



Mappings of the Finite and Infinite Zero Structures and Invertibility Structures of General Linear Multivariable Systems Under the Bilinear Transformation*

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Key Words—Linear system theory; continuous-time systems; discrete-time systems; bilinear transformation.

Abstract—This paper presents a comprehensive picture of the mapping of structural properties associated with general linear multivariable systems under bilinear transformation. While the mapping of poles of linear multivariable systems under such a transformation is well known, the question of how the structural invariant properties of a given system are mapped remains unanswered. This paper addresses this question. More specifically, we investigate how the finite and infinite zero structures, as well as invertibility structures, of a general continuous-time (discrete-time) linear time-invariant multivariable system are mapped to those of its discrete-time (continuous-time) counterpart under the bilinear (inverse bilinear) transformation. We demonstrate that the structural invariant indices lists \mathcal{S}_2 and \mathcal{S}_3 of Morse remain invariant under the bilinear transformation, while the structural invariant indices lists \mathcal{S}_1 and \mathcal{S}_4 of Morse are, in general, changed. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION AND PROBLEM STATEMENT

The need to perform continuous-time to discrete-time model conversions arises in a range of engineering contexts, including sampled-data control system design, and digital signal processing. As a consequence, numerous discretization procedures exist, including zero- and first-order hold input approximation, impulse invariant transformation, and bilinear transformation (see, for example Åström *et al.*, 1984; Franklin *et al.*, 1980). Despite of the widespread use of the bilinear transform, however, a comprehensive treatment is lacking which details how key structural properties of continuous-time systems, such as the finite and infinite zero structures, and invertibility properties, are inherited by their discrete-time counterparts. Given

the important role played by the infinite and finite zero structures in control system design, a clear understanding of the zero structures under bilinear transformation would be useful in the design of sampled-data control systems, and would complement existing results on the mapping of finite and infinite zero structures under zero-order hold sampling (see, for example, Åström *et al.*, 1984; Grizzle and Shor, 1988). In this paper, we present a comprehensive study of how the structures, i.e. the finite and infinite zero structures, as well as invertibility structures, of a general continuous-time (discrete-time) linear time-invariant system are mapped to those of its discrete-time (continuous-time) counterpart under the well-known bilinear (inverse bilinear) transformations

$$s = a \left(\frac{z-1}{z+1} \right) \quad \text{and} \quad z = \frac{a+s}{a-s}, \quad (1)$$

respectively. Since the bilinear and inverse bilinear transformations find widespread use in digital control and signal processing, the results obtained have immediate applications. In particular, some of the results have already been applied to solve discrete-time algebraic Riccati equations (see for example Chen *et al.*, 1994), and in the solution of certain H_∞ control problems (see for example, Chen, 1996; and Chen *et al.*, 1996).

We consider in this paper general linear time-invariant systems characterized by

$$\Sigma: \begin{cases} \delta x = Ax + Bu, \\ y = Cx + Du, \end{cases} \quad (2)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and A , B , C and D are matrices of appropriate dimensions. In equation (2), δ is an operator defined as follows: $\delta x = \dot{x}$ if Σ is a continuous-time system, and $\delta x = x(k+1)$ if Σ is a discrete-time system. Without loss of any generality, we assume throughout this paper that both matrices $[C \ D]$ and $[B' \ D']$ are of full rank.

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The paper is organized as follows: In Section 2, we recall the special coordinate basis of linear systems, which is instrumental to the development and derivation of the main results. Section 3 contains the main results of this paper, namely, in Section 3.1 we present the mapping of finite and infinite zero structures of a continuous-time linear time-invariant system to those of its discrete-time counterpart under the bilinear transformation, while in Section 3.2, we give corresponding results for the mapping from discrete time to continuous time. Section 4 gives the detailed proof of our main results. Finally, concluding remarks are made in Section 5.

Throughout this paper, X' denotes the transpose of matrix X . I denotes an identity matrix of appropriate dimension, while I_k denotes a $k \times k$ identity matrix. \mathbb{R} and \mathbb{C} are, respectively, the sets of real and complex numbers. $\lambda(X)$ is the set of eigenvalues of a real square matrix X .

2. BACKGROUND MATERIALS

In this section, we recall the special coordinate basis of linear systems from Sannuti and Saberi (1987) and Saberi and Sannuti (1990), which is instrumental to the main results developed in subsequent sections. We will also recall some basic concepts of finite and infinite zero structures of linear time-invariant systems. First, it can be easily shown using the singular-value decomposition that one can always find nonsingular matrices U and V that put the direct feedthrough matrix D of equation (2) into the following form:

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \quad (3)$$

where m_0 is the rank of D . It is well known that these nonsingular transformations V and U leave the finite and infinite zero structures of Σ unaffected. Hence, without loss of generality, one can assume that the matrix D in equation (2) has the form as shown in equation (3). Thus, the system in equation (2) can be rewritten as

$$\begin{aligned} \delta &= Ax + [B_0 \quad B_1] \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}. \end{aligned} \quad (4)$$

where B_0 , B_1 , C_0 and C_1 are the matrices of appropriate dimensions. We have the following theorem.

Theorem 2.1. For any given system Σ of equation (2), there exist

- (1) coordinate-free nonnegative integers $n_a, n_b, n_c, n_d, m_d \leq m - m_0$ and $q_i, i = 1, 2, \dots, m_d$;
- (2) nonsingular state, output, and input transformations Γ_s, Γ_o and Γ_i which take the given Σ into a special coordinate basis that explicitly displays both finite and infinite zero structures of Σ .

The special coordinate basis is described by the following set of equations:

$$u = \Gamma_i \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \quad x = \Gamma_s \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad y = \Gamma_o \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix}, \quad (5)$$

$$x_d = \begin{pmatrix} x_1 \\ \vdots \\ x_{m_d} \end{pmatrix}, \quad u_d = \begin{pmatrix} u_1 \\ \vdots \\ u_{m_d} \end{pmatrix}, \quad y_d = \begin{pmatrix} y_1 \\ \vdots \\ y_{m_d} \end{pmatrix} \quad (6)$$

and

$$\delta x_a = A_{aa}x_a + B_{a0}y_0 + L_{ad}y_d + L_{ab}y_b, \quad (7)$$

$$\delta x_b = A_{bb}x_b + B_{b0}y_0 + L_{bd}y_d, \quad y_b = C_b x_b, \quad (8)$$

$$\begin{aligned} \delta x_c &= A_{cc}x_c + B_{c0}y_0 + L_{cd}y_d + L_{cb}y_b \\ &\quad + B_c E_{ca}x_a + B_c u_c, \end{aligned} \quad (9)$$

$$y_0 = C_{0a}x_a + C_{0b}x_b + C_{0c}x_c + C_{0d}x_d + u_0 \quad (10)$$

and for $i = 1, 2, \dots, m_d$,

$$\begin{aligned} \delta x_i &= A_{q_i}x_i + L_{i0}y_0 + L_{id}y_d \\ &\quad + B_{q_i}[u_i + E_{ia}x_a + E_{ib}x_b + E_{ic}x_c + E_{id}x_d], \end{aligned} \quad (11)$$

$$y_i = C_{q_i}x_i, \quad y_d = C_d x_d. \quad (12)$$

Here the states x_a, x_b, x_c and x_d are, respectively, of dimensions n_a, n_b, n_c and $n_d = \sum_{i=1}^{m_d} q_i$, while x_i is of dimension q_i for each $i = 1, 2, \dots, m_d$. The control vectors u_0, u_d and u_c are, respectively, of dimensions m_0, m_d and $m_c = m - m_0 - m_d$ while the output vectors y_0, y_d and y_b are, respectively, of dimensions m_0, m_d and $p_b = p - m_0 - m_d$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:

$$\begin{aligned} A_{q_i} &= \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_{q_i} &= [1 \quad 0 \quad \dots \quad 0]. \end{aligned} \quad (13)$$

Furthermore, the pair (A_{c_i}, B_c) is controllable and the pair (A_{b_i}, C_b) is observable.

Proof. The original work of the special coordinate basis was reported in Sannuti and Saberi (1987) and

Saberi and Sannuti (1990). Software packages that realize the above special coordinate basis are available either in LAS by Chen (1988) or in MATLAB by Lin (1989). \square

For future use, we rewrite the special coordinate basis given in Theorem 2.1 in a more compact form as a system characterized by the quadruple $(A_{SCB}, B_{SCB}, C_{SCB}, D_{SCB})$, where

$$A_{SCB} := \Gamma_s^{-1}(A - B_0 C_0) \Gamma_s = \begin{bmatrix} A_{aa} & L_{ab} C_b & 0 & L_{ad} C_d \\ 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca} & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da} & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix}, \quad (14)$$

$$B_{SCB} := \Gamma_s^{-1} [B_0 \quad B_1] \Gamma_i = \begin{bmatrix} B_{a0} & 0 & 0 \\ B_{b0} & 0 & 0 \\ B_{c0} & 0 & B_c \\ B_{d0} & B_d & 0 \end{bmatrix} \quad (15)$$

and

$$C_{SCB} := \Gamma_o^{-1} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \Gamma_s = \begin{bmatrix} C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & C_d \\ 0 & C_s & 0 & 0 \end{bmatrix},$$

$$D_{SCB} := \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (16)$$

In what follows, we recall some properties of the special coordinate basis that are pertinent to the present work. The first property we will recall deals with the invariant zeros and their structures of the given systems (see also MacFarlane and Karcnias, 1976; Saberi *et al.*, 1991; for details).

Property 2.1. Invariant zeros of Σ are the eigenvalues of A_{aa} , i.e. $\lambda(A_{aa})$.

In order to display various multiplicities of invariant zeros, let X_a be a nonsingular transformation matrix such that

$$X_a^{-1} A_{aa} X_a = J = \text{Block diag}\{J_1, J_2, \dots, J_k\}, \quad (17)$$

where J_i , $i = 1, 2, \dots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_i = \text{diag}\{\alpha_i, \alpha_i, \dots, \alpha_i\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}. \quad (18)$$

For any given $\alpha \in \lambda(A_{aa})$, let there be τ_α Jordan blocks of A_{aa} associated with α . Let $n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}$ be the dimensions of the corresponding Jordan blocks. Then we say α is an invariant zero of Σ with multiplicity structure $S_\alpha^*(\Sigma)$ (see also Saberi *et al.*, 1991),

$$S_\alpha^*(\Sigma) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}. \quad (19)$$

The geometric multiplicity of α is then simply given by τ_α , and the algebraic multiplicity of α is given by $\sum_{i=1}^{\tau_\alpha} n_{\alpha,i}$. Here we should note that the invariant zeros together with their structures of Σ are related to the structural invariant indices list $\mathcal{I}_1(\Sigma)$ of Morse (1973).

The next property concerns the infinite structure of the given system, which is related to the structural invariant indices list $\mathcal{I}_4(\Sigma)$ of Morse (see, for example, Commault and Dion, 1982; Morse, 1973; Owens, 1978; Saberi *et al.*, 1995; for relevant definitions).

Property 2.2. Σ has $m_0 = \text{rank}(D)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ is given by

$$S_\infty^*(\Sigma) = \{q_1, q_2, \dots, q_{m_0}\}. \quad (20)$$

That is, each q_i corresponds to an infinite zero of Σ of order q_i .

Here we would note that the index S_∞^* of equation (20) is not exactly the same as \mathcal{I}_4 of Morse. However, they are related and can be converted to one another in a trivial manner (see Saberi *et al.*, 1995).

The special coordinate basis can also exhibit the invertibility structure of a given system Σ . The definitions of right and left invertibility of a linear system can be found in Moylan (1977).

Property 2.3. The given system Σ is right invertible if and only if x_b (and hence y_b) are non-existent, left invertible if and only if x_c (and hence u_c) are non-existent, and invertible if and only if both x_b and x_c are non-existent. Moreover, Σ is degenerate if and only if it is neither left nor right invertible, i.e., both x_b and x_c are present.

The special coordinate basis can also be modified to obtain the structural invariant indices lists \mathcal{I}_2 and \mathcal{I}_3 of Morse (1973) of the given system Σ . In order to display $\mathcal{I}_2(\Sigma)$, we let X_c and X_i be nonsingular matrices such that the controllable pair (A_{cc}, B_c) is transformed into Brunovsky

canonical form, i.e.

$$X_c^{-1}A_{cc}X_c = \begin{bmatrix} 0 & I_{l_1-1} & \cdots & 0 & 0 \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{l_{m_c}-1} \\ * & * & \cdots & * & * \end{bmatrix},$$

$$X_c^{-1}B_cX_i = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \quad (21)$$

where *'s denote constant scalars or row vectors. Then we have

$$\mathcal{J}_2(\Sigma) = \{\ell_1, \dots, \ell_{m_c}\} \quad (22)$$

which is also called the controllability index of (A_{cc}, B_c) . Similarly, we have

$$\mathcal{J}_3(\Sigma) = \{\mu_1, \dots, \mu_{p_b}\}, \quad (23)$$

where $\{\mu_1, \dots, \mu_{p_b}\}$ is the controllability index of the controllable pair (A_{bb}, C'_b) .

3. A COMPLETE STRUCTURAL PICTURE OF THE BILINEAR TRANSFORMATION

We present our main results in this section. More specifically, in Section 3.1, we study how the finite and infinite zero structures, as well as invertibility properties, of a continuous-time linear time-invariant system are mapped to those of its discrete-time counterpart under bilinear transformation. In Section 3.2, we present corresponding results for the mapping from a discrete time to continuous time. Two examples are included to illustrate the results obtained in this section. The detailed proof of our main results, i.e. Theorem 3.1, will be given in the next section for the sake of clarity.

3.1. From continuous time to discrete time

The bilinear transformation has been widely applied to solve problems in the areas of signal processing, communications and control system design. It is therefore important to study its structural behavior in order to have a clear understanding of this technique. In this section, we will consider a continuous-time linear time-invariant system Σ_c characterized by

$$\Sigma_c: \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \quad (24)$$

where Σ_c has transfer function

$$G_c(s) = C(sI - A)^{-1}B + D. \quad (25)$$

Let us apply a bilinear transformation to the above continuous-time system, by replacing s in equation (25) with

$$s = \frac{2}{T} \left(\frac{z-1}{z+1} \right) = a \left(\frac{z-1}{z+1} \right), \quad (26)$$

where $T = 2/a$ is the sampling period. As presented in equation (26), the bilinear transformation is often called Tustin's approximation, while the choice

$$a = \frac{\omega_1}{\tan(\omega_1 T/2)} \quad (27)$$

yields the pre-warped Tustin approximation, in which the frequency responses of the continuous-time system and its discrete-time counterpart are matched at frequency ω_1 . In this way, we obtain a discrete-time system

$$G_d(z) = C \left(a \frac{z-1}{z+1} I - A \right)^{-1} B + D. \quad (28)$$

The following lemma provides a direct state-space realization of $G_d(z)$. Its proof is straightforward and hence is omitted.

Lemma 3.1. A state-space realization of $G_d(z)$, the discrete-time counterpart of the continuous-time system Σ_c of equation (24) under the bilinear transformation (26), is given by

$$\Sigma_d: \begin{cases} x(k+1) = \tilde{A}x(k) + \tilde{B}u(k), \\ y(k) = \tilde{C}x(k) + \tilde{D}u(k), \end{cases} \quad (29)$$

where

$$\begin{aligned} \tilde{A} &= (aI + A)(aI - A)^{-1}, \\ \tilde{B} &= \sqrt{2a}(aI - A)^{-1}B, \\ \tilde{C} &= \sqrt{2a}C(aI - A)^{-1}, \\ \tilde{D} &= D + C(aI - A)^{-1}B. \end{aligned} \quad (30)$$

Here we clearly assume that matrix A has no eigenvalues at a . \square

The following theorem establishes the interconnection of the structural properties of Σ_c and Σ_d , and forms the major contribution of this paper.

Theorem 3.1. Consider the continuous-time system Σ_c of equation (24) characterized by the quadruple (A, B, C, D) with matrix A having no eigenvalues at a , and its discrete-time counterpart under the bilinear transformation (26), i.e. Σ_d of equation (24)

characterized by the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. We have the following properties:

- (1) Controllability (stabilizability) and observability (detectability) of Σ_d :
 - (a) Σ_d is controllable (stabilizable) if and only if Σ_c is controllable (stabilizable).
 - (b) Σ_d is observable (detectable) if and only if Σ_c is observable (detectable).
- (2) Effects of nonsingular state, output and input transformations, together with state feedback and output injection laws:
 - (a) For any given nonsingular state, output and input transformations T_s, T_o and T_i , the quadruple $(T_s^{-1}\tilde{A}T_s, T_s^{-1}\tilde{B}T_i, T_o^{-1}\tilde{C}T_s, T_o^{-1}\tilde{D}T_i)$, is the discrete-time counterpart of the continuous time system $(T_s^{-1}AT_s, T_s^{-1}BT_i, T_o^{-1}CT_s, T_o^{-1}DT_i)$ under the transformation (26).
 - (b) For any $F \in \mathbb{R}^{m \times n}$ with $A + BF$ having no eigenvalues at a , define a nonsingular matrix

$$\begin{aligned} \tilde{T}_i &:= I + F(aI - A - BF)^{-1}B \\ &= [I - F(aI - A)^{-1}B]^{-1} \in \mathbb{R}^{m \times m}. \end{aligned} \quad (31)$$

and a constant matrix

$$\tilde{F} := \sqrt{2a}F(aI - A - BF)^{-1} \in \mathbb{R}^{m \times n}. \quad (32)$$

Then a continuous-time system Σ_{cF} characterized by $(A + BF, B, C + DF, D)$ is mapped to a discrete-time system Σ_{dF} , characterized by $(\tilde{A} + \tilde{B}\tilde{F}, \tilde{B}\tilde{T}_i, \tilde{C} + \tilde{D}\tilde{F}, \tilde{D}\tilde{T}_i)$, under the bilinear transformation (26).

- (c) For any $K \in \mathbb{R}^{n \times p}$ with $A + KC$ having no eigenvalues at a , define a nonsingular matrix

$$\tilde{T}_o := [I + C(aI - A - KC)^{-1}K]^{-1} \in \mathbb{R}^{p \times p} \quad (33)$$

and a constant matrix

$$\tilde{K} := \sqrt{2a}(aI - A - KC)^{-1}K. \quad (34)$$

Then a continuous-time system Σ_{cK} characterized by $(A + KC, B + KD, C, D)$ is mapped to a discrete-time system Σ_{dK} , characterized by $(\tilde{A} + \tilde{K}\tilde{C}, \tilde{B} + \tilde{K}\tilde{D}, \tilde{T}_o^{-1}\tilde{C}, \tilde{T}_o^{-1}\tilde{D})$, under the bilinear transformation (26).

- (3) Invertibility and structural invariant indices lists \mathcal{I}_2 and \mathcal{I}_3 of Σ_d :
 - (a) $\mathcal{I}_2(\Sigma_d) = \mathcal{I}_2(\Sigma_c)$, and $\mathcal{I}_3(\Sigma_d) = \mathcal{I}_3(\Sigma_c)$.
 - (b) Σ_d is left (right) invertible if and only if Σ_c is left (right) invertible.
 - (c) Σ_d is invertible (degenerate) if and only if Σ_c is invertible (degenerate).

- (4) The invariant zeros of Σ_d and their associated structures consist of the following two parts:
 - (a) Let the infinite zero structure (of order greater than 0) of Σ_c be given by $S_\infty^*(\Sigma_c) = \{q_1, q_2, \dots, q_{m_d}\}$. Then $z = -1$ is an invariant zero of Σ_d with multiplicity structure $S_{-1}^*(\Sigma_d) = \{q_1, q_2, \dots, q_{m_d}\}$.
 - (b) Let $s = \alpha \neq a$ be an invariant zero of Σ_c with multiplicity structure $S_\alpha^*(\Sigma_c) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}$. Then $z = \beta = (a + \alpha)/(a - \alpha)$ is an invariant zero of Σ_d with multiplicity structure $S_\beta^*(\Sigma_d) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}$.
- (5) The infinite zero structure of Σ_d consists of the following two parts:
 - (a) Let m_0 be the number of infinite zeros of Σ_c of order 0, i.e. $m_0 = \text{rank}(D)$, and let m_d be the total number of infinite zeros of Σ_c of order greater than 0. Also, let τ_a be the geometric multiplicity of the invariant zero of Σ_c at $s = a$. Then the total number of infinite zeros of Σ_d of order 0, i.e. $\text{rank}(\tilde{D})$, is equal to $m_0 + m_d - \tau_a$.
 - (b) Let $s = a$ be an invariant zero of the given continuous-time system Σ_c with a multiplicity structure $S_a^*(\Sigma_c) = \{n_{a,1}, n_{a,2}, \dots, n_{a,\tau_a}\}$. Then Σ_d has an infinite zero (of order greater than 0) structure $S_\infty^*(\Sigma_d) = \{n_{a,1}, n_{a,2}, \dots, n_{a,\tau_a}\}$.

We have the following two interesting propositions. The first is with regard to the minimum phase and nonminimum phase properties of Σ_d , while the second concerns the asymptotic behavior of Σ_d as the sampling period T tends to zero (or, equivalently, as $a \rightarrow \infty$).

Observation 3.1. Consider a general continuous-time system Σ_c and its discrete-time counterpart Σ_d under the transformation (26). It follows from 4(a) and 4(b) of Theorem 3.1 that

- (1) Σ_d has all its invariant zeros inside the unit circle if and only if Σ_c has all its invariant zeros in the open left-half plane and has no infinite zeros of order higher than 0;
- (2) Σ_d has invariant zeros on the unit circle if and only if Σ_c has invariant zeros on the imaginary axis, and/or Σ_c has at least one infinite zero of order greater than 0;
- (3) Σ_d has invariant zeros outside the unit circle if and only if Σ_c has invariant zeros in the open right-half plane.

Observation 3.2. Consider a general continuous-time system Σ_c and its discrete-time counterpart Σ_d under the bilinear transformation (26). Then

a consequence of Theorem 3.1, Σ_d has the following asymptotic properties as the sampling period T tends to zero (but not equal to zero):

- (1) Σ_d has no infinite zeros of order greater than 0, i.e. no delays from the input to the output;
- (2) Σ_d has one invariant zero at $z = -1$ with an appropriate multiplicity structure if Σ_c has any infinite zeros of order greater than 0; and
- (3) The remaining invariant zeros of Σ_d , if any, tend to the point $z = 1$. More interestingly, the invariant zeros of Σ_d corresponding to the stable invariant zeros of Σ_c are always stable, and approach the point $z = 1$ from inside the unit circle. Conversely, the invariant zeros of Σ_d corresponding to the unstable invariant zeros of Σ_c are always unstable, and approach the point $z = 1$ from outside the unit circle. Finally, those associated with the imaginary axis invariant zeros of Σ_c are always mapped onto the unit circle and move towards to the point $z = 1$.

3.2. From discrete time to continuous time

The inverse bilinear transformation has found an important application in the solution of discrete-time H_∞ control problems. Namely, one can design a discrete-time H_∞ suboptimal controller by (i) first transforming the discrete-time model to continuous time, using an inverse bilinear transformation; (ii) designing a continuous-time H_∞ suboptimal controller for the continuous-time model so obtained; and then (iii) the discrete-time H_∞ suboptimal controller is given by the bilinear transformation of the obtained continuous-time H_∞ suboptimal controller. The inverse bilinear transformation has also been applied to solve general discrete-time Riccati equations (see Chen *et al.*, 1994).

We present in this section a similar result as in the previous subsection, but for the inverse bilinear transformation mapping a discrete-time system to a continuous-time system. We begin with a discrete-time linear time-invariant system $\tilde{\Sigma}_d$ characterized by

$$\tilde{\Sigma}_d: \begin{cases} x(k+1) = \tilde{A}x(k) + \tilde{B}u(k), \\ y(k) = \tilde{C}x(k) + \tilde{D}u(k) \end{cases} \quad (35)$$

and with corresponding transfer function

$$H_d(z) = \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} + \tilde{D}. \quad (36)$$

The inverse bilinear transformation corresponding to equation (26) replaces z in equation (36) with

$$z = \frac{a+s}{a-s} \quad (37)$$

to obtain the following continuous-time system:

$$H_c(s) = \tilde{C} \left(\frac{a+s}{a-s} I - \tilde{A} \right)^{-1} \tilde{B} + \tilde{D}. \quad (38)$$

The following lemma is analogous to Lemma 3.1, and provides a state-space realization of $H_c(s)$.

Lemma 3.2. A state-space realization of $H_c(s)$, the continuous-time counterpart of the discrete-time system $\tilde{\Sigma}_d$ of equation (35) under the inverse bilinear transformation (37), is given by

$$\tilde{\Sigma}_c: \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du \end{cases} \quad (39)$$

where

$$\begin{aligned} A &= a(\tilde{A} + I)^{-1}(\tilde{A} - I), \\ B &= \sqrt{2a}(\tilde{A} + I)^{-1}\tilde{B}, \\ C &= \sqrt{2a}\tilde{C}(\tilde{A} + I)^{-1}, \\ D &= \tilde{D} - \tilde{C}(\tilde{A} + I)^{-1}\tilde{B}. \end{aligned} \quad (40)$$

Here we clearly assume that the matrix \tilde{A} has no eigenvalues at -1 .

The following theorem is analogous to Theorem 3.1.

Theorem 3.2. Consider the discrete-time system $\tilde{\Sigma}_d$ of equation (35) characterized by the quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with matrix \tilde{A} having no eigenvalues at -1 , and its continuous-time counterpart under the inverse bilinear transformation (37), i.e. $\tilde{\Sigma}_c$ of equation (39) characterized by the quadruple (A, B, C, D) . We have the following properties:

- (1) Controllability (stabilizability) and observability (detectability) of $\tilde{\Sigma}_c$:
 - (a) $\tilde{\Sigma}_c$ is controllable (stabilizable) if and only if $\tilde{\Sigma}_d$ is controllable (stabilizable).
 - (b) $\tilde{\Sigma}_c$ is observable (detectable) if and only if $\tilde{\Sigma}_d$ is observable (detectable).
- (2) Effects of nonsingular state, output and input transformations, together with state feedback and output injection laws:
 - (a) For any given nonsingular state, output and input transformations T_s , T_o and T_i , the quadruple $(T_s^{-1}AT_s, T_s^{-1}BT_i, T_o^{-1}CT_s, T_o^{-1}DT_i)$ is the continuous-time counterpart of the discrete-time system $(T_s^{-1}\tilde{A}T_s, T_s^{-1}\tilde{B}T_i, T_o^{-1}\tilde{C}T_s, T_o^{-1}\tilde{D}T_i)$ under the inverse bilinear transformation (37).
 - (b) For any $\tilde{F} \in \mathbb{R}^{m \times n}$ with $\tilde{A} + \tilde{B}\tilde{F}$ having no eigenvalues at -1 , define a nonsingular matrix

$$T_i := I - \tilde{F}(I + \tilde{A} + \tilde{B}\tilde{F})^{-1}\tilde{B} \in \mathbb{R}^{m \times m} \quad (41)$$

and a constant matrix

$$F := \sqrt{2a} \tilde{F}(I + \tilde{A} + \tilde{B}\tilde{F})^{-1} \in \mathbb{R}^{m \times n}. \quad (42)$$

Then a discrete-time system $\tilde{\Sigma}_{dF}$ characterized by $(\tilde{A} + \tilde{B}\tilde{F}, \tilde{B}, \tilde{C} + \tilde{D}\tilde{F}, \tilde{D})$ is mapped to a continuous-time counterpart $\tilde{\Sigma}_{cF}$ characterized by $(A + BF, BT_i, C + DF, DT_i)$ under the inverse bilinear transformation (37).

- (c) For any $\tilde{K} \in \mathbb{R}^{n \times p}$ with $\tilde{A} + \tilde{K}\tilde{C}$ having no eigenvalues at -1 , define a nonsingular matrix

$$T_o := [I - \tilde{C}(I + \tilde{A} + \tilde{K}\tilde{C})^{-1}\tilde{K}]^{-1} \in \mathbb{R}^{p \times p} \quad (43)$$

and a constant matrix

$$K := \sqrt{2a}(I + \tilde{A} + \tilde{K}\tilde{C})^{-1}\tilde{K}. \quad (44)$$

Then a discrete-time system $\tilde{\Sigma}_{dK}$ characterized by $(\tilde{A} + K\tilde{C}, \tilde{B} + \tilde{K}\tilde{D}, \tilde{C}, \tilde{D})$ is mapped to a continuous-time $\tilde{\Sigma}_{cK}$, characterized by $(A + KC, B + KD, T_o^{-1}C, T_o^{-1}D)$, under the inverse bilinear transformation (37).

- (3) Invertibility and structural invariant indices lists \mathcal{I}_2 and \mathcal{I}_3 of $\tilde{\Sigma}_c$:

- (a) $\mathcal{I}_2(\tilde{\Sigma}_c) = \mathcal{I}_2(\tilde{\Sigma}_d)$, and $\mathcal{I}_3(\tilde{\Sigma}_c) = \mathcal{I}_3(\tilde{\Sigma}_d)$.
 (b) $\tilde{\Sigma}_c$ is left (right) invertible if and only if $\tilde{\Sigma}_d$ is left (right) invertible.
 (c) $\tilde{\Sigma}_c$ is invertible (degenerate) if and only if $\tilde{\Sigma}_d$ is invertible (degenerate).

- (4) Invariant zeros of $\tilde{\Sigma}_c$ and their structures consist of the following two parts:

- (a) Let the infinite zero structure (of order greater than 0) of $\tilde{\Sigma}_d$ be given by $S_\infty^*(\tilde{\Sigma}_d) = \{q_1, q_2, \dots, q_{m_d}\}$. Then $s = a$ is an invariant zero of $\tilde{\Sigma}_c$ with multiplicity structure $S_a(\tilde{\Sigma}_c) = \{q_1, q_2, \dots, q_{m_d}\}$.
 (b) Let $z = \alpha \neq -1$ be an invariant zero of $\tilde{\Sigma}_d$ with multiplicity structure $S_\alpha^*(\tilde{\Sigma}_d) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}$. Then $s = \beta = a(\alpha - 1)/(\alpha + 1)$ is an invariant zero of $\tilde{\Sigma}_c$ with multiplicity structure $S_\beta^*(\tilde{\Sigma}_c) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}$.

- (5) The infinite zero structure of $\tilde{\Sigma}_c$ consists of the following two parts:

- (a) Let m_0 be the number of infinite zeros of $\tilde{\Sigma}_d$ of order 0, i.e. $m_0 = \text{rank}(\tilde{D})$, and let m_d be the total number of infinite zeros of $\tilde{\Sigma}_d$ of order greater than 0. Also, let τ_{-1} be the geometric multiplicity of the invariant zero of $\tilde{\Sigma}_d$ at $z = -1$. Then the total number of infinite zeros of $\tilde{\Sigma}_c$ of order 0, i.e. $\text{rank}(D)$, is equal to $m_0 + m_d - \tau_{-1}$.
 (b) Let $z = -1$ be an invariant zero of the given discrete-time system $\tilde{\Sigma}_d$ with multiplicity structure $S_{-1}^*(\tilde{\Sigma}_d) = \{n_{-1,1}, n_{-1,2}, \dots, n_{-1,\tau_{-1}}\}$. Then $\tilde{\Sigma}_d$ has an infinite zero (of order greater than 0) structure $S_\infty^*(\tilde{\Sigma}_d) = \{n_{-1,1}, n_{-1,2}, \dots, n_{-1,\tau_{-1}}\}$.

Finally, we wrap up this section with Fig. 1, which summarizes in graphical form the structural mappings associated with the bilinear and inverse bilinear transformations.

4. PROOF OF MAIN RESULTS

We give in this section a detailed proof of Theorem 3.1 of the previous section. The derivation of Theorem 3.2 follows very similar lines to that of Theorem 3.1 and thus is omitted.

Proof of Theorem 3.1. For the sake of simplicity of presentation, and without loss of any generality, we assume that $a = 2/T = 1$ throughout the proof.

1(a). Let β be an eigenvalue of \tilde{A} , i.e. $\beta \in \lambda(\tilde{A})$. It is straightforward to verify that $\beta \neq -1$, provided A has no eigenvalues at $a = 1$ and $\alpha = (\beta - 1)/(\beta + 1)$ is an eigenvalue of A , i.e. $\alpha \in \lambda(A)$. Next, consider the matrix pencil

$$\begin{aligned} & [\beta I - \tilde{A} \quad \tilde{B}] \\ &= [\beta I - (I - A)^{-1}(I + A) \quad \sqrt{2}(I - A)^{-1}B] \\ &= (I - A)^{-1}[\beta(I - A) - (I + A) \quad \sqrt{2}B] \\ &= (I - A)^{-2}[\beta - 1)I - (\beta + 1)A \quad \sqrt{2}B] \\ &= (I - A)^{-2}[\alpha I - A \quad B] \begin{bmatrix} (\beta + 1)I_n & 0 \\ 0 & \sqrt{2}I_m \end{bmatrix}. \end{aligned}$$

Clearly, $\text{rank}[\beta I - \tilde{A} \tilde{B}] = \text{rank}[\alpha I - A \quad B]$, and the result 1(a) follows. \square

1(b) is the dual of 1(a) and 2(a) is trivial. \square

2(b). It follows from Lemma 3.1 that the discrete-time counterpart Σ_{dF} of the bilinear transformation (with $a = 1$) of Σ_{cF} , characterized by $(A + BF, B, C + DF, D)$, is given by $(\tilde{A}_F, \tilde{B}_F, \tilde{C}_F, \tilde{D}_F)$ with

$$\begin{aligned} \tilde{A}_F &= (I + A + BF)(I - A - BF)^{-1}, \\ \tilde{B}_F &= \sqrt{2}(I - A - BF)^{-1}B, \\ \tilde{C}_F &= \sqrt{2}(C + DF)(I - A - BF)^{-1}, \\ \tilde{D}_F &= D + (C + DF)(I - A - BF)^{-1}B. \end{aligned} \quad (45)$$

We first recall from the appendix of Kailath (1980) the following matrix identities that are frequently used in the derivation of our result:

$$(I + XY)^{-1}X = X(I + YX)^{-1} \quad (46)$$

and

$$[I + X(sI - Z)^{-1}Y]^{-1} = I - X(sI - Z + XY)^{-1}Y. \quad (47)$$

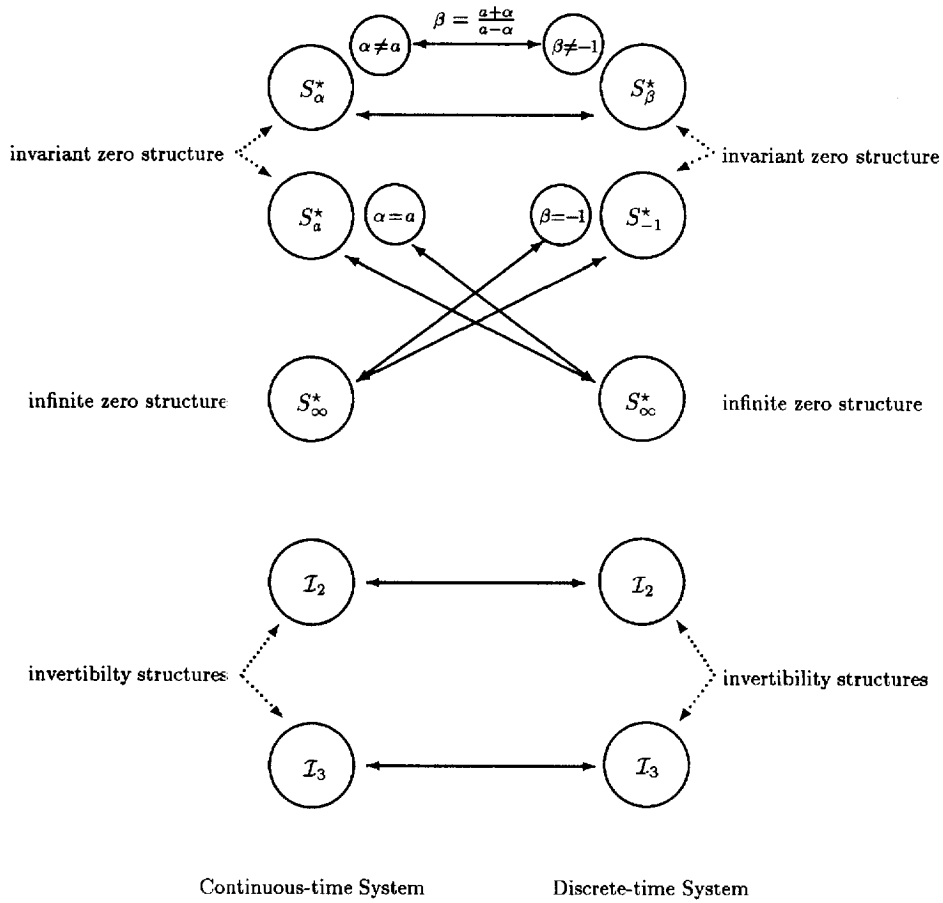


Fig. 1. Structural mappings of the bilinear and inverse bilinear transformations.

Next, we note that

$$\begin{aligned}
 \tilde{A}_F &= (I + A + BF)(I - A - BF)^{-1} \\
 &= (I + A + BF)(I - A)^{-1}[I - BF(I - A)^{-1}]^{-1} \\
 &= [\tilde{A} + BF(I - A)^{-1}][I - BF(I - A)^{-1}]^{-1} \\
 &= [\tilde{A} + BF(I - A)^{-1}][I + BF(I - A - BF)^{-1}] \\
 &= \tilde{A} + \tilde{A}BF(I - A - BF)^{-1} \\
 &\quad + BF(I - A)^{-1}[I + BF(I - A - BF)^{-1}] \\
 &= \tilde{A} + \tilde{A}BF(I - A - BF)^{-1} \\
 &\quad + BF(I - A)^{-1}(I - A)(I - A - BF)^{-1} \\
 &= \tilde{A} + \tilde{A}BF(I - A - BF)^{-1} \\
 &\quad + BF(I - A - BF)^{-1} \\
 &= \tilde{A} + (\tilde{A} + I)BF(I - A - BF)^{-1} \\
 &= \tilde{A} + 2(I - A)^{-1}BF(I - A - BF)^{-1} = \tilde{A} + \tilde{B}\tilde{F}, \\
 \tilde{B}_F &= \sqrt{2}(I - A - BF)^{-1}B \\
 &= \sqrt{2}[I - (I - A)^{-1}BF]^{-1}(I - A)^{-1}B \\
 &= \sqrt{2}(I - A)^{-1}B[I - F(I - A)^{-1}B]^{-1} = \tilde{B}\tilde{T}_i, \\
 \tilde{C}_F &= \sqrt{2}(C + DF)(I - A - BF)^{-1} \\
 &= \sqrt{2}(C + DF)(I - A)^{-1}[I - BF(I - A)^{-1}]^{-1}
 \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{2}(C + DF)(I - A)^{-1}[I + BF(I - A - BF)^{-1}] \\
 &= \sqrt{2}C(I - A)^{-1} + \sqrt{2}DF(I - A)^{-1} \\
 &\quad + \sqrt{2}(C + DF)(I - A)^{-1}BF(I - A - BF)^{-1} \\
 &= \tilde{C} + \sqrt{2}[DF(I - A)^{-1}(I - A - BF) \\
 &\quad + (C + DF)(I - A)^{-1}BF](I - A - BF)^{-1} \\
 &= \tilde{C} + \sqrt{2}[DF - DF(I - A)^{-1}BF \\
 &\quad + C(I - A)^{-1}BF \\
 &\quad + DF(I - A)^{-1}BF](I - A - BF)^{-1} \\
 &= C + [D + C(I - A)^{-1}B]\sqrt{2}F(I - A - BF)^{-1} \\
 &= \tilde{C} + \tilde{D}\tilde{F},
 \end{aligned}$$

$$\begin{aligned}
 \tilde{D}_F &= D + (C + DF)(I - A - BF)^{-1}B \\
 &= D + (C + DF)[I - (I - A)^{-1}BF]^{-1}(I - A)^{-1}B \\
 &= D + (C + DF)(I - A)^{-1}B[I - F(I - A)^{-1}B]^{-1} \\
 &= \{D[I - F(I - A)^{-1}B] + (C + DF)(I - A)^{-1}B\}\tilde{T}_i \\
 &= \{D - DF(I - A)^{-1}B + C(I - A)^{-1}B \\
 &\quad + DF(I - A)^{-1}B\}\tilde{T}_i = \tilde{D}\tilde{T}_i,
 \end{aligned}$$

which completes the proof of 2(b). \square

2(c). Dual of 2(b). \square

With the benefit of properties of 2(a)–2(c), the remainder of the proof is considerably simplified. It is well known that the structural invariant indices lists of Morse, which correspond precisely to the structures of finite and infinite zeros as well as invertibility, are invariant under nonsingular state, output and input transformations, state feedback laws and output injections. We can thus apply appropriate nonsingular state, output and input transformations, as well as state feedback and output injection, to Σ_c and so obtain a new system, say Σ_c^* . If this new system has Σ_d^* as its discrete-time counterpart under bilinear transformation, then from properties 2(a)–2(c), it follows that Σ_d^* and Σ_d have the same structural invariant properties. It is therefore sufficient for the remainder of the proof that we show 3(a)–5(b) are properties of Σ_d^* .

Let us first apply nonsingular state, output and input transformations Γ_s , Γ_o and Γ_i to Σ_c such that the resulting system is in the form of the special coordinate basis as in Theorem 2.1, or, equivalently, the compact form in equations (14)–(16). We will assume that A_{aa} is already in the Jordan form of equations (17) and (18), and that matrices A_{aa} , L_{ad} , B_{a0} , E_{da} , C_{0a} , E_{ca} and L_{ab} are partitioned as follows:

$$A_{aa} = \begin{bmatrix} A_{aa}^a & 0 \\ 0 & A_{aa}^* \end{bmatrix}, \quad L_{ad} = \begin{bmatrix} L_{ad}^a \\ L_{ad}^* \end{bmatrix},$$

$$B_{a0} = \begin{bmatrix} B_{a0}^a \\ B_{a0}^* \end{bmatrix}, \quad L_{ab} = \begin{bmatrix} L_{ab}^a \\ L_{ab}^* \end{bmatrix}, \quad (48)$$

$$E_{da} = [E_{da}^a], \quad C_{0a} = [C_{0a}^a \ C_{0a}^*],$$

$$E_{ca} = [E_{ca}^a \ E_{ca}^*], \quad (49)$$

where matrix A_{aa}^a has all its eigenvalues at $a = 1$, i.e.

$$A_{aa}^a = I + \begin{bmatrix} 0 & I_{n_{a,1}-1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{n_{a,r_a}-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (50)$$

and A_{aa}^* contains the remaining invariant zeros of Σ_c . Furthermore, we assume that the pair (A_{cc}, B_c) is in the Brunovsky canonical form of equation (21), as is the pair (A_{bb}^*, C_b^*) . Next, define a state feedback gain matrix

$$F = -\Gamma_i \begin{bmatrix} C_{0a}^a - C_2^a & C_{0a}^* & C_{0b} & C_{0c} & C_{0d} \\ E_{da}^a - C_1^a & E_{da}^* & E_{db} & E_{dc} & E_{dd} \\ E_{ca}^a & E_{ca}^* & 0 & E_{cc} & 0 \end{bmatrix} \Gamma_s^{-1} \quad (51)$$

and an output injection gain matrix

$$K = -\Gamma_s \begin{bmatrix} B_{a0}^a - B_2^a & L_{ad}^a - B_1^a & L_{ab}^a \\ B_{a0}^* & L_{ad}^* & L_{ab}^* \\ B_{b0} & L_{bd} & L_{bb} \\ B_{c0} & L_{cd} & L_{cb} \\ B_{d0} & L_{dd} & 0 \end{bmatrix} \Gamma_o^{-1}. \quad (52)$$

Here, E_{cc} is chosen such that all *'s in equation (21) are cleaned out, i.e. $A_{cc}^* := A_{cc} - B_c E_{cc}$ is in Jordan form with all diagonal elements equal to 0. Similarly, L_{bb} is chosen such that $(A_{bb}^*) := (A_{bb} - L_{bb} C_b)$ is in Jordan form with all diagonal elements equal to 0. Likewise, E_{da} and L_{ad} are chosen such that $A_{da}^* := A_{da} - L_{ad} C_a - B_d E_{da}$ is in Jordan form with all diagonal elements equal to 0, which in turn implies

$$C_d (I - A_{da}^*)^{-1} B_d = I_{m_d}. \quad (53)$$

The matrices B_1^a , B_2^a , C_1^a and C_2^a are chosen in conformity with A_{aa}^a of equation (50) as follows:

$$B^a := [B_2^a \ B_1^a] := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$C^a := \begin{bmatrix} C_2^a \\ C_1^a \end{bmatrix} := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \quad (54)$$

This can always be done, as a consequence of the assumption that the matrix A has no eigenvalues at $a = 1$, which implies that the invariant zero at $a = 1$ of Σ_c is completely controllable and observable.

Finally, we obtain a continuous-time system Σ_c^* characterized by the quadruple (A^*, B^*, C^*, D^*) , where

$$A^* = P^{-1} \Gamma_s^{-1} (A + BF + KC + KDF) \Gamma_s P$$

$$= \begin{bmatrix} A_{aa}^* & 0 & 0 & 0 & 0 \\ 0 & A_{bb}^* & 0 & 0 & 0 \\ 0 & 0 & A_{cc}^* & 0 & 0 \\ 0 & 0 & 0 & A_{dd}^* & B_d C_1^a \\ 0 & 0 & 0 & B_1^a C_d & A_{aa}^a + B_2^a C_2^a \end{bmatrix}, \quad (55)$$

$$B^* = P^{-1} \Gamma_s^{-1} (B + KD) \Gamma_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_c \\ 0 & B_d & 0 \\ B_2^a & 0 & 0 \end{bmatrix} \quad (56)$$

and

$$C^* = \Gamma_o^{-1}(C + DF)\Gamma_s P = \begin{bmatrix} 0 & 0 & 0 & 0 & C_2^a \\ 0 & 0 & 0 & C_d & 0 \\ 0 & C_b & 0 & 0 & 0 \end{bmatrix},$$

$$D^* = \Gamma_o^{-1}D\Gamma_i = \begin{bmatrix} I_{m_o} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (57)$$

where P is a permutation matrix that transforms A_{aa}^a from its original position, i.e. block (1, 1), to block (5, 5) in equation (55).

Next, define a subsystem (A_s, B_s, C_s, D_s) with

$$A_s := \begin{bmatrix} A_{dd}^* & B_d C_1^a \\ B_1^a C_d & A_{aa}^a + B_2^a C_2^a \end{bmatrix},$$

$$B_s := \begin{bmatrix} 0 & B_d \\ B_2^a & 0 \end{bmatrix} \quad (58)$$

and

$$C_s := \begin{bmatrix} 0 & C_2^a \\ C_d & 0 \end{bmatrix}, \quad D_s := \begin{bmatrix} I_{m_o} & 0 \\ 0 & 0 \end{bmatrix}. \quad (59)$$

It is straightforward to verify that with the choice of B^a and C^a as in equation (54), A_s has no eigenvalues at $a = 1$. Hence A^* has no eigenvalues at $a = 1$ either, since both A_{bb}^* and A_{cc}^* have all eigenvalues at 0, and A_{aa}^* contains only the invariant zeros of Σ_c which are not equal to $a = 1$. Applying the bilinear transformation (26) to Σ_c^* , it follows from Lemma 3.1 that we obtain a discrete-time system Σ_d^* , characterized by $(\tilde{A}^*, \tilde{B}^*, \tilde{C}^*, \tilde{D}^*)$, with

$$\tilde{A}^* = \begin{bmatrix} (I + A_{aa}^*)(I - A_{aa}^*)^{-1} & 0 & 0 & 0 \\ 0 & (I + A_{bb}^*)(I - A_{bb}^*)^{-1} & 0 & 0 \\ 0 & 0 & (I + A_{cc}^*)(I - A_{cc}^*)^{-1} & 0 \\ 0 & 0 & 0 & (I + A_s)(I + A_s)^{-1} \end{bmatrix}, \quad (60)$$

$$\tilde{B}^* = \sqrt{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & (I - A_{cc}^*)^{-1} B_c \\ (I - A_s)^{-1} B_s & 0 \end{bmatrix}, \quad (61)$$

and

$$\tilde{C}^* = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 & C_s(I - A_s)^{-1} \\ 0 & C_b(I - A_s)^{-1} & 0 & 0 \end{bmatrix},$$

$$\tilde{D}^* = \begin{bmatrix} D_s + C_s(I - A_s)^{-1} B_s & 0 \\ 0 & 0 \end{bmatrix}. \quad (62)$$

Our next task is to find appropriate transformations, state feedback, and output injection laws, so as to transform the above system into the form of the special coordinate basis displaying the properties 3(a)–5(b).

To simplify the presentation, we first focus on the subsystem $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$ with

$$\tilde{A}_s := (I + A_s)(I - A_s)^{-1}, \quad (63)$$

$$\tilde{B}_s := \sqrt{2}(I - A_s)^{-1} B_s$$

and

$$\tilde{C}_s := \sqrt{2} C_s(I - A_s)^{-1}, \quad (64)$$

$$\tilde{D}_s := D_s + C_s(I - A_s)^{-1} B_s.$$

Using equation (53) in conjunction with Appendix A.22 of Kailath (1980), it is straightforward to compute

$$\tilde{A}_s = \begin{bmatrix} X \\ 2(I - A_{aa}^a - B^a C^a)^{-1} B_1^a C_d (I - A_{dd}^*)^{-1} \\ 2(I - A_{dd}^*)^{-1} B_d C_1^a (I - A_{aa}^a - B^a C^a)^{-1} \\ (I + A_{aa}^a + B^a C^a)(I - A_{aa}^a - B^a C^a)^{-1} \end{bmatrix}, \quad (65)$$

where

$$X = (I + A_{dd}^*)(I - A_{dd}^*)^{-1} + 2(I - A_{dd}^*)^{-1} B_d C_1^a \\ \times (I - A_{aa}^a - B^a C^a)^{-1} B_1^a C_d (I - A_{dd}^*)^{-1},$$

$$\tilde{B}_s = \sqrt{2} \begin{bmatrix} (I - A_{dd}^*)^{-1} B_d C_1^a (I - A_{dd}^* - B^a C^a)^{-1} B_2^a \\ (I - A_{aa}^a - B^a C^a)^{-1} B_2^a \\ (I - A_{dd}^*)^{-1} B_d [I + C_1^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a] \\ (I - A_{aa}^a - B^a C^a)^{-1} B_1^a \end{bmatrix} \quad (66)$$

$\tilde{C}_s =$

$$\sqrt{2} \begin{bmatrix} C_2^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a C_d (I - A_{dd}^*)^{-1} \\ [I + C_1^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a] C_d (I - A_{dd}^*)^{-1} \\ C_2^a (I - A_{aa}^a - B^a C^a)^{-1} \\ C_1^a (I - A_{aa}^a - B^a C^a)^{-1} \end{bmatrix} \quad (67)$$

and

$$\tilde{D}_s = \begin{bmatrix} I + C_2^a (I - A_{aa}^a - B^a C^a)^{-1} B_2^a \\ C_1^a (I - A_{aa}^a - B^a C^a)^{-1} B_2^a \\ C_2^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a \\ I + C_1^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a \end{bmatrix}. \quad (68)$$

Noting the structure of A_{aa}^a in equation (50), and the structures of B^a and C^a in equation (54), we have

$$(I - A_{aa} - B^a C^a)^{-1} = \begin{bmatrix} 0 & -1 & \cdots & 0 & 0 \\ -I_{n_{a,1}-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -I_{n_{a,\tau_a}-1} & 0 \end{bmatrix}, \quad (69)$$

$$\begin{aligned} C_1^a(I - A_{aa} - B^a C^a)^{-1} B_2^a &= 0, \\ C_2^a(I - A_{aa} - B^a C^a)^{-1} B_1^a &= 0 \end{aligned} \quad (70)$$

and

$$C^a(I - A_{aa} - B^a C^a)^{-1} B^a = \begin{bmatrix} 0 & 0 \\ 0 & -\mathcal{I}_{\tau_a} \end{bmatrix}. \quad (71)$$

Thus, \tilde{B}_s , \tilde{C}_s and \tilde{D}_s are reduced to the following forms:

$$\tilde{B}_s = \sqrt{2} \begin{bmatrix} 0 & (I - A_{dd}^*)^{-1} B_d [I + C_1^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a] \\ (I - A_{aa}^a - B^a C^a)^{-1} B_2^a & (I - A_{aa}^a - B^a C^a)^{-1} B_1^a \end{bmatrix}, \quad (72)$$

$$\tilde{C}_s = \sqrt{2} \begin{bmatrix} 0 & C_2^a (I - A_{aa}^a - B^a C^a)^{-1} \\ [I + C_1^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a] C_d (I - A_{dd}^*)^{-1} & C_1^a (I - A_{aa}^a - B^a C^a)^{-1} \end{bmatrix} \quad (73)$$

and

$$\tilde{D}_s = \begin{bmatrix} I + C_2^a (I - A_{aa}^a - B^a C^a)^{-1} B_2^a & 0 \\ 0 & I + C_1^a (I - A_{aa}^a - B^a C^a)^{-1} B_1^a \end{bmatrix} \quad (74)$$

Next, define

$$\tilde{F}_s := \sqrt{2} \begin{bmatrix} 0 & 0 \\ -C_d (I - A_{dd}^*)^{-1} & 0 \end{bmatrix} \quad (75)$$

and

$$\tilde{K}_s := \sqrt{2} \begin{bmatrix} 0 & -(I - A_{dd}^*)^{-1} B_d \\ 0 & 0 \end{bmatrix}, \quad (76)$$

from which it follows that

$$\begin{aligned} \tilde{A}_{sc} &= \tilde{A}_s + \tilde{B}_s \tilde{F}_s + \tilde{K}_s \tilde{C}_s + \tilde{K}_s \tilde{D}_s \tilde{F}_s \\ &= \begin{bmatrix} \tilde{A}_{aa}^{**} & 0 \\ 0 & (I + A_{aa}^a + B^a C^a)(I - A_{aa}^a - B^a C^a)^{-1} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \tilde{A}_{aa}^{**} &:= (I + A_{dd}^*)(I - A_{dd}^*)^{-1} \\ &\quad - 2(I - A_{dd}^*)^{-1} B_d C_d (I - A_{dd}^*)^{-1}, \end{aligned} \quad (77)$$

$$\tilde{B}_{sc} = \tilde{B}_s + \tilde{K}_s \tilde{D}_s =$$

$$\sqrt{2} \begin{bmatrix} 0 & 0 \\ (I - A_{aa}^a - B^a C^a)^{-1} B_2^a & (I - A_{aa}^a - B^a C^a)^{-1} B_1^a \end{bmatrix}$$

and

$$\begin{aligned} \tilde{C}_{sc} &= \tilde{C}_s + \tilde{D}_s \tilde{F}_s \\ &= \sqrt{2} \begin{bmatrix} 0 & C_2^a (I - A_{aa}^a - B^a C^a)^{-1} \\ 0 & C_1^a (I - A_{aa}^a - B^a C^a)^{-1} \end{bmatrix}. \end{aligned}$$

Next, repartition B^a and C^a of equation (54) as follows:

$$B^a = [0 \quad \tilde{B}_a] \quad \text{and} \quad C^a = \begin{bmatrix} 0 \\ \tilde{C}_a \end{bmatrix}, \quad (78)$$

where both \tilde{B}_a and \tilde{C}_a are of maximal rank. We thus obtain

$$\tilde{A}_{sc} = \begin{bmatrix} \tilde{A}_{aa}^{**} & 0 \\ 0 & (I + A_{aa}^a + \tilde{B}_a \tilde{C}_a)(I - A_{aa}^a - \tilde{B}_a \tilde{C}_a)^{-1} \end{bmatrix},$$

$$\tilde{B}_{sc} = \sqrt{2} \begin{bmatrix} 0 & 0 \\ 0 & (I - A_{aa} - \tilde{B}_a \tilde{C}_a)^{-1} \tilde{B}_a \end{bmatrix},$$

and

$$\tilde{C}_{sc} = \sqrt{2} \begin{bmatrix} 0 & 0 \\ 0 & \tilde{C}_a (I - A_{aa} - \tilde{B}_a \tilde{C}_a)^{-1} \end{bmatrix},$$

$$\tilde{D}_{sc} = \tilde{D}_s = \begin{bmatrix} I_{m_0 + m_d - \tau_a} & 0 \\ 0 & 0 \end{bmatrix}.$$

Using equations (50) and (69), straightforward manipulations yield

$$(I + A_{aa}^a + \tilde{B}_a \tilde{C}_a)(I - A_{aa}^a - \tilde{B}_a \tilde{C}_a)^{-1} = \begin{bmatrix} \begin{bmatrix} 0 & -2 \\ -2I_{n_{a,1}-1} & 0 \end{bmatrix} - I_{n_{a,1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{bmatrix} 0 & -2 \\ -2I_{n_{a,\tau_a}-1} & 0 \end{bmatrix} - I_{n_{a,\tau_a}} \end{bmatrix}$$

$$(I - A_{aa}^a - \tilde{B}_a \tilde{C}_a)^{-1} \tilde{B}_a = - \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{bmatrix}$$

and

$$\tilde{C}_a (I - A_{aa}^a - \tilde{B}_a \tilde{C}_a)^{-1} = - \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Moreover, it can be readily verified that each subsystem $(\tilde{A}_{ai}, \tilde{B}_{ai}, \tilde{C}_{ai})$, $i = 1, \dots, \tau_a$, with

$$\tilde{A}_{ai} = -I_{n_{a,i}} + \begin{bmatrix} 0 & -2 \\ -2I_{n_{a,i-1}} & 0 \end{bmatrix},$$

and

$$\begin{aligned} \tilde{C}_d &= \Gamma_{oa}^{-1} [\tilde{C}_a (I - A_{aa}^a - \tilde{B}_a \tilde{C}_a)^{-1}] \Gamma_{sa} \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \end{aligned} \quad (81)$$

Now, let us return to Σ_d^* characterized by $(\tilde{A}^*, \tilde{B}^*, \tilde{C}^*, \tilde{D}^*)$ as in equations (60)–(62). Using the properties of the subsystem $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$ just derived, we are in a position to define appropriate state feedback and output injection gain matrices, say \tilde{F}^* and \tilde{K}^* , together with nonsingular state, output and input transformations $\tilde{\Gamma}_s^*$, $\tilde{\Gamma}_o^*$ and $\tilde{\Gamma}_i^*$, such that

$$\begin{aligned} \tilde{A}_{\text{SCB}}^* &:= (\tilde{\Gamma}_s^*)^{-1} (\tilde{A}^* + \tilde{B}^* \tilde{F}^* + \tilde{K}^* \tilde{C}^* + \tilde{K}^* \tilde{D}^* \tilde{F}^*) \tilde{\Gamma}_s^* \\ &= \begin{bmatrix} (I + A_{aa}^*) (I - A_{aa}^*)^{-1} & 0 & 0 & 0 & 0 \\ 0 & (I + A_{bb}^*) (I - A_{bb}^*)^{-1} & 0 & 0 & 0 \\ 0 & 0 & (I + A_{cc}^*) (I - A_{cc}^*)^{-1} & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}_{aa}^{**} & 0 \\ 0 & 0 & 0 & 0 & \tilde{A}_d \end{bmatrix} \end{aligned} \quad (82)$$

$$\tilde{B}_{ai} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \tilde{C}_{ai} = [0 \quad -1],$$

has the following properties:

$$\tilde{C}_{ai} \tilde{B}_{ai} = \tilde{C}_{ai} \tilde{A}_{ai} \tilde{B}_{ai} = \cdots = \tilde{C}_{ai} (\tilde{A}_{ai})^{n_{a,i}-2} \tilde{B}_{ai} = 0$$

and

$$\tilde{C}_{ai} (\tilde{A}_{ai})^{n_{a,i}-1} \tilde{B}_{ai} \neq 0.$$

It follows from Theorem 2.1 that there exist nonsingular transformations Γ_{sa} , Γ_{oa} and Γ_{ia} such that

$$\tilde{A}_d = \Gamma_{sa}^{-1} [(I + A_{aa}^a + \tilde{B}_a \tilde{C}_a) (I - A_{aa}^a - \tilde{B}_a \tilde{C}_a)^{-1}] \Gamma_{sa}$$

$$= \begin{bmatrix} * & I_{n_{a,1}-1} & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & I_{n_{a,\tau_a}-1} \\ 0 & 0 & \cdots & * & * \end{bmatrix}, \quad (79)$$

$$\tilde{B}_d = \Gamma_{sa}^{-1} [(I - A_{aa}^a - \tilde{B}_a \tilde{C}_a)^{-1} \tilde{B}_a] \Gamma_{ia}$$

$$= \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix} \quad (80)$$

with \tilde{A}_{aa}^{**} given by equation (77), and

$$\tilde{B}_{\text{SCB}}^* := (\tilde{\Gamma}_s^*)^{-1} (\tilde{B}^* + \tilde{K}^* \tilde{D}^*) \tilde{\Gamma}_i^*$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (I - A_{cc}^*)^{-1} B_c \\ 0 & 0 & 0 \\ 0 & \tilde{B}_d & 0 \end{bmatrix}, \quad (83)$$

$$\tilde{C}_{\text{SCB}}^* := (\tilde{\Gamma}_o^*)^{-1} (\tilde{C}^* + \tilde{D}^* \tilde{F}^*) \tilde{\Gamma}_s^*$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & C_b (I - A_{bb}^*)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{C}_d \end{bmatrix} \quad (84)$$

and

$$\tilde{D}_{\text{SCB}}^* := (\tilde{\Gamma}_o^*)^{-1} \tilde{D}^* \tilde{\Gamma}_i^* = \begin{bmatrix} I_{m_o+m_d-\tau_a} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (85)$$

Clearly, Σ_{SCB}^* characterized by $(\tilde{A}_{\text{SCB}}^*, \tilde{B}_{\text{SCB}}^*, \tilde{C}_{\text{SCB}}^*, \tilde{D}_{\text{SCB}}^*)$ has the same structural invariant indices lists as does Σ_d^* , which in turn has the same structural

invariant indices lists as Σ_d . Most importantly, however, Σ_{SCB}^* is in the form of the special coordinate basis, and we are now ready to prove properties 3(a)–5(b) of the theorem.

3(a). First, we note that $\mathcal{S}_2(\Sigma_d) = \mathcal{S}_2(\Sigma_{SCB}^*)$. From equations (82) to (85) and the properties of the special coordinate basis, we know that $\mathcal{S}_2(\Sigma_{SCB}^*)$ is given by the controllability index of the pair

$$\{(I + A_{cc}^*)(I - A_{cc}^*)^{-1}, (I - A_{cc}^*)^{-1}B_c\}$$

or

$$(I + A_{cc}^*)(I - A_{cc}^*)^{-1}, B_c\}.$$

It is straightforward to verify that the controllability index of $\{(I + A_{cc}^*)(I - A_{cc}^*)^{-1}, B_c\}$ is the same as that of (A_{cc}^*, B_c) , and thus $I_2(\Sigma_d) = I_2(\Sigma_c)$. Likewise, the proof that $I_3(\Sigma_d) = I_3(\Sigma_c)$ follows along similar lines. \square

3(b)–3(c). Follow directly from 3(a). \square

4(a). It follows from the properties of the special coordinate basis that the invariant zero structure of Σ_{SCB}^* , or equivalently Σ_d , is given by the eigenvalues of \tilde{A}_{aa}^{**} and $(I + A_{aa}^*)(I - A_{aa}^*)^{-1}$, together with their associated Jordan blocks. Property 4(a) corresponds with the eigenvalues of \tilde{A}_{aa}^{**} of equation (77), together with their associated Jordan blocks. First, we note that for any $z \in \mathbb{C}$,

$$zI - \tilde{A}_{aa}^{**} = [(z - 1)I - (z + 1)A_{aa}^* + 2(I - A_{aa}^*)^{-1}B_a C_a](I - A_{aa}^*)^{-1}. \quad (86)$$

It can be shown that

$$(z - 1)I - (z + 1)A_{aa}^* + 2(I - A_{aa}^*)^{-1}B_a C_a = \text{Blockdiag}\{Q_1(z), \dots, Q_i(z)\},$$

where $Q_i(z) \in \mathbb{C}^{n_i \times n_i}$ is given by

$$Q_i(z) = \begin{bmatrix} z + 1 & -(z + 1) & 0 & \dots & 0 & 0 \\ 2 & z - 1 & -(z + 1) & \dots & 0 & 0 \\ 2 & 0 & z - 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & \dots & z - 1 & -(z + 1) \\ 2 & 0 & 0 & \dots & 0 & z - 1 \end{bmatrix} \quad (87)$$

for $i = 1, \dots, m_d$. It follows from (86) that the eigenvalue of \tilde{A}_{aa}^{**} is the scalar z that causes the rank of $\text{Block diag}\{Q_1(z), \dots, Q_{m_d}(z)\}$ to drop below $n_d = \sum_{i=1}^{m_d} q_i$. Using the particular form of $Q_i(z)$, it is clear that the only such scalar $z \in \mathbb{C}$ which causes $Q_i(z)$ to drop rank is $z = -1$. Moreover, $\text{rank}\{Q_i(-1)\} = n_{q_i} - 1$, i.e. $Q_i(-1)$ has only one

linearly independent eigenvector. Hence, $z = -1$ is the eigenvalue of \tilde{A}_{aa}^{**} , or equivalently the invariant zero of Σ_d , with multiplicity structure $S_{-1}^*(\Sigma_d) = \{q_1, \dots, q_{m_d}\} = S_{\infty}^*(\Sigma_c)$, thereby proving 4(a). \square

4(b). This part of the infinite zero structure corresponds to the invariant zeros of the matrix $(I + A_{aa}^*)(I - A_{aa}^*)^{-1}$. With A_{aa}^* in Jordan form, Property 4(b) follows by straightforward manipulations. \square

5(a) follows directly from equation (85) and 5(b) follows from the structure of $(\tilde{A}_d, \tilde{B}_d, \tilde{C}_d)$ in equation (79) to equation (81), in conjunction with Property 2 of the special coordinate basis. \square

This concludes the proof of Theorem 3.1.

5 CONCLUSIONS

We have presented in this paper a comprehensive study of the structural mappings, and invertibility properties, associated with the bilinear transformation of general linear time-invariant multivariable systems. It has been demonstrated that, in general, the structural invariant indices lists \mathcal{S}_1 and \mathcal{S}_4 of Morse (1973) are changed under bilinear transformation, while the structural invariant indices lists \mathcal{S}_2 and \mathcal{S}_3 of Morse are preserved. Finally, it is worth pointing out that the Jordan canonical form, the Brunovsky canonical form, and the Special Coordinate Basis of Sannuti and Saberi, are truly remarkable tools that can be utilized to characterize all structural properties of interest for general linear time-invariant multivariable systems, as demonstrated in the proof of the main results. Future research will investigate the applicability of these same tools to the analysis of the structural

properties of sampled-data systems generated by zero-order hold (ZOH) sampling.

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