Jrl Syst Sci & Complexity (2007) 20: 198–214

STRUCTURAL DECOMPOSITION AND ITS PROPERTIES OF LINEAR MULTIVARIABLE SINGULAR SYSTEMS

Minghua HE · Ben M. CHEN · Zongli LIN

Received: 28 December 2006

Abstract We present in this paper a structural decomposition for linear multivariable singular systems. Such a decomposition has a distinct feature of capturing and displaying all the structural properties, such as the finite and infinite zero structures, invertibility structures, and redundant dynamics of the given system. As its counterpart for non-singular systems, we believe that the technique is a powerful tool in solving control problems for singular systems.

Key words Descriptor systems, singular systems, structural invariants, system decomposition.

1 Introduction

Singular systems, also commonly called generalized or descriptor systems in the literature, appear in many practical situations including engineering systems, economic systems, network analysis, and biological systems (see, e.g., Lewis[1], Dai[2] and Kuijper[3]). In fact, many systems in the real life are singular in nature. They are usually simplified as or approximated by non-singular models because of the lack of efficient tools for dealing with singular systems. The structural analysis of linear singular systems, using either algebraic or geometric approach, has attracted considerable attention from many researchers during the past three decades (see, e.g., Lewis[1], Chu and Ho[4], Chu and Mehrmann[5], Fliess[6], Geerts[7], Lewis and Ozcaldiran[8], Loiseau[9], Malabre[10], Misra et al.[11], Van Dooren[12,13], Verghese[14], Zhou et al.[15], and the references cited therein). Generally speaking, almost all the research works dealing with singular systems are the natural extensions of their non-singular system counterparts, although these extensions are usually non-trivial.

It has been extensively demonstrated and proven for non-singular systems that the system structural properties, such as the finite and infinite zero structures and the invertibility structures, play a very important role in solving various control problems including H_2, H_∞ control and disturbance decoupling (see, e.g., Chen[16] and Saberi et al.[17]). The structural properties

Minghua HE
School of Electrical Engineering and Automation, Fuzhou University, Fuzhou 350002, *China.*
Email: mhhe@fzu.edu.cn.
Ben M. CHEN
Department of Electrical and Computer Engineering, The National University of Singapore, Singapore 117576, *Republic of Singapore.* Email: bmchen@nus.edu.sg.
Zongli LIN
Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, Charlottesville, VA 22904-4743, *USA.* Email: zl5y@virginia.edu.

of singular systems and their applications to the control problems of singular systems are however less emphasized in the literature. In their recent work, He and Chen[18] have developed a technique that gives a structural decomposition for single-input and single-output (SISO) singular systems. The technique is capable of revealing all the structural properties, including the finite and infinite zero structures. In this paper, we present a structural decomposition technique for linear multivariable singular systems. Again, such a technique can be used to capture and display the structural properties of singular systems. Our work generalizes the result of He and Chen[18]. It can also be regarded as a natural extension and counterpart of the work of Sannuti and Saberi[19] for non-singular systems. However, it will be seen shortly that the structural decomposition of a multivariable singular system is much more involved. Such a decomposition technique is expected to be a powerful tool for solving a wide range of control problems for singular systems.

To be more specific, we consider a linear time-invariant system Σ characterized by

$$\Sigma : \begin{cases} E\dot{x} = Ax + Bu, \quad x(0) = x_0, \\ \quad y = Cx + Du, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are respectively the state, input and output of the system, and E, A, B, C and D are constant matrices of appropriate dimensions. The system Σ is said to be singular if $\text{rank}(E) < n$. As usual, in order to avoid any ambiguity in the solutions to the system, we assume throughout this paper that the given singular system Σ is regular, i.e., $\det(sE - A) \not\equiv 0$, for $s \in \mathbb{C}$. Traditionally, the Kronecker canonical form, a classical form of matrix pencils under strictly equivalent transformation, has been used extensively in the structural analysis of singular systems. Malabre[10] presents a geometric approach and introduces structural invariants of singular systems. In that paper, some definitions are shown to be consistent with others that can be directly deduced from matrix pencil tools. It extends many geometric and structural results from the non-singular systems to singular systems. In this paper, our focus is not on the computation of the invariant indices, but to derive a constructive algorithm that decomposes the state space of the given system into several distinct parts, which are directly associated with the finite and infinite zero dynamics, as well as the invertibility structures of the given system. It is interesting to note that our decomposition will automatically and explicitly separate the redundant dynamics of the system, which cannot be captured at all in the Kronecker canonical form. As mentioned earlier, it is expected that the technique presented in this paper will play a similar role in solving a variety of control problems for singular systems as its counterpart has played for non-singular systems.

The outline of the remainder of the paper is as follows: In Section 2, we present our main results, i.e., the structural decomposition of multivariable linear singular systems and all its structural properties. The constructive proof for the structural decomposition and proofs for its properties will be given in Section 3. A numerical example will be presented in Section 4 to illustrate the proposed decomposition technique and to show how the structural properties of a multivariable singular system can easily be revealed under such a decomposition. Finally, Section 5 draws a conclusion to the paper.

Throughout this paper, the following notation will be used: I denotes an identity matrix of appropriate dimensions; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$, $\mathbb{C}^0$, $\mathbb{C}^-$ and $\mathbb{C}^+$ represent respectively the set of all complex numbers, the imaginary axis, the open left-half plane and the open right-half plane; $\lambda(X)$ is the set of eigenvalues of a real square matrix X; and $u^{(v)}$ denotes the v-th derivative of u, where v is an integer. Finally, with a slight abuse of notation, we occasionally write $u^{(v)}$ as $s^v u$ when it is clear from the context. Here, s can be regarded as a differentiation operator or the operator used in Laplace transform.

2 Structural Decomposition and Its Properties

We first summarize the structural decomposition of multivariable singular systems in the following main theorem. All its properties will also be given. The constructive algorithm for the structural decomposition and proofs of all these properties will be given in Section 3 for clarity of presentation. We have the following theorem.

Theorem 1 *Consider the singular system Σ of* (1) *satisfying* $\det(sE-A)\not\equiv 0$ *for* $s\in\mathbb{C}$. *Then,*

1) *there exist coordinate-free nonnegative integers* $n_{\rm z}$, $n_{\rm e}$, $n_{\rm a}$, $n_{\rm b}$, $n_{\rm c}$, $n_{\rm d}$, $m_{\rm d}$, m_0, $m_{\rm c}$, $p_{\rm b}$, *and positive integers* q_i, $i=1,2,\cdots,m_{\rm d}$, *if* $m_{\rm d}>0$*; and*
2) *there exist nonsingular state and output constant transformations* $\Gamma_{\rm s}\in\mathbb{R}^{n\times n}$ *and* $\Gamma_{\rm o}\in\mathbb{R}^{p\times p}$, *as well as an* $m\times m$ *nonsingular input transformation* $\Gamma_{\rm i}(s)$, *whose inverse's elements are polynomials in* s *(i.e., its inverse contains various differentiation operators), and an* $n\times n$ *nonsingular transformation* $\Gamma_{\rm e}(s)$, *whose elements are polynomials of* s, *which together give a structural decomposition of* Σ *and display explicitly its structural properties.*

The structural decomposition of Σ *can be described by the following set of equations:*

$$x=\Gamma_{\rm s}\widetilde{x},\quad y=\Gamma_{\rm o}\widetilde{y},\quad u=\Gamma_{\rm i}(s)\widetilde{u}, \tag{2}$$

and

$$\widetilde{x}=\begin{pmatrix}x_{\rm z}\\x_{\rm e}\\x_{\rm a}\\x_{\rm b}\\x_{\rm c}\\x_{\rm d}\end{pmatrix},\ \widetilde{y}=\begin{pmatrix}y_0\\y_{\rm d}\\y_{\rm b}\end{pmatrix},\ \widetilde{u}=\begin{pmatrix}u_0\\u_{\rm d}\\u_{\rm c}\end{pmatrix},\ x_{\rm d}=\begin{pmatrix}x_{\rm d1}\\x_{\rm d2}\\\vdots\\x_{{\rm d}m_{\rm d}}\end{pmatrix},\ y_{\rm d}=\begin{pmatrix}y_{\rm d1}\\y_{\rm d2}\\\vdots\\y_{{\rm d}m_{\rm d}}\end{pmatrix},\ u_{\rm d}=\begin{pmatrix}u_{\rm d1}\\u_{\rm d2}\\\vdots\\u_{{\rm d}m_{\rm d}}\end{pmatrix}, \tag{3}$$

and

$$J_{n_{\rm z}}\dot{x}_{\rm z}=x_{\rm z}, \tag{4}$$

where $J_{n_{\rm z}}\in\mathbb{R}^{n_{\rm z}\times n_{\rm z}}$ *has all its eigenvalues at* 0,

$$x_{\rm e}=B_{\rm e0}u_0+B_{\rm ec}u_{\rm c}+B_{\rm ed}u_{\rm d}+sN_{\rm ez}(s)x_{\rm z}, \tag{5}$$
$$\dot{x}_{\rm a}=A_{\rm aa}x_{\rm a}+B_{\rm 0a}y_0+L_{\rm ad}y_{\rm d}+L_{\rm ab}y_{\rm b}+sL_{\rm az}(s)x_{\rm z}, \tag{6}$$
$$\dot{x}_{\rm b}=A_{\rm bb}x_{\rm b}+B_{\rm 0b}y_0+L_{\rm bd}y_{\rm d}+sL_{\rm bz}(s)x_{\rm z}, \tag{7}$$
$$y_{\rm b}=C_{\rm b}x_{\rm b}+C_{\rm bz}x_{\rm z}+sC_{\rm bzs}(s)x_{\rm z}, \tag{8}$$
$$\dot{x}_{\rm c}=A_{\rm cc}x_{\rm c}+B_{\rm 0c}y_0+L_{\rm cd}y_{\rm d}+L_{\rm cb}y_{\rm b}+B_{\rm c}M_{\rm ca}x_{\rm a}+B_{\rm c}u_{\rm c}+sL_{\rm cz}(s)x_{\rm z}, \tag{9}$$
$$y_0=C_{\rm 0a}x_{\rm a}+C_{\rm 0b}x_{\rm b}+C_{\rm 0c}x_{\rm c}+C_{\rm 0d}x_{\rm d}+u_0+C_{\rm 0z}x_{\rm z}+sC_{\rm 0zs}(s)x_{\rm z}, \tag{10}$$

and for each $i=1,2,\cdots,m_{\rm d}$,

$$\begin{aligned}\dot{x}_{{\rm d}i}=&A_{q_i}x_{{\rm d}i}+L_{i0}y_0+L_{i{\rm d}}y_{\rm d}+sL_{i{\rm z}}(s)x_{\rm z}\\&+B_{q_i}\Big(u_{{\rm d}i}+M_{i{\rm a}}x_{\rm a}+M_{i{\rm b}}x_{\rm b}+M_{i{\rm c}}x_{\rm c}+\sum_{j=1}^{m_{\rm d}}M_{ij}x_{{\rm d}j}\Big),\end{aligned} \tag{11}$$

$$y_{{\rm d}i}=C_{q_i}x_i+C_{q_i{\rm z}}x_{\rm z}+sC_{q_i{\rm zs}}(s)x_{\rm z},\ y_{\rm d}=C_{\rm d}x_{\rm d}+C_{\rm dz}x_{\rm z}+sC_{\rm dzs}(s)x_{\rm z}, \tag{12}$$

for some constant matrices of appropriate dimensions and some matrices whose elements are polynomials of s. *Here the states* $x_{\rm z}$, $x_{\rm e}$, $x_{\rm a}$, $x_{\rm b}$, $x_{\rm c}$ *and* $x_{\rm d}$ *are of dimensions* $n_{\rm z}$, $n_{\rm e}$, $n_{\rm a}$, $n_{\rm b}$,

$n_{\rm c}$ and $n_{\rm d} = \sum_{i=1}^{m_{\rm d}} q_i$, respectively, while $x_{{\rm d}i}$ is of dimension q_i for each $i = 1, 2, \cdots, m_{\rm d}$. The control vectors u_0, $u_{\rm d}$ and $u_{\rm c}$ are of dimensions m_0, $m_{\rm d}$ and $m_{\rm c} = m - m_0 - m_{\rm d}$, respectively, while the output vectors y_0, $y_{\rm d}$ and $y_{\rm b}$ are respectively of dimensions m_0, $m_{\rm d}$, and $p_{\rm b} = p - m_0 - m_{\rm d}$. The pair $(A_{\rm bb}, C_{\rm b})$ is observable, the pair $(A_{\rm cc}, B_{\rm c})$ is controllable, and the triple $(A_{q_i}, B_{q_i}, C_{q_i})$ has the form

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [\,1 \quad 0 \quad \cdots \quad 0\,]. \tag{13}$$

Assuming that $x_i, i = 1, 2, \cdots, m_{\rm d}$, are arranged such that $q_i \le q_{i+1}$, the matrix $L_{i{\rm d}}$ will be in the particular form of

$$L_{i{\rm d}} = [\,L_{i1} \quad L_{i2} \quad \cdots \quad L_{ii-1} \quad 0 \quad \cdots \quad 0\,], \tag{14}$$

with its last row being all zeros.

A constructive proof of the structural decomposition in Theorem 1 will be given in the next section. The following corollaries of Theorem 1 give a compact matrix form for the structural decomposition and establish its equivalence to the original system.

Corollary 1 *The structural decomposition of Σ of Theorem 1 can be represented in the following form:*

$$\begin{aligned} \widetilde{E} &= \Gamma_{\rm e}(s) E \Gamma_{\rm s} = E_{\rm s} - E_{\rm z}(s) + \Psi(s) \\ &= \begin{bmatrix} J_{n_{\rm z}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_{\rm a}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{\rm b}} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_{\rm c}} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\rm d}} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ N_{\rm ez}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{\rm az}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{\rm bz}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{\rm cz}(s) & 0 & 0 & 0 & 0 & 0 \\ L_{\rm dz}(s) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \Psi(s), \end{aligned} \tag{15}$$

$$\begin{aligned} \widetilde{A} &= \Gamma_{\rm e}(s) A \Gamma_{\rm s} = A_{\rm s} + s\Psi(s) \\ &= \left(\begin{bmatrix} 0 \\ 0 \\ B_{\rm 0a} \\ B_{\rm 0b} \\ B_{\rm 0c} \\ B_{\rm 0d} \end{bmatrix} C_0 + \begin{bmatrix} I_{n_{\rm z}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_{\rm e}} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{\rm aa} & L_{\rm ab}C_{\rm b} & 0 & L_{\rm ad}C_{\rm d} \\ 0 & 0 & 0 & A_{\rm bb} & 0 & L_{\rm bd}C_{\rm d} \\ 0 & 0 & B_{\rm c}M_{\rm ca} & L_{\rm cb}C_{\rm b} & A_{\rm cc} & L_{\rm cd}C_{\rm d} \\ 0 & 0 & B_{\rm d}M_{\rm da} & B_{\rm d}M_{\rm db} & B_{\rm d}M_{\rm dc} & A_{\rm dd} \end{bmatrix} \right) + s\Psi(s), \end{aligned} \tag{16}$$

$$\widetilde{B} = \Gamma_{\rm e}(s) B \Gamma_{\rm i}(s) = B_{\rm s} = \begin{bmatrix} 0 & 0 & 0 \\ B_{\rm 0e} & B_{\rm de} & B_{\rm ce} \\ B_{\rm 0a} & 0 & 0 \\ B_{\rm 0b} & 0 & 0 \\ B_{\rm 0c} & 0 & B_{\rm c} \\ B_{\rm 0d} & B_{\rm d} & 0 \end{bmatrix}, \tag{17}$$

$$\widetilde{C} = \Gamma_{\rm o}^{-1} C \Gamma_{\rm s} = C_{\rm s} + \Psi_{\rm c} = \begin{bmatrix} C_{\rm 0z} & 0 & C_{\rm 0a} & C_{\rm 0b} & C_{\rm 0c} & C_{\rm 0d} \\ C_{\rm dz} & 0 & 0 & 0 & 0 & C_{\rm d} \\ C_{\rm bz} & 0 & 0 & C_{\rm b} & 0 & 0 \end{bmatrix} + \Psi_{\rm c}, \tag{18}$$

$$\widetilde{D} = \Gamma_{\rm o}^{-1} D \Gamma_{\rm i}(s) = D_{\rm s} + \Psi_{\rm d}(s) = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \Psi_{\rm d}(s), \tag{19}$$

where $\Psi(s)$ is an $n \times n$ matrix with entries being some polynomials of s,

$$C_0 = [\,0 \quad 0 \quad C_{\rm 0a} \quad C_{\rm 0b} \quad C_{\rm 0c} \quad C_{\rm 0d}\,], \quad \Psi_{\rm c}\widetilde{x} + \Psi_{\rm d}(s)\widetilde{u} = \Psi_{R}(s) x_{\rm z}, \tag{20}$$

and where $\Psi_R(s)$ *is a matrix with its elements being some polynomials of* s.

Corollary 2 *Let* Σ_s *be a singular system characterized by* $(E_s, A_s, B_s, C_s, D_s)$*, which has a transfer function*

$$H_s(s) = C_s(sE_s - A_s)^{-1}B_s + D_s. \tag{21}$$

Let $H(s)$ *be the transfer function of the original singular system* (1)*. Then,*

$$H(s) = C(sE - A)^{-1}B + D = \Gamma_o H_s(s)\Gamma_i^{-1}(s), \tag{22}$$

which shows that the transfer functions of the original system Σ *and the system characterized by* Σ_s *are related by some nonsingular transformations.*

Next, we would like to note that it does not lose too much generality to assume that the state variable of Σ, $x(t)$, to be a continuous function of t at $t = 0$, which simply means that there is no sudden jump from $x(0^-)$ to $x(0^+)$. Then, it is straightforward to show that (4) implies that $x_z = 0$, for all t. We summarize below the physical features of the state variables in our structural decomposition under such a minor assumption:

1) The state x_z is purely static and identically zero for all time t. It can neither be controlled by the system input nor be affected by other states.
2) The state x_e is again static and contains a linear combination of the input variables of the system and their derivatives of appropriate orders.
3) The state x_a is neither directly controlled by the system input nor does it directly affect the system output.
4) The output y_b and the state x_b are not directly influenced by any input, although they could be indirectly controlled through the output y_d. Moreover, (A_{bb}, C_b) forms an observable pair. This implies that the state x_b is observable.
5) The state x_c is directly controlled by the input u_c, but it does not directly affect any output. (A_{cc}, B_c) forms a controllable pair. This implies that the state x_c is controllable.
6) The variables u_{di} control the output y_{di} through a stack of q_i integrators. Furthermore, all the states x_{di} are both controllable and observable.

It is simple and interesting to observe from the structural decomposition of Σ of Theorem 1 that there are redundant state variables associated with the given system. Thus, an immediate application of such a technique is the reduction of a singular system to an equivalent proper system as the state variable x_z is identically zero, and the state variable x_e is simply a linear combination of the system input variables and their derivatives. As such, from the input-output behavior point of view, the given singular system can be equivalently reduced to the following proper system:

$$\dot{x}_a = A_{aa}x_a + B_{0a}y_0 + L_{ad}y_d + L_{ab}y_b, \tag{23}$$

$$\dot{x}_b = A_{bb}x_b + B_{0b}y_0 + L_{bd}y_d, \quad y_b = C_b x_b, \tag{24}$$

$$\dot{x}_c = A_{cc}x_c + B_{0c}y_0 + L_{cd}y_d + L_{cb}y_b + B_c M_{ca}x_a + B_c u_c, \tag{25}$$

$$y_0 = C_{0a}x_a + C_{0b}x_b + C_{0c}x_c + C_{0d}x_d + u_0, \tag{26}$$

and for each $i = 1, 2, \cdots, m_d$,

$$\begin{aligned}\dot{x}_{di} = {} & A_{q_i}x_{di} + L_{i0}y_0 + L_{id}y_d \\ & + B_{q_i}\left(u_{di} + M_{ia}x_a + M_{ib}x_b + M_{ic}x_c + \sum_{j=1}^{m_d} M_{ij}x_{dj}\right),\end{aligned} \tag{27}$$

$$y_{di} = C_{q_i}x_i, \quad y_d = C_d x_d. \tag{28}$$

Naturally, we can expect that many results related to systems and control theory of proper systems can be extended to singular systems without too much difficulty.

We note our structural decomposition of Theorem 1 is a natural extension of the special coordinate basis of [19] for non-singular systems and is capable of capturing the unique features of singular systems. In what follows, we study how the system properties of Σ, such as the stabilizability, detectability, finite and infinite zero structures, can be obtained from the decomposition of the system.

We first recall the definitions of stability, stabilizability and detectability of linear singular systems from the literature (see, e.g., Dai[2]).

Definition 1 (Stability, Stabilizability, and Detectability) The system Σ of (1) is said to be stable if its characteristic polynomial $\det(sE - A)$ has all roots in $\mathbb{C}^-$. It is said to be stabilizable if there exists a constant matrix F of appropriate dimensions such that all roots of $\det(sE - A - BF)$ are in $\mathbb{C}^-$. Similarly, it is said to be detectable if there exists a constant matrix K of appropriate dimensions such that all roots of $\det(sE - A - KC)$ are in $\mathbb{C}^-$.

The definition of invariant zeros of singular systems can be made in a similar way as that for proper systems or in the Kronecker canonical form associated with Σ (see, e.g., Malabre[10]).

Definition 2 (Invariant Zeros) Let

$$P_\Sigma(s) = \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}. \tag{29}$$

A complex scalar $\alpha \in \mathbb{C}$ is said to be an invariant zero of the singular system Σ of (1) if

$$\operatorname{rank}\{P_\Sigma(\alpha)\} < n + \operatorname{normrank}\{H(s)\}, \tag{30}$$

where

$$H(s) = C(sE - A)^{-1}B + D, \tag{31}$$

and normrank$\{(H(s)\}$ denotes the normal rank of $H(s)$, which is defined as its rank over the field of rational functions of s with real coefficients. Alternatively and physically, the invariant zero dynamics of Σ can also be defined as the dynamics of Σ, when $y \equiv 0$.

The following properties show that the stabilizability, detectability, and the invariant zeros of Σ can be obtained through the structural decomposition in a trivial manner.

Property 1 (Stabilizability and Detectability) The given system Σ of (1) is stabilizable if and only if $(A_{\rm con}, B_{\rm con})$ is stabilizable, and is detectable if and only if $(A_{\rm obs}, C_{\rm obs})$ is detectable, where

$$A_{\rm con} := \begin{bmatrix} A_{\rm aa} & L_{\rm ab}C_{\rm b} \\ 0 & A_{\rm bb} \end{bmatrix}, \quad B_{\rm con} := \begin{bmatrix} B_{\rm 0a} & L_{\rm ad} \\ B_{\rm 0b} & L_{\rm bd} \end{bmatrix}, \tag{32}$$

and

$$A_{\rm obs} := \begin{bmatrix} A_{\rm aa} & 0 \\ B_{\rm c}M_{\rm ca} & A_{\rm cc} \end{bmatrix}, \quad C_{\rm obs} := \begin{bmatrix} C_{\rm 0a} & C_{\rm 0c} \\ M_{\rm da} & M_{\rm dc} \end{bmatrix}. \tag{33}$$

Property 2 (Invariant Zeros) The invariant zeros of Σ are the eigenvalues of $A_{\rm aa}$.

Our structural decomposition can also exhibit the invertibility structure of a given singular system Σ. Basically, for the usual case when matrices $[\,B' \quad D'\,]$ and $[\,C \quad D\,]$ are of maximal rank, the system Σ or equivalently $H(s)$ is said to be left invertible if there exists a rational matrix function $L(s)$ such that $L(s)H(s) = I_m$. The system Σ is right invertible if there exists a rational matrix function $R(s)$ such that $H(s)R(s) = I_p$. Moreover, Σ is said to be invertible if it is both left and right invertible, and Σ is noninvertible, or degenerate, if it is neither left nor right invertible.

Property 3 (Invertibility) Σ is right invertible if and only if $x_{\rm b}$ and hence $y_{\rm b}$ are nonexistent, left invertible if and only if $x_{\rm c}$ and hence $u_{\rm c}$ is nonexistent, and invertible if and only if both $x_{\rm b}$ and $x_{\rm c}$ are nonexistent.

The infinite zero structure of the given system Σ can be defined as the infinite elementary divisors associated with the Kronecker canonical form of $P_\Sigma(s)$. It can also be defined using the well-known Smith–McMillan form. It basically represents the numbers of integrator chains between the control input channels and output channels. Unfortunately, due to the differentiating actions performed on the system input, the infinite zero structures of Σ and the transformed system $\Sigma_{\rm s}$ are slightly different. Nonetheless, the infinite zero structure of $\Sigma_{\rm s}$ is given by

$$S_\infty^\star(\Sigma) = \{q_1, q_2, \cdots, q_{m_{\rm d}}\}, \tag{34}$$

i.e., for each $i = 1, 2, \cdots, m_{\rm d}$, $\Sigma_{\rm s}$ has an infinite zero of order q_i, respectively.

3 Proofs of Main Results

We first present a constructive proof for Theorem 1. The following is a step-by-step algorithm for the structural decomposition of multivariable singular systems.

Step 1 Preliminary decomposition

This step, adopted from [2], is to separate the given descriptor system into a proper subsystem and a special descriptor subsystem. First, we note that the regularity assumption on the given system (1) implies the existence of a real scalar β such that $\det(\beta E + A) \neq 0$. Next, we define

$$\widehat{E} = (\beta E + A)^{-1} E. \tag{35}$$

It follows from the well known real Jordan canonical decomposition that there exists a nonsingular transformation $T \in \mathbb{R}^{n\times n}$ such that

$$T\widehat{E}T^{-1} = \begin{bmatrix} \widehat{E}_1 & 0 \\ 0 & \widehat{E}_2 \end{bmatrix}, \tag{36}$$

where $\widehat{E}_1 \in \mathbb{R}^{n_1\times n_1}$ is a nonsingular matrix and $\widehat{E}_2 \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix. Lastly, we let

$$P = \begin{bmatrix} \widehat{E}_1^{-1} & 0 \\ 0 & (I_{n_2} - \beta\widehat{E}_2)^{-1} \end{bmatrix} T(\beta E + A)^{-1}, \quad Q = T^{-1}. \tag{37}$$

It is then straightforward to verify that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix},$$

where $N = (I_{n_2} - \beta\widehat{E}_2)^{-1}\widehat{E}_2$ is a nilpotent matrix, and $A_1 = \widehat{E}_1^{-1}(I_{n_1} - \beta\widehat{E}_1)$. We also partition accordingly

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CQ = [\, C_1 \quad C_2 \,]. \tag{38}$$

Equivalently, Σ can be decomposed into the following two subsystems:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u, \\ y_1 = C_1 x_1 + D u, \end{cases} \tag{39}$$

$$\Sigma_2 : \begin{cases} N\dot{x}_2 = x_2 + B_2 u, \\ y_2 = C_2 x_2, \end{cases} \tag{40}$$

where $T^{-1}x = (x_1^{\mathrm{T}}\ x_2^{\mathrm{T}})^{\mathrm{T}}$, and $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$, and $y = y_1 + y_2$.

Step 2 Decomposition of x_{z} and x_{e}

The key idea is to separate the controllable and uncontrollable parts of the pair (N, B_2) in Σ_2. It follows from Theorems 2.3.1 and 2.3.2 of [16] that there exist nonsingular coordinate transformations

$$x_2 = T_{\mathrm{s}}\widehat{x}_2, \quad u = T_{\mathrm{i}}\widehat{u}, \tag{41}$$

such that

$$\widehat{x}_2 = \begin{pmatrix} x_{\mathrm{v}} \\ x_{\mathrm{z}} \end{pmatrix}, \quad x_{\mathrm{v}} = \begin{pmatrix} x_{\mathrm{v}1} \\ x_{\mathrm{v}2} \\ \vdots \\ x_{\mathrm{v}n_{\mathrm{e}}} \end{pmatrix}, \quad x_{\mathrm{z}} \in \mathbb{R}^{n_{\mathrm{z}}}, \quad \widehat{u} = \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \vdots \\ \widehat{u}_{n_{\mathrm{e}}} \\ \widehat{u}_* \end{pmatrix}, \tag{42}$$

where

$$x_{\mathrm{v}i} \in \mathbb{R}^{p_i}, \quad x_{\mathrm{v}i} = \begin{pmatrix} x_{\mathrm{v}i,1} \\ x_{\mathrm{v}i,2} \\ \vdots \\ x_{\mathrm{v}i,p_i} \end{pmatrix}, \quad i = 1, 2, \cdots, n_{\mathrm{e}},$$

$$\widehat{N} = T_{\mathrm{s}}^{-1}NT_{\mathrm{s}} = \begin{bmatrix} J_{\mathrm{v}} & N_{\mathrm{zv}} \\ 0 & J_{n_{\mathrm{z}}} \end{bmatrix} = \begin{bmatrix} J_{\mathrm{v}1} & 0 & \cdots & 0 & N_{1\mathrm{z}} \\ 0 & J_{\mathrm{v}2} & \cdots & 0 & N_{2\mathrm{z}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{\mathrm{v}n_{\mathrm{e}}} & N_{n_{\mathrm{e}}\mathrm{z}} \\ 0 & 0 & \cdots & 0 & J_{n_{\mathrm{z}}} \end{bmatrix},$$

$$\widehat{B}_2 = T_{\mathrm{s}}^{-1}B_2T_{\mathrm{i}} = \begin{bmatrix} B_{\mathrm{v}} \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n_{\mathrm{e}}} & B_{1\mathrm{z}} \\ 0 & B_{22} & \cdots & B_{2n_{\mathrm{e}}} & B_{2\mathrm{z}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{n_{\mathrm{e}}n_{\mathrm{e}}} & B_{n_{\mathrm{e}}\mathrm{z}} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and where $(J_{\mathrm{v}}, B_{\mathrm{v}})$ is controllable. Moreover, the fact that N is nilpotent implies that $J_{\mathrm{v}i}$ and $J_{n_{\mathrm{z}}}$ have all their eigenvalues at 0, and $J_{\mathrm{v}i}$, $N_{i\mathrm{z}}$, $B_{i\mathrm{z}}$ and B_{ij} have the following forms,

$$J_{\mathrm{v}i} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ N_{i\mathrm{z}} = \begin{bmatrix} \eta_{i\mathrm{z},1} \\ \vdots \\ \eta_{i\mathrm{z},p_i-1} \\ \eta_{i\mathrm{z},p_i} \end{bmatrix}, \ B_{ii} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ B_{i\mathrm{z}} = \begin{bmatrix} b_{i\mathrm{z},1} \\ \vdots \\ b_{i\mathrm{z},p_i-1} \\ 0 \end{bmatrix}, \ B_{ij} = \begin{bmatrix} b_{ij,1} \\ \vdots \\ b_{ij,p_i-1} \\ 0 \end{bmatrix}.$$

As such, by the transformation of (41), Σ_2 is decomposed into the subsystems

$$J_{n_{\mathrm{z}}}\dot{x}_{\mathrm{z}} = x_{\mathrm{z}}, \tag{43}$$

and for $i = 1,\, 2, \cdots, n_{\mathrm{e}}$,

$$J_{\mathrm{v}i}\dot{x}_{\mathrm{v}i} + N_{i\mathrm{z}}\dot{x}_{\mathrm{z}} = x_{\mathrm{v}i} + B_{ii}\widehat{u}_i + \sum_{j=i+1}^{n_{\mathrm{e}}} B_{ij}\widehat{u}_j + B_{i\mathrm{z}}\widehat{u}_*, \tag{44}$$

which is equivalent to

$$J_{\mathrm{v}i}\dot{x}_{\mathrm{v}i} = x_{\mathrm{v}i} + B_{ii}\widehat{u}_i + \sum_{j=i+1}^{n_{\mathrm{e}}} B_{ij}\widehat{u}_j + B_{i\mathrm{z}}\widehat{u}_* - \Big(N_{i\mathrm{z}}\dot{x}_{\mathrm{z}}\Big). \tag{45}$$

Because of the special structure of $J_{\mathrm{v}i}$, we have, for $i = 1, 2, \cdots, n_{\mathrm{e}}$,

$$\left.\begin{aligned}
\dot{x}_{\mathrm{v}i,2} &= x_{\mathrm{v}i,1} + \sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,1}\widehat{u}_j + b_{i\mathrm{z},1}\widehat{u}_* - \eta_{i\mathrm{z},1}\dot{x}_{\mathrm{z}},\\
\dot{x}_{\mathrm{v}i,3} &= x_{\mathrm{v}i,2} + \sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,2}\widehat{u}_j + b_{i\mathrm{z},2}\widehat{u}_* - \eta_{i\mathrm{z},2}\dot{x}_{\mathrm{z}},\\
&\vdots\\
\dot{x}_{\mathrm{v}i,p_i} &= x_{\mathrm{v}i,p_i-1} + \sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,p_i-1}\widehat{u}_j + b_{i\mathrm{z},p_i-1}\widehat{u}_* - \eta_{i\mathrm{z},p_i-1}\dot{x}_{\mathrm{z}},
\end{aligned}\right\} \tag{46}$$

$$\widehat{u}_i = -x_{\mathrm{v}i,p_i} + \eta_{i\mathrm{z},p_i}\dot{x}_{\mathrm{z}}. \tag{47}$$

Repeatedly differentiating $\widehat{u}_i$ of (47), we obtain

$$x_{\mathrm{v}i,1} = -\widehat{u}_i^{(p_i-1)} - \sum_{k=0}^{p_i-2}\sum_{j=i+1}^{n_{\mathrm{e}}} b_{ij,k+1}\widehat{u}_j^{(k)} - \sum_{k=0}^{p_i-2} b_{i\mathrm{z},k+1}\widehat{u}_*^{(k)} + \sum_{k=1}^{p_i}\eta_{i\mathrm{z},k}x_{\mathrm{z}}^{(k)}. \tag{48}$$

Let us define a new input variable

$$\breve{u}_i = -x_{\mathrm{v}i,1} + \sum_{k=1}^{p_i}\eta_{i\mathrm{z},k}x_{\mathrm{z}}^{(k)} = \psi_i(s)\widehat{u}, \tag{49}$$

for an appropriate vector $\psi_i(s)$ whose elements are polynomials of s. Then, we can rewrite (46) as follows:

$$\left.\begin{aligned}
\dot{x}_{\mathrm{v}i,2} &= -\sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,1}x_{\mathrm{v}j,p_j} + \sum_{j=i+1\ \&\ p_j=1}^{n_{\mathrm{e}}} b_{ij,1}\breve{u}_j + b_{i\mathrm{z},1}\widehat{u}_*\\
&\quad - \breve{u}_i + \sum_{k=2}^{p_i}\eta_{i\mathrm{z},k}x_{\mathrm{z}}^{(k)} + \sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,1}\eta_{j\mathrm{z},p_j}\dot{x}_{\mathrm{z}},\\
\dot{x}_{\mathrm{v}i,3} &= x_{\mathrm{v}i,2} - \sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,2}x_{\mathrm{v}j,p_j} + \sum_{j=i+1\ \&\ p_j=1}^{n_{\mathrm{e}}} b_{ij,2}\breve{u}_j\\
&\quad + b_{i\mathrm{z},2}\widehat{u}_* - \eta_{i\mathrm{z},2}\dot{x}_{\mathrm{z}} + \sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,2}\eta_{j\mathrm{z},p_j}\dot{x}_{\mathrm{z}},\\
&\vdots\\
\dot{x}_{\mathrm{v}i,p_i} &= x_{\mathrm{v}i,p_i-1} - \sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,p_i-1}x_{\mathrm{v}j,p_j} + \sum_{j=i+1\ \&\ p_j=1}^{n_{\mathrm{e}}} b_{ij,p_i-1}\breve{u}_j\\
&\quad + b_{i\mathrm{z},p_i-1}\widehat{u}_* - \eta_{i\mathrm{z},p_i-1}\dot{x}_{\mathrm{z}} + \sum_{j=i+1\ \&\ p_j>1}^{n_{\mathrm{e}}} b_{ij,p_i-1}\eta_{j\mathrm{z},p_j}\dot{x}_{\mathrm{z}}.
\end{aligned}\right\} \tag{50}$$

Next, define

$$\check{u}_{\rm e} = \begin{pmatrix} \check{u}_1 \\ \check{u}_2 \\ \vdots \\ \check{u}_{n_{\rm e}} \end{pmatrix} = \begin{pmatrix} -x_{{\rm v}1,1} + \sum_{k=1}^{p_1} \eta_{1{\rm z},k} x_{\rm z}^{(k)} \\ -x_{{\rm v}2,1} + \sum_{k=1}^{p_2} \eta_{2{\rm z},k} x_{\rm z}^{(k)} \\ \vdots \\ -x_{{\rm v}n_{\rm e},1} + \sum_{k=1}^{p_{n_{\rm e}}} \eta_{n_{\rm e}{\rm z},k} x_{\rm z}^{(k)} \end{pmatrix}, \tag{51}$$

and

$$x_{\rm e} = \begin{pmatrix} x_{{\rm v}1,1} \\ x_{{\rm v}2,1} \\ \vdots \\ x_{{\rm v}n_{\rm e},1} \end{pmatrix} = -\check{u}_{\rm e} + \begin{pmatrix} \sum_{k=1}^{p_1} \eta_{1{\rm z},k} x_{\rm z}^{(k)} \\ \sum_{k=1}^{p_2} \eta_{2{\rm z},k} x_{\rm z}^{(k)} \\ \vdots \\ \sum_{k=1}^{p_{n_{\rm e}}} \eta_{n_{\rm e}{\rm z},k} x_{\rm z}^{(k)} \end{pmatrix} = -\check{u}_{\rm e} + sN_{\rm ez}(s)x_{\rm z}, \tag{52}$$

where $N_{\rm ez}(s)$ is a matrix whose elements are polynomials of s. It is now straightforward to verify that the transformed system of Σ_2 as given in (40) can be rearranged into the form

$$\begin{cases} J_{n_{\rm z}}\dot{x}_{\rm z} = x_{\rm z}, \\ x_{\rm e} = -\check{u}_{\rm e} + sN_{\rm ez}(s)x_{\rm z}, \\ \dot{\check{x}}_2 = \check{A}_2\check{x}_2 + \check{B}_{2{\rm e}}\check{u}_{\rm e} + \check{B}_{2*}\widehat{u}_* + s\check{B}_{2{\rm z}}(s)x_{\rm z}, \\ y_2 = \check{C}_2\check{x}_2 + \check{D}_{2{\rm e}}\check{u}_{\rm e} + [s\check{D}_{2{\rm z}}(s) + \check{C}_{\rm z}]x_{\rm z}, \end{cases} \tag{53}$$

where $\check{x}_2$ consists of all the state variables of $x_{\rm v}$ that are not contained in $x_{\rm e}$, and $\check{A}_2$, $\check{B}_{2{\rm e}}$, $\check{B}_{2*}$, $\check{C}_2$, $\check{D}_{2{\rm e}}$ and $\check{C}_{\rm z}$ are constant matrices of appropriate dimensions, and $\check{B}_{2{\rm z}}(s)$ and $\check{D}_{2{\rm z}}(s)$ are matrices with their entries being some polynomials of s. Furthermore, Σ_1 of (39) can be rewritten as follows:

$$\begin{cases} \dot{x}_1 = A_1x_1 + \check{A}_{12}\check{x}_2 + \check{B}_{1{\rm e}}\check{u}_{\rm e} + \check{B}_{1*}\widehat{u}_* + s\check{B}_{1{\rm z}}(s)x_{\rm z}, \\ y_1 = C_1x_1 + \check{C}_{12}\check{x}_2 + \check{D}_{1{\rm e}}\check{u}_{\rm e} + \check{D}_{1*}\widehat{u}_* + s\check{D}_{1{\rm z}}(s)x_{\rm z}, \end{cases} \tag{54}$$

for some constant matrices $\check{A}_{12}$, $\check{B}_{1{\rm e}}$, $\check{B}_{1*}$, $\check{C}_{12}$, $\check{D}_{1{\rm e}}$ and $\check{D}_{1*}$ of appropriate dimensions, and for some matrices $\check{B}_{1{\rm z}}(s)$ and $\check{D}_{1{\rm z}}(s)$, whose elements are polynomials in s.

Step 3 Formation of a proper system and final decomposition

The key idea is to form a proper system from the subsystems (53) and (54), and then apply the result of proper systems to obtain a structural decomposition for the original system given in (1). Following (53) and (54), we obtain

$$\overline{\Sigma}: \begin{cases} \dot{\overline{x}} = \overline{A}\,\overline{x} + \overline{B}\,\overline{u} + \overline{B}_{\rm z}(s)x_{\rm z}, \\ y = \overline{C}\,\overline{x} + \overline{D}\,\overline{u} + \overline{D}_{\rm z}(s)x_{\rm z}, \end{cases} \tag{55}$$

where

$$\overline{x} = \begin{pmatrix} x_1 \\ \check{x}_2 \end{pmatrix}, \ \overline{u} = \begin{pmatrix} \check{u}_{\rm e} \\ \widehat{u}_* \end{pmatrix}, \ \overline{A} = \begin{bmatrix} A_1 & \check{A}_{12} \\ 0 & \check{A}_2 \end{bmatrix}, \ \overline{B} = \begin{bmatrix} \check{B}_{1{\rm e}} & \check{B}_{1*} \\ \check{B}_{2{\rm e}} & \check{B}_{2*} \end{bmatrix}, \ \overline{B}_{\rm z}(s) = \begin{bmatrix} s\check{B}_{1{\rm z}}(s) \\ s\check{B}_{2{\rm z}}(s) \end{bmatrix}, \tag{56}$$

and

$$\overline{C} = [\, C_1 \quad \check{C}_2 + \check{C}_{12} \,] \,,\ \overline{D} = [\, \check{D}_{1\mathrm{e}} + \check{D}_{2\mathrm{e}} \quad \check{D}_{1*} \,] \,,\ \overline{D}_{\mathrm{z}}(s) = \check{C}_{\mathrm{z}} + s\check{D}_{1\mathrm{z}}(s) + s\check{D}_{2\mathrm{z}}(s). \tag{57}$$

It then follows from the result of [19] (see also, Theorem 2.4.1 of [16] and [20]) that there exist nonsingular transformations $\overline{\varGamma}_{\mathrm{s}} \in \mathbb{R}^{\overline{n}\times\overline{n}}$, where $\overline{n} = n - n_{\mathrm{e}} - n_{\mathrm{z}}$, $\overline{\varGamma}_{\mathrm{o}} \in \mathbb{R}^{p\times p}$ and $\overline{\varGamma}_{\mathrm{i}} \in \mathbb{R}^{m\times m}$ such that when they are applied to $\overline{\varSigma}$, i.e.,

$$\overline{x} = \overline{\varGamma}_{\mathrm{s}} \begin{pmatrix} x_{\mathrm{a}} \\ x_{\mathrm{b}} \\ x_{\mathrm{c}} \\ x_{\mathrm{d}} \end{pmatrix}, \quad y = \overline{\varGamma}_{\mathrm{o}}\widetilde{y} = \overline{\varGamma}_{\mathrm{o}} \begin{pmatrix} y_0 \\ y_{\mathrm{d}} \\ y_{\mathrm{b}} \end{pmatrix}, \quad \overline{u} = \overline{\varGamma}_{\mathrm{i}}\widetilde{u} = \overline{\varGamma}_{\mathrm{i}} \begin{pmatrix} u_0 \\ u_{\mathrm{d}} \\ u_{\mathrm{c}} \end{pmatrix}, \tag{58}$$

where $x_{\mathrm{a}} \in \mathbb{R}^{n_{\mathrm{a}}}$, $x_{\mathrm{b}} \in \mathbb{R}^{n_{\mathrm{b}}}$, $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, $x_{\mathrm{d}} \in \mathbb{R}^{n_{\mathrm{d}}}$, $u_0 \in \mathbb{R}^{n_0}$, $u_{\mathrm{c}} \in \mathbb{R}^{m_{\mathrm{c}}}$, $u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}$, $y_0 \in \mathbb{R}^{n_0}$, $y_{\mathrm{b}} \in \mathbb{R}^{p_{\mathrm{b}}}$, $y_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}$,

$$x_{\mathrm{d}} = \begin{pmatrix} x_{\mathrm{d}1} \\ x_{\mathrm{d}2} \\ \vdots \\ x_{\mathrm{d}m_{\mathrm{d}}} \end{pmatrix}, \quad y_{\mathrm{d}} = \begin{pmatrix} y_{\mathrm{d}1} \\ y_{\mathrm{d}2} \\ \vdots \\ y_{\mathrm{d}m_{\mathrm{d}}} \end{pmatrix}, \quad u_{\mathrm{d}} = \begin{pmatrix} u_{\mathrm{d}1} \\ u_{\mathrm{d}2} \\ \vdots \\ u_{\mathrm{d}m_{\mathrm{d}}} \end{pmatrix}, \tag{59}$$

we have

$$\dot{x}_{\mathrm{a}} = A_{\mathrm{aa}}x_{\mathrm{a}} + B_{0\mathrm{a}}y_0 + L_{\mathrm{ad}}y_{\mathrm{d}} + L_{\mathrm{ab}}y_{\mathrm{b}} + sL_{\mathrm{az}}(s)x_{\mathrm{z}}, \tag{60}$$

$$\dot{x}_{\mathrm{b}} = A_{\mathrm{bb}}x_{\mathrm{b}} + B_{0\mathrm{b}}y_0 + L_{\mathrm{bd}}y_{\mathrm{d}} + sL_{\mathrm{bz}}(s)x_{\mathrm{z}}, \tag{61}$$

$$y_{\mathrm{b}} = C_{\mathrm{b}}x_{\mathrm{b}} + C_{\mathrm{bz}}x_{\mathrm{z}} + sC_{\mathrm{bzs}}(s)x_{\mathrm{z}}, \tag{62}$$

$$\dot{x}_{\mathrm{c}} = A_{\mathrm{cc}}x_{\mathrm{c}} + B_{0\mathrm{c}}y_0 + L_{\mathrm{cd}}y_{\mathrm{d}} + L_{\mathrm{cb}}y_{\mathrm{b}} + B_{\mathrm{c}}\left[u_{\mathrm{c}} + M_{\mathrm{ca}}x_{\mathrm{a}}\right] + sL_{\mathrm{cz}}(s)x_{\mathrm{z}}, \tag{63}$$

$$y_0 = C_{0\mathrm{a}}x_{\mathrm{a}} + C_{0\mathrm{b}}x_{\mathrm{b}} + C_{0\mathrm{c}}x_{\mathrm{c}} + C_{0\mathrm{d}}x_{\mathrm{d}} + u_0 + C_{0\mathrm{z}}x_{\mathrm{z}} + sC_{0\mathrm{zs}}(s)x_{\mathrm{z}}, \tag{64}$$

and

$$\dot{x}_{\mathrm{d}i} = A_{q_i}x_{\mathrm{d}i} + L_{i0}y_0 + L_{i\mathrm{d}}y_{\mathrm{d}} + sL_{i\mathrm{z}}(s)x_{\mathrm{z}} + B_{q_i}\left(u_{\mathrm{d}i} + M_{i\mathrm{a}}x_{\mathrm{a}} + M_{i\mathrm{b}}x_{\mathrm{b}} + M_{i\mathrm{c}}x_{\mathrm{c}} + \sum_{j=1}^{m_{\mathrm{d}}} M_{ij}x_{\mathrm{d}j}\right), \tag{65}$$

$$y_{\mathrm{d}i} = C_{q_i}x_{\mathrm{d}i} + C_{q_i\mathrm{z}}x_{\mathrm{z}} + sC_{q_i\mathrm{zs}}(s)x_{\mathrm{z}},\ y_{\mathrm{d}} = C_{\mathrm{d}}x_{\mathrm{d}} + C_{\mathrm{dz}}x_{\mathrm{z}} + sC_{\mathrm{dzs}}(s)x_{\mathrm{z}}, \tag{66}$$

with $(A_{q_i}, B_{q_i}, C_{q_i})$ having the special form as given in (13).

This completes the proof of Theorem 1. ∎

Next, we note that the results of Corollaries 1 and 2 follow from the above construction procedures and some tedious manipulations.

Noting that all transformations involved in the structural decomposition in Theorem 1 are non-singular (with some differentiations), the result of Property 1 follows similarly as that given in [21] for proper systems. Assuming that $\varSigma$ is continuous at the initial state and setting $y \equiv 0$, it is straightforward to show that the dynamics of $\varSigma$ under such a situation are reduced to

$$\dot{x}_{\mathrm{a}} = A_{\mathrm{aa}}x_{\mathrm{a}}, \tag{67}$$

$$\dot{x}_{\mathrm{c}} = A_{\mathrm{cc}}x_{\mathrm{c}} + B_{\mathrm{c}}(u_{\mathrm{c}} + M_{\mathrm{ca}}x_{\mathrm{a}}). \tag{68}$$

Clearly, the dynamics associated with x_{c} are variant under the system input. Thus, the invariant zero dynamics of $\varSigma$, which are invariant with respect to the system input, are given by (67).

Hence, $\lambda(A_{\rm aa})$ are the invariant zeros of Σ, and Property 2 follows. Finally, the invertibility of Σ, i.e., Property 3, follows directly from Corollary 2 and the result for proper systems reported in [21].

4 An Illustrative Example

We now present a numerical example to illustrate the structural decomposition technique and its properties. We consider a descriptor system of (1) characterized by

$$E = \left[\begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 0 & 2 \\ 0 & 0 & -1 & 1 & -3 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad A = I_7, \quad B = \left[\begin{array}{ccc} 1 & 1 & 0 \\ \hline -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{array}\right], \tag{69}$$

and

$$C = \left[\begin{array}{c|cccccc} 1 & -2 & 0 & 0 & 1 & 2 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 & 0 \end{array}\right], \quad D = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{70}$$

Step 1 Preliminary decomposition

It is simple to note that the given system is already in the forms of (38) with $n_1 = 1$, $n_2 = 6$,

$$\Sigma_1 : \begin{cases} \dot{x}_1 &= x_1 + [1 \quad 1 \quad 0]\, u, \\ y_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u, \end{cases}$$

and Σ_2 being characterized by

$$N\dot{x}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 2 \\ 0 & -1 & 1 & -3 & 0 & 3 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{x}_2 = x_2 + B_2 u = x_2 + \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} u,$$

and

$$y_2 = C_2 x_2 = \begin{bmatrix} -2 & 0 & 0 & 1 & 2 & -1 \\ -1 & -1 & 1 & 0 & 1 & 0 \end{bmatrix} x_2.$$

Step 2 Decomposition of $x_{\rm z}$ and $x_{\rm e}$

Using the toolkit of [22], we obtain two nonsingular transformations

$$T_{\rm s} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad T_{\rm i} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

which transform Σ_2 into the canonical form

$$x_2 = T_{\rm s}\begin{pmatrix} x_{\rm v} \\ x_{\rm z}\end{pmatrix}, \quad x_{\rm v} = \begin{pmatrix} x_{\rm v1} \\ x_{\rm v2}\end{pmatrix}, \quad u = T_{\rm i}\begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_* \end{pmatrix},$$

$$T_{\rm s}^{-1}NT_{\rm s} = \left[\begin{array}{ccc|cc|c} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad T_{\rm s}^{-1}B_2T_{\rm i} = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad C_2T_{\rm s} = \left[\begin{array}{ccc|cc|c} 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

The transformed system of Σ_2 can then be written as

$$\begin{cases} \dot{x}_{\rm v1,2} = x_{\rm v1,1} + \widehat{u}_* - \dot{x}_{\rm z}, \\ \dot{x}_{\rm v1,3} = x_{\rm v1,2} + \widehat{u}_2 + \widehat{u}_* - \dot{x}_{\rm z}, \\ \quad \widehat{u}_1 = -x_{\rm v1,3} + \dot{x}_{\rm z}, \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_{\rm v2,2} = x_{\rm v2,1} + \widehat{u}_* - \dot{x}_{\rm z}, \\ \quad \widehat{u}_2 = -x_{\rm v2,2} + \dot{x}_{\rm z}, \end{cases} \tag{71}$$

and

$$0 \cdot \dot{x}_{\rm z} = x_{\rm z} \quad \Rightarrow \quad x_{\rm z} = 0, \tag{72}$$

as well as

$$y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{\rm v1,3} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_{\rm v2,2} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_z\,. \tag{73}$$

Hence, we have $n_{\rm z} = 1$, $n_{\rm e} = 2$, $p_1 = 3$, $p_2 = 2$,

$$x_{\rm v1,1} = -\ddot{\widehat{u}}_1 - \dot{\widehat{u}}_2 - \left(\widehat{u}_* + \dot{\widehat{u}}_*\right) + \left(\dot{x}_{\rm z} + \ddot{x}_{\rm z} + x_{\rm z}^{(3)}\right),$$
$$x_{\rm v2,1} = -\dot{\widehat{u}}_2 - \widehat{u}_* + (\dot{x}_{\rm z} + \ddot{x}_{\rm z}).$$

Next, define

$$\check{u}_{\rm e} = \begin{pmatrix} \check{u}_1 \\ \check{u}_2 \end{pmatrix} = \begin{pmatrix} \ddot{\widehat{u}}_1 + \dot{\widehat{u}}_2 + \dot{\widehat{u}}_* + \widehat{u}_* \\ \dot{\widehat{u}}_2 + \widehat{u}_* \end{pmatrix} = \begin{bmatrix} s^2 & s & s+1 \\ 0 & s & 1 \end{bmatrix} \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_* \end{pmatrix},$$

and

$$x_{\rm e} = \begin{pmatrix} x_{\rm v1,1} \\ x_{\rm v2,1} \end{pmatrix} = -\check{u}_{\rm e} + \begin{pmatrix} \dot{x}_{\rm z} + \ddot{x}_{\rm z} + x_{\rm z}^{(3)} \\ \dot{x}_{\rm z} + \ddot{x}_{\rm z} \end{pmatrix} = -\check{u}_{\rm e} + sN_{\rm ez}(s)x_{\rm z},$$

where

$$N_{\rm ez}(s) = \begin{bmatrix} s^2 + s + 1 \\ s + 1 \end{bmatrix}.$$

Then, (71) can be rewritten as

$$\dot{x}_{\rm v1,2} = -\check{u}_1 + \widehat{u}_* + \left(\ddot{x}_z + x_{\rm z}^{(3)}\right),$$
$$\dot{x}_{\rm v1,3} = x_{\rm v1,2} - x_{\rm v2,2} + \widehat{u}_*,$$
$$\dot{x}_{\rm v2,2} = -\check{u}_2 + \widehat{u}_* + \ddot{x}_{\rm z},$$

or in the matrix form

$$\dot{\check{x}}_2 = \begin{pmatrix} \dot{x}_{\mathrm{v}1,2} \\ \dot{x}_{\mathrm{v}1,3} \\ \dot{x}_{\mathrm{v}2,2} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \check{x}_2 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \check{u}_{\mathrm{e}} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \widehat{u}_* + s \begin{bmatrix} s^2+s \\ 0 \\ s \end{bmatrix} x_{\mathrm{z}}. \tag{74}$$

Also, (73) can be rewritten as

$$y_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \check{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_{\mathrm{z}}. \tag{75}$$

Further, we have

$$B_1 T_{\mathrm{i}} = [\,2 \quad 1 \quad 0\,], \qquad DT_{\mathrm{i}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{76}$$

In view of (71) and (76), we can rewrite Σ_1 as

$$\dot{x}_1 = x_1 + [\,0 \quad -2 \quad -1\,]\, \check{x}_2 + 3 s x_{\mathrm{z}}, \tag{77}$$

$$y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \check{x}_2 + s \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_{\mathrm{z}}. \tag{78}$$

Step 3 Formation of a proper system and final decomposition
Combining (74), (75), (77) and (78), We obtain an auxiliary proper system

$$\begin{cases} \dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}\overline{u} + \overline{B}_{\mathrm{z}}(s) x_{\mathrm{z}}, \\ y = \overline{C}\overline{x} + \overline{D}\overline{u} + \overline{D}_{\mathrm{z}}(s) x_{\mathrm{z}}, \end{cases}$$

with

$$\overline{x} = \begin{pmatrix} x_1 \\ \check{x}_2 \end{pmatrix}, \quad \overline{u} = \begin{pmatrix} \check{u}_{\mathrm{e}} \\ \widehat{u}_* \end{pmatrix},$$

$$\overline{A} = \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \overline{B}_{\mathrm{z}}(s) = s \begin{bmatrix} 3 \\ s^2+s \\ 0 \\ s \end{bmatrix},$$

$$\overline{C} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{D}_{\mathrm{z}}(s) = \begin{bmatrix} s+1 \\ 0 \end{bmatrix}.$$

Again, using the toolkit of [22], we obtain

$$\overline{\Gamma}_{\mathrm{s}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \overline{\Gamma}_{\mathrm{i}} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad \overline{\Gamma}_{\mathrm{o}} = I_2,$$

$$n_{\mathrm{a}} = 1, \quad n_{\mathrm{b}} = 0, \quad n_{\mathrm{c}} = 1, \quad n_{\mathrm{d}} = 2,$$

$$\overline{x} = \overline{\Gamma}_{\mathrm{s}} \begin{pmatrix} x_{\mathrm{a}} \\ x_{\mathrm{c}} \\ x_{\mathrm{d}1} \\ x_{\mathrm{d}2} \end{pmatrix}, \quad \overline{u} = \overline{\Gamma}_{\mathrm{i}} \begin{pmatrix} u_{\mathrm{d}} \\ u_{\mathrm{c}} \end{pmatrix} = \overline{\Gamma}_{\mathrm{i}} \begin{pmatrix} u_{\mathrm{d}1} \\ u_{\mathrm{d}2} \\ u_{\mathrm{c}} \end{pmatrix}, \quad y = \overline{\Gamma}_{\mathrm{o}} \begin{pmatrix} y_{\mathrm{d}1} \\ y_{\mathrm{d}2} \end{pmatrix},$$

$$\overline{\Gamma}_{\mathrm{s}}^{-1} \overline{A} \overline{\Gamma}_{\mathrm{s}} = \begin{bmatrix} 2 & 0 & -1 & 2 \\ 1 & -1 & 2 & -1 \\ -2 & 0 & 1 & -2 \\ -3 & 1 & -1 & -1 \end{bmatrix}, \quad \overline{\Gamma}_{\mathrm{s}}^{-1} \overline{B} \overline{\Gamma}_{\mathrm{i}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\overline{\Gamma}_{\rm s}^{-1}\overline{B}_{\rm z}(s) = s\begin{bmatrix} -3 \\ s^2+s \\ s+3 \\ s+3 \end{bmatrix}, \quad \overline{\Gamma}_{\rm o}^{-1}\overline{C\Gamma}_{\rm s} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{\Gamma}_{\rm o}^{-1}\overline{D}_{\rm z}(s) = \begin{bmatrix} s+1 \\ 0 \end{bmatrix}.$$

Finally, the structural decomposition of the given descriptor system is given by

$$\begin{aligned}
&0\cdot \dot{x}_{\rm z} = x_{\rm z}, \\
&x_{\rm e} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{\rm c} - \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} u_{\rm d} + s\begin{bmatrix} s^2+s+1 \\ s+1 \end{bmatrix} x_{\rm z}, \quad u_{\rm d} = \begin{pmatrix} u_{\rm d1} \\ u_{\rm d2} \end{pmatrix}, \\
&\dot{x}_{\rm a} = 2x_{\rm a} + [\,-1 \quad 2\,]\, y_{\rm d} - 3s x_{\rm z}, \\
&\dot{x}_{\rm c} = -x_{\rm c} + x_{\rm a} + [\,2 \quad -1\,]\, y_{\rm d} + u_{\rm c} + s(s^2+s)x_{\rm z}, \\
&\begin{pmatrix} \dot{x}_{\rm d1} \\ \dot{x}_{\rm d2} \end{pmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} x_{\rm a} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_{\rm c} + \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} y_{\rm d} + u_{\rm d} + s\begin{bmatrix} s+3 \\ s+3 \end{bmatrix} x_{\rm z}, \\
&y_{\rm d} = \begin{pmatrix} y_{\rm d1} \\ y_{\rm d2} \end{pmatrix} = \begin{pmatrix} x_{\rm d1} \\ x_{\rm d2} \end{pmatrix} + \begin{bmatrix} s+1 \\ 0 \end{bmatrix} x_{\rm z}.
\end{aligned}$$

It is now simple to see from the above decomposition that the given system is right invertible with one invariant zero at $s = 2$ and two infinite zeros of order 1. The given system has one state variable, which is identically zero, and two state variables, which are nothing but the linear combination of the system inputs and their derivatives. These state variables are actually redundant in the system dynamics. For completeness, we give below all the necessary transformation matrices:

$$\Gamma_{\rm e}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & s^2-s & -s^2 & s^2-s & 0 & s-s^2 \\ 0 & s & 0 & 0 & -s-1 & -s & s \\ -1 & -1 & -2 & 2 & -1 & 1 & 1 \\ 0 & -1 & s^2-s-1 & -s^2 & s^2-s & 1 & s-s^2 \\ 1 & 1+s & 2 & -2 & 1-s & -1-s & s-1 \\ 1 & 2+s & 3 & -2 & 1-s & -2-s & s-1 \end{bmatrix},$$

$$\Gamma_{\rm s} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Gamma_{\rm i}^{-1}(s) = \begin{bmatrix} s & -s & 0 \\ s-1 & -s & 1 \\ -s^2+2s+1 & -s & -s-1 \end{bmatrix}, \quad \Gamma_{\rm o} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We note that the s-dependent input transformation $\Gamma_{\rm i}(s)$ simply implies that

$$\begin{pmatrix} u_{\rm c} \\ u_{\rm d1} \\ u_{\rm d2} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \ddot{u} + \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix} \dot{u} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} u.$$

The compact form of the structural decomposition of Σ (see Corollary 1) is given by

$$E_s = \left[\begin{array}{c|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad A_s = \left[\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 & 1 & -1 & -1 \end{array}\right], \quad B_s = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 2 & -1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right],$$

$$C_s = \left[\begin{array}{c|cc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad D_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_z(s) = \left[\begin{array}{c|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline s^2+s+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ s+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ s^2+s & 0 & 0 & 0 & 0 & 0 & 0 \\ s+3 & 0 & 0 & 0 & 0 & 0 & 0 \\ s+3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\Psi_c = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Psi_d(s) = \begin{bmatrix} -\frac{1}{s} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Psi(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & s+1 & -s-1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & s+1 & -s-1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

It is straightforward to verify that

$$\Psi_c \widetilde{x} + \Psi_d(s)\widetilde{u} = \begin{bmatrix} s \\ 0 \end{bmatrix} x_z,$$

$$H(s) = C(sE - A)^{-1}B + D = \Gamma_o \Big[C_s(sE_s - A_s)^{-1}B_s + D_s \Big] \Gamma_i^{-1}(s).$$

5 Conclusions

We have presented in this paper a structural decomposition technique for linear singular systems, which has a distinct feature of explicitly capturing and displaying the structural properties, such as the finite and infinite zero structures, of the given system. As its counterpart in non-singular systems, the technique is expected to play an important role in solving many control problems related to linear multivariable singular systems. This will be the subject of our future research.

Acknowledgment

The authors would like to thank Xinmin Liu of the University of Virginia for carefully proofreading the whole manuscript and for many suggestions that help to improve the results and the presentation of our work.

References

[1] F. L. Lewis, A survey of linear singular systems, *Circuits, Systems, and Signal Processing*, 1986, **5**: 3–36.

[2] L. Dai, *Singular Control System*, Springer-Verlag, Berlin, 1989.

[3] M. Kuijper, *First Order Representations of Linear Systems*, Birkhauser, Boston, 1994.

[4] D. L. Chu and D. W. C. Ho, Necessary and sufficient conditions for the output feedback regularization of descriptor systems, *IEEE Transactions on Automatic Control*, 1999, **44**: 405–412.

[5] D. L. Chu and V. Mehrmann, Disturbance decoupling for descriptor systems by state feedback, *SIAM Journal on Control and Optimization*, 2000, **38**: 1830–1858.

[6] M. Fliess, Some basic structural properties of generalized linear systems, *Systems & Control Letters*, 1990, **15**: 391–396.

[7] T. Geerts, Invariant subspaces and invertibility properties for singular systems: The general case, *Linear Algebra and Its Applications*, 1993, **183**: 61–88.

[8] F. L. Lewis and K. Ozcaldiran, Geometric structures and feedback in singular systems, *IEEE Transactions on Automatic Control*, 1989, **34**: 450–455.

[9] J. J. Loiseau, Some geometric considerations about the Kronecker normal form, *International Journal of Control*, 1985, **42**: 1411–1431.

[10] M. Malabre, Generalized linear systems: Geometric and structural approaches, *Linear Algebra and its Applications*, 1989, **122–123**: 591–621.

[11] P. Misra, P. Van Dooren, and A. Varga, Computation of structural invariants of generalized state-space systems, *Automatica*, 1994, **30**: 1921–1936.

[12] P. Van Dooren, The generalized eigenstructure problem in linear system theory, *IEEE Transactions on Automatic Control*, 1981, **26**: 111–129.

[13] P. Van Dooren, The eigenstructure of an arbitrary polynomial matrix: Computational aspects, *Linear Algebra and its Applications*, 1983, **50**: 545–579.

[14] G. Verghese, *Infinite Frequency Behavior in Generalized Dynamical Systems*, PhD Dissertation, Stanford University, 1978.

[15] Z. Zhou, M. A. Shayman, and T. J. Tarn, Singular systems: A new approach in the time domain, *IEEE Transactions on Automatic Control*, 1987, **32**: 42–50.

[16] B. M. Chen, *Robust and H_∞ Control*, Springer, New York, 2000.

[17] A. Saberi, P. Sannuti, and B. M. Chen, *H_2 Optimal Control*, Prentice Hall, London, 1995.

[18] M. He and B. M. Chen, Structural decomposition of linear singular systems: The single-input and single-output case, *Systems & Control Letters*, 2002, **47**: 327–334.

[19] P. Sannuti and A. Saberi, A special coordinate basis of multivariable linear systems – Finite and infinite zero structure, squaring down and decoupling, *International Journal of Control*, 1987, **45**: 1655–1704.

[20] D. L. Chu, X. Liu, and R. C. E. Tan, On the numerical computation of a structural decomposition in systems and control, *IEEE Transactions on Automatic Control*, 2002, **47**: 1786–1799.

[21] B. M. Chen, On the properties of the special coordinate basis of linear systems, *International Journal of Control*, 1998, **71**: 981–1003.

[22] Z. Lin, B. M. Chen, and X. Liu, *Linear Systems Toolkit*. URL: http://www.linearsystemskit.net or http://hdd.ece.nus.edu.sg/~bmchen/linsyskit/index.html, available online since 2004.