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Brief Paper

Solutions to disturbance decoupling problem with constant measurement feedback for linear systems☆

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Abstract

We study in this paper the problem of disturbance decoupling with constant (i.e., static) measurement feedback (DDPCM) for linear systems. For a class of systems which have a left invertible transfer function from the control input to the controlled output or a right invertible transfer function from the disturbance input to the measurement output, we obtain a complete characterization of all solutions to the DDPCM. For a system that does not satisfy the above invertibility condition, we use the special co-ordinate basis to obtain a reduced-order system. Then a complete characterization of all possible solutions to the DDPCM for the given system can be explicitly obtained, if the obtained reduced-order system itself satisfies the invertibility condition. The main advantage of these solutions is that the solutions are given in a set of linear equations. This resolves the well known difficulty in solving non-linear equations associated with the DDPCM.

Keywords: Linear systems; Disturbance decoupling; Static measurement feedback control

1. Introduction

The problem of disturbance decoupling with or without internal stability by either state feedback or measurement feedback is well known and has been extensively studied for the last three decades (see e.g., Basile & Marro, 1968; Wonham, 1979; Akashi & Imai, 1979; Schumacher, 1980; Imai & Akashi, 1981; Willems & Commault, 1981). It actually motivated the development of the geometric approach to linear systems, and has played a key role in a number of problems, such as decentralized control, non-interacting control, model reference tracking control, and H_∞ optimal control. For the problem of disturbance decoupling with constant or static measurement feedback (DDPCM), there have been only a few results in the literature. Hamano and Furuta (1975) formulated the problem as finding a geometric subspace that only covers some special solutions. Recently, Chen (1997) obtained a set of constructive conditions for the solvability of the DDPCM and characterized all the possible solutions for a class of systems which have a left invertible transfer function from the control input to the controlled output. A similar result for this class of system has also been reported by Koumboulis and Tzierakis (1998).

Consider the following linear-time-invariant system Σ:

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew, \\ y &= C_1 x + D_1 w, \\ z &= C_2 x + D_2 u + D_{22} w, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measured output, $w \in \mathbb{R}^q$ is the disturbance, $z \in \mathbb{R}^p$ is the controlled output, and $A, B, E, C_1, D_1, C_2, D_2$ and D_{22} are constant matrices of appropriate dimensions. Define Σ_P and Σ_Q respectively as the quadruples characterized by (A, B, C_2, D_2) and (A, E, C_1, D_1). Then, the DDPCM is to find a constant measurement feedback control law

$$u = Ky, \tag{2}$$

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with $K \in \mathbb{R}^{m\times \ell}$, such that the transfer function $H_{zw}(s)$ from w to z of the closed-loop system is zero, i.e.,

$$H_{zw}(s) = (C_2 + D_2KC_1)(sI - A - BKC_1)^{-1}(E + BKD_1) + (D_{22} + D_2KD_1) = 0. \tag{3}$$

Furthermore, the problem of disturbance decoupling with constant measurement feedback and with internal stability (DDPCMS) is to find a constant measurement feedback control in the form (2) such that (3) is satisfied and the closed loop system state matrix $A + BKC_1$ is stable.

The main contribution of this paper is a characterization of all solutions to the DDPCM for a class of linear systems by a set of linear equations. This resolves the difficulty in solving the non-linear equations associated with the DDPCM. The solutions to the DDPCM which are characterized by a set of linear equations require that Σ_P is left invertible or Σ_Q is right invertible. For systems which do not satisfy this condition, we use the special coordinate basis to obtain an irreducible reduced-order system. Then, the DDPCM can also be solved from a set of linear equations if the obtained reduced-order system itself satisfies the above invertibility condition. For systems whose reduced-order system does not satisfy the invertibility condition, the reduced-order system also simplifies the solution to the DDPCM.

Throughout this paper, $X^\dagger$ denotes the pseudo-inverse of X, $\mathrm{Ker}(X)$ denotes the kernel of X, $\mathrm{Im}(X)$ denotes the image of X, $\langle A|\mathscr{X}\rangle$ denotes the smallest A-invariant subspace containing $\mathscr{X}$ which itself is a subspace, and $C^{-1}\{\mathscr{X}\} := \{x \mid Cx \in \mathscr{X}\}$, where $\mathscr{X}$ is a subspace and C is a constant matrix. We also use the following definitions of geometric subspaces: Given an nth-order system Σ characterized by (A, B, C, D), we define (1) the weakly unobservable subspaces of Σ, $\mathscr{V}^*(\Sigma)$, to be the maximal subspace of $\mathbb{R}^n$ which is $(A + BF)$-invariant and is contained in $\mathrm{Ker}(C + DF)$ such that the eigenvalues of $(A + BF)|\mathscr{V}^*$ are contained in $\mathbb{C}$ for some constant matrix F; and (2) the strongly controllable subspaces of Σ, $\mathscr{S}^*(\Sigma)$, to be the minimal $(A + KC)$-invariant subspace of $\mathbb{R}^n$ containing $\mathrm{Im}(B + KD)$ such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/\mathscr{S}^*$ are contained in $\mathbb{C}$ for some constant matrix K.

2. Solvability conditions for DDPCM and characterizations of its solutions

This section presents the main results of the paper. For clarity of presentation, we provide the proofs of the main results in appendices. The following two theorems give the necessary conditions for the existence of solutions to the DDPCM, which can be derived from the result of Stoorvogel and van der Woude (1991). Due to space limitation, the proofs of these two theorems are omitted.

Theorem 1. *Consider the given system* Σ *of* (1). *If the problem of disturbance decoupling with constant measurement feedback* (*DDPCM*) *for* Σ *is solvable, then* Σ *must satisfy the following conditions*:

(i) $D_{22} + D_2SD_1 = 0$, *where* $S := -(D_2'D_2)^\dagger D_2'D_{22}D_1'(D_1D_1')^\dagger$;
(ii) $\mathrm{Im}(E + BSD_1) \subseteq \mathscr{V}^*(\Sigma_P) + B\mathrm{Ker}(D_2)$;
(iii) $\mathrm{Ker}(C_2 + D_2SC_1) \supseteq \mathscr{S}^*(\Sigma_Q) \cap C_1^{-1}\{\mathrm{Im}(D_1)\}$;
(iv) $\mathscr{S}^*(\Sigma_Q) \subseteq \mathscr{V}^*(\Sigma_P)$.

Theorem 2. *Consider the system* Σ *in* (1). *Let* X *and* Y *be any full rank constant matrices such that* $\mathrm{Ker}(X) = \mathscr{V}^*(\Sigma_P)$ *and* $\mathrm{Im}(Y) = \mathscr{S}^*(\Sigma_Q)$. *If the problem of disturbance decoupling with constant measurement feedback* (*DDPCM*) *for* Σ *is solvable, then the following equation has at least one solution* N,

$$\begin{bmatrix} XB \\ D_2 \end{bmatrix} N [C_1Y \quad D_1] + \begin{bmatrix} XAY & XE \\ C_2Y & D_{22} \end{bmatrix} = 0. \tag{4}$$

Let $\mathscr{N}$ *be the set of all the solutions of* (4). *Then, any constant measurement feedback law* $u = Ky$, *which solves the DDPCM for* Σ, *satisfies* $K \in \mathscr{N}$, *i.e.,* K *is a solution of* (4).

We now use these results to present a necessary and sufficient condition for the solvability of the DDPCM and the DDPCMS for a special class of systems.

Corollary 3. *Consider the given system* Σ *of* (1). *Assume that both* $[C_1 \quad D_1]$ *and* $[B' \quad D_2']$ *are full rank matrices,* Σ_P *is left invertible and* Σ_Q *is right invertible. Then, the problem of disturbance decoupling with constant measurement feedback* (*DDPCM*) *for* Σ *is solvable if and only if*

$$(C_2 + D_2NC_1)(sI - A - BNC_1)^{-1}(E + BND_1) + (D_{22} + D_2ND_1) \equiv 0, \tag{5}$$

where N *is a known constant matrix and is given by*

$$N = -(B'X'XB + D_2'D_2)^{-1}[B'X' \ D_2'] \begin{bmatrix} XAY & XE \\ C_2Y & D_{22} \end{bmatrix} \times \begin{bmatrix} Y'C_1' \\ D_1' \end{bmatrix} (C_1YY'C_1' + D_1D_1')^{-1}. \tag{6}$$

Also, the problem of disturbance decoupling with constant measurement feedback and with internal stability (*DDPCMS*) *for* Σ *is solvable if and only if* (5) *holds, and* $A + BNC_1$ *is stable. Furthermore, both solutions to the*

DDPCM and DDPCMS for the given Σ, if existent, are identical. They are uniquely given by $u = Ny$.

The proof of the above corollary follows from the properties of $\mathscr{V}^*$ and $\mathscr{S}^*$, and some simple algebras. Next, we will proceed to tackle the case when a given system does not satisfy the conditions posed in Corollary 3. We will partition the given system Σ of (1) into some subsystems using the special coordinate basis of Sannuti and Saberi (1987) (see also Chen (1998) for the detailed proofs of its properties). From now on, we will assume that the necessary conditions for the solvability of the DDPCM in Theorem 1 are satisfied. The following is a step-by-step algorithm to obtain a reduced-order system, which can be used to simplify the solution to the DDPCM.

Step 1: Compute

$$N = -(B'X'XB + D_2'D_2)^{\dagger}[B'X' \ \ D_2']\begin{bmatrix} XAY & XE \\ C_2Y & D_{22}\end{bmatrix} \times \begin{bmatrix} Y'C_1' \\ D_1'\end{bmatrix}(C_1YY'C_1' + D_1D_1')^{\dagger}, \tag{7}$$

and then apply a pre-output feedback $u = Ny + v$ to the given system Σ, which yields the following new system,

$$\begin{aligned} \dot{x} &= (A + BNC_1)x + Bv + (E + BND_1)w, \\ y &= C_1x + D_1w, \\ z &= (C_2 + D_2NC_1)x + D_2v + 0w. \end{aligned} \tag{8}$$

Furthermore, it follows from the proof of Theorem 2 that $\mathrm{Im}(E + BND_1) \subseteq \mathscr{V}^*(\Sigma_{\mathrm{P}})$.

Step 2: Find a non-singular transformation Γ_m such that

$$y = \Gamma_m y_m = \Gamma_m\begin{pmatrix} y_0 \\ y_1\end{pmatrix}, \qquad C_{1m} := \Gamma_m^{-1}C_1 = \begin{bmatrix} C_{1,0} \\ C_{1,1}\end{bmatrix},$$

$$D_{1m} := \Gamma_m^{-1}D_1 = \begin{bmatrix} D_{1,0} \\ 0\end{bmatrix}, \tag{9}$$

where $D_{1,0}$ is of maximal row rank.

Step 3: Utilize the special coordinate basis of Sannuti and Saberi (1987) (see also Chen, 1998) to find the non-singular transformations Γ_s, Γ_i and Γ_o, i.e., let

$$x = \Gamma_s\begin{pmatrix} x_c \\ x_a \\ x_b \\ x_d\end{pmatrix}, \quad v = \Gamma_i\begin{pmatrix} v_0 \\ v_d \\ v_c \\ v_*\end{pmatrix}, \quad z = \Gamma_o\begin{pmatrix} z_0 \\ z_d \\ z_b \\ z_*\end{pmatrix}, \tag{10}$$

which yields the following transformed system

$$\left\{\begin{aligned}
\begin{pmatrix} \dot{x}_a \\ \dot{x}_c \\ \dot{x}_b \\ \dot{x}_d\end{pmatrix} &= \left(\begin{bmatrix} A_{cc} & B_cE_{ca} & L_{cb}C_b & L_{cd}C_d \\ 0 & A_{aa} & L_{ab}C_b & L_{ad}C_d \\ 0 & 0 & A_{bb} & L_{bd}C_d \\ B_dE_{dc} & B_dE_{da} & B_dE_{db} & A_{dd}\end{bmatrix} + A_0\right)\begin{pmatrix} x_c \\ x_a \\ x_b \\ x_d\end{pmatrix} \\
&\quad + \begin{bmatrix} B_{c0} & 0 & B_c & 0 \\ B_{a0} & 0 & 0 & 0 \\ B_{b0} & 0 & 0 & 0 \\ B_{d0} & B_d & 0 & 0\end{bmatrix}\begin{pmatrix} v_0 \\ v_d \\ v_c \\ v_*\end{pmatrix} + \begin{bmatrix} E_c \\ E_a \\ E_b \\ E_d\end{bmatrix}w \\
y_m &= \begin{bmatrix} C_{1,0c} & C_{1,0a} & C_{1,0b} & C_{1,0d} \\ C_{1,1c} & C_{1,1a} & C_{1,1b} & C_{1,1d}\end{bmatrix}\begin{pmatrix} x_c \\ x_a \\ x_b \\ x_d\end{pmatrix} + \begin{bmatrix} D_{1,0} \\ 0\end{bmatrix}w \\
\begin{pmatrix} z_0 \\ z_b \\ z_d \\ z_*\end{pmatrix} &= \begin{bmatrix} C_{0c} & C_{0a} & C_{0b} & C_{0d} \\ 0 & 0 & 0 & C_d \\ 0 & 0 & C_b & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix} x_c \\ x_a \\ x_b \\ x_d\end{pmatrix} \\
&\quad + \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix} v_0 \\ v_d \\ v_c \\ v_*\end{pmatrix}
\end{aligned}\right\} \tag{11}$$

where

$$A_0 := \begin{bmatrix} B_{c0} \\ B_{a0} \\ B_{b0} \\ B_{d0}\end{bmatrix}[C_{0c} \quad C_{0a} \quad C_{0b} \quad C_{0d}],$$

$$\begin{bmatrix} E_c \\ E_a \\ E_b \\ E_d\end{bmatrix} := \Gamma_s^{-1}(E + BND_1), \tag{12}$$

and

$$[C_{1c} \quad C_{1a} \quad C_{1b} \quad C_{1d}] := C_1\Gamma_s. \tag{13}$$

Let Σ_{N} be characterized by $(A+BNC_1, B, C_2+DNC_1, D_2)$. We further note that the decomposition in (11) has the following properties: The pair (A_{cc}, B_c) is completely controllable and Σ_{N} is left invertible if x_c is non-existent; (A_{bb}, C_b) is completely observable and Σ_{N} is right invertible if x_c is non-existent; Σ_{N} is invertible if both x_c and x_b are non-existent; (A_{dd}, B_d, C_d) is square invertible and is free of invariant zeros; the eigenvalues of A_{aa} are the invariant zeros of Σ_N; and finally,

$$\mathscr{V}^*(\Sigma_N) = \mathrm{Im}\left\{\Gamma_s \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

and

$$\mathscr{S}^*(\Sigma_N) = \mathrm{Im}\left\{\Gamma_s \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}\right\}. \tag{14}$$

It is simple to verify that under the conditions of Theorem 1, we have $E_b = 0$, $E_d = 0$. Moreover, the DDPCM for (1) is equivalent to that for the transformed system (11).

Step 4: Let Γ_a be a non-singular transformation such that

$$\Gamma_a^{-1} A_{aa} \Gamma_a = \begin{bmatrix} A_{aa}^{cc} & A_{aa}^{c\bar{c}} \\ 0 & A_{aa}^{\bar{c}\bar{c}} \end{bmatrix}, \qquad \Gamma_a^{-1} E_a = \begin{bmatrix} E_a^c \\ 0 \end{bmatrix},$$

$$\Gamma_a^{-1} B_{a0} = \begin{bmatrix} B_{a0}^c \\ B_{a0}^{\bar{c}} \end{bmatrix}, \tag{15}$$

$$\Gamma_a^{-1} L_{ab} = \begin{bmatrix} L_{ab}^c \\ L_{ab}^{\bar{c}} \end{bmatrix}, \qquad \Gamma_a^{-1} L_{ad} = \begin{bmatrix} L_{ad}^c \\ L_{ad}^{\bar{c}} \end{bmatrix},$$

$$\begin{bmatrix} C_{1,0a} \\ C_{1,1a} \end{bmatrix} \Gamma_a = \begin{bmatrix} C_{1,0a}^c & C_{1,0a}^{\bar{c}} \\ C_{1,1a}^c & C_{1,1a}^{\bar{c}} \end{bmatrix}, \tag{16}$$

and

$$C_{0a}\Gamma_a = [C_{0a}^c \quad C_{0a}^{\bar{c}}], \qquad E_{da}\Gamma_a = [E_{da}^c \quad E_{da}^{\bar{c}}],$$
$$E_{ca}\Gamma_a = [E_{ca}^c \quad E_{ca}^{\bar{c}}] \tag{17}$$

where (A_{aa}^{cc}, E_a^c) is controllable.

Step 5: Define a reduced-order auxiliary system Σ_R as follows:

$$\begin{aligned} \dot{x}_R &= A_R x_R + B_R u_R + E_R w, \\ y_R &= C_{1R} x_R + D_{1R} w, \\ z_R &= C_{2R} x_R + D_{2R} u_R, \end{aligned} \tag{18}$$

where

$$A_R = \begin{bmatrix} A_{cc} & B_c E_{ca}^c \\ 0 & A_{aa}^{cc} \end{bmatrix} + \begin{bmatrix} B_{c0} \\ B_{a0}^c \end{bmatrix} [C_{0c} \quad C_{0a}^c], \quad E_R = \begin{bmatrix} E_c \\ E_a^c \end{bmatrix}, \tag{19}$$

$$B_R = \begin{bmatrix} B_{c0} & 0 & B_c & 0 \\ B_{a0}^c & 0 & 0 & 0 \end{bmatrix} \Gamma_i^{-1},$$

$$D_{2R} = \begin{bmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & I_{m_d} & 0 & 0 \end{bmatrix} \Gamma_i^{-1}, \tag{20}$$

and

$$C_{1R} = \Gamma_m \begin{bmatrix} C_{1,0c} & C_{1,0a}^c \\ C_{1,1c} & C_{1,1a}^c \end{bmatrix}, \qquad D_{1R} = \Gamma_m \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix},$$

$$C_{2R} = \begin{bmatrix} C_{0c} & C_{0a}^c \\ E_{dc} & E_{da}^c. \end{bmatrix}. \tag{21}$$

This concludes the algorithm.

Let n_x be the dimension of the space spanned by x_R. Apparently, n_x can be in general considerably smaller than n the dimension of the original system (1). Furthermore, it is simple to see that $(A_R, B_R, C_{2R}, D_{2R})$ is right invertible without infinite zeros.

For the given system Σ of (1) and the reduced-order system Σ_R of (18), we define

$$\mathscr{K} := \{K \mid u = Ky \text{ solves the DDPCM for } \Sigma\}, \tag{22}$$

and

$$\mathscr{K}_R := \{K_R \mid u_R = K_R y_R \text{ solves the DDPCM for } \Sigma_R\}. \tag{23}$$

We now establish an equivalence between the DDPCM for the original system Σ in (1) and that for the reduced-order system Σ_R in the following theorem.

Theorem 4. *Consider the given system* Σ *of* (1). *Assume that conditions* 1–4 *of Theorem* 1 *are satisfied. Then, we have*

$$\mathscr{K} = \{K_R + N \mid K_R \in \mathscr{K}_R\}, \tag{24}$$

where N is given by (7). *Thus, the solvability of the DDPCM for* Σ *of* (1) *and that for* Σ_R *of* (18) *are equivalent.*

Proof. See Appendix A.

The following corollaries deal with some special cases for which we are able to obtain complete solutions for the DDPCM. The proofs of these corollaries follow the lines of that given in Chen (1997).

Corollary 5. *Consider the given system* Σ (1) *with* (A, B, C_2, D_2) *or* Σ_P *being left invertible. Then, the problem of disturbance decoupling with constant measurement feedback is solvable for* Σ *if and only if conditions* 1–4 *of Theorem* 1 *are satisfied and*

$$\mathrm{Ker}(C_{1,1a}^c) \subset \mathrm{Ker}\left\{\begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix}\right\}, \tag{25}$$

and all solutions to the DDPCM for this class of systems are characterized by

$$\mathscr{K} := \left\{ \Gamma_i \begin{bmatrix} 0 & K_{01} \\ 0 & K_{d1} \\ K_{*0} & K_{*1} \end{bmatrix} \Gamma_m^{-1} \middle| K \in \mathbb{R}^{m \times \ell}, \right.$$
$$\left. K_{*0}, K_{*1} \text{ are free and } \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} + \begin{bmatrix} K_{01} \\ K_{d1} \end{bmatrix} C_{1,1a}^c = 0 \right\}. \quad (26)$$

We note that for the case when Σ_P is left invertible, the necessary and sufficient conditions for the solvability of DDPCM can be simplified as follows:

$$E_c = \emptyset, \quad E_b = 0, \quad E_d = 0,$$
$$\operatorname{Ker}(C_{1,1a}^c) \subset \operatorname{Ker}\left\{ \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} \right\}, \quad (27)$$

where E_c, E_b and E_d are as defined in (12). These conditions are quite transparent and can be easily verified using the special coordinate basis decomposition.

Next, recall the given system Σ in (1). We define a transposed system of Σ as

$$\dot{x} = A'x + C_1'u + C_2'w,$$
$$y = B'x + D_2'w,$$
$$z = E'x + D_1'u + D_{22}'w. \quad (28)$$

It is apparent that the DDPCM for Σ is solvable if and only if the DDPCM for the above transposed system is solvable. Furthermore, if Σ_Q is right invertible, then the transposed system satisfies the condition of Corollary 5. Thus, we have the following corollary.

Corollary 6. *Consider the given system* Σ (1). *If* Σ_Q *is right invertible, then the set of all possible solutions to the DDPCM for* Σ *can be obtained by applying the result of Corollary* 5 *to the transposed system* (28).

From Corollaries 5 and 6, we see how to solve the DDPCM for Σ when either Σ_P is left invertible or Σ_Q is right invertible or both. It is very interesting to note that the solutions can be solved from a set of linear equations in the form (26). Thus, the solutions can be easily computed. We now further tackle the case when Σ_P is not left invertible and Σ_Q is not right invertible. For this case, we use the following algorithm to obtain an irreducible reduced-order system, which considerably simplifies the solution to the DDPCM. The basic idea is as follows: it is clear from Theorem 4 that the DDPCM for the original system is equivalent to that for a much smaller dimensional auxiliary system Σ_R, which is taken from a subset in $\mathscr{V}^*$ of the original system. We then dualize this auxiliary system and apply a similar reduction on it to obtain a new auxiliary system whose dynamical order is further reduced. We keep repeating this process until we reach a system, which is irreducible. We have the following step-by-step algorithm.

Step A: For the given system Σ, whose Σ_P is not left invertible and Σ_Q is not right invertible, we apply Steps 1–5 of the previous algorithm to obtain a constant matrix N and a reduced-order system Σ_R. Let $\Sigma_{\alpha,R} := \Sigma_R$ and $N_\alpha := N$ with $\alpha = 1$. Furthermore, we append a subscript α to all the matrices of $\Sigma_{\alpha,R}$.

Step B: For $\Sigma_{\alpha,R}$, we define an auxiliary system $\Sigma_{\alpha,R}^\star$ as follows:

$$\dot{x}_{\alpha,R} = A_{\alpha,R}'x_{\alpha,R} + C_{\alpha,1R}'u_{\alpha,R} + C_{\alpha,2R}'w_{\alpha,R},$$
$$y_{\alpha,R} = B_{\alpha,R}'x_{\alpha,R} + D_{\alpha,2R}'w_{\alpha,R},$$
$$z_{\alpha,R} = E_{\alpha,R}'x_{\alpha,R} + D_{\alpha,1R}'u_{\alpha,R}. \quad (29)$$

If the above system $\Sigma_{\alpha,R}^\star$ does not satisfy conditions 1–4 of Theorem 1, then the DDPCM for Σ has no solution and the procedure stops. If the above system $\Sigma_{\alpha,R}^\star$ cannot be further reduced, we let $\tilde{\alpha} := \alpha$, $\tilde{n}_x$ be the dynamic order of $\Sigma_{\alpha,R}$ and stop the algorithm. Otherwise, go to Step C.

Step C: Apply Steps 1–5 of the previous algorithm to $\Sigma_{\alpha,R}^\star$ to find another matrix N (rename it as $N_{\alpha+1}$ for future use) and another reduced-order system, say $\Sigma_{\alpha+1,R}$, characterized by

$$\dot{x}_{\alpha+1,R} = A_{\alpha+1,R}x_{\alpha+1,R} + B_{\alpha+1,R}u_{\alpha+1,R} + E_{\alpha+1,R}w_{\alpha+1,R},$$
$$y_{\alpha+1,R} = C_{\alpha+1,1R}x_{\alpha+1,R} + D_{\alpha+1,1R}w_{\alpha+1,R},$$
$$z_{\alpha+1,R} = C_{\alpha+1,2R}x_{\alpha+1,R} + D_{\alpha+1,2R}u_{\alpha+1,R}. \quad (30)$$

If $(A_{\alpha+1,R}, B_{\alpha+1,R}, C_{\alpha+1,2R}, D_{\alpha+1,2R})$, which is always right invertible, is also invertible, we let $\tilde{\alpha} := \alpha + 1$ and stop the algorithm. Otherwise, let $\alpha := \alpha + 1$ and then go back to Step B. This concludes the algorithm.

Consider the given system (1) with Σ_P being not left invertible and Σ_Q being not right invertible, and assume that conditions 1–4 of Theorem 1 are satisfied. We use the results of Theorem 4, Corollaries 5 and 6 to obtain the following theorem.

Theorem 7. *If the quadruple* $(A_{\tilde{\alpha},R}, B_{\tilde{\alpha},R}, C_{\tilde{\alpha},2R}, D_{\tilde{\alpha},2R})$ *is invertible, then the DDPCM for* Σ *can be solved using the result of Corollary* 5. *Specifically, if we let* $\mathscr{K}_{\tilde{\alpha},R}$ *be the set of all solutions to the DDPCM to* $\Sigma_{\tilde{\alpha},R}$, *then all the solutions to the DDPCM for* Σ *are given by*

$$\mathscr{K} = \{K_{\tilde{\alpha},R} + N_1 + N_2' + \cdots + N_{\tilde{\alpha}} \mid K_{\tilde{\alpha},R} \in \mathscr{K}_{\tilde{\alpha},R}\} \quad (31)$$

if $\tilde{\alpha}$ *is an odd integer, or*

$$\mathscr{K} = \{K_{\tilde{\alpha},R}' + N_1 + N_2' + \cdots + N_{\tilde{\alpha}}' \mid K_{\tilde{\alpha},R} \in \mathscr{K}_{\tilde{\alpha},R}\} \quad (32)$$

if $\tilde{\alpha}$ *is an even integer. Obviously, if* $\mathscr{K}_{\tilde{\alpha},R}$ *is empty,* $\mathscr{K}$ *is empty, i.e., the DDPCM to* Σ *has no solutions at all.*

Finally, we note that for the case where $(A_{\tilde{x},\mathrm{R}}, B_{\tilde{x},\mathrm{R}}, C_{\tilde{x},2\mathrm{R}}, D_{\tilde{x},2\mathrm{R}})$ is not invertible, in principle, solutions to the DDPCM can be carried out through the use of QEPCAD in Collins (1996). This is a finite step computation problem, but the emerging conditions could be hard to interpret. If we are interested in a purely numerical characterization, then the method of Grobner bases combined with QEPCAD may be applied to the reduced-order system to find all possible solutions for DDPCM and DDPCMS (see, e.g., Cox, Little & O'Shea, 1992).

3. Conclusion

We have studied in this paper the problem of disturbance decoupling with constant measurement feedback (DDPCM) for linear systems, and obtained solutions to the DDPCM for a class of systems which can be explicitly solved from a set of linear equations. The solutions are firstly obtained for systems which have a left invertible transfer function from the control input to the controlled output or a right invertible transfer function from the disturbance input to the measurement output. For systems that do not satisfy the above invertibility property, we have used the special coordinate basis to obtain a reduced-order auxiliary system. The solutions to the DDPCM for the original system can then be generated from a set of linear equations, if the obtained reduced-order system itself satisfies the invertibility condition.

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Appendix A. Proof of Theorem 4

Without loss of generality but for simplicity of presentation, we assume that the non-singular transformations $\Gamma_s = I, \Gamma_i = I, \Gamma_o = I$ and $\Gamma_m = I$ as all of them do not affect the solutions to the DDPCM at all. We will prove the theorem in two stages:

Stage 1: Assume that the feedback control $u_\mathrm{R} = K_\mathrm{R} y_\mathrm{R}$ is a solution for the DDPCM of the system Σ_R. Let K_R be partitioned as follows:

$$K_\mathrm{R} = \begin{bmatrix} K_{00} & K_{01} \\ K_{d0} & K_{d1} \\ K_{c0} & K_{c1} \\ K_{*0} & K_{*1} \end{bmatrix}. \tag{A.1}$$

Then, $K_\mathrm{R} \in \mathscr{K}_\mathrm{R}$ implies that $D_{2\mathrm{R}} K_\mathrm{R} D_{1\mathrm{R}} = 0$, which implies that $K_{00} = 0$ and $K_{d0} = 0$. Thus, we have

$$A_{\mathrm{R}x} := A_\mathrm{R} + B_\mathrm{R} K_\mathrm{R} C_{1\mathrm{R}} = \begin{bmatrix} X_1 & X_2 \\ B^c_{a0}(C_{0c} + K_{01} C_{1,1c}) & A^{cc}_{aa} + B^c_{a0}(C^c_{0a} + K_{01} C^c_{1,1a}) \end{bmatrix}, \tag{A.2}$$

where

$$X_1 := A_{cc} + B_c(K_{c0} C_{1,0c} + K_{c1} C_{1,1c}) + B_{c0}(C_{0c} + K_{01} C_{1,1c}), \tag{A.3}$$

$$X_2 := B_{c0}(C^c_{0a} + K_{01} C^c_{1,1a}) + B_c(E^c_{ca} + K_{c0} C^c_{1,0a} + K_{c1} C^c_{1,1a}), \tag{A.4}$$

$$E_{\mathrm{R}x} := E_\mathrm{R} + B_\mathrm{R} K_\mathrm{R} D_{1\mathrm{R}} = \begin{bmatrix} E_c + B_c K_{c0} D_{1,0} \\ 0 \end{bmatrix}, \tag{A.5}$$

and

$$C_{\mathrm{R}x} := \begin{bmatrix} C_{\mathrm{R}x0} \\ C_{\mathrm{R}xd} \end{bmatrix} := C_{2\mathrm{R}} + D_{2\mathrm{R}} K_\mathrm{R} C_{1\mathrm{R}} = \begin{bmatrix} C_{0c} + K_{01} C_{1,1c} & C^c_{0a} + K_{01} C^c_{1,1a} \\ E_{dc} + K_{d1} C_{1,1c} & E^c_{da} + K_{d1} C^c_{1,1a} \end{bmatrix}. \tag{A.6}$$

Note that $K_\mathrm{R} \in \mathscr{K}_\mathrm{R}$ implies that $C_{\mathrm{R}x}(sI - A_{\mathrm{R}x})^{-1} E_{\mathrm{R}x} \equiv 0$, for all s, or equivalently,

$$C_{\mathrm{R}x} A^i_{\mathrm{R}x} E_{\mathrm{R}x} = \begin{bmatrix} C_{\mathrm{R}x0} \\ C_{\mathrm{R}xd} \end{bmatrix} A^i_{\mathrm{R}x} E_\mathrm{R} = 0, \tag{A.7}$$

for $i = 0, 1, \ldots, n-1$. Since $E_b = 0$ and $E_d = 0$, we have

$$E_x =: E + BND_1 + BK_\mathrm{R} D_1 = \begin{bmatrix} E_{\mathrm{R}x} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A.8}$$

For system (11), we apply the constant measurement feedback control $u = K_\mathrm{R} y$ to obtain

$$A_x := A + BNC_1 + BK_\mathrm{R} C_1 = \begin{bmatrix} A_{\mathrm{R}x} & \star & \star & \star \\ B^{\bar{c}}_{a0} C_{\mathrm{R}x0} & \star & \star & \star \\ B_{b0} C_{\mathrm{R}x0} & \star & \star & \star \\ B_{d0} C_{\mathrm{R}x0} + B_d C_{\mathrm{R}xd} & \star & \star & \star \end{bmatrix} \tag{A.9}$$

and

$$C_x := C_2 + D_2 N C_1 + D_2 K_\mathrm{R} C_1 = \begin{bmatrix} C_{\mathrm{R}x0} & \star & \star & \star \\ 0 & \star & \star & \star \end{bmatrix}, \tag{A.10}$$

where ★'s are some matrices of not much interest. It is now straightforward to verify that

$$C_x E_x = \begin{bmatrix} C_{\mathrm{R}x0} E_{\mathrm{R}x} \\ 0 \end{bmatrix} = 0, \tag{A.11}$$

$$C_x A_x E_x = C_x \begin{bmatrix} A_{\mathrm{R}x} E_{\mathrm{R}x} \\ B_{a0}^{\bar{c}} C_{\mathrm{R}x0} E_{\mathrm{R}x} \\ B_{b0} C_{\mathrm{R}x0} E_{\mathrm{R}x} \\ B_{d0} C_{\mathrm{R}x0} E_{\mathrm{R}} + B_d C_{\mathrm{R}xd} E_{\mathrm{R}x} \end{bmatrix}$$

$$= C_x \begin{bmatrix} A_{\mathrm{R}x} E_{\mathrm{R}x} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_{\mathrm{R}x0} A_{\mathrm{R}x} E_{\mathrm{R}x} \\ 0 \end{bmatrix} = 0. \tag{A.12}$$

It follows that $C_x A_x^i E_x = 0$, for all $i = 2, \ldots, n-1$. Thus, $C_x(sI - A_x)^{-1} E_x \equiv 0$, for all s. Moreover, it is simple to check that $D_x := D_{22} + D_2(N + K_{\mathrm{R}})D_1 = 0$. Hence, $u = (N + K_{\mathrm{R}})y$ is a solution for the DDPCM of the original system Σ and $N + K_{\mathrm{R}}$ is an element of $\mathscr{K}$ or equivalently

$$\{K_{\mathrm{R}} + N \,|\, K_{\mathrm{R}} \in \mathscr{K}_{\mathrm{R}}\} \subseteq \mathscr{K}. \tag{A.13}$$

Stage 2: Suppose that $u = Ky$ is a solution for the DDPCM of the original system Σ. It follows from (3) that

$$D_{22} + D_2 K D_1 = 0. \tag{A.14}$$

Next, define the smallest $(A + BKC_1)$-invariant subspace containing $\mathrm{Im}(E + BKD_1)$ as

$$\mathscr{W} := \langle A + BKC_1 \,|\, \mathrm{Im}(E + BKD_1) \rangle. \tag{A.15}$$

We note that this subspace $\mathscr{W}$ is well defined as both $A + BKC_1$ and $E + BKD_1$ are constant matrices. Then Eqs. (3) and (A.14) imply that $\mathscr{W} \subset \mathrm{Ker}(C_2 + D_2 K C_1)$ and by definition

$$\mathscr{W} \subset \mathscr{V}^*(\Sigma_{\mathrm{P}}) = \mathscr{V}^*(\Sigma_{\mathrm{N}}) = \mathrm{span}\left\{ \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}. \tag{A.16}$$

Hence, there exists a similarity transformation T such that

$$T^{-1}(A + BKC_1)T = \begin{bmatrix} A^{cc} & A^{c\bar{c}} \\ 0 & A^{\bar{c}\bar{c}} \end{bmatrix},$$

$$T^{-1}(E + BKD_1) = \begin{bmatrix} E^c \\ 0 \end{bmatrix}, \tag{A.17}$$

and

$$(C_2 + D_2 K C_1)T = [0 \;\, C^{\bar{c}}], \quad \mathscr{W} = \mathrm{span}\left\{ T \begin{bmatrix} I \\ 0 \end{bmatrix} \right\}, \tag{A.18}$$

where (A^{cc}, E^c) is controllable. It is now straightforward to verify that T can be chosen in the following form:

$$T = \begin{bmatrix} T_* & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{A.19}$$

where T_* is of dimension $n_x \times n_x$. Next, we let $K_{\mathrm{R}} := K - N$. Clearly, K_{R} is a solution to the DDPCM for the system in (11). We further partition the subsystem associated with x_a into the form (15)–(17), and partition K_{R} as

$$K_{\mathrm{R}} = \begin{bmatrix} K_{00} & K_{01} \\ K_{d0} & K_{d1} \\ K_{c0} & K_{c1} \\ K_{*0} & K_{*1} \end{bmatrix}. \tag{A.20}$$

It is simple to show that $D_{22} + D_2 K D_1 = 0$ implies $K_{00} = 0$ and $\mathrm{Im}(E + BKD_1) \subseteq \mathscr{V}^*(\Sigma_{\mathrm{P}})$ implies $K_{d0} = 0$. Thus, we have $D_{2\mathrm{R}} K_{\mathrm{R}} D_{1\mathrm{R}} = 0$. Also, (A.17)–(A.20) imply that

$$T_*^{-1} A_{\mathrm{R}x} T_* = \begin{bmatrix} A^{cc} & A_*^{c\bar{c}} \\ 0 & A_*^{\bar{c}\bar{c}} \end{bmatrix}, \quad T_*^{-1} E_{\mathrm{R}x} = \begin{bmatrix} E^c \\ 0 \end{bmatrix}, \tag{A.21}$$

and

$$\begin{bmatrix} C_{\mathrm{R}x0} \\ B_d C_{\mathrm{R}xd} \end{bmatrix} T_* = \begin{bmatrix} 0 & \bigstar \\ 0 & \bigstar \end{bmatrix}, \tag{A.22}$$

where $A_{\mathrm{R}x}, E_{\mathrm{R}x}, C_{\mathrm{R}x0}$ and $C_{\mathrm{R}xd}$ are as defined in (A.2)–(A.6), and a ★ again denotes a matrix of not much interest. Since (A_{dd}, B_d, C_d) is invertible, which implies that B_d is of full column rank, Eq. (A.22) is equivalent to

$$C_{\mathrm{R}x} T_* = \begin{bmatrix} C_{\mathrm{R}x0} \\ C_{\mathrm{R}xd} \end{bmatrix} T_* = [0 \;\, \bigstar]. \tag{A.23}$$

Eqs. (A.21) and (A.23) together yield $C_{\mathrm{R}x}(sI - A_{\mathrm{R}x})^{-1} E_{\mathrm{R}x} \equiv 0$, for all s. Hence, $u_{\mathrm{R}} = K_{\mathrm{R}} y_{\mathrm{R}}$ is a solution for the DDPCM of the reduced-order system Σ_{R}, which implies that

$$\mathscr{K} \subseteq \{K_{\mathrm{R}} + N \,|\, K_{\mathrm{R}} \in \mathscr{K}_{\mathrm{R}}\}. \tag{A.24}$$

Eqs. (A.13) and (A.24) imply $\mathscr{K} = \{K_{\mathrm{R}} + N \,|\, K_{\mathrm{R}} \in \mathscr{K}_{\mathrm{R}}\}$.

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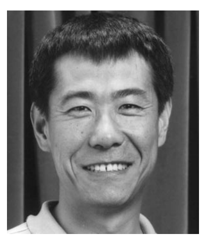

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