Pergamon

0005-1098(93)E0065-C

Automatica, Vol. 30, No. 10, pp. 1617-1624, 1994

0005-1098/94 $7.00 + 0.00

Brief Paper

Construction and Parameterization of all Static and Dynamic H_2-optimal State Feedback Solutions for Discrete-time Systems*

B. M. CHEN,† A. SABERI,‡§ Y. SHAMASH|| and P. SANNUTI¶

Key Words—Optimal control; control theory; discrete-time systems.

Abstract—This paper considers an H_2 optimization problem via state feedback for discrete-time systems. The class of problems dealt with here has a left invertible transfer matrix function from the control input to the controlled output. The paper constructs and parameterizes all the static and dynamic H_2-optimal state feedback solutions. Moreover, all the eigenvalues of an optimal closed-loop system are characterized. All optimal closed-loop systems share a set of eigenvalues which are termed the *optimal fixed modes.* Every H_2-optimal controller must assign among the closed-loop eigenvalues the set of optimal fixed modes. This set of optimal fixed modes includes a set of optimal fixed decoupling zeros which shows the minimum absolutely necessary number and locations of pole-zero cancellations present in any H_2-optimal design. Most of the results presented here are analogous to, but not quite the same as, those for continuous-time systems. In fact, there are some fundamental differences between the continuous and discrete-time systems reflecting mainly the inherent nature and characteristics of these systems.

1. *Introduction*

OPTIMIZATION THEORY IS one of the corner stones of modern control theory. In a typical control design, the given specifications are at first transformed into a performance index, and then control laws are sought which would minimize some norm, say the H_2 or H_∞ norm, of the performance index. This paper considers discrete-time systems, and focuses on H_2-optimal control theory or otherwise known as Linear Quadratic Gaussian (LQG) control theory. For discrete-time systems, optimal control theory based on the H_2 norm was heavily studied in the 70s and early 80s (see, e.g. Athans, 1971; Dorato and Levis, 1971; Kucera, 1972; Molinari, 1975; Pappas *et al.*, 1980). Some of these aspects of discrete-time H_2-optimal control theory can be found in most graduate text books on control (see, e.g. Anderson and Moore, 1979; Kwakernaak and Sivan, 1972). Although a lot of research effort has been spent during the 70s and 80s, the conditions for the existence of optimal solutions for a general discrete-time H_2-optimal control problem, and a way of determining an optimal solution if it exists, were not known until the very recent work of Trentelman and Stoorvogel (1992). Trentelman and Stoorvogel, not only obtained a set of necessary and sufficient conditions for the existence of optimal solutions to a general discrete-time H_2-optimal control problem, but also constructed one such solution. This paper explores, among other things, several issues associated with the construction of all H_2-optimal solutions for a general discrete-time H_2-optimal control problem while utilizing state feedback controllers.

The motivation for the present work comes from the recent work of Chen *et al.* (1993) on continuous-time systems. The work done by Chen *et al.* concentrates on three different aspects of H_2-optimal control problems:

(1) it parameterizes and then constructs the set of all static as well as dynamic H_2-optimal state feedback controllers;
(2) it introduces the notion of optimal fixed modes which are the complex numbers that every optimal state feedback controller must assign among the closed-loop eigenvalues. Moreover, it identifies and constructs the set of all optimal fixed modes; and
(3) it introduces the notion of optimal fixed decoupling zeros which are either the input or the output decoupling zeros (or both) of an H_2-optimal closed-loop transfer function for every optimal controller the given system uses. Moreover, it identifies and constructs the set of all optimal fixed decoupling zeros.

Evidently, all three aspects of H_2-optimal control problems described above need to be studied for discrete-time systems as well. The intention of this paper is to do exactly this. The development given here for discrete-time systems is analogous to, but not quite the same as, that for continuous-time systems. In order to preserve the thought process and the conceptual analogy, we use here the same notation as in continuous-time systems for several objects such as vectors, matrices, subsystems etc. However, the objects introduced here are different from those in continuous-time systems in the sense that they are produced differently and, moreover, might have different properties. Undoubtedly, such differences reflect the specific nature and characteristics of discrete-time systems. That is, due to the nature of discrete-time systems, an H_2 optimization problem for them is bound to be different from that of continuous-time systems.

The paper is organized as follows. Section 2 gives a clear mathematical statement of the problem, while Section 3 deals with some preliminary results. Section 4 recalls the necessary

* Received 2 February 1993; revised 24 August 1993; received in final form 16 November 1993. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Kenko Uchida under the direction of Editor Tamer Başar. Corresponding author Dr Ben M. Chen. Tel. +65 772 2289; Fax +65 779 1103; E-mail bmchen@ee.nus.sg.

† Department of Electrical Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511.

‡ School of Electrical Engineering & Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.

§ The work of A. Saberi is supported in part by NASA Langley Research Center under grant contract NAG-1-1210.

|| College of Engineering and Applied Science, State University of New York at Stony Brook, Stony Brook, NY 11794, U.S.A.

¶ Department of Electrical & Computer Engineering, P.O. Box 909, Rutgers University, Piscataway, NJ 08855-0909, U.S.A.

and sufficient conditions under which a static or a dynamic H_2-optimal state feedback controller exists. Section 5 considers static feedback controllers. Here an algorithm called 'optimal gains and fixed modes' (abbreviated as OGFM) is developed. Section 6 considers the case of dynamic controllers, where the well known Q-parameterization technique is used to characterize all the possible H_2-optimal solutions. Finally, the conclusions are given in Section 7.

Throughout the paper, A' denotes the transpose of A, I denotes an identity matrix while I_k denotes the identity matrix of dimension $k \times k$. Also, $\mathbb{C}$, $\mathbb{C}^{\odot}$, $\mathbb{C}^{\circ}$ and $\mathbb{C}^{\otimes}$, respectively denote the whole complex plane, the open unit disc, the unit circle, and the set of complex numbers outside the unit circle. A matrix is said to be stable if all its eigenvalues are in $\mathbb{C}^{\odot}$. Im $[V]$ denotes the image of V. Given a stable transfer function $G(z)$, as usual, its H_2 norm is defined by

$$\|G\|_2^2 := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\{G'(e^{-j\omega})G(e^{j\omega})\}\,d\omega.$$

Also, RH^s denotes the set of real-rational transfer functions which are stable and strictly proper. RH^∞ denotes the set of real-rational transfer functions which are stable and proper.

2. *Problem statement*

Consider the following discrete-time system Σ characterized by,

$$\Sigma:\begin{cases} x(k+1) = Ax(k) + Bu(k) + Ew(k) \\ y(k) = x(k) \\ z(k) = Cx(k) + Du(k) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^l$ is the unknown disturbance and $z \in \mathbb{R}^q$ is the controlled output. Also, consider an arbitrary proper controller

$$u = \mathrm{F}(z)x. \tag{2}$$

A controller $u = \mathrm{F}(z)x$ is said to be admissible if it provides internal stability of the resulting closed-loop system. Let $T_{zw}(\mathrm{F})$ denote the closed-loop transfer function from w to z after applying an admissible controller $u = \mathrm{F}(z)x$ to Σ. Then the H_2 optimization state feedback problem for Σ is to find an admissible state feedback controller $\mathrm{F}(z)$ which minimizes $\|T_{zw}(\mathrm{F})\|_2$.

For future use, we also define another system Σ_* related to Σ,

$$\Sigma_*:\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ z(k) = Cx(k) + Du(k). \end{cases} \tag{3}$$

The following definitions will be convenient in the sequel.

Definition 2.1. (*The infimum of* H_2 *optimization.*) For a given system Σ, the infimum of the H_2 norm of the closed-loop transfer function $T_{zw}(\mathrm{F})$ over all the stabilizing proper controllers $\mathrm{F}(z)$ is denoted by γ^*, namely

$$\gamma^* := \inf\{\|T_{zw}(\mathrm{F})\|_2 \mid u = \mathrm{F}(z)x \text{ internally stabilizes } \Sigma\}. \tag{4}$$

Definition 2.2. (*The* H_2 *optimal controllers.*) A stabilizing proper controller $\mathrm{F}(z)$ is said to be an H_2-optimal controller if $\|T_{zw}(\mathrm{F})\|_2 = \gamma^*$. The sets of all optimal static and dynamic state feedback controllers are, respectively, denoted by F_s^* and F_d^*. Obviously, $F_s^* \subset F_d^*$.

Definition 2.3. (*The* H_2 *optimal fixed modes.*) A scalar $\lambda \in \mathbb{C}^{\odot}$ is said to be an H_2-optimal fixed mode if λ is a pole of the closed-loop system for every H_2-optimal controller of a particular type, say static or dynamic, that one uses. The sets of all the H_2-optimal fixed modes corresponding to the static and the dynamic controllers are, respectively, denoted by Ω_s^* and Ω_d^*.

Definition 2.4. (*The* H_2 *optimal fixed decoupling zeros.*) A scalar $\lambda \in \mathbb{C}^{\odot}$ is said to be an H_2-optimal fixed decoupling zero if λ is either an input decoupling zero, or an output decoupling zero, or an input–output decoupling zero (Rosenbrock, 1970) of the closed-loop system for every H_2-optimal controller of a particular type, say static or dynamic, that one uses. The sets of all the H_2-optimal fixed decoupling zeros corresponding to the static and the dynamic controllers are, respectively, denoted by Λ_s^* and Λ_d^*.

Throughout this paper, for simplicity of presentation, we assume that Σ_* is left invertible. As in Chen *et al.* (1993), the goals of this paper are:

1. to present explicit design methods to determine the sets of static and dynamic optimal controllers F_s^* and F_d^*;
2. to determine the sets of optimal fixed modes Ω_s^* and Ω_d^*;
3. to determine the sets of optimal fixed decoupling zeros Λ_s^* and Λ_d^*.

3. *Preliminaries*

In this section we recall the special coordinate basis for a linear time-invariant nonstrictly proper system. Such a coordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structures of a given system as well as other system geometric properties. It is instrumental in the derivation of the method described in Section 5.

Consider the system Σ_* as in (3). It can be easily shown that using a singular value decomposition one can always find an orthogonal transformation U and a nonsingular matrix V that render the direct feedthrough matrix d into the following form,

$$\bar{D} = UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where m_0 is the rank of D. Without loss of generality one can assume that the matrix D in equation (3) has the form as shown in equation (5). Thus the system in (3) can be rewritten as

$$\begin{cases} x(k+1) = Ax(k) + [B_0 \;\; B_1]\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix}, \\ \begin{pmatrix} z_0(k) \\ z_1(k) \end{pmatrix} = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}x(k) + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_0(k) \\ u_1(k) \end{pmatrix}, \end{cases} \tag{6}$$

where B_0, B_1, C_0 and C_1 are matrices of appropriate dimensions. Note that the inputs u_0 and u_1, and the outputs z_0 and z_1 are those of the transformed system. Namely,

$$u = V\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = Uz.$$

We have the following theorem.

Theorem 3.1. (*SCB.*) Let Σ_* be left invertible. Then, there exist nonsingular transformations Γ_s, Γ_o and Γ_i such that

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \Gamma_i\begin{pmatrix} u_0 \\ u_d \end{pmatrix}, \quad x = \Gamma_s\begin{pmatrix} x_a \\ x_a^0 \\ x_a^+ \\ x_b \\ x_d \end{pmatrix}, \quad \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \Gamma_o\begin{pmatrix} z_0 \\ z_d \\ z_b \end{pmatrix}$$

and

$$\Gamma_s^{-1}(A - B_0C_0)\Gamma_s = \begin{bmatrix} A_{aa} & 0 & 0 & L_{ab}C_b & L_{ab}C_d \\ 0 & A_{aa}^0 & 0 & L_{ab}^0C_b & L_{ad}^0C_d \\ 0 & 0 & A_{aa}^+ & L_{ab}^+C_b & L_{ad}^+C_d \\ 0 & 0 & 0 & A_{bb} & L_{bd}C_d \\ B_dE_{da}^- & B_dE_{da}^0 & B_dE_{da}^+ & B_dE_{db} & A_d \end{bmatrix}, \tag{7}$$

$$\Gamma_s^{-1}[B_0, B_1]\Gamma_i = \begin{bmatrix} B_{a0} & 0 \\ B_{a0}^0 & 0 \\ B_{a0}^+ & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix}, \tag{8}$$

$$\Gamma_o^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0d} \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & C_b & 0 \end{bmatrix}, \qquad (9)$$

and

$$\Gamma_o^{-1}\begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}\Gamma_i = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (10)$$

where $\lambda(A_{aa}^-) \in \mathbb{C}^{\odot}$, $\lambda(A_{aa}^0) \in \mathbb{C}^{\circ}$, $\lambda(A_{aa}^+) \in \mathbb{C}^{\otimes}$. Also, (A_{bb}, C_b) is observable and the subsystem characterized by (A_d, B_d, C_d) is invertible with no invariant zeros.

The proof of this theorem can be found in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). In what follows, we state some important properties of the SCB which are pertinent to our present work.

Property 3.1. System Σ_* is invertible if and only if x_b is nonexistent.

Property 3.2. $\lambda(A_{aa}^-) \cup \lambda(A_{aa}^0) \cup \lambda(A_{aa}^+)$ are the invariant zeros of Σ_*. We note that $\lambda(A_{aa}^-)$ are the stable and $\lambda(A_{aa}^+)$ are the unstable invariant zeros of Σ_* while $\lambda(A_{aa}^0)$ are on the unit circle.

Property 3.3. The pair (A, B) is stabilizable if and only if $(A_{\mathrm{con}}, B_{\mathrm{con}})$ is stabilizable where

$$A_{\mathrm{con}} = \begin{bmatrix} A_{aa}^0 & 0 & L_{ab}^0 C_b \\ 0 & A_{aa}^+ & L_{ab}^+ C_b \\ 0 & 0 & A_{bb} \end{bmatrix}, \quad B_{\mathrm{con}} = \begin{bmatrix} B_{a0}^0 & L_{ad}^0 \\ B_{a0}^+ & L_{ad}^+ \\ B_{b0} & L_{bd} \end{bmatrix}. \qquad (11)$$

Let us next recall the following definition of a weakly unobservable subspace (Hautus and Silverman, 1983; Wonham, 1985).

Definition 3.1. We define the stabilizable weakly unobservable subspace $\mathcal{V}_g(\Sigma_*)$ as the largest subspace $\mathcal{V}$ for which there exists a mapping F such that the following subspace inclusions are satisfied,

$$(A + BF)\mathcal{V} \subseteq \mathcal{V}, \qquad (12)$$

$$(C + DF)\mathcal{V} = \{0\}, \qquad (13)$$

and such that $A + BF \mid \mathcal{V}$ is asymptotically stable.

Property 3.4. x_a^- spans $\mathcal{V}_g(\Sigma_*)$.

4. *Existence of optimal controllers*

One of our goals in this paper is to parameterize and to construct all the optimal static and dynamic state feedback controllers. Before we do so, we need to know the conditions under which an optimal controller exists. In this regard in their recent paper, Trentelman and Stoorvogel (1992), consider a general discrete-time H_2-optimal control problem while utilizing proper output feedback controllers, and establish for the first time the necessary and sufficient conditions under which an optimal controller exists. In fact, under those conditions, Trentelman and Stoorvogel also construct an optimal controller. However, as stated earlier, among other things one of our goals here is to parameterize and to construct the set of all optimal controllers rather than a single one, for the special case when the measured output coincides with the state.

To begin with, we particularize the necessary and sufficient conditions given by Trentelman and Stoorvogel (1992) for the special case when state feedback controllers are used. These conditions are stated in terms of an auxiliary system Σ_{au} derived from the data of the given H_2-optimal control problem. It turns out that Σ_{au} itself is prescribed in terms of a solution of a linear matrix inequality (LMI). To define the LMI, consider a linear matrix function $F(P)$,

$$F(P) := \begin{bmatrix} A'PA - P + C'C & A'PB + C'D \\ B'PA + D'C & B'PB + D'D \end{bmatrix}. \qquad (14)$$

Let P be the largest real symmetric solution of the LMI,

$$F(P) \geq 0. \qquad (15)$$

A similar LMI is used in continuous-time systems as well. As in its continuous-time counterpart, whenever (A, B) is stabilizable, it can be shown that the largest real symmetric solution P of the LMI (15) exists and is unique. In fact, later on in Section 5, we construct explicitly such a solution. Knowing the largest real symmetric solution P of $F(P) \geq 0$, one can next define two matrices C_{P} and D_{P} satisfying,

$$F(P) = \begin{bmatrix} C_{\mathrm{P}}' \\ D_{\mathrm{P}}' \end{bmatrix}[C_{\mathrm{P}} \quad D_{\mathrm{P}}]. \qquad (16)$$

We can now define the following auxiliary system Σ_{au}:

$$\Sigma_{\mathrm{au}}: \begin{cases} x_{\mathrm{P}}(k+1) = Ax_{\mathrm{P}}(k) + Bu_{\mathrm{P}}(k) + Ew_{\mathrm{P}}(k) \\ y_{\mathrm{P}}(k) = x_{\mathrm{P}}(k) \\ z_{\mathrm{P}}(k) = C_{\mathrm{P}}x_{\mathrm{P}}(k) + D_{\mathrm{P}}u_{\mathrm{P}}(k). \end{cases} \qquad (17)$$

We are now ready to state the following theorem which gives the conditions for the existence of optimal static as well as dynamic state feedback controllers.

Theorem 4.1. Consider the given system Σ as in (1), and the auxiliary system Σ_{au} as in (17). Define a subsystem Σ_{P} of Σ_{au} as that characterized by the quadruple $(A, B, C_{\mathrm{P}}, D_{\mathrm{P}})$. We have the following results:

1. (*Existence of an optimal static state feedback controller.*) The infimum, γ^*, can be attained by a static stabilizing state feedback controller if and only if the pair (A, B) is stabilizable and $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_{\mathrm{P}})$.
2. (*Existence of an optimal proper dynamic state feedback controller.*) The infimum, γ^*, can be attained by a proper dynamic stabilizing state feedback controller if and only if the pair (A, B) is stabilizable and $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_{\mathrm{P}})$.

Moreover, the infimum, γ^*, is given by

$$\gamma^* = \sqrt{\mathrm{tr}(E'PE)}, \qquad (18)$$

where P is the maximal solution of $F(P) \geq 0$ for $F(P)$ as in (14).

Proof. This is an extract of Theorem 4.10 of Trentelman and Stoorvogel (1992) for the special case when state feedback controllers are utilized. ■

Remark 4.1. Whenever Σ_* has no invariant zeros on the unit circle $\mathbb{C}^{\circ}$, it is simple to verify that $\mathcal{V}_g(\Sigma_{\mathrm{P}}) = \mathbb{R}^n$, and thus the condition $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_{\mathrm{P}})$ is automatically satisfied. Hence, an optimal static as well as a proper dynamic state feedback controller for Σ always exists whenever the pair (A, B) is stabilizable and whenever Σ_* has no invariant zeros on $\mathbb{C}^{\circ}$.

As in continuous-time systems, a key concept which led to Theorem 4.1 is the formulation of an interrelationship between the given discrete-time H_2-optimal control problem and a Disturbance Decoupling Problem with internal Stability (DDPS) for the auxiliary system Σ_{au}. This interrelationship also plays an important role in the development of our algorithm OGFM which is to be given in the following section. As such, we like to recall here this interrelationship. To do so, let us next consider a general controller $\mathrm{F}(z)$ and let it be applied to both Σ and Σ_{au}. That is, let $u = \mathrm{F}(z)x$ and $u_{\mathrm{P}} = \mathrm{F}(z)x_{\mathrm{P}}$. It turns out that a controller $\mathrm{F}(z)$ solves the given H_2 optimization problem when applied to Σ if and only if it solves the DDPS problem with internal stability when applied to Σ_{au}. This is emphasized by recalling the following lemma.

Lemma 4.1. The following two statements are equivalent.

(i) The controller $u = \mathrm{F}(z)x$ when applied to the given system Σ is internally stabilizing and the resulting closed-loop transfer function from w to z has the H_2 norm γ^*.

(ii) The controller $u_P = F(z)x_P$ when applied to the new system Σ_{au} is internally stabilizing and the resulting closed-loop transfer function from w_P to z_P has the H_2 norm 0.

Proof. It follows from Lemma 3.4 of Trentelman and Stoorvogel (1992). ■

The next remark points out some fundamental differences between the continuous- and discrete-time systems.

Remark 4.2. As indicated in the introduction, the development given here and throughout this paper for discrete-time systems is analogous to, but not exactly the same as, that for continuous-time systems. Readers familiar with Chen *et al.* (1993) recognize that the condition $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$ given in Theorem 4.1 for the existence of either a static or a dynamic optimal state feedback controller, is the same as the one given in Chen *et al.* for the continuous-time case. However, one has to be careful. That is, one has to note that although notationally the LMI $F(P) \geq 0$, the auxiliary system Σ_{au}, and the subsystem Σ_P of Σ_{au}, are introduced for both the continuous- and discrete-time systems, the definitions and hence the properties for the two cases are quite different. In particular, it turns out that the zero structure of Σ_P for discrete- and continuous-time systems is quite different. For example, for all discrete-time H_2-optimal control problems whenever Σ_* is left invertible, the matrix D_P defined here is always nonsingular, and consequently the Σ_P defined here has no infinite zeros at all. This is in contrast to the continuous-time case where the matrix D_P is in general singular, and moreover Σ_P there has the same infinite zero structure as that of Σ_*. As Σ_P plays a crucial role in our development, these differences between the discrete- and continuous-time systems are reflected in many places throughout the paper.

5. *Design of optimal static state feedback controllers*

For discrete-time systems, so far in the literature, no procedure exists to construct the set of all H_2-optimal static feedback controllers F_s^*, the set of all optimal fixed modes Ω_s^* as well as the set of all optimal fixed decoupling zeros Λ_s^*. The goal of this section is to rectify this situation. In this regard, we present here an explicit design procedure to determine the set F_s^*, Ω_s^* and Λ_s^*.

In view of Theorem 4.1, the infimum, γ^*, can be attained by a static state feedback law if and only if (A, B) is stabilizable and $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Next, it is important to comment on the need to calculate the sets F_s^*, Ω_s^* and Λ_s^*. Constructing the set of all H_2-static optimal state feedback controllers F_s^* is necessary for a number of reasons. In practice, H_2 optimality is not the sole design goal. Some other criteria come into play in the final stages of a design. For instance, as in Rotea and Khargonekar (1991), one would like to minimize the H_2 norm of a certain performance index while keeping, say the H_∞ norm of some other performance index below a certain value. Solutions to such simultaneous H_2/H_∞ problems can be tracked easily by knowing the parameterization and construction of all H_2-optimal controllers. Next, an optimal design indicates that certain poles of the closed-loop system are always located at some fixed locations in the complex plane. Constructing the set Ω_s^*, which is the set of all such optimal fixed modes, helps a designer to learn the flexibility one has in assigning the closed-loop poles. In this regard, let us note that it is a common, but in general a mistaken, notion that one should place all the stable and the mirror images† of all the unstable invariant zeros of Σ_* in Ω_s^*. It is not necessary to do so in general. To illustrate this, consider one extreme case when $\mathrm{Im}(E) = 0$. In this case, any controller which guarantees the closed-loop stability is an H_2-optimal controller, and hence the set Ω_s^* is an empty set. That is, in this case, none of the stable invariant zeros and none of the mirror images of the unstable invariant zeros of Σ_* need to be in Ω_s^*. On the other hand, consider another extreme case when $\mathrm{Im}(E) = \mathcal{V}_g(\Sigma_P)$. In this case, as will be seen shortly, all the stable invariant zeros and all the mirror images of the unstable invariant zeros of Σ_* must be in Ω_s^*. However, in general when $\mathrm{Im}(E)$ is strictly included in $\mathcal{V}_g(\Sigma_P)$, only some, but not all, of the stable invariant zeros and only some, but not all, of the mirror images of the unstable invariant zeros need to be in Ω_s^*. Let us also emphasize that as will be computed later on, besides those related to the invariant zeros of Σ_*, there could also be other elements in Ω_s^*. Next, let us emphasize one more aspect of H_2-optimal control. Namely, one encounters often certain pole–zero cancellations in an optimal design. In practice, one tries to avoid pole–zero cancellations close to the unit circle. Thus, constructing the set Λ_s^* which shows the minimum absolutely necessary number and locations of pole–zero cancellations present in any H_2-optimal design, is of immense importance in practice.

As in Chen *et al.* (1993), a basic component of our design procedure here is an algorithm called 'optimal gains and fixed modes' which is abbreviated as OGFM. The matrix quintuple (A, B, C, D, E) is a set of input parameters to OGFM, while the outputs of OGFM are, F_s^*, Ω_s^*, Λ_s^*, and the infimum, γ^*. Besides these, OGFM also calculates the maximal solution P of the inequality $F(P) \geq 0$, and checks whether the condition, $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$, is satisfied by the given problem or not. The leading component of the algorithm is the isolation of a pair of matrices (A_z, B_z) from the input data (A, B, C, D, E). A gain matrix F_z which renders $A_z - B_z F_z$ asymptotically stable, is a parameter by varying which appropriately the entire set of static optimal feedback gains is constructed. In what follows, a step-by-step description of the algorithm OGFM is given. As in Chen *et al.* (1993), a key tool used for all the main calculations in OGFM is the construction of appropriate SCBs of some subsystems. A software tool box for constructing SCBs is given by Lin *et al.* (1992). Also, a software package in Matlab for the OGFM algorithm is developed by Chen *et al.* (1991).

Step 1. (*Computation of the pair* (A_z, B_z)): In this step, we compute a pair (A_z, B_z) which leads to the parameterization of the set of all optimal static state feedback gains. Our computations are divided into several substeps.

Step 1(*a*). *Construction of the SCB of* Σ_*: Transform the subsystem Σ_* into the SCB as given in (7)–(10) of Section 3. For future development, let us compute

$$\Gamma_s^{-1}E = [(E_a^-)', (E_a^0)', (E_a^+)', (E_b)', (E_d)']' \quad (19)$$

and define $E_x = [(E_a^+)', (E_b)', (E_d)']'$.

Step 1(*b*). *Construction of the subsystem* Σ_P: An explicit construction of the subsystem Σ_P is purused in this step. Define a matrix quadruple,

$$A_x := \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b & L_{ad}^+ C_d \\ 0 & A_{bb} & L_{bd} C_d \\ B_d E_{da}^+ & B_d E_{db} & A_d \end{bmatrix},$$

$$B_x := \begin{bmatrix} B_{a0}^+ & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix} \quad C_x := \Gamma_o \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C_d \\ 0 & C_b & 0 \end{bmatrix}, \quad (20)$$

$$D_x := \Gamma_o \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then solve the following algebraic Riccati equation,

$$P_x = A_x' P_x A_x + C_x' C_x - (B_x' P_x A_x + D_x' C_x)' \times (D_x' D_x + B_x' P_x B_x)^{-1} (B_x' P_x A_x + D_x' C_x) \quad (21)$$

for $P_x > 0$. Note that such a solution P_x always exists because the quadruple (A_x, B_x, C_x, D_x) is left invertible and has no invariant zeros in the closed disc $\mathbb{C}^\circ \cup \mathbb{C}^{\bigcirc}$. As a matter of fact, a non-recursive procedure that solves the above Riccati equation can be found in Chen *et al.* (1993).

† For continuous-time systems, the mirror image of a complex number $\alpha + j\beta$ is defined as $-\alpha + j\beta$, whereas in discrete-time systems, the mirror image of a complex number $re^{j\theta}$ is defined as $(1/r)e^{j\theta}$.

Now, by some algebraic manipulations, it can be shown that the maximum solution of LMI $F(P) \geq 0$ is given by,

$$P = (\Gamma_s^{-1})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_x \end{bmatrix} \Gamma_s^{-1}. \tag{22}$$

Thus, we have

$$\gamma^* = \sqrt{\operatorname{tr}(E'PE)} \tag{23}$$

and

$$\begin{aligned} C_P &= (B'PB + D'D)^{-1/2}(B'PA + D'C), \\ D_P &= (B'PB + D'D)^{1/2}. \end{aligned} \tag{24}$$

Step 1(*c*). *Construction of the SCB of* Σ_P: Here we transform the system Σ_P into an SCB as given in Section 3. Following the procedure of Chen *et al.* (1992), we note that Σ_P is invertible with no infinite zeros (i.e. D_P is nonsingular), and moreover Σ_P does not have any invariant zeros in $\mathbb{C}^{\otimes}$. Notationally, to all the submatrices and transformations in the SCB of Σ_P, we append a subscript $_P$ to signify their relation to the system Σ_P. To facilitate the construction of the SCB of Σ_P, first compute the orthogonal transformation matrix U_P and a nonsingular transformation matrix V_P such that

$$BV_P = B_{0P}, \quad U_P C_P = C_{0P} \quad \text{and} \quad U_P D_P V_P = I. \tag{25}$$

Then construct the nonsingular transformations Γ_{sP} such that

$$\Gamma_{sP}^{-1}(A - B_{0P}C_{0P})\Gamma_{sP} = \begin{bmatrix} A_{aaP}^- & 0 \\ 0 & A_{aaP}^0 \end{bmatrix},$$

$$\Gamma_{sP}^{-1}B_{0P} = \begin{bmatrix} B_{a0P}^- \\ B_{a0P}^0 \end{bmatrix}, \quad C_{0P}\Gamma_{sP} = [C_{0aP}^- \quad C_{0aP}^0].$$

From the property of SCB (e.g. Property 3.4), it is simple to see that $\operatorname{Im}(E) \subseteq \mathscr{V}_g(\Sigma_P)$ implies that

$$\Gamma_{sP}^{-1}E = \begin{bmatrix} E_{aP}^- \\ 0 \end{bmatrix}. \tag{26}$$

If $\Gamma_{sP}^{-1}E$ is not of the form (26), the infimum γ^* is not attainable and the procedure of OGFM stops at this point. Otherwise it continues to the next step.

Step 1(*d*). *Decomposition of* Σ_{au}: In this step, Σ_{au} is decomposed into two parts, one part being controllable via the disturbance w and the other not. Consider the pair (A_{aaP}^-, E_{aP}^-). This pair need not be controllable, that is, the disturbance w with its coefficient matrix as E_{aP}^- need not affect all the modes of A_{aaP}^-. Compute a nonsingular transformation T_{aP} such that

$$T_{aP}^{-1}A_{aaP}^-T_{aP} = \begin{bmatrix} A_{aaP}^{11} & A_{aaP}^{12} \\ 0 & A_{aaP}^{22} \end{bmatrix}$$

and

$$T_{aP}^{-1}E_{aP}^- = \begin{bmatrix} E_{aP}^1 \\ 0 \end{bmatrix}, \tag{27}$$

where the pair (A_{aaP}^{11}, E_{aP}^1) is completely controllable. Also, let us partition

$$T_{aP}^{-1}B_{a0P}^- = \begin{bmatrix} B_{a0P}^1 \\ B_{a0P}^2 \end{bmatrix} \quad \text{and} \quad C_{0aP}^-T_{aP} = [C_{0aP}^1 \quad C_{0aP}^2].$$

Finally, form the matrices A_z and B_z as follows:

$$A_z := \begin{bmatrix} A_{aaP}^{22} & 0 \\ 0 & A_{aaP}^0 \end{bmatrix}, \quad B_z := \begin{bmatrix} B_{a0P}^2 \\ B_{a0P}^0 \end{bmatrix}. \tag{28}$$

Again, from the property of SCB (e.g. Property 3.3), it is simple to verify that the pair (A_z, B_z) is stabilizable if and only if the pair (A, B) is stabilizable. Thus, whenever (A, B) is stabilizable, a gain F_z exists such that $\lambda(A_z - B_zF_z) \subset \mathbb{C}^{\odot}$.

Step 2. (*Parameterization and construction of the sets* F_s^*, Ω_s^* *and* Λ_s^*): in this step F_s^*, Ω_s^* and Λ_s^* are parameterized in terms of F_z which renders $A_z - B_zF_z$ asymptotically stable. Let us define the set

$$\mathscr{F}_z := \{F_z \mid \lambda(A_z - B_zF_z) \subset \mathbb{C}^{\odot}\}. \tag{29}$$

Let us also partition $F_z \in \mathscr{F}_z$ to be compatible with the partitions of A_z and B_z as,

$$F_z = [F_{a0P}^2 \quad F_{a0P}^0]. \tag{30}$$

Let

$$F = -V_P[C_{0aP}^1 \quad C_{0aP}^2 + F_{a0P}^2 \quad C_{0aP}^0 + F_{a0P}^0]T_{sP}^{-1}, \tag{31}$$

where

$$T_{sP} = \Gamma_{sP}\begin{bmatrix} T_{aP} & 0 \\ 0 & I \end{bmatrix}.$$

Also, let

$$F_s^* := \{F \in \mathbb{R}^{m \times n} \mid F \text{ is given by (31) with } F_z \in \mathscr{F}_z\}, \tag{32}$$

$$\Omega_s^* := \lambda(A_{aaP}^{11}) \cup \{\text{input decoupling zeros of } (A_z, B_z)\}, \tag{33}$$

and

$$\begin{aligned} \Lambda_s^* := \{&\lambda(A_{aaP}^{11}) \cap \lambda(A_{aa}^-)\} \\ &\cup \{\text{input decoupling zeros of } (A_z, B_z)\}. \end{aligned} \tag{34}$$

This concludes the description of OGFM.

We have the following theorem.

Theorem 5.1. Consider the given system Σ as in (1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\operatorname{Im}(E) \subseteq \mathscr{V}_g(\Sigma_P)$. Then we have:

(1). (*Optimal static state feedback controllers*). Any member of the set F_s^* is an optimal state feedback controller, i.e. the state feedback law $u = Fx$ where F is of the form (31) with $F_z \in \mathscr{F}_z$, when applied to Σ is stabilizing and the closed-loop H_2 norm is equal to γ^*. Conversely, any state feedback law $u = Fx$ which is stabilizing and yields a closed-loop H_2 norm equal to γ^* is such that F is of the form (31) with $F_z \in \mathscr{F}_z$.

(2). (*Optimal fixed modes*). The set of all H_2-optimal fixed modes under a static state feedback is given by Ω_s^*. That is, any optimal static state feedback controller must assign the elements of Ω_s^* among the closed-loop eigenvalues. The rest of the closed-loop eigenvalues can be assigned arbitrarily in $\mathbb{C}^{\odot}$ as long as they are symmetric with respect to the real axis, by an appropriate selection of a static state feedback controller from F_s^*.

(3). (*Optimal fixed decoupling zeros*). The set of all H_2-optimal fixed decoupling zeros under a static state feedback is given by Λ_s^*. That is, regardless of the choice of F from F_s^*, the absolutely minimum number and locations of pole–zero cancellations in the optimal closed-loop transfer functions are given by the set Λ_s^*.

(4). (*Other pole–zero cancellations in optimal closed-loops*). For any $F \in F_s^*$, define

$$\Lambda_{idz}(F) := \lambda(A + BF)/\Omega_s^*.$$

Then for any $\lambda \in \Lambda_{idz}(F)$, λ is an input decoupling zero of the closed-loop system comprising of Σ and the static state feedback controller $u = Fx$. Moreover, by varying F over the set $\mathscr{F}_s^*$, the elements of $\Lambda_{idz}(F)$ can be assigned arbitrarily in $\mathbb{C}^{\odot}$ as long as they are symmetric with respect to the real axis.

Proof. It is omitted due to space limitations. Conceptually, it follows along the same lines as the proof of Theorem 4.1 of Chen *et al.* (1993) when specific characteristics of discrete-time systems are taken into account. Details can be found in an extended version of this paper (Chen *et al.*, 1993). ■

Given the quintuple (A, B, C, D, E), it is clear that the algorithm OGFM explicitly yields the sets F_s^*, Ω_s^*, and Λ_s^*. Let us note that in the given quintuple (A, B, C, D, E), the quadruple (A, B, C, D) prescribes the dynamic model Σ_* of the given plant, while the matrix E prescribes how the disturbance w is coupled to the plant. Obviously, for any fixed dynamic model Σ_* of the plant, the sets F_s^*, Ω_s^* and Λ_s^* have a definite relationship with the matrix E. To examine this, let us recall first the condition for the existence of an H_2-optimal state feedback controller, namely $\mathrm{Im}(E) \subseteq \mathscr{V}_g(\Sigma_P)$. It is interesting to note that the size of the set F_s^* decreases while the size of the set Ω_s^* grows as $\mathrm{Im}(E)$ varies from $\{0\}$ to $\mathscr{V}_g(\Sigma_P)$. Hence, the sizes of F_s^* and Ω_s^* obtained for $\mathrm{Im}(E) = \{0\}$ are, respectively, the largest and the smallest possible ones, while the sizes of the same obtained for $\mathrm{Im}(E) = \mathscr{V}_g(\Sigma_P)$ are, respectively, the smallest and the largest possible ones. Moreover, both the sets F_s^* and Ω_s^* have a *nested* property as stated and formalized in the following proposition.

Proposition 5.1. Consider two different values for E, say E_1 and E_2, and let $\mathrm{Im}(E_1) \subseteq \mathrm{Im}(E_2) \subseteq \mathscr{V}_g(\Sigma_P)$. Let F_{s1}^* and Ω_{s1}^* be the sets corresponding to F_s^* and Ω_s^* for the case when $E = E_1$. Similarly, let F_{s2}^* and Ω_{s2}^* be the sets corresponding to F_s^* and Ω_s^* for the case when $E = E_2$. Then, we have

$$F_{s2}^* \subseteq F_{s1}^* \quad \text{and} \quad \Omega_{s1}^* \subseteq \Omega_{s2}^*. \tag{35}$$

Proof. Let us first consider the proof of property $F_{s2}^* \subseteq F_{s1}^*$. Given that $\mathrm{Im}(E_1) \subseteq \mathrm{Im}(E_2)$, we note that there exists a matrix X such that $E_1 = E_2 X$. Then for any $F \in F_{s2}^*$, i.e.

$$(C_P + D_P F)(zI - A - BF)^{-1} E_2 = 0,$$

we have

$$(C_P + D_P F)(zI - A - BF)^{-1} E_1 = (C_P + D_P F)(zI - A - BF)^{-1} E_2 X = 0,$$

which implies that $F \in F_{s1}^*$. Hence, $F_{s2}^* \subseteq F_{s1}^*$.

Let us next consider the proof of property $\Omega_{s1}^* \subseteq \Omega_{s2}^*$. By definition, we know that the input decoupling zeros of (A, B) are always the optimal fixed modes for any E. Thus, for the sake of simplicity but without loss of any generality, we assume that the pair (A, B) is controllable. Also, let A_{aaP1}^{11} and A_{aaP2}^{11} be the corresponding A_{aaP}^{11} defined in (27) for $E = E_1$ and $E = E_2$, respectively. Then it is trivial to see from Step 1(d) of the OGFM algorithm that $\lambda(A_{aaP1}^{11}) \subseteq \lambda(A_{aaP2}^{11})$. Hence the result. ■

Next, let us consider the set of optimal fixed decoupling zeros Λ_s^*. It turns out that for the general case, Λ_s^* does not have any kind of nested property as F_s^* and Ω_s^* do. To see this, let us first consider a simple example characterized by the quadruple $(A, B, C, D) = (0, 0, 1, 0)$ with $E_1 = 0$ and $E_2 = 1$. It is then easy to show that the corresponding $\Lambda_{s1}^* = \{0\}$ and $\Lambda_{s2}^* = \emptyset$. As such, here $\Lambda_{s2}^* \subset \Lambda_{s1}^*$ for $\mathrm{Im}(E_1) \subseteq \mathrm{Im}(E_2)$. Next consider another simple example characterized by the quadruple $(A, B, C, D) = (1, 1, 1, 1)$ with $E_1 = 0$ and $E_2 = 1$. Again, it is easy to show that the corresponding $\Lambda_{s1}^* = \emptyset$ and $\Lambda_{s2}^* = \{0\}$. As such, here $\Lambda_{s1}^* \subset \Lambda_{s2}^*$ for $\mathrm{Im}(E_1) \subseteq \mathrm{Im}(E_2)$. This shows that in general Λ_s^* does not have any nested property. However, Λ_s^* does have the nested property for a special case when (A, B) is controllable. The following proposition deals with such a special case.

Proposition 5.2. Assume that (A, B) is controllable. Consider two different values for E, say E_1 and E_2, and let $\mathrm{Im}(E_1) \subseteq \mathrm{Im}(E_2) \subseteq \mathscr{V}_g(\Sigma_P)$. Also, let Λ_{s1}^* and Λ_{s2}^* be the sets of optimal fixed decoupling zeros corresponding to $E = E_1$ and $E = E_2$, respectively. Then, we have $\Lambda_{s1}^* \subseteq \Lambda_{s2}^*$.

Proof. Let A_{aaP1}^{11} and A_{aaP2}^{11} be the corresponding A_{aaP}^{11} defined in (27) for $E = E_1$ and $E = E_2$, respectively. Then, under the assumption that (A, B) is controllable, it is trivial to see from Step 1(d) of the OGFM algorithm that $\lambda(A_{aaP1}^{11}) \subseteq \lambda(A_{aaP2}^{11})$. Hence the result. ■

Having noted the results of Propositions 5.1 and 5.2, we now move on to study Ω_s^* and Λ_s^* for two extreme values of E, namely for $\mathrm{Im}(E) = \mathscr{V}_g(\Sigma_P)$ and $\mathrm{Im}(E) = \{0\}$. For the case when $\mathrm{Im}(E) = \mathscr{V}_g(\Sigma_P)$, Ω_s^* contains all the stable invariant zeros of Σ_* and moreover all the mirror images of the unstable invariant zeros of Σ_*. In fact, in this case, Ω_s^* is the union of all the stable invariant zeros of Σ_* and $\lambda(A_x - B_x F_x)$ where

$$F_x := (B_x' P_x B_x + D_x' D_x)^{-1}(B_x' P_x A_x + D_x' C_x). \tag{36}$$

Here, we note that $(A_x - B_x F_x)$ has $n_x := n - n_a^- - n_a^0$ eigenvalues where n_a^- and n_a^0 are, respectively, the number of stable invariant zeros of Σ_* and the number of invariant zeros of Σ_* which are on the unit circle $\mathbb{C}^{\circ}$. Furthermore, we note that $\lambda(A_x - B_x F_x)$ includes the mirror images of all the unstable invariant zeros of Σ_*. Similarly, for the case when $\mathrm{Im}(E) = \mathscr{V}_g(\Sigma_P)$, $\Lambda_s^* = \lambda(A_{aa}^-)$, i.e. Λ_s^* consists of all the stable invariant zeros of Σ_*. On the other hand, for the case when $\mathrm{Im}(E) = \{0\}$, both the sets Ω_s^* and Λ_s^* contain only the input decoupling zeros of Σ_*. That is, for the case when $\mathrm{Im}(E) = \{0\}$, Ω_s^* and Λ_s^* do not necessarily contain all the stable invariant zeros of Σ_*, and moreover Ω_s^* and Λ_s^* never contain any mirror images of the unstable invariant zeros of Σ_*.

Next, following the same lines of Propositions 5.1 and 5.2, it is easy to show that, in general whenever $\mathrm{Im}(E) \subset \mathscr{V}_g(\Sigma_P)$, Ω_s^* consists of only some but not necessarily all the stable invariant zeros, and only some but not necessarily all the mirror images of the unstable invariant zeros of Σ_*. Similarly, in general whenever $\mathrm{Im}(E) \subset \mathscr{V}_g(\Sigma_P)$, Λ_s^* consists of only some but not necessarily all the stable invariant zeros of Σ_*.

As formalized in Theorem 5.1, the algorithm OGFM constructs the set of all static state feedback controllers F_s^*. An important question that arises next is under what conditions F_s^* is a singleton. The following proposition gives these conditions.

Proposition 5.3. (*Uniqueness of a static state feedback solution*). Consider the given system Σ as in (1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\mathrm{Im}(E) \subset \mathscr{V}_g(\Sigma_P)$. Then, an H_2-optimal static state feedback law is unique if and only if Σ_* left invertible with no invariant zeros on the unit circle, and the pair $(A - BD_P^{-1}C_P, E)$ is completely controllable. Moreover, under these conditions,

(1) $F_s^* = \{-D_P^{-1}C_P\}$, which is a singleton;
(2) $\Lambda_s^* = \{\text{stable invariant zeros of } \Sigma_*\}$; and
(3) $\Omega_s^* = \lambda(A - BD_P^{-1}C_P)$ which is the union of all the stable invariant zeros of Σ_* and $\lambda(A_x - B_x F_x)$. Note that $\lambda(A_x - B_x F_x)$ contains the mirror images of all the unstable invariant zeros of Σ_*.

Proof. It is simple to see under the conditions given in the proposition that the matrices A_z and B_z as in (28) of the OGFM algorithm are nonexistent. Thus the result is obvious from the construction procedure of OGFM. ■

Remark 5.1. In continuous-time systems, it is necessary (though not sufficient) that the given H_2-optimal control problem be a regular one (i.e. D is injective and (A, B, C, D) do not have any invariant zeros on the imaginary axis) in order that an H_2-optimal static state feedback law be unique. For discrete-time systems, it is not necessary that D be injective in order that an H_2-optimal static state feedback law be unique.

Remark 5.2. An interesting case is when $\mathrm{Im}(E) = \mathbb{R}^n$. In this case, since $(A - BD_P^{-1}C_P, E)$ is completely controllable, an H_2-optimal static state feedback solution is always unique. However, as Proposition 5.3 alludes, there are cases when a solution to the H_2 optimization problem is unique even if $\mathrm{Im}(E) \neq \mathbb{R}^n$.

6. *Design of optimal dynamic state feedback controllers*

In this section, we characterize all the possible H_2-optimal dynamic state feedback control laws using the well-known Q-parameterization technique. From Lemma 4.1, it follows that there exists an H_2-optimal state feedback law for Σ if and only if there exists a state feedback law which when applied to Σ_{au} of (17) achieves disturbance decoupling.

Then, in view of the necessary and sufficient conditions given in Theorem 4.1 under which the disturbance decoupling problem with internal stability (DDPS) of Σ_{au} is solvable, we know that whenever an optimal solution to the original system exists, there exists a constant gain F such that $A + BF$ is stable and that

$$(C_P + D_P F)(zI - A - BF)^{-1}E \equiv 0. \tag{37}$$

Next, following the results of Chen *et al.* (1993), it can be shown easily that any proper dynamic controller F(z) that stabilizes the system Σ_{au} can be written in the following form,

$$\begin{cases} \xi(k+1) = (A+BF)\xi(k) + By_1(k), \\ u(k) = Fx_p(k) + y_1(k), \end{cases} \tag{38}$$

where

$$y_1(k) = Q(z)[x_P(k) - \xi(k)], \tag{39}$$

for some proper and stable $Q(z)$, i.e. $Q(z) \in \mathrm{RH}^\infty$, with appropriate dimensions. The following theorem qualifies $Q(z)$ so that the controller F(z) is H_2-optimal for the given system Σ.

Theorem 6.1. Consider the given system Σ as in (1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Define a set $\mathcal{Q}$ as,

$$\mathcal{Q} := \{Q(z) \in \mathrm{RH}^\infty \mid Q(z) = W(z)(I - EE^\dagger) \times (zI - A - BF),\, W(z) \in \mathrm{RH}^s\}, \tag{40}$$

where $E^\dagger$ is the generalized inverse of E, i.e. $EE^\dagger E = E$. Then a proper controller F(z) stabilizes Σ and achieves the infimum, γ^*, i.e. $\mathrm{F}(z) \in F^*_{\mathrm{d}}$, if and only if F(z) can be written in the form of (38) and (39) for some $Q(z) \in \mathcal{Q}$.

Proof. It is omitted due to space limitations. Conceptually, it follows along the same lines as the proof of Theorem 5.1 of Chen *et al.* (1993) when specific characteristics of discrete-time systems are taken into account. Details can be found in an extended version of this paper (Chen *et al.*, 1993). ■

We have the following result regarding the uniqueness of a dynamic state feedback solution.

Proposition 6.1. (*Uniqueness of a dynamic state feedback solution*). Consider the given system Σ as in (1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Then, an H_2-optimal dynamic state feedback law is unique if and only if Σ_* is left invertible with no invariant zeros on the unit circle and $\mathrm{Im}(E) = \mathbb{R}^n$. Moreover, under these conditions,
(1) $F^*_{\mathrm{d}} = \{-D_P^{-1}C_P\}$, which is a singleton;
(2) $\Lambda^*_s = \{$stable invariant zeros of $\Sigma_*\}$; and
(3) $\Omega^*_s = \lambda(A - BD_P^{-1}C_P)$ which is the union of all the stable invariant zeros of Σ_* and $\lambda(A_x - B_xF_x)$. Note that $\lambda(A_x - B_xF_x)$ contains the mirror images of the unstable invariant zeros of Σ_*.

Proof. The fact that F^*_{d} is a singleton implies that $Q(z) \equiv 0$. By (40), we have $\mathrm{Im}(E) = \mathbb{R}^n$. It is then simple to show that this and the condition $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$ imply that Σ_* is left invertible with no invariant zeros on the unit circle and $\mathrm{Im}(E) = \mathbb{R}^n$. Next, the converse part is obvious. Also, the remaining results follow directly from Proposition 5.3. ■

The following remarks are in order.

Remark 6.1. It is interesting to note that the condition under which an optimal dynamic state feedback controller is unique, is stronger than the condition for which a static optimal control law is unique (see Proposition 5.3).

Remark 6.2. It is easy to show that an H_2-optimal closed-loop transfer function from z to w, denoted here as $T^*_{zw}(z)$, is unique whatever the type of optimal controller used, i.e. whether the controller is an element of F^*_s or F^*_{d}. Moreover, $T^*_{zw}(z)$ is given by

$$T^*_{zw}(z) = (C_x - D_xF_x)(zI - A_x + B_xF_x)^{-1}E_x. \tag{41}$$

It is simple to verify that the set of poles of the irreducible transfer function $T^*_{zw}(z)$ is equal to Ω^*_s/Λ^*_s, i.e. the optimal fixed modes that are not the optimal fixed decoupling zeros.

Next, the following theorem shows that the optimal fixed modes and the fixed decoupling zeros remain unchanged regardless of what type of controller is used.

Theorem 6.2. Consider the given system Σ as in (1). Let Σ_* be left invertible. Also, assume that the pair (A, B) is stabilizable, and that $\mathrm{Im}(E) \subseteq \mathcal{V}_g(\Sigma_P)$. Then we have (i) $\Omega^*_{\mathrm{d}} = \Omega^*_s$, and (ii) $\Lambda^*_{\mathrm{d}} = \Lambda^*_s$.

Proof. It follows along the same lines as the proof of Theorem 5.2 of Chen *et al.* (1993). ■

7. *Conclusions*

This paper is a continuation of our earlier work (Chen *et al.*, 1993) which concerns itself with continuous-time systems. Here discrete-time problems are considered. As in our earlier work, all the static and dynamic H_2-optimal state feedback solutions are explicitly constructed and parameterized. Moreover, the necessary and sufficient conditions for the uniqueness of an H_2-optimal solution for both the cases of static and dynamic state feedback controllers are established. Also, it turns out that all the optimal controllers must include certain fixed modes among the closed-loop eigenvalues, and moreover, must inherently involve certain pole–zero cancellations. The set of optimal fixed modes, which is a set of complex numbers that every optimal state feedback controller must assign among the closed-loop eigenvalues, is identified and constructed. Similarly, the set of optimal fixed decoupling zeros, which shows the minimum absolutely necessary number and locations of pole–zero cancellations present in any H_2-optimal solution is identified and constructed. It is also seen that the sets of optimal fixed modes and optimal fixed decoupling zeros do not vary depending upon whether the static or the dynamic controllers are used. Most of the results presented here are analogous to, but not quite the same as, those of our earlier results for continuous-time systems. In fact, there are some fundamental differences between the continuous and discrete-time systems and these are pointed out.

References

Anderson, B. D. O. and J. B. Moore (1979). *Optimal Filtering*. Prentice-Hall, Englewood Cliffs, NJ.

Athans, M. (1971). The role and use of the stochastic linear quadratic Gaussian problem in control system design. *IEEE Trans. Autom. Control*, **16**, 529–552.

Chen, B. M., A. Saberi and Z.-L. Lin (1991). Linear Control Toolbox. Technical Report No. EE/CS 0098, Washington State University.

Chen, B. M., Saberi and P. Sannuti (1992). Explicit expressions for cascade factorization of general nonminimum phase systems. *IEEE Trans. Autom. Control*, **37**(3), 358–363.

Chen, B. M., A. Saberi, P. Sannuti and Y. Shamash (1993). Construction and parameterization of all static and dynamic H_2 optimal state feedback solutions, optimal fixed modes and fixed decoupling zeros. *IEEE Trans. Autom. Control*, **38**(2), 248–261.

Chen, B. M., A. Saberi and Y. Shamash (1993). A non-recursive method for solving the general discrete time algebraic Riccati equation related to the H_∞ control problem. In *Proceedings of* 1993 *American Control Conference*, San Francisco, California. Also, to appear in *Int. J. Robust Nonlinear Control.*

Chen, B. M., A. Saberi, Y. Shamash and P. Sannuti (1993). Construction and parameterization of all static and dynamic H_2 optimal state feedback solutions for discrete-time systems. Technical Report No. EE/CS 93-018, Washington State University.

Dorato, P. and A. H. Levis (1971). Optimal linear regulators: the discrete-time case. *IEEE Trans. Autom. Control,* **16,** 613–620.

Hautus, M. L. J. and L. M. Silverman (1983). System structure and singular control. *Linear Algebraic and its Applications,* **50,** 369–402.

Kucera, V. (1972). The discrete Riccati equation of optimal control. *Kybernetica,* **8**(5), 430–477.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems.* John Wiley, New York.

Lin, Z. L., A. Saberi and B. M. Chen (1992). Linear systems toolbox: system analysis and control design in the Matlab environment. In *Proceedings of the 1st IEEE Conference on Control Applications,* Dayton, Ohio, pp. 659–664.

Molinari, B. P. (1975). The stabilizing solution of the discrete algebraic Riccati equation. *IEEE Trans. Autom. Control,* **20**(3), 396–399.

Pappas, T., A. J. Laub and N. R. Sandell Jr (1980). On the numerical solution of the discrete algebraic Riccati equation. *IEEE Trans. Autom. Control,* **25**(4), 631–641.

Rosenbrock, H. H. (1970). *State-space and Multivariable Theory.* Nelson, London.

Rotea, M. A. and P. P. Khargonekar (1991). H_2 optimal control with an H_∞ constraint: the state feedback case. *Automatica,* **27,** 307–316.

Saberi, A. and P. Sannuti (1990). Squaring down of non-strictly proper systems. *Int. J. Control,* **51**(3), 621–629.

Sannuti, P. and A. Saberi (1987). A special coordinate basis of multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling. *Int. J. Control,* **45**(5), 1655–1704.

Trentelman, H. L. and A. A. Stoorvogel (1992). Sampled-data and discrete-time H_2 optimal control, Preprint.

Wonham, W. M. (1985). *Linear Multivariable Control.* Springer, Berlin.