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Simultaneous H_2/H_∞ Optimal Control: The State Feedback Case*

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Abstract—A simultaneous H_2/H_∞ control problem is considered. This problem seeks to minimize the H_2 norm of a closed-loop transfer matrix while simultaneously satisfying a prescribed H_∞ norm bound on some other closed-loop transfer matrix by utilizing dynamic state feedback controllers. Such a problem was formulated earlier by Rotea and Khargonekar (*Automatica*, **27**, 307–316, 1991) who considered only so called regular problems. Here, for a class of singular problems, necessary and sufficient conditions are established so that the posed simultaneous H_2/H_∞ problem is solvable by using state feedback controllers. The class of singular problems considered have a left invertible transfer function matrix from the control input to the controlled output which is used for the H_2 norm performance measure. This class of problems subsumes the class of regular problems.

1. *Introduction*

IN MODERN CONTROL THEORY, optimization of some performance measure is a standard design tool. In this regard two performance measures, H_2 and H_∞ norms, are popular. Recently, in order to guarantee robust performance, i.e. guarantee performance in the face of plant uncertainty, some optimal control problems, dealing with both the H_2 and H_∞ norm measures, have been formulated (see e.g. Bernstein and Haddad, 1989; Doyle *et al*, 1989; Khargonekar and Rotea, 1991; Rotea and Khargonekar, 1991). Such problems include (a) constrained optimization problems of minimizing the H_2 norm of a closed-loop transfer matrix subject to an H_∞ norm constraint on another closed-loop transfer matrix, (b) mixed H_2/H_∞ control problems of minimizing a mixed H_2/H_∞ performance measure on a closed-loop transfer matrix subject to an H_∞ norm constraint on another closed-loop transfer matrix, and (c) simultaneous H_2/H_∞ optimal control problems of minimizing the H_2 norm of a closed-loop transfer matrix while simultaneously satisfying a prescribed H_∞ norm bound on another closed-loop transfer matrix. Our interest here is in simultaneous H_2/H_∞ optimal control problems. Recently, Rotea and Khargonekar (1991) considered such simultaneous H_2/H_∞ optimal control problems and developed the necessary and sufficient conditions under which they are solvable by utilizing dynamic state feedback controllers. However, they dealt with only so called regular problems (see Definition 2.2). The intent of this paper is to develop the necessary and sufficient conditions under which a simultaneous H_2/H_∞ optimal control problem is solvable for a class of singular problems (see Definition 2.3). The class of problems considered have a left invertible transfer function matrix from the control input to the controlled output which is used for the H_2 norm performance measure. This class of problems subsumes the class of regular problems.

Throughout the paper, $\mathrm{Ker}[V]$ and $\mathrm{Im}[V]$ denote the kernel and the image of V respectively. Also, $\rho(M)$ denotes the spectral radius of matrix M, while normal rank denotes the rank of a matrix with entries in the field of rational functions. Given a stable and strictly proper transfer function $G(s)$, as usual, its H_2 norm is denoted by $\|G\|_2$; and given a proper stable transfer function $G(s)$, its H_∞ norm is denoted by $\|G\|_\infty$. Also, RH^2 denotes the set of real-rational transfer functions which are stable and strictly proper. Similarly, RH^∞ denotes the set of real-rational transfer functions which are stable and proper.

2. *Problem Statement and Definitions*

Consider the following system:

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu + E_2 w_2 + E_\infty w_\infty \\ y = x \\ z_2 = C_2 x + D_2 u \\ z_\infty = C_\infty x + D_\infty u \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w_2 \in \mathbb{R}^{l_2}$ and $w_\infty \in \mathbb{R}^{l_\infty}$ are the unknown disturbance inputs, and $z_2 \in \mathbb{R}^{q_2}$ and $z_\infty \in \mathbb{R}^{q_\infty}$ are the controlled outputs. Also, consider an arbitrary proper controller,

$$u = \mathsf{F}(s)x. \tag{2.2}$$

A controller $u = \mathsf{F}(s)x$ is said to be admissible if it provides internal stability of the resulting closed-loop system. Let $T_2(\mathsf{F})$ and $T_\infty(\mathsf{F})$ denote the closed-loop transfer functions from w_2 to z_2 and from w_∞ to z_∞, respectively, under the feedback control law $u = \mathsf{F}(s)x$. Moreover, let the infimum of the H_2 norm of the closed-loop transfer function $T_2(\mathsf{F})$ over all the stabilizing proper controllers $\mathsf{F}(s)$ be denoted by γ_2^*, that is,

$$\gamma_2^* := \inf\{\|T_2(\mathsf{F})\|_2 \mid u = \mathsf{F}(s)x \text{ internally stabilizes } \Sigma\}. \tag{2.3}$$

The simultaneous H_2/H_∞ optimal control problem is defined as follows.

Definition 2.1 (Simultaneous H_2/H_∞ optimal control problem). For the given plant Σ and a scalar $\gamma > 0$, find a stabilizing proper controller $\mathsf{F}(s)$ such that $\|T_2(\mathsf{F})\|_2 = \gamma_2^*$ and $\|T_\infty(\mathsf{F})\|_\infty < \gamma$.

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The following definitions will also be convenient in the sequel.

Definition 2.2 (regular simultaneous H_2/H_∞ optimal control problem). A regular simultaneous H_2/H_∞ state feedback optimal control problem refers to a problem in which the given plant Σ satisfies:
(i) D_i, $i = 2, \infty$, are injective, and
(ii) (A, B, C_i, D_i), $i = 2$ and ∞, have no invariant zeros on the $j\omega$ axis.

Definition 2.3 (singular simultaneous H_2/H_∞ optimal control problem). A singular simultaneous H_2/H_∞ state feedback optimal control problem refers to a problem in which the given plant Σ does not satisfy either one or both of the conditions (i) and (ii) in Definition 2.2.

Definition 2.4 (H_2 optimal controller). A stabilizing proper controller $\mathsf{F}(s)$ is said to be an H_2 optimal controller if $\|T_2(\mathsf{F})\|_2 = \gamma_2^*$.

Definition 2.5 (H_∞ γ-suboptimal controller). A stabilizing proper controller $\mathsf{F}(s)$ is said to be an H_∞ γ-suboptimal controller if $\|T_\infty(\mathsf{F})\|_\infty < \gamma$.

Definition 2.6 (stabilizable weakly unobservable subspace). Given a system Σ_* characterized by a matrix quadruple (A, B, C, D), we define the stabilizable weakly unobservable subspace $\mathcal{V}_g(\Sigma_*)$ as the largest subspace $\mathcal{V}$ for which there exists a mapping F such that the following subspace inclusions are satisfied:

$$(A + BF)\mathcal{V} \subseteq \mathcal{V} \quad \text{and} \quad (C + DF)\mathcal{V} = \{0\},$$

and such that $A + BF \mid \mathcal{V}$ is asymptotically stable.

Our goal in this paper is to derive a set of necessary and/or sufficient conditions under which the simultaneous H_2/H_∞ optimal control problem is solvable.

3. *Preliminaries*

In this section, we recall several preliminary results needed to establish the necessary and/or sufficient conditions under which the simultaneous H_2/H_∞ optimal control problem is solvable, while at the same time we also introduce some new results.

3.1. *Review of H_2-optimal Control*. In this subsection, we recall from Stoorvogel *et al.* (1992) the necessary and sufficient conditions under which an H_2-optimal state feedback control law of either static or dynamic type exists. We also recall a recent result of Chen *et al.* (1993) which characterizes all the possible H_2 optimal state feedback laws.

The conditions under which an optimal controller exists for the system

$$\Sigma_2: \begin{cases} \dot{x} = Ax + Bu + E_2 w_2 \\ y = x \\ z_2 = C_2 x + D_2 u \end{cases} \tag{3.1}$$

can be formulated in terms of an auxiliary system Σ_{au2} constructed from the data of (3.1). The auxiliary system Σ_{au2} is as given below:

$$\Sigma_{au2}: \begin{cases} \dot{x}_P = Ax_P + Bu_P + E_2 w_2, \\ y_P = x_P, \\ z_P = C_P x_P + D_P u_P. \end{cases} \tag{3.2}$$

Here C_P and D_P satisfy

$$\tilde{F}_2(P_2) = \begin{bmatrix} C'_P \\ D'_P \end{bmatrix} [C_P \quad D_P],$$

where

$$\tilde{F}_2(P_2) := \begin{bmatrix} A'P_2 + P_2 A + C_2' C_2 & P_2 B + C_2' D_2 \\ B'P_2 + D_2' C_2 & D_2' D_2 \end{bmatrix}, \tag{3.3}$$

and where P_2 is the largest solution of the matrix inequality $\tilde{F}_2(P_2) \geq 0$. It is known that under the condition that (A, B) is stabilizable, such a solution P_2 exists and is unique.

We have the following theorem.

Theorem 3.1. Consider the given system Σ_2 as in (3.1), and the auxiliary system Σ_{au2} as in (3.2). Define a subsystem Σ_P of Σ_{au2} as that characterized by the quadruple (A, B, C_P, D_P). Then, the infimum, γ_2^*, can be attained by a static as well as by a dynamic stabilizing state feedback controller if and only if the pair (A, B) is stabilizable and $\operatorname{Im}(E_2) \subseteq \mathcal{V}_g(\Sigma_P)$.

Proof. See Stoorvogel *et al.* (1992).

We know that whenever an optimal solution to the original H_2 problem exists, there exits a constant gain F such that $A_F := A + BF$ is stable and that

$$\|(C_2 + D_2 F)(sI - A_F)^{-1} E_2\|_2 = \gamma_2^*. \tag{3.4}$$

Next, following the result of Rotea and Khargonekar (1991), it can be shown easily that any proper dynamic controller $\mathsf{F}(s)$ that stabilizes the system Σ_{au2} (which is not necessarily regular) can be written in the following form:

$$\begin{cases} \dot{\xi} = A_F \xi + B y_1, \\ u = Fx + y_1, \end{cases} \tag{3.5}$$

where

$$y_1 = Q(s)(x - \xi) \tag{3.6}$$

for some proper and stable $Q(s)$, i.e. $Q(s) \in \mathrm{RH}^\infty$, with appropriate dimensions. The following theorem qualifies $Q(s)$ so that the controller $\mathsf{F}(s)$ is H_2 optimal for the given system Σ_2.

Theorem 3.2. Consider the given system Σ_2 as in (3.1). Let the system characterized by the matrix quadruple (A, B, C_2, D_2) be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\operatorname{Im}(E_2) \subseteq \mathcal{V}_g(\Sigma_P)$. Define a set $\mathbf{Q}$ as

$$\mathbf{Q} := \{Q(s) \in \mathrm{RH}^\infty \mid Q(s) = W(s) (I - E_2 E_2^\dagger)(sI - A_F),\ W(s) \in \mathrm{RH}^2\}, \tag{3.7}$$

where $E_2^\dagger$ is the generalized inverse of E_2, i.e. $E_2 E_2^\dagger E_2 = E_2$. Then a proper dynamic controller $\mathsf{F}(s)$ stabilizes Σ_2 and achieves the infimum, γ_2^*, if and only if $\mathsf{F}(s)$ can be written in the form of (3.5) and (3.6) for some $Q(s) \in \mathbf{Q}$.

Proof. See Chen *et al.* (1993).

Remark 3.1. It is worth noting that if (A_w, B_w, C_w) is a state space realization of $W(s)$, then $Q(s) = W(s)(I - E_2 E_2^\dagger)(sI - A_F)$ can be written as

$$Q(s) = C_w (sI - A_w)^{-1} [A_w B_w (I - E_2 E_2^\dagger) - B_w (I - E_2 E_2^\dagger) A_F] + C_w B_w (I - E_2 E_2^\dagger). \tag{3.8}$$

3.2. *Existence of H_∞-suboptimal Controllers*. We recall in this subsection a theorem of Stoorvogel (1992a) which gives a set of necessary and sufficient conditions under which the following auxiliary system has an H_∞ γ-suboptimal controller:

$$\Sigma_{au\infty}: \begin{cases} \dot{x} = Ax + Bu + E_\infty w_\infty \\ y = Cx + Dw_\infty \\ z_\infty = C_\infty x + D_\infty u \end{cases}. \tag{3.9}$$

Before we introduce Stoorvogel's theorem, let us define the following quadratic matrices:

$$\tilde{F}_\gamma(P_\infty) := \begin{bmatrix} A'P_\infty + P_\infty A + C'_\infty C_\infty + \gamma^{-2} P_\infty E_\infty E'_\infty P_\infty & P_\infty B + C'_\infty D_\infty \\ B'P_\infty + D'_\infty C_\infty & D'_\infty D_\infty \end{bmatrix} \tag{3.10}$$

and

$$\tilde{G}_\gamma(Q_\infty) := \begin{bmatrix} AQ_\infty + Q_\infty A' + E_\infty E'_\infty + \gamma^{-2} Q_\infty C'_\infty C_\infty Q_\infty & Q_\infty C' + E_\infty D' \\ CQ_\infty + DE'_\infty & DD' \end{bmatrix}. \tag{3.11}$$

It should be noted that the above matrices are dual to each other. In addition to these two matrices, we define two polynomial matrices whose roles are again completely dual to each other:

$$L_\gamma(P_\infty; s) := [sI - A - \gamma^{-2}E_\infty E_\infty' P_\infty \quad -B] \tag{3.12}$$

and

$$M_\gamma(Q_\infty, s) := \begin{bmatrix} sI - A - \gamma^{-2}Q_\infty C_\infty' C_\infty \\ -C \end{bmatrix}. \tag{3.13}$$

Next, we recall Stoorvogel's theorem.

Theorem 3.3. Consider the auxiliary system $\Sigma_{au\infty}$ as in (3.9). Assume that two systems, one characterized by $(A, B, C_\infty, D_\infty)$ and the other by (A, E_∞, C, D) have no invariant zeros on $\mathbb{C}^0$. Then the following statements are equivalent:

(1) There exists a linear, time-invariant and proper dynamic compensator $\mathsf{F}_0(s)$ such that when the control law $u(s) = \mathsf{F}_0(s)y(s)$ is applied to $\Sigma_{au\infty}$, the resulting closed-loop system is internally stable. Moreover, the H_∞ norm of the closed-loop transfer function from the disturbance input w_∞ to the controlled output z_∞ is less than γ.
(2) There exist positive semi-definite solutions P_∞, Q_∞ of the quadratic matrix inequalities $\bar{F}_\gamma(P_\infty) \geq 0$ and $\bar{G}_\gamma(Q_\infty) \geq 0$ satisfying $\rho(P_\infty Q_\infty) < \gamma^2$, such that the following rank conditions are satisfied:
   (a) rank $\{\bar{F}_\gamma(P_\infty)\}$ = normal rank $\{G_\infty(s)\}$,
   (b) rank $\{\bar{G}_\gamma(Q_\infty)\}$ = normal rank $\{G(s)\}$,
   (c) rank $\begin{bmatrix} L_\gamma(P_\infty, s) \\ \bar{F}_\gamma(P_\infty) \end{bmatrix} = n$ + normal rank $\{G_\infty(s)\}$, $\forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$,
   (d) rank $[M_\gamma(Q_\infty, s),\ \bar{G}_\gamma(Q_\infty)] = n$ + normal rank $\{G(s)\}$, $\forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$,

   where $G(s) = C(sI - A)^{-1}E_\infty + D$ and $G_\infty(s) = C_\infty(sI - A)^{-1}B + D_\infty$.

Proof. See Stoorvogel (1992a).

4. *The simultaneous H_2/H_∞ problem*

In this section, we give our main result regarding the simultaneous H_2/H_∞ problem. We have the following theorem.

Theorem 4.1. Consider the given system Σ as in (2.1). Assume that the pair (A, B) is stabilizable and the quadruple (A, B, C_2, D_2) is left invertible. Also, assume that the quadruple $(A, B, C_\infty, D_\infty)$ has no invariant zeros on the $j\omega$ axis. Then there exists an internally stabilizing control law $u = \mathsf{F}(s)x$ such that $\|T_2(\mathsf{F})\|_2 = \gamma_2^*$ and $\|T_\infty(\mathsf{F})\|_\infty < \gamma$ if and only if the following conditions hold:

(1) $\mathrm{Im}(E_2) \subseteq \mathscr{V}_g(\Sigma_P)$, which is equivalent to the fact that there exists an F such that $A_F := A + BF$ is stable and (3.4) holds.
(2) There exists a positive semi-definite solution P_∞ of the quadratic matrix inequality $\bar{F}_\gamma(P_\infty) \geq 0$ such that
   (a) rank $\{\bar{F}_\gamma(P_\infty)\}$ = normal rank $\{G_\infty(s)\}$,
   (b) rank $\begin{bmatrix} L_\gamma(P_\infty, s) \\ \bar{F}_\gamma(P_\infty) \end{bmatrix} = n$ + normal rank $\{G_\infty(s)\}$, $\forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$,

   where $\bar{F}_\gamma(P_\infty)$, $L_\gamma(P_\infty, s)$ and $G_\infty(s)$ are as defined in Subsection 3.2.
(3) There exists a positive semi-definite solution Q_∞ of the quadratic matrix inequality $\bar{G}_\gamma(Q_\infty) \geq 0$ such that
   (a) rank $\{\bar{G}_\gamma(Q_\infty)\}$ = rank $\{(I - E_2E_2^\dagger)E_\infty\}$,
   (b) rank $[M_\gamma(Q_\infty, s),\ \bar{G}_\gamma(Q_\infty)] = n$ + rank $\{(I - E_2E_2^\dagger)E_\infty\}$, $\forall s \in \mathbb{C}^0 \cup \mathbb{C}^+$,

   where $\bar{G}_\gamma(Q_\infty)$ and $M_\gamma(Q_\infty, s)$ are as defined in Subsection 3.2 with A, C_∞, C and D being replaced by $A_F := A + BF$, $C_{\infty F} := C_\infty + D_\infty F$, 0, and $V_\infty := (I - E_2E_2^\dagger)E_\infty$, respectively.
(4) $\rho(P_\infty Q_\infty) < \gamma^2$.

Proof. At first, let us note that $T_\infty(\mathsf{F})$, the closed-loop transfer function from w_∞ to z_∞ under the controller of (3.5) and (3.6) with $Q(s) \in \mathbf{Q}$, is given by

$$T_\infty(\mathsf{F}) = C_{\infty F}(sI - A_F)^{-1}E_\infty + [C_{\infty F}(sI - A_F)^{-1}B + D_\infty]W(s)V_\infty. \tag{4.1}$$

It can be simply verified that $T_\infty(\mathsf{F})$ is equivalent to the closed-loop transfer function from w_∞ to z_∞ of the following auxiliary feedback system:

$$\Sigma_\infty: \begin{cases} \dot{x} = A_Fx + Bu + E_\infty w_\infty \\ y = V_\infty w_\infty \\ z_\infty = C_{\infty F}x + D_\infty u \end{cases} \tag{4.2}$$

$$u = W(s)y. \tag{4.3}$$

Furthermore, let us observe that the system characterized by the matrix quadruple $(A_F, E_\infty, 0, V_\infty)$ has no invariant zeros on the $j\omega$ axis due to the fact that A_F is stable. We are now ready to prove the theorem.

For the given system Σ, if there exists a stabilizing proper controller $u = \mathsf{F}(s)x$ such that the corresponding $\|T_2(\mathsf{F})\|_2 = \gamma_2^*$ and $\|T_\infty(\mathsf{F})\|_\infty < \gamma$, then by Theorem 3.1 we have $\mathrm{Im}(E_2) \subseteq \mathscr{V}_g(\Sigma_P)$, which is equivalent to the fact that there exists a constant gain F such that $A_F := A + BF$ is stable and (3.4) holds. Next, $\|T_\infty(\mathsf{F})\|_\infty < \gamma$ implies that there exists a $Q(s) \in \mathbf{Q}$ such that the corresponding $W(s)$ is an H_∞ suboptimal controller to the auxiliary system Σ_∞ of (4.2). We also observe that Condition 2 in Theorem 4.1 is the condition under which there exists a state feedback H_∞ suboptimal law to the following system:

$$\begin{cases} \dot{x} = Ax + Bu + E_\infty w_\infty \\ y = x \\ z_\infty = C_\infty x + D_\infty u. \end{cases} \tag{4.4}$$

Then, from Theorem 3.3 and some simple algebra, it follows that Conditions 2–4 hold. Conversely, we assume that Conditions 1–4 in Theorem 4.1 hold. Then Conditions 2–4 imply that there exists a proper controller $W(s)$ such that when it is applied to Σ_∞ the resulting closed-loop transfer function from w_∞ to z_∞ has H_∞ norm less than γ. We first note that due to the special structure of Σ_∞, all the internally stabilizing controllers must themselves be stable. Hence $w(s)$ is stable. Next, it is shown in Stoorvogel (1992a) that in fact $W(s)$ can be chosen to be a full order observer-based controller for an auxiliary system and hence $W(s)$ can be chosen to be strictly proper, i.e. $W(s) \in \mathrm{RH}^2$. Then it is straightforward to verify that the controller (3.5) and (3.6) with $Q(s) = W(s)(I - E_2E_2^\dagger)(sI - A_F)$ achieves $\|T_2(\mathsf{F})\|_2 = \gamma_2^*$ and $\|T_\infty(\mathsf{F})\|_\infty < \gamma$. This completes the proof of Theorem 4.1.

The following remarks are in order.

Remark 4.1. Theorem 4.1 generalizes the result of Rotea and Khargonekar (1991). In fact, it is easy to verify that for the regular simultaneous H_2/H_∞ optimal control problem, conditions 1–4 of Theorem 4.1 reduce to those conditions of the main theorem (i.e. Theorem 2) in Rotea and Khargonekar (1991).

Remark 4.2. The proof of the above theorem is constructive. In fact, by utilizing the construction procedure in the proof of Theorem 4.1 and the design algorithms in Chen *et al.* (1993) and Saberi *et al.* (1991), one can easily compute the controller (whenever it exists) that solve the simultaneous H_2/H_∞ problem. The design procedure has been implemented in a Matlab software environment by Chen *et al.* (1991).

5. *Conclusion*

For a class of singular problems, necessary and sufficient conditions are established so that a simultaneous H_2/H_∞ problem, originally formulated by Rotea and Khargonekar (1991), is solvable by using dynamic state feedback controllers. The class of singular problems considered have a left invertible transfer function matrix from the control input to the controlled output which is used for the H_2 norm performance measure. This class of problems subsumes the class of regular H_2/H_∞ problems for the solvability of which

Rotea and Khargonekar (1991) established earlier the necessary and sufficient conditions.

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