mathrm{Im}(wI - A)$, *where q is any integer. Then, there exists a $\chi \ge 0$, independent of ε, such that*

$$|Q^{-1}(\varepsilon)E| \le \chi, \quad \varepsilon \in (0,1], \tag{A.23}$$

and, if we partition $Q^{-1}(\varepsilon)E$ according to that of $J(\varepsilon)$ as

$$Q^{-1}(\varepsilon)E = \begin{bmatrix} E_0(\varepsilon) \\ E_1(\varepsilon) \\ \vdots \\ E_l(\varepsilon) \end{bmatrix}, \quad E_0(\varepsilon) = \begin{bmatrix} E_{01}(\varepsilon) \\ E_{02}(\varepsilon) \\ \vdots \\ E_{0n_0}(\varepsilon) \end{bmatrix}_{n_0 \times q},$$

$$E_i(\varepsilon) = \begin{bmatrix} E_{i1}(\varepsilon) \\ E_{i2}(\varepsilon) \\ \vdots \\ E_{in_i}(\varepsilon) \end{bmatrix}_{2n_i \times q}, \tag{A.24}$$

then, there exists a $\beta \ge 0$, independent of ε, such that, for each $i = 0$ to l,

$$|E_{in_i}(\varepsilon)| \le \beta\varepsilon. \tag{A.25}$$

4. *Let* $S(\varepsilon) = \mathrm{blkdiag}\{S_{-1}(\varepsilon), S_{+1}(\varepsilon), S_1(\varepsilon), S_2(\varepsilon), \ldots, S_l(\varepsilon)\}$, *where* $S_{-1}(\varepsilon) = \mathrm{diag}\{\varepsilon^{n_{-1}-1}, \varepsilon^{n_{-1}-2}, \ldots, \varepsilon, 1\}$, $S_{+1}(\varepsilon) = \mathrm{diag}\{\varepsilon^{n_{+1}-1}, \varepsilon^{n_{+1}-2}, \ldots, \varepsilon, 1\}$, *and for each $i = 1$ to l,* $S_i(\varepsilon) = \mathrm{blkdiag}\ \{\varepsilon^{n_i-1} I_2, \varepsilon^{n_i-2} I_2, \ldots, \varepsilon I_2, I_2\}$. *Then,*
   (a) $S(\varepsilon)J(\varepsilon)S^{-1}(\varepsilon) = \tilde{J}(\varepsilon)$

$$:= \mathrm{blkdiag}\{\tilde{J}_{-1}(\varepsilon), \tilde{J}_{+1}(\varepsilon), \tilde{J}_1(\varepsilon), \ldots, \tilde{J}_l(\varepsilon)\} \tag{A.26}$$

where

$$\tilde{J}_{-1}(\varepsilon) = \begin{bmatrix} -(1-\varepsilon) & \varepsilon & & \\ & \ddots & \ddots & \\ & & -(1-\varepsilon) & \varepsilon \\ & & & -(1-\varepsilon) \end{bmatrix}_{n_{-1} \times n_{-1}}, \tag{A.27}$$

$$\tilde{J}_{+1}(\varepsilon)=\begin{bmatrix}(1-\varepsilon) & \varepsilon & & \\ & \ddots & \ddots & \\ & & (1-\varepsilon) & \varepsilon \\ & & & (1-\varepsilon)\end{bmatrix}_{n_{+1}\times n_{+1}}, \quad (A.28)$$

and for each $i = 1$ to l,

$$\tilde{J}_i(\varepsilon)=\begin{bmatrix}J_i^{\star}(\varepsilon) & \varepsilon I_2 & & \\ & \ddots & \ddots & \\ & & J_i^{\star}(\varepsilon) & \varepsilon I_2 \\ & & & J_i^{\star}(\varepsilon)\end{bmatrix}_{2n_i\times 2n_i},$$

$$J_i^{\star}(\varepsilon)=(1-\varepsilon)\begin{bmatrix}\alpha_i & \beta_i \\ -\beta_i & \alpha_i\end{bmatrix} \quad (A.29)$$

with $\beta_i > 0$ for all $i = 1$ to l and $\beta_i \neq \beta_j$ for $i \neq j$;
(b) *The unique positive-definite solution $\tilde{P}(\varepsilon)$ to the Lyapunov equation*

$$\tilde{J}'(\varepsilon)\tilde{P}\tilde{J}(\varepsilon)-\tilde{P}=-\varepsilon I \quad (A.30)$$

is bounded, i.e., there exist positive-definite matrices $\tilde{P}_1$ and $\tilde{P}_2$, independent of ε, such that

$$\tilde{P}_1 \le \tilde{P}(\varepsilon) \le \tilde{P}_2, \quad \forall \varepsilon \in (0,\varepsilon^*] \quad (A.31)$$

for some $\varepsilon^* \in (0,1]$.

5. *There exist $\alpha,\beta \ge 0$, independent of ε, such that, for all $\varepsilon \in (0,1]$,*

$$|F(\varepsilon)Q(\varepsilon)S^{-1}(\varepsilon)| \le \alpha\varepsilon, \quad (A.32)$$

$$|F(\varepsilon)AQ(\varepsilon)S^{-1}(\varepsilon)| \le \beta\varepsilon. \quad (A.33)$$

We now define the following second state transformation on the closed-loop system:

$$\tilde{x}_a^- = \bar{x}_a^-, \quad \tilde{x}_{abd}^+ = \bar{x}_{abd}^+, \quad (A.34)$$

$$\tilde{x}_a^0 = [(\tilde{x}_{a1}^0)', (\tilde{x}_{a2}^0)', \dots, (\tilde{x}_{al}^0)']' = S_a(\varepsilon)Q_a^{-1}(\varepsilon)(\Gamma_{sa}^0)^{-1}\bar{x}_a^0,$$

$$S_a(\varepsilon) = \mathrm{blkdiag}\{S_{a1}(\varepsilon), S_{a2}(\varepsilon), \dots, S_{al}(\varepsilon)\},$$

$$Q_a(\varepsilon) = \mathrm{blkdiag}\{Q_{a1}(\varepsilon), Q_{a2}(\varepsilon), \dots, Q_{al}(\varepsilon)\}, \quad (A.35)$$

$$\tilde{x}_c = \varepsilon\bar{x}_c, \quad (A.36)$$

where $Q_{ai}(\varepsilon)$ and $S_{ai}(\varepsilon)$ are the $Q(\varepsilon)$ and $S(\varepsilon)$ of Lemmas A.1 for the triple (A_i, B_i, F_i). Hence, the properties of Lemma A.1 all apply. In these new state variables, the closed-loop system becomes

$$\delta\tilde{x}_a^- = A_{aa}^-\tilde{x}_a^- + A_{aabd+}^-\tilde{x}_{abd}^+ + B_{0a}^-F_{a0}^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon)S_a^{-1}(\varepsilon)\tilde{x}_a^0 + E_a^- w, \quad (A.37)$$

$$\delta\tilde{x}_{abd}^+ = A_{abd}^{+c}\tilde{x}_{abd}^+ + [B_{0abd}^+, B_{abd}^+]F_a^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon)S_a^{-1}(\varepsilon)\tilde{x}_a^0, \quad (A.38)$$

$$\delta\tilde{x}_a^0 = \tilde{J}_a(\varepsilon)\tilde{x}_a^0 + \tilde{B}(\varepsilon)\tilde{x}_a^0 + \tilde{E}_a^0(\varepsilon)w, \quad (A.39)$$

$$\delta\tilde{x}_c = A_{cc}^c\tilde{x}_c + \varepsilon[A_{cabd+}\tilde{x}_{abd}^+ + B_{0c}F_{a0}^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon)S_a^{-1}(\varepsilon)\tilde{x}_a^0 + E_c w], \quad (A.40)$$

$$h_0 = [F_{a0}^+, F_{b0}, F_{d0}]x_{abd}^+ + F_{a0}^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon)S_a^{-1}(\varepsilon)\tilde{x}_a^0, \quad (A.41)$$

$$h_b = [0_{m_b\times n_a^+}, C_b, 0_{m_b\times n_d}]x_{abd}^+, \quad (A.42)$$

$$h_d = [0_{m_b\times n_a^+}, 0_{m_b\times n_b}, C_d]x_{abd}^+, \quad (A.43)$$

where

$$\tilde{J}_a(\varepsilon) = \mathrm{blkdiag}\{\varepsilon\tilde{J}_{a1}(\varepsilon), \varepsilon\tilde{J}_{a2}(\varepsilon), \dots, \varepsilon\tilde{J}_{al}(\varepsilon)\}, \quad (A.44)$$

$$\tilde{B}(\varepsilon)=\begin{bmatrix}0 & \tilde{B}_{12}(\varepsilon) & \tilde{B}_{13}(\varepsilon) & \cdots & \tilde{B}_{1l}(\varepsilon) \\ 0 & 0 & \tilde{B}_{23}(\varepsilon) & \cdots & \tilde{B}_{2l}(\varepsilon) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}, \quad (A.45)$$

$$\tilde{B}_{ij}(\varepsilon) = S_{ai}(\varepsilon)Q_{ai}^{-1}(\varepsilon)B_{ij}F_j(\varepsilon)Q_{aj}(\varepsilon)S_{aj}^{-1}(\varepsilon),$$
$$i = 1,2,\dots,l, \; j = i+1, i+2, \dots, l \quad (A.46)$$

$$\tilde{E}_a^0(\varepsilon) = S_a(\varepsilon)Q_a^{-1}(\varepsilon)(\Gamma_{sa}^0)^{-1}E_a^0,$$

$$\tilde{E}_a^0(\varepsilon) = [(\tilde{E}_{a1}^0(\varepsilon))' \quad (\tilde{E}_{a2}^0(\varepsilon))' \quad \cdots \quad (\tilde{E}_{al}^0(\varepsilon))']' \quad (A.47)$$

and where, for $i = 1$ to l, $\tilde{J}_{ai}(\varepsilon)$ is the $\tilde{J}(\varepsilon)$ of Lemma A.1 for the triple (A_i, B_i, F_i).

By Lemma A.1, we have that, for all $\varepsilon \in (0,1]$,

$$|F_a^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon)S_a^{-1}(\varepsilon)| \le \tilde{f}_{a0}^0 \quad (A.48)$$

for $i = 1$ to l,

$$|\tilde{E}_{ai}^0(\varepsilon)| \le \tilde{e}_a^0\varepsilon, \quad (A.49)$$

and finally, for $i = 1$ to l, $j = i + 1$ to l,

$$|\tilde{B}_{ij}(\varepsilon)| \le \tilde{b}_{ij}\varepsilon, \quad (A.50)$$

where $\tilde{f}_{a0}^0$, $\tilde{e}_a^0$ and $\tilde{b}_{ij}$ are some positive constants, independent of ε.

We next construct a Lyapunov function for the closed-loop system (A.37)–(A.43). We do this by composing Lyapunov functions for the subsystems. For the subsystem of $\tilde{x}_a^-$, we choose

$$V_a^-(\tilde{x}_a^-) = (\tilde{x}_a^-)'P_a^-\tilde{x}_a^-, \quad (A.51)$$

where $P_a^- > 0$ is the unique solution to the Lyapunov equation

$$(A_{aa}^-)'P_a^-A_{aa}^- - P_a^- = -I \quad (A.52)$$

and for the subsystem of $\tilde{x}_{abd}^+$, choose a Lyapunov function

$$V_{abd}^+(\tilde{x}_{abd}^+) = (\tilde{x}_{abd}^+)'P_{abd}^+\tilde{x}_{abd}^+, \quad (A.53)$$

where $P_{abd}^+ > 0$ is the unique solution to the Lyapunov equation

$$(A_{abd}^{+c})'P_{abd}^+A_{abd}^{+c} - P_{abd}^+ = -I. \quad (A.54)$$

The existence of such P_a^- and P_{ab}^+ is guaranteed by the fact that both A_{aa}^- and A_{abd}^{+c} are asymptotically stable. For the subsystem of $\tilde{x}_a^0 = [(\tilde{x}_{a1}^0)',(\tilde{x}_{a2}^0)',\ldots,(\tilde{x}_{al}^0)']'$, we choose

$$V_a^0(\tilde{x}_a^0) = \sum_{i=1}^{l} \frac{(\alpha_a^0)^{i-1}}{\varepsilon}(\tilde{x}_{ai}^0)' P_{ai}^0(\varepsilon)\tilde{x}_{ai}^0, \tag{A.55}$$

where α_a^0 is a positive scalar, whose value is to be determined later, and each $P_{ai}^0(\varepsilon)$ is the unique solution to the Lyapunov equation

$$\tilde{J}_{ai}(\varepsilon)' P_{ai}^0 \tilde{J}_{ai}(\varepsilon) - P_{ai}^0 = -\varepsilon I, \tag{A.56}$$

which, by Lemma A.1, satisfies,

$$P_{ai}(\varepsilon) \le \bar{P}_{ai} \tag{A.57}$$

for some $\bar{P}_{ai}$ independent of ε. Similarly, for the subsystem $\tilde{x}_c$, choose a Lyapunov function

$$V_c(\tilde{x}_c) = \tilde{x}_c' P_c \tilde{x}_c, \tag{A.58}$$

where $P_c > 0$ is the unique solution to the Lyapunov equation,

$$(A_{cc}^c)' P_c A_{cc}^c - P_c = -I. \tag{A.59}$$

The existence of such a P_c is again guaranteed by the fact that A_{cc}^c is asymptotically stable.

We now construct a Lyapunov function for the closed-loop system (A.37)–(A.43) as follows:

$$\begin{aligned} V(\tilde{x}_a^-,\tilde{x}_{abd}^+,\tilde{x}_a^0,\tilde{x}_c) = {} & V_a^-(\tilde{x}_a^-) + \alpha_{abd}^+ V_{abd}^+(\tilde{x}_{abd}^+) \\ & + V_a^0(\tilde{x}_a^0) + V_c(\tilde{x}_c), \end{aligned} \tag{A.60}$$

where $\alpha_{abd}^+ = 2|P_a^{-1}|^2|A_{aa}^-|^2 + 1$.

Let us first consider the difference of $V_a^0(\tilde{x}_a^0)$ along the trajectories of the subsystem $\tilde{x}_a^0$ and obtain that

$$\begin{aligned} \Delta V_a^0 = \sum_{i=1}^{l} \Big[& -(\alpha_a^0)^{i-1}(\tilde{x}_{ai}^0)'\tilde{x}_{ai}^0 \\ & + 2\sum_{j=i+1}^{l} \frac{(\alpha_a^0)^{i-1}}{\varepsilon}(\tilde{x}_{ai}^0)'\tilde{J}_{ai}'(\varepsilon)P_{ai}^0(\varepsilon)[\tilde{B}_{ij}(\varepsilon)\tilde{x}_{aj}^0 + \tilde{E}_{ai}^0(\varepsilon)w] \\ & + \frac{(\alpha_a^0)^{i-1}}{\varepsilon}\left(\sum_{j=i+1}^{l} \tilde{B}_{ij}(\varepsilon)\tilde{x}_{aj}^0(\varepsilon) + \tilde{E}_{ai}^0(\varepsilon)w\right)' P_{ai}^0(\varepsilon) \\ & \times\left(\sum_{j=i+1}^{l} \tilde{B}_{ij}(\varepsilon)\tilde{x}_{aj}^0(\varepsilon) + \tilde{E}_{ai}^0(\varepsilon)w\right)\Big]. \end{aligned} \tag{A.61}$$

Using (A.49), (A.50) and Lemma A.1, it is straightforward to show that, there exists an $\alpha_a^0 > 0$ such that

$$\Delta V_a^0 \le -\tfrac{3}{4}|\tilde{x}_a^0|^2 + \alpha_1 |w|^2 \tag{A.62}$$

for some nonnegative constants α_1, independent of ε.

In view of (A.62), the difference of V along the trajectory of the closed-loop system (A.37)–(A.43) can be evaluated as follows:

$$\begin{aligned} \Delta V \le {} & -(\tilde{x}_a^-)'\tilde{x}_a^- + 2(\tilde{x}_a^-)'(A_{aa}^-)'P_a^-[A_{aabd}^-(\varepsilon)\tilde{x}_{abd}^+ \\ & + B_{0a}^- F_{a0}^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon) S_a^{-1}(\varepsilon)\tilde{x}_a^0 + E_a^- w] \\ & - \alpha_{abd}^+(\tilde{x}_{abd}^+)'\tilde{x}_{abd}^+ + 2\alpha_{abd}^+(\tilde{x}_{abd}^+)'(A_{abd}^{+c})'P_{abd}^+ \\ & \times [B_{0abd}^+, B_{abd}^+] F_a^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon) S_a^{-1}(\varepsilon)\tilde{x}_a^0 - \tfrac{3}{4}|\tilde{x}_a^0|^2 \\ & + \alpha_1|w|^2 - \tilde{x}_c'\tilde{x}_c + 2\varepsilon\tilde{x}_c'(A_{cc}^{+c})'P_c[A_{cabd+}\tilde{x}_{abd}^+ \\ & + B_{0c}F_{a0}^0(\varepsilon)\Gamma_{sa}^0 Q_a(\varepsilon)S_a^{-1}(\varepsilon)\tilde{x}_a^0 + E_c w]. \end{aligned} \tag{A.63}$$

Using (A.48) and noting the definition of α_{ab}^+ (A.60), we can easily verify that, there exists an $\varepsilon_1^* \in (0,1]$ such that, for all $\varepsilon \in (0, \varepsilon_1^*]$,

$$\Delta V \le -\tfrac{1}{2}|\tilde{x}_a^-|^2 - \tfrac{1}{2}|\tilde{x}_{ab}^+|^2 - \tfrac{1}{2}|\tilde{x}_a^0|^2 - \tfrac{1}{2}|\tilde{x}_c|^2 + \alpha_2|w|^2 \tag{A.64}$$

for some positive constant α_2, independent of ε.

From (A.64), it follows that the closed-loop system in the absence of disturbance w is asymptotically stable. It remains to show that, for any given $\gamma > 0$, there exists an $\varepsilon^* \in (0, \varepsilon_1^*]$ such that, for all $\varepsilon \in (0, \varepsilon^*]$,

$$\|h\|_{l_2} \le \gamma \|w\|_{l_2}. \tag{A.65}$$

To this end, we sum both sides of (A.63) from 0 to ∞. Noting that $V(k) \ge 0$ and $V(0) = 0$, we have

$$\|\tilde{x}_a^0\|_{l_2} \le (\sqrt{2\alpha_3})\|w\|_{l_2}, \tag{A.66}$$

which, when used together with (A.48) in (A.38), results in

$$\|\tilde{x}_{abd}^+\|_{l_2} \le \alpha_3 \varepsilon \|w\|_{l_2} \tag{A.67}$$

for some positive constant α_3, independent of ε.

Finally, recalling that

$$h = \Gamma_{\mathrm{oP}}[h_0', h_d', h_b']', \tag{A.68}$$

where h_0, h_d and h_b are as defined in the closed-loop system (A.37)–(A.43), we have

$$\|h\|_{l_2} \le \alpha_4 |\Gamma_{\mathrm{oP}}| \varepsilon \|w\|_{l_2} \tag{A.69}$$

for some positive constant α_4 independent of ε.

To complete the proof, we choose $\varepsilon^* \in (0, \varepsilon_1^*]$ such that,

$$\alpha_4 |\Gamma_{\mathrm{oP}}| \varepsilon \le \gamma. \tag{A.70}$$

Finally, for the use in the proof of Theorem 4.1, it is straightforward to verify from the closed-loop system equations (A.37)–(A.43) that the transfer function from $E_a^0 w$ to h is given by

$$T_{a0}^0(z,\varepsilon) = T_{a0}(z,\varepsilon)[sI - A_{aa}^0 - B_a^0 F_a^0(\varepsilon)]^{-1}, \tag{A.71}$$

where $T_{a0}(z,\varepsilon) \to 0$ pointwise in z as $\varepsilon \to 0$. □

Appendix B — Proof of Theorem 3.2

Without loss of generality, we assume that the matrix quadruple (A, B, C_2, D_2) is in the SCB form. It is simple to verify that if condition (b) of Theorem 2.1 holds, we have

$$D_{22} + D_2 F_w = D_{22} + D_2 S - \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (B_d' B_d)^{-1} B_d' E_d \\ 0 \end{bmatrix} = 0. \tag{B.1}$$

Also, condition (c) of Theorem 2.1 implies that

$$E + BS = [(E_a^-)' \quad (E_a^0)' \quad 0 \quad 0 \quad (E_c)' \quad (B_d X_d)']' \tag{B.2}$$

with an appropriately dimensional X_d, and $E_a^0 = Y_a^0 X_a^0$, where Y_a^0 is a matrix whose columns span $\bigcap_{\alpha \in \lambda(A_{aa}^0)} \mathrm{Im}(\alpha I - A_{aa}^0)$ and X_a^0 is of appropriate dimension. It is simple to verify that

$$E + BF_w = \begin{bmatrix} E_a^- \\ E_a^0 \\ 0 \\ 0 \\ E_c \\ B_d X_d - B_d (B_d' B_d)^{-1} B_d' B_d X_d \end{bmatrix} = \begin{bmatrix} E_a^- \\ E_a^0 \\ 0 \\ 0 \\ E_c \\ 0 \end{bmatrix}. \tag{B.3}$$

Hence, $\mathrm{Im}(E + BF_w) \subset \mathscr{V}^{\odot}(\Sigma_\mathrm{P}) \cap \{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\Sigma_\mathrm{P})\}$, and the result follows from Theorem 3.1. □

Appendix C — Proof of Theorem 4.1

Let us apply a pre-output feedback law

$$u = (S + N)y + \tilde{u} \tag{C.1}$$

to system (1). We obtain another new system,

$$\begin{aligned} \delta_x &= [A + B(S+N)C_1]x + B\tilde{u} + [E + B(S+N)D_1]w, \\ y &= C_1 x + D_1 w, \\ h &= [C_2 + D_2(S+N)D_1]x + D_2 \tilde{u} + 0w. \end{aligned} \tag{C.2}$$

Clearly, it is sufficient to prove Theorem 4.1 by showing the following controller

$$\tilde{\Sigma}_{\mathrm{FC}}(\varepsilon): \quad \begin{aligned} \delta x_c &= A_{\mathrm{FC}}(\varepsilon) x_c + B_{\mathrm{FC}}(\varepsilon) y, \\ \tilde{u} &= C_{\mathrm{FC}}(\varepsilon) x_c + 0y \end{aligned} \tag{C.3}$$

with $A_{\mathrm{FC}}(\varepsilon)$, $B_{\mathrm{FC}}(\varepsilon)$ and $C_{\mathrm{FC}}(\varepsilon)$ being given as in (82), solves the H_∞-ADDPMS for (C.2). For simplicity of presentation, we denote $\tilde{\Sigma}_\mathrm{P}$ the subsystem,

$$(\tilde{A}, B, \tilde{C}_2, D_2) := (A + B(S+N)C_1, B, C_2 + D_2(S+N)C_1, D_2) \tag{C.4}$$

and denote $\tilde{\Sigma}_\mathrm{Q}$ the subsystem,

$$(\tilde{A}, \tilde{E}, C_1, D_1) := (A + B(S+N)C_1, E + B(S+N)D_1, C_1, D_1). \tag{C.5}$$

It is simple to see that $(\tilde{A}, B, C_1)$ remains stabilizable and detectable. Also, it is trivial to show the stability of the closed-loop system comprising the given plant (C.2) and the controller (C.3). The closed-loop poles are given by $\lambda\{\tilde{A} + BF_\mathrm{P}(\varepsilon)\}$, which are in $\mathbb{C}^{\odot}$ for sufficiently small ε as shown in Theorem 3.1, and $\lambda\{\tilde{A} + K_\mathrm{Q}(\varepsilon)C_1\}$, which can be dually shown to be in $\mathbb{C}^{\odot}$ for sufficiently small ε. In what follows, we will show that controller (C.3) achieves the H_∞ almost disturbance decoupling for (C.2), under all the conditions of Theorem 2.2. Following the result of Stoorvogel and van der Woude (1991) and some algebraic manipulations, one can show that conditions (d)–(f) of Theorem 2.2 are equivalent to the following conditions:

($\tilde{\mathrm{d}}$) $\mathrm{Im}(\tilde{E}) \subset \mathscr{V}^{\odot}(\tilde{\Sigma}_\mathrm{P}) \cap \{\bigcap_{|\lambda|=1} \mathscr{S}_\lambda(\tilde{\Sigma}_\mathrm{P})\}$;
($\tilde{\mathrm{e}}$) $\mathrm{Ker}(\tilde{C}_2) \supset \mathscr{S}^{\odot}(\tilde{\Sigma}_\mathrm{Q}) \cup \{\bigcup_{|\lambda|=1} \mathscr{V}_\lambda(\tilde{\Sigma}_\mathrm{Q})\}$;
($\tilde{\mathrm{f}}$) $\mathscr{S}^{\odot}(\tilde{\Sigma}_\mathrm{Q}) \subset \mathscr{V}^{\odot}(\tilde{\Sigma}_\mathrm{P})$; and
($\tilde{\mathrm{g}}$) $\tilde{A}\mathscr{S}^{\odot}(\tilde{\Sigma}_\mathrm{Q}) \subset \mathscr{V}^{\odot}(\tilde{\Sigma}_\mathrm{P})$.

Next, without of loss generality, we assume throughout the rest of the proof that the subsystem $\tilde{\Sigma}_\mathrm{P}$, i.e., the quadruple $(\tilde{A}, B, \tilde{C}_2, D_2)$, has already been transformed into the special coordinate basis as given in Theorem 2.4. To be more specific, we re-write its SCB in following compact form:

$$\tilde{A} = B_0 C_{2,0} + \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix}$$

$$:= B_0 C_{2,0} + \bar{A}, \tag{C.6}$$

$$B = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix}, \tag{C.7}$$

$$\tilde{C}_2 = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix}, \tag{C.8}$$

$$C_{2,0} = [C_{0a}^- \quad C_{0a}^0 \quad C_{0a}^+ \quad C_{0b} \quad C_{0c} \quad C_{0d}],$$

$$D_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{C.9}$$

and

$$\mathscr{V}^{\odot}(\tilde{\Sigma}_{\mathrm{P}}) = \mathrm{Im}\left\{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}\right\}. \tag{C.10}$$

It is simple to note that condition ($\tilde{\mathrm{d}}$) implies that

$$\tilde{E} = [(E_a^-)' \quad (E_a^0)' \quad 0 \quad 0 \quad (E_c)' \quad 0]'. \tag{C.11}$$

Next, for any $\zeta \in \mathscr{V}_\lambda(\tilde{\Sigma}_{\mathrm{Q}})$ with $\lambda \in \mathbb{C}^\circ$, we partition ζ as follows:

$$\zeta = [(\zeta_a^-)' \quad (\zeta_a^0)' \quad (\zeta_a^+)' \quad (\zeta_b)' \quad (\zeta_c)' \quad (\zeta_d)']'. \tag{C.12}$$

Then, condition ($\tilde{\mathrm{e}}$) implies that $\tilde{C}_2\zeta = 0$, or equivalently

$$C_{2,0}\zeta = 0, \quad C_b\zeta_b = 0 \quad \text{and} \quad C_d\zeta_d = 0. \tag{C.13}$$

By Definition 2.2, we have

$$\begin{bmatrix} \tilde{A} - \lambda I & \tilde{E} \\ C_1 & D_1 \end{bmatrix}\begin{pmatrix} \zeta \\ \eta \end{pmatrix} = 0 \tag{C.14}$$

for some appropriate vector η. Clearly, (C.14) and (C.11) imply that

$$(\tilde{A} - \lambda I)\zeta = -\tilde{E}\eta = (\bigstar \quad \bigstar \quad 0 \quad 0 \quad \bigstar \quad 0)', \tag{C.15}$$

where $\bigstar$'s are some vectors of not much interests. Note that (C.13) implies

$$(\tilde{A} - \lambda I)\zeta = (B_0 C_{2,0} + \bar{A} - \lambda I)\zeta = (\bar{A} - \lambda I)\zeta$$

$$= \begin{bmatrix} \bigstar \\ \bigstar \\ (A_{aa}^+ - \lambda I)\zeta_a^+ + L_{ab}^+ C_b\zeta_b + L_{ad}^+ C_d\zeta_d \\ (A_{bb} - \lambda I)\zeta_b + L_{bd}C_d\zeta_d \\ \bigstar \\ (A_{dd} - \lambda I)\zeta_d + B_d\zeta_x \end{bmatrix}$$

$$= \begin{bmatrix} \bigstar \\ \bigstar \\ (A_{aa}^+ - \lambda I)\zeta_a^+ \\ (A_{bb} - \lambda I)\zeta_b \\ \bigstar \\ (A_{dd} - \lambda I)\zeta_d + B_d\zeta_x \end{bmatrix}, \tag{C.16}$$

where

$$\zeta_x = E_{da}^-\zeta_a^- + E_{da}^0\zeta_a^0 + E_{da}^+\zeta_a^+ + E_{db}\zeta_b + E_{dc}\zeta_c. \tag{C.17}$$

(C.15) and (C.16) imply

$$(A_{aa}^+ - \lambda I)\zeta_a^+ = 0, \quad (A_{bb} - \lambda I)\zeta_b = 0 \tag{C.18}$$

and

$$(A_{dd} - \lambda I)\zeta_d + B_d\zeta_x = 0. \tag{C.19}$$

Since A_{aa}^+ has all its eigenvalues in $\mathbb{C}^{\otimes}$, $(A_{aa}^+ - \lambda I)\zeta_a^+ = 0$ implies that $\zeta_a^+ = 0$. Similarly, since (A_{bb}, C_b) is completely observable, $(A_{bb} - \lambda I)\zeta_b = 0$ and $C_b\zeta_b = 0$ imply $\zeta_b = 0$. Also, (C.19) and $C_d\zeta_d = 0$ imply that

$$\begin{bmatrix} A_{dd} - \lambda I & B_d \\ C_d & 0 \end{bmatrix}\begin{pmatrix} \zeta_d \\ \zeta_x \end{pmatrix} = 0. \tag{C.20}$$

Because the triple (A_{dd}, B_d, C_d) is invertible and is free of invariant zeros, (C.20) implies that $\zeta_d = 0$ and $\zeta_x = 0$. Thus, we have

$$\zeta \in \mathrm{Ker}\{B_d[E_{da}^- \quad E_{da}^0 \quad E_{da}^+ \quad E_{db} \quad E_{dc} \quad 0]\} \tag{C.21}$$

and hence

$$\mathscr{V}_\lambda(\tilde{\Sigma}_{\mathrm{Q}}) \subset \mathrm{Ker}\{B_d[E_{da}^- \quad E_{da}^0 \quad E_{da}^+ \quad E_{db} \quad E_{dc} \quad 0]\}. \tag{C.22}$$

Moreover, ζ has the following property:

$$\zeta = [(\zeta_a^-)' \quad (\zeta_a^0)' \quad 0 \quad 0 \quad (\zeta_c)' \quad 0]' \in \mathscr{V}^{\odot}(\tilde{\Sigma}_{\mathrm{P}}). \tag{C.23}$$

Obviously, (C.23) together with condition ($\tilde{\mathrm{f}}$) imply

$$\mathscr{V}^{\odot}(\tilde{\Sigma}_{\mathrm{P}}) \supset \mathscr{S}^{\odot}(\tilde{\Sigma}_{\mathrm{Q}}) \cup \left\{\bigcup_{\lambda \in \mathbb{C}s^\circ} \mathscr{V}_\lambda(\tilde{\Sigma}_{\mathrm{Q}})\right\}. \tag{C.24}$$

Similarly, for any $\xi \in \mathscr{S}^{\odot}(\tilde{\Sigma}_{\mathrm{Q}})$, conditions ($\tilde{\mathrm{e}}$) and ($\tilde{\mathrm{g}}$) imply that $\tilde{C}_2\xi = 0$ and

$$\tilde{A}\xi = (\bigstar \quad \bigstar \quad 0 \quad 0 \quad \bigstar \quad 0)'. \tag{C.25}$$

Now, it is straightforward to show that

$$\xi \in \mathrm{Ker}\{B_d[E_{da}^- \quad E_{da}^0 \quad E_{da}^+ \quad E_{db} \quad E_{dc} \quad 0]\} \tag{C.26}$$

and hence

$$\mathscr{S}^{\odot}(\tilde{\Sigma}_{\mathrm{Q}}) \subset \mathrm{Ker}\{B_d[E_{da}^- \quad E_{da}^0 \quad E_{da}^+ \quad E_{db} \quad E_{dc} \quad 0]\}. \tag{C.27}$$

Eqs. (C.22) and (C.27) imply that

$$\mathrm{Ker}\{B_d[E_{da}^- \quad E_{da}^0 \quad E_{da}^+ \quad E_{db} \quad E_{dc} \quad 0]\} \supset \mathscr{S}^{\odot}(\tilde{\Sigma}_{\mathrm{Q}}) \cup \left\{ \bigcup_{\lambda \in \mathbb{C}^{\circ}} \mathscr{V}_{\lambda}(\tilde{\Sigma}_{\mathrm{Q}}) \right\}. \tag{C.28}$$

Next, we partition $\tilde{A} - zI$ as follows,

$$\tilde{A} - zI = X_1 + X_2 C_2 + X_3 + X_4 + X_5, \tag{C.29}$$

where

$$X_1 := \begin{bmatrix} A_{aa}^- - zI & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} - zI & L_{cd} C_d \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C.30}$$

$$X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{aa}^+ - zI & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{bb} - zI & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{dd} - zI \end{bmatrix}, \tag{C.31}$$

$$X_2 = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & L_{ad}^0 & L_{ab}^0 \\ B_{0a}^+ & L_{ad}^+ & L_{ab}^+ \\ B_{0b} & L_{bd} & 0 \\ B_{0c} & 0 & 0 \\ B_{0d} & 0 & 0 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{aa}^0 - zI & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{C.32}$$

and

$$X_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & 0 \end{bmatrix}. \tag{C.33}$$

It is simple to see that

$$\mathrm{Im}(X_1) \subset \mathscr{V}^{\odot}(\tilde{\Sigma}_{\mathrm{P}}) \cap \left\{ \bigcap_{|\lambda|=1} \mathscr{S}_{\lambda}(\tilde{\Sigma}_{\mathrm{P}}) \right\}, \tag{C.34}$$

$$\mathrm{Ker}(X_3) \supset \mathscr{V}^{\odot}(\tilde{\Sigma}_{\mathrm{P}}) \supset \mathscr{S}^{\odot}(\tilde{\Sigma}_{\mathrm{Q}}) \cup \left\{ \bigcup_{|\lambda|=1} \mathscr{V}_{\lambda}(\tilde{\Sigma}_{\mathrm{Q}}) \right\}. \tag{C.35}$$

Also, (C.28) implies that

$$\mathrm{Ker}(X_5) \supset \mathscr{S}^{\odot}(\tilde{\Sigma}_{\mathrm{Q}}) \cup \left\{ \bigcup_{|\lambda|=1} \mathscr{V}_{\lambda}(\tilde{\Sigma}_{\mathrm{Q}}) \right\}. \tag{C.36}$$

It follows from the proof of Theorem 3.1 that as $\varepsilon \to 0$

$$\|[\tilde{C}_2 + D_2 F_{\mathrm{P}}(\varepsilon)][zI - \tilde{A} - BF_{\mathrm{P}}(\varepsilon)]^{-1}\|_\infty < \kappa_{\mathrm{P}}, \tag{C.37}$$

where κ_{P} is a finite positive constant independent of ε. Moreover, under condition ($\tilde{\mathrm{d}}$), we have

$$[\tilde{C}_2 + D_2 F_{\mathrm{P}}(\varepsilon)][zI - \tilde{A} - BF_{\mathrm{P}}(\varepsilon)]^{-1}\tilde{E} \to 0, \tag{C.38}$$

and

$$[\tilde{C}_2 + D_2 F_{\mathrm{P}}(\varepsilon)][zI - \tilde{A} - BF_{\mathrm{P}}(\varepsilon)]^{-1}X_1 \to 0, \tag{C.39}$$

pointwise in z as $\varepsilon \to 0$. By (A.70), we have

$$[\tilde{C}_2 + D_2 F_{\mathrm{P}}(\varepsilon)][zI - \tilde{A} - BF_{\mathrm{P}}(\varepsilon)]^{-1}X_4 \to 0, \tag{C.40}$$

pointwise in z as $\varepsilon \to 0$. Dually, one can show that

$$\|[zI - \tilde{A} - K_{\mathrm{Q}}(\varepsilon)C_1]^{-1}[\tilde{E} + K_{\mathrm{Q}}(\varepsilon)D_1]\|_\infty < \kappa_{\mathrm{Q}}, \tag{C.41}$$

where κ_{Q} is a finite positive constant independent of ε. If condition ($\tilde{\mathrm{e}}$) is satisfied, the following results hold:

$$\tilde{C}_2[zI - \tilde{A} - K_{\mathrm{Q}}(\varepsilon)C_1]^{-1}[\tilde{E} + K_{\mathrm{Q}}(\varepsilon)D_1] \to 0, \tag{C.42}$$

$$X_3[zI - \tilde{A} - K_{\mathrm{Q}}(\varepsilon)C_1]^{-1}[\tilde{E} + K_{\mathrm{Q}}(\varepsilon)D_1] \to 0 \tag{C.43}$$

and

$$X_5[zI - \tilde{A} - K_{\mathrm{Q}}(\varepsilon)C_1]^{-1}[\tilde{E} + K_{\mathrm{Q}}(\varepsilon)D_1] \to 0, \tag{C.44}$$

pointwise in z as $\varepsilon \to 0$.

Finally, it is simple to verify that the transfer matrix from the disturbance w to the controlled output h of the closed-loop system comprising system (C.2) and controller (C.3) is given by

$$\begin{aligned} T_{hw}(z,\varepsilon) = {} & [\tilde{C}_2 + D_2 F_{\mathrm{P}}(\varepsilon)][zI - \tilde{A} - BF_{\mathrm{P}}(\varepsilon)]^{-1}\tilde{E} \\ & + \tilde{C}_2[zI - \tilde{A} - K_{\mathrm{Q}}(\varepsilon)C_1]^{-1}[\tilde{E} + K_{\mathrm{Q}}(\varepsilon)D_1] \\ & + [\tilde{C}_2 + D_2 F_{\mathrm{P}}(\varepsilon)][zI - \tilde{A} - BF_{\mathrm{P}}(\varepsilon)]^{-1} \\ & \times (\tilde{A} - zI)[zI - \tilde{A} - K_{\mathrm{Q}}(\varepsilon)C_1]^{-1}[\tilde{E} + K_{\mathrm{Q}}(\varepsilon)D_1]. \end{aligned}$$

Using (C.29), we can re-write $T_{hw}(z,\varepsilon)$ as

$$\begin{aligned} T_{hw}(z,\varepsilon) = {} & [\tilde{C}_2 + D_2 F_{\mathrm{P}}(\varepsilon)][zI - \tilde{A} - BF_{\mathrm{P}}(\varepsilon)]^{-1}\tilde{E} \\ & + \tilde{C}_2[zI - \tilde{A} - K_{\mathrm{Q}}(\varepsilon)C_1]^{-1}[\tilde{E} + K_{\mathrm{Q}}(\varepsilon)D_1] \end{aligned}$$

$$+\,[\tilde{C}_2 + D_2F_P(\varepsilon)][zI - \tilde{A} - BF_P(\varepsilon)]^{-1} \times (X_1 + X_2C_2 + X_3 + X_4 + X_5) \times [zI - \tilde{A} - K_Q(\varepsilon)C_1]^{-1}[\tilde{E} + K_Q(\varepsilon)D_1].$$

Following (C.37)–(C.44), and some simple manipulations, it is straightforward to show that as $\varepsilon \to 0$, $T_{hw}(z,\varepsilon) \to 0$, pointwise in z, which is equivalent to $\|T_{hw}\|_\infty \to 0$ as $\varepsilon \to 0$. Hence, the full order output feedback controller (81) solves the H_∞-ADDPMS for the given plant (1), provided that all the conditions of Theorem 2.2 are satisfied. □

Appendix D — Proof of Theorem 4.2

It is sufficient to show Theorem 4.2 by showing that the following controller:

$$\tilde{\Sigma}_{RC}(\varepsilon)\colon \begin{array}{l} \delta x_c = A_{RC}(\varepsilon)x_c + B_{RC}(\varepsilon)y, \\ \tilde{u} = C_{RC}(\varepsilon)x_c + \tilde{D}_{RC}(\varepsilon)y \end{array} \tag{D.1}$$

with $A_{RC}(\varepsilon)$, $B_{RC}(\varepsilon)$, $C_{RC}(\varepsilon)$ being given as in (95), and

$$\tilde{D}_{RC}(\varepsilon) = [0, \quad F_{P1}(\varepsilon) - F_{P2}(\varepsilon)K_{R1}(\varepsilon)], \tag{D.2}$$

solves the H_∞-ADDPMS for (C.2).

Again, it is trivial to show the stability of the closed-loop system comprising with (C.2) and the controller (D.1) as the closed-loop poles are given by $\lambda\{\tilde{A} + BF_P(\varepsilon)\}$ and $\lambda\{A_R + K_R(\varepsilon)C_R\}$, which are asymptotically stable for sufficiently small ε. Next, it is easy to compute the corresponding closed-loop transfer matrix from the disturbance w to the controlled output h,

$$\begin{aligned} T_{hw}(z,\varepsilon) = {} & \tilde{C}_2\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}[zI - A_R - K_R(\varepsilon)C_R]^{-1} \\ & \times [E_R + K_R(\varepsilon)D_R] + [\tilde{C}_2 + D_2F_P(\varepsilon)] \\ & \times [zI - \tilde{A} - BF_P(\varepsilon)]^{-1}(\tilde{A} - zI)\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix} \\ & \times [zI - A_R - K_R(\varepsilon)C_R]^{-1}[E_R + K_R(\varepsilon)D_R] \\ & + [\tilde{C}_2 + D_2F_P(\varepsilon)][zI - \tilde{A} - BF_P(\varepsilon)]^{-1}\tilde{E}. \end{aligned}$$

Following the result of Chen (1991) (i.e., Proposition 2.2.1), one can show that

$$\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}\mathscr{S}^{\odot}(\Sigma_{QR}) = \mathscr{S}^{\odot}(\tilde{\Sigma}_Q) \cap C_1^{-1}\{\mathrm{Im}(D_1)\} \tag{D.3}$$

and

$$\begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}\bigcup_{|\lambda|=1}\mathscr{V}_\lambda(\Sigma_{QR}) = \bigcup_{|\lambda|=1}\mathscr{V}_\lambda(\tilde{\Sigma}_Q). \tag{D.4}$$

Hence, we have

$$\begin{aligned} & \begin{pmatrix} 0 \\ I_{n-k} \end{pmatrix}\left(\mathscr{S}^{\odot}(\Sigma_{QR}) \cup \left\{\bigcup_{|\lambda|=1}\mathscr{V}_\lambda(\Sigma_{QR})\right\}\right) \\ & = \{\mathscr{S}^{\odot}(\tilde{\Sigma}_Q) \cap C_1^{-1}\{\mathrm{Im}(D_1)\}\} \cup \left\{\bigcup_{|\lambda|=1}\mathscr{V}_\lambda(\tilde{\Sigma}_Q)\right\} \\ & \subset \mathscr{S}^{\odot}(\tilde{\Sigma}_Q) \cup \left\{\bigcup_{|\lambda|=1}\mathscr{V}_\lambda(\tilde{\Sigma}_Q)\right\}. \end{aligned} \tag{D.5}$$

The rest of the proof follows from the same lines as those of Theorem 4.1. □

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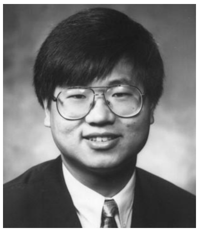

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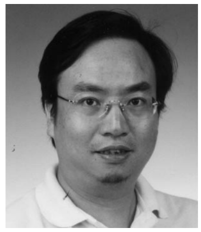

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