Leader-Following Output Consensus of Discrete-Time Heterogeneous Systems

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Abstract Motivated by the importance of physical constraints of multi-agent systems, we investigate in this work the semi-global leader-following output consensus control of discrete-time heterogeneous linear systems subject to position and rate-limited actuators and directed switching networks. Based on a distributed observer, we designed a distributed control law for each follower. It is shown that the problem is solvable by the control law if the switching networks are directed and every time disconnected. The problem is then solved by utilizing the theory of output regulation and low gain feedback design techniques. The result is successfully demonstrated by a numerical example.

1 Introduction

The past decade saw the tremendous development of multi-agent systems because of their wide applications in output regulation [1, 2], consensus [3], synchronization [4], and formation control [5], to name just a few, among which consensus control is one of the fundamental problems, and the consensus problem include the leaderless type and the leader-following type. The leaderless consensus control aims to drive the states/outputs of agents converge to a same value, while the objective of the leader-following consensus control is to drive the state/output of the follower systems converge to the state/output of the leader.

All real-life systems, be it aircraft, spacecraft, ground vehicles and underwater vehicles, have physical constraints. The physical constraints include but not limited to actuator saturation and limits of velocities and accelerations. In most results considering physical constraints, only input saturation is studied (see, for example, [6–10]).

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Specifically, Refs. [7, 8] solved the semi-global asymptotically stabilization problem for respectively a continuous-time linear system and a discrete-time linear system with input saturation. References [9, 10] tackled the semi-global leader-following output consensus problem of a group of linear systems over continuous-time domain and discrete-time domain, respectively.

For multi-agent systems with position-limited actuators, a few results on global consensus were obtained, see for example Zhao and Lin for continuous time systems [11] and discrete-time systems [12], in which both a state feedback nonlinear control law and an output feedback nonlinear control law were designed. Event-triggered approach is also used in [13, 14] for establishing global consensus control laws. In the work of Zhou and Chen [15] which studied the formation-containment control of Euler–Lagrange systems, the leaders are subject to position-limited actuators, while the followers are not.

Besides input saturation, rate saturation is also an inevitable part of actuators. For instance, rate saturation has been identified as a contributing factor to the mishaps of YF-22 [16] and Gripen [17]. Destabilizing effects of actuator rate saturation was further discussed in [18]. It shows that rate-limited actuators may lead to more severe cases. Motivated by the above facts, Lin [19] studied the position and rate-limited case in 1997 for the semi-global stabilization problem of a linear system. In [20], the discrete-time counterpart of [19], semi-global stabilization was achieved by a linear feedback control law if a discrete-time linear system is asymptotically null controllable. The result of [19] is extended to solve the semi-global leader-following consensus [21] and containment problem [22] of multiple linear systems in continuous time domain.

In the above literature, the communication topology is assumed to be static. The communication topology can also be switching. The jointly connected switching graph is the mildest condition because it allows the graph to be disconnected at every time instant. Relevant results about multi-agent systems over jointly connected switching graph include [23–26], in which the authors has tackled the problem from different angles for different classes of systems.

In this chapter, we study the problem of the leader-following output consensus control of heterogeneous linear systems with actuator position saturation and rate saturation, and the graph is directed jointly connected switching. Based on a distributed observer, we designed a distributed control law. The difficulty lies in the process to prove the stability of the overall closed-loop system or equivalently the convergence of a corresponding Lyapunov function.

Throughout this chapter, for a time constant $T \geq 0$ and a signal $x(t): \mathbb{R}_+ \rightarrow \mathbb{R}^s$, $x(t) = [x_1(t), x_2(t), \ldots, x_s(t)]^{\mathrm{T}}$ is a column vector, $|x(t)|$ denotes the Euclidean norm at time t, $\|x(t)\|_\infty = \max_i |x_i(t)|$, and $\|x(t)\|_{T,\infty} = \sup_{t \geq T} |x(t)|$. We denote $x(t)$, $t \in \mathbb{Z}^+$ by a shorthand notation x when no confusion will occur. The symbol $\otimes$ denotes the Kronecker product of matrices, $\mathbb{Z}^+$ represents the set of nonnegative integers, $\mathbf{1}_N \in \mathbb{R}^N$ denotes a $N \times 1$ vector with all elements being 1, and $\mathbf{0}$ represents a vector or matrix of zero with appropriate dimension. Let $\sigma : \mathbb{Z}^+ \rightarrow \mathcal{P}$, where

$\mathcal{P} = \{1, 2, \ldots, \rho\}$ denote a discrete-time switching signal in the sense that there exists a subsequence k_i, $i = 0, 1, \ldots$, of $\{k | k \in \mathbb{Z}^+\}$, called switching instants, such that $\sigma(k)$ is a constant for $k_i \le k < k_{i+1}$.

2 Preliminaries

This chapter considers the leader-following output consensus problem of the following class of discrete-time heterogeneous linear systems which are subject to actuator position and rate saturation:

$$\begin{cases} x_i(t+1) = A_i x_i + B_i \mathrm{sat}_{\Delta_1}(v_i) + W_i w \\ v_i(t+1) = v_i + \mathrm{sat}_{\Delta_2}\big((\alpha - 1)v_i + u_i\big), \quad |\alpha| < 1 \\ \quad y_i(t) = C_i x_i \\ \quad e_i(t) = C_i x_i + Qw, \quad i = 1, \ldots, N, \ k \in \mathbb{Z}^+ \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^m$, $u_i \in \mathbb{R}^{q_i}$ are respectively, the plant state, output and control input of the ith follower. The second equation denotes the actuator dynamics with state $v_i \in \mathbb{R}^{q_i}$. Without loss of generality, we assume that all actuators have the same time constant α. $e_i \in \mathbb{R}^m$ denotes output tracking error between the ith follower and the leader. The leader generates both the output to be tracked y_0 and the disturbances to be rejected $W_i w$. Here, $\mathrm{sat}_{\Delta_1}(\cdot)$, $\mathrm{sat}_{\Delta_2}(\cdot) : \mathbb{R}^{q_i} \to \mathbb{R}^{q_i}$ represent vector valued saturation functions. For $v_i = [v_{i,1}, v_{i,2}, \ldots, v_{i,q_i}]^{\mathrm{T}}$, $\mathrm{sat}_{\Delta_1}(v_i) = [\mathrm{sat}_{\Delta_1}(v_{i,1}), \mathrm{sat}_{\Delta_1}(v_{i,2}), \ldots, \mathrm{sat}_{\Delta_1}(v_{i,q_i})]^{\mathrm{T}}$. For each $j = 1, 2, \ldots, q_i$, $\mathrm{sat}_{\Delta_1}(v_{i,j}) = \mathrm{sgn}(v_{i,j}) \min\{|v_{i,j}|, \Delta_1\}$ is the standard saturation function. Like in [27], the leader system takes the following form:

$$\begin{cases} w(t+1) = Sw, \\ \quad y_0(t) = -Qw, \end{cases} \tag{2}$$

where $w \in \mathbb{R}^s$, $y_0 \in \mathbb{R}^m$ are the state and output, respectively.

As in [23], associated with (1) and (2), and a given switching signal $\sigma(\cdot)$, we can define a switching digraph $\bar{\mathcal{G}}_{\sigma(t)} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}_{\sigma(t)})$, where $\bar{\mathcal{V}} = \{0, 1, \ldots, N\}$ and $(j, i) \in \bar{\mathcal{E}}_{\sigma(t)}$, $i = 1, \ldots, N$, $j = 0, 1, \ldots, N$, if and only if the control u_i can make use of the information of node j at time t. Denote by $\bar{\mathcal{A}}_{\sigma(t)} = [a_{ij}(t)]_{i,j=0}^{N} \in \mathbb{R}^{(N+1)\times(N+1)}$ the weighted adjacency matrix of $\bar{\mathcal{G}}_{\sigma(t)}$, in which $a_{ii}(t) = 0$. For $i, j = 0, 1, \ldots, N$, let $a_{ij}(t) > 0$ if u_i can make use of the information of node j at time t, otherwise, $a_{ij}(t) = 0$.

The following assumptions on matrices of agents (1) and (2) are made.

Assumption 1 The matrix S is neutrally stable, which means all eigenvalues of S are semi-simple with modulus 1.

Remark 1 Assumption 1 can be relaxed to the assumption that S is marginally stable, i.e., all eigenvalues of S are inside the unit circle and those eigenvalues with modulus 1 are semi-simple.

Assumption 2 For each $i = 1, \ldots, N$, (A_i, B_i) is stabilizable, and all eigenvalues of A_i are located inside or on the unit circle.

Lemma 1 (See Saberi et al. [28]) *Let Assumption 2 hold. Then, for any $\epsilon > 0$, for $i = 1, \ldots, N$, there exists a unique positive definite matrix $P_i(\epsilon) \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, N$, of the following parametric algebraic Riccati equation (ARE):*

$$P_i(\epsilon) = A_i^{\mathrm{T}} P_i(\epsilon) A_i - A_i^{\mathrm{T}} P_i(\epsilon) B_i \left(B_i^{\mathrm{T}} P_i(\epsilon) B_i + I\right)^{-1} B_i^{\mathrm{T}} P_i(\epsilon) A_i + \epsilon I \tag{3}$$

and $A_i - B_i\left(B_i^{\mathrm{T}} P_i(\epsilon) B_i + I\right)^{-1} B_i^{\mathrm{T}} P_i(\epsilon) A_i$ is Schur. Moreover, $\lim_{\epsilon \to 0} P_i(\epsilon) = 0$.

Here we note that the above discrete-time ARE can be solved non-recursively using the approach given in Chen et al. [29]. For convenience, we denote $P_i := P_i(\epsilon)$ hereafter. We further define

$$K_i = (B_i^{\mathrm{T}} P_i B_i + I)^{-1} B_i^{\mathrm{T}} P_i A_i. \tag{4}$$

Since $\lim_{\epsilon \to 0} P_i(\epsilon) = 0$, it is obvious that $\lim_{\epsilon \to 0} K_i(\epsilon) = 0$. Moreover, by (3), it can be verified that

$$(A_i - B_i K_i)^{\mathrm{T}} P_i (A_i - B_i K_i) - P_i = \epsilon I - K_i^{\mathrm{T}} K_i. \tag{5}$$

Assumption 3 For each $i = 1, \ldots, N$, the following regulator Equation

$$\begin{aligned} \Pi_i S &= A_i \Pi_i + B_i \Gamma_i + W_i \\ C_i \Pi_i &+ Q = 0 \end{aligned} \tag{6}$$

have a pair of solutions $\Pi_i \in \mathbb{R}^{n_i \times s}$ and $\Gamma_i \in \mathbb{R}^{q_i \times s}$.

Assumption 4 For each $i = 1, \ldots, N$, there exists a time $T \geq 0$ such that

$$\|\Gamma_i w\|_{T,\infty} < \Delta_1 \quad \text{and} \quad \|\Gamma_i S w\|_{T,\infty} < \Delta_2. \tag{7}$$

Remark 2 Under Assumption 1, $\|w\|$ is bounded. $\Gamma_i w$ and $\Gamma_i S w$ can be viewed as the generalized actuator position and rate of the leader. If Assumption 4 is not satisfied, it is impossible for the followers to catch up the leader when it moves at its maximal pace. Note that $\Gamma_i w(t+1) = \Gamma_i S w(t)$, therefore, $\|\Gamma_i w\|_{T,\infty} = \|\Gamma_i S w\|_{T,\infty}$ for a time $T \geq 0$. For simplicity, we denote $\Delta = \min\{\Delta_1, \Delta_2\}$.

For the switching digraph, we made the following assumption.

Assumption 5 There exists $\bar{T} \geq 0$ such that for all $t \in \mathbb{Z}^+$, every node i, $i = 1, \ldots, N$, is reachable from node 0 in the union digraph $\cup_{p=0}^{\bar{T}} \bar{\mathcal{G}}_{\sigma(t+p)}$.

Remark 3 Similar assumptions to Assumption 5 have also been used in [25, 30]. It is the mildest assumption about a leader-follower digraph because it allows the digraph to be disconnected at any time instant.

We now describe the problem as follows.

Discrete-time leader-following output consensus problem: Given a multi-agent system consisting of the leader system (2), the follower systems (1) and a switching digraph $\bar{\mathcal{G}}_{\sigma(t)}$, design a distributed state feedback control law such that, for a priori given bounded sets $\mathcal{X}_{i,0} \subset \mathbb{R}^{n_i}$, $\mathcal{V}_{i,0} \subset \mathbb{R}^{q_i}$, $\mathcal{W}_0 \subset \mathbb{R}^s$, and

$$[x_i^{\mathrm{T}}(0), v_i^{\mathrm{T}}(0), w^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_{i,0} \times \mathcal{V}_{i,0} \times \mathcal{W}_0,$$

the leader-following output consensus is achieved, that is, for any $i = 1, \ldots, N$,

$$\lim_{t \to \infty} e_i = 0. \tag{8}$$

3 Main Result

This section aims to design a distributed control law for the discrete-time leader-following output consensus problem .

For $i, j = 0, 1, \ldots, N$, let

$$w_{ij}(t) = \begin{cases} \dfrac{1}{1 + \sum_{j=0}^{N} a_{ij}(t)}, & \text{if } i = j, \\ \dfrac{a_{ij}(t)}{1 + \sum_{j=0}^{N} a_{ij}(t)}, & \text{otherwise.} \end{cases} \tag{9}$$

Then, we design a dynamic compensator as follows:

$$\eta_i(t+1) = S\eta_i + S \sum_{j=0}^{N} w_{ij}(t)(\eta_j - \eta_i), \quad i = 1, \ldots, N \tag{10}$$

where $\eta_0 = w$. Based on (10), we construct the state feedback consensus control law for each follower:

$$u_i = -K_i A_i (x_i - \Pi_i \eta_i) - K_i B_i (v_i - \Gamma_i \eta_i) + \Gamma_i S \eta_i - \alpha v_i, \tag{11}$$

where Π_i and Γ_i are a pair of solutions of the regulator Eq. (6), and K_i is defined in (4).

We now have the following theorem.

Theorem 1 *Consider a multi-agent system consisting of the leader (2) and the followers (1). Assume that Assumptions 1–5 hold. The state feedback consensus protocol (11) solves the discrete-time leader-following output consensus problem. That is, for a priori given bounded sets* $\mathcal{X}_{i,0} \subset \mathbb{R}^{n_i}$, $\mathcal{V}_{i,0} \subset \mathbb{R}^{q_i}$, $\mathcal{W}_0 \subset \mathbb{R}^s$, *and*

$$[x_i^{\mathrm{T}}(0), v_i^{\mathrm{T}}(0), w^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_{i,0} \times \mathcal{V}_{i,0} \times \mathcal{W}_0,$$

there exists an $\epsilon^* \in (0, 1]$ *such that for each* $\epsilon \in (0, \epsilon^*]$, *the output consensus error satisfies*

$$\lim_{t\to\infty} e_i = 0.$$

Proof Denote $\tilde{\eta}_i = \eta_i - w$ as the leader state estimation error for the ith follower. Then Theorem 1 of [25] shows $\lim_{t\to\infty} \tilde{\eta}_i = 0, i = 1, \ldots, N$.

Denote $\tilde{x}_i = x_i - \Pi_i w$, one have

$$\begin{aligned}\tilde{x}_i(t+1) &= A_i x_i + B_i \mathrm{sat}_{\Delta_1}(v_i) + W_i w - \Pi_i S w \\ &= A_i \tilde{x}_i + B_i \mathrm{sat}_{\Delta_1}(v_i) - B_i \Gamma_i w,\end{aligned} \tag{12}$$

where the last equality holds due to Assumption 4.

The Lyapunov function candidate we defined is as follows

$$V = \sum_{i=1}^{N} \Big[\tilde{x}_i^{\mathrm{T}} P_i \tilde{x} + 4\big(K_i \tilde{x}_i + v_i - \Gamma_i w\big)^{\mathrm{T}} \big(K_i \tilde{x}_i + v_i - \Gamma_i w\big) \Big]. \tag{13}$$

Then, along the trajectories of (1), (2), and (12),

$$\begin{aligned}\Delta V &= V(t+1) - V(t) \\ &= 4\sum_{i=1}^{N} \Big(\big[K_i \tilde{x}_i(t+1) + v_i(t+1) - \Gamma_i w(t+1)\big]^{\mathrm{T}} \big[K_i \tilde{x}_i(t+1) + v_i(t+1) - \Gamma_i w(t+1)\big] \\ &\quad - (K_i \tilde{x}_i + v_i - \Gamma_i w)^{\mathrm{T}} (K_i \tilde{x}_i + v_i - \Gamma_i w) \Big) + \sum_{i=1}^{N} \Big(\tilde{x}_i^{\mathrm{T}}(t+1) P_i \tilde{x}_i(t+1) - \tilde{x}_i^{\mathrm{T}} P_i \tilde{x}_i \Big).\end{aligned} \tag{14}$$

Denote

$$\begin{aligned}\Theta_{i,1} &= \tilde{x}_i^{\mathrm{T}}(t+1) P_i \tilde{x}_i(t+1) - \tilde{x}_i^{\mathrm{T}} P_i \tilde{x}_i, \\ \Theta_{i,2} &= \big[K_i \tilde{x}_i(t+1) + v_i(t+1) - \Gamma_i w(t+1)\big]^{\mathrm{T}} \big[K_i \tilde{x}_i(t+1) + v_i(t+1) - \Gamma_i w(t+1)\big] \\ &\quad - (K_i \tilde{x}_i + v_i - \Gamma_i w)^{\mathrm{T}} (K_i \tilde{x}_i + v_i - \Gamma_i w).\end{aligned} \tag{15}$$

We have

$$
\begin{aligned}
\Theta_{i,1} &= \tilde{x}_i^{\mathrm{T}}(t+1)P_i\tilde{x}_i(t+1) - \tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i \\
&= \tilde{x}_i^{\mathrm{T}}A_i^{\mathrm{T}}P_i\tilde{x}_i - \tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i + \tilde{x}_i^{\mathrm{T}}A_i^{\mathrm{T}}P_iB_i\left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w\right] \\
&\quad + \left[B_i\mathrm{sat}_{\Delta_1}(v_i) - B_i\Gamma_i w\right]^{\mathrm{T}} P_i\left[A_i\tilde{x}_i + B_i\mathrm{sat}_{\Delta_1}(v_i) - B_i\Gamma_i w\right] \\
&= \tilde{x}_i^{\mathrm{T}}\big(-\epsilon I - K_i^{\mathrm{T}}K_i + A_i^{\mathrm{T}}P_iB_iK_i + K_i^{\mathrm{T}}B_i^{\mathrm{T}}P_iA_i - K_i^{\mathrm{T}}B_i^{\mathrm{T}}P_iB_iK_i\big)\tilde{x}_i \\
&\quad + \tilde{x}_i^{\mathrm{T}}A_i^{\mathrm{T}}P_iB_i\left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w\right] \\
&\quad + \left[B_i\mathrm{sat}_{\Delta_1}(v_i) - B_i\Gamma_i w\right]^{\mathrm{T}} P_i\left[A_i\tilde{x}_i + B_i\mathrm{sat}_{\Delta_1}(v_i) - B_i\Gamma_i w\right] \qquad (16) \\
&= -\epsilon\tilde{x}_i^{\mathrm{T}}\tilde{x}_i - \tilde{x}_i^{\mathrm{T}}K_i^{\mathrm{T}}K_i\tilde{x}_i + 2\tilde{x}_i^{\mathrm{T}}(A_i - B_iK_i)^{\mathrm{T}}P_iB_i\left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w + K_i\tilde{x}_i\right] \\
&\quad + \left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w + K_i\tilde{x}_i\right]^{\mathrm{T}} B_i^{\mathrm{T}}P_iB_i\left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w + K_i\tilde{x}_i\right] \\
&= -\epsilon\tilde{x}_i^{\mathrm{T}}\tilde{x}_i - \tilde{x}_i^{\mathrm{T}}K_i^{\mathrm{T}}K_i\tilde{x}_i + 2\tilde{x}_i^{\mathrm{T}}K_i^{\mathrm{T}}\left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w + K_i\tilde{x}_i\right] \\
&\quad + \left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w + K_i\tilde{x}_i\right]^{\mathrm{T}} B_i^{\mathrm{T}}P_iB_i\left[\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_i w + K_i\tilde{x}_i\right], \qquad (17)
\end{aligned}
$$

where (16) and (17) hold because of (5) and the definition of K_i in (4), respectively. In addition,

$$
\begin{aligned}
\Theta_{i,2} &= \left[K_i\tilde{x}_i(t+1) + v_i(t+1) - \Gamma_i w(t+1)\right]^{\mathrm{T}}\left[K_i\tilde{x}_i(t+1) + v_i(t+1) - \Gamma_i w(t+1)\right] \\
&\quad - \big(K_i\tilde{x}_i + v_i - \Gamma_i w\big)^{\mathrm{T}}\big(K_i\tilde{x}_i + v_i - \Gamma_i w\big) \\
&= \left[K_iA_i\tilde{x}_i + K_iB_i\mathrm{sat}_{\Delta_1}(v_i) - K_iB_i\Gamma_i w + v_i + \mathrm{sat}_{\Delta_2}(u_i + \alpha v_i - v_i) - \Gamma_i Sw\right]^{\mathrm{T}} \\
&\quad \cdot\left[K_iA_i\tilde{x}_i + K_iB_i\mathrm{sat}_{\Delta_1}(v_i) - K_iB_i\Gamma_i w + v_i + \mathrm{sat}_{\Delta_2}(u_i + \alpha v_i - v_i) - \Gamma_i Sw\right] \\
&\quad - \big(K_i\tilde{x}_i + v_i - \Gamma_i w\big)^{\mathrm{T}}\big(K_i\tilde{x}_i + v_i - \Gamma_i w\big), \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathrm{sat}_{\Delta_2}(u_i + \alpha v_i - v_i) \\
&\quad = \mathrm{sat}_{\Delta_2}\Big(-K_iA_i(x_i - \Pi_i\eta_i) - K_iB_i(v_i - \Gamma_i\eta_i) + \Gamma_i S\eta_i - v_i\Big) \\
&\quad = \mathrm{sat}_{\Delta_2}\Big(-K_iA_i\tilde{x}_i - K_iB_iv_i + K_iB_i\Gamma_i w + \Gamma_i Sw - v_i + K_i(A_i\Pi_i\tilde{\eta}_i + B_i\Gamma_i\tilde{\eta}_i) + \Gamma_i S\tilde{\eta}_i\Big).
\end{aligned}
$$

Recall that $\lim_{t\to\infty}\tilde{\eta}_i = 0$, there exists a time T_1 such that

$$
\|K_i(A_i\Pi_i + B_i\Gamma_i)\tilde{\eta}_i\|_{T_1,\infty} \le \frac{1}{60}\Delta, \qquad \|\Gamma_i S\tilde{\eta}_i\|_{T_1,\infty} \le \frac{1}{60}\Delta. \qquad (19)
$$

For any $[x_i^{\mathrm{T}}(0), v_i^{\mathrm{T}}(0), w^{\mathrm{T}}(0), \eta_i^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_{i,0} \times \mathcal{V}_{i,0} \times \mathcal{W}_0 \times \mathcal{Z}_{i,0}$, $\tilde{x}_i(T_1)$, $v_i(T_1)$, $w(T_1)$ and $\tilde{\eta}_i(T_1)$ belong to bounded sets $\tilde{\mathcal{X}}_{i,T_1}$, $\mathcal{V}_{i,T_1}$, $\mathcal{W}_{T_1}$ and $\mathcal{Z}_{i,T_1}$ respectively, independent of ϵ, since they are determined by bounded control inputs. Let $c > 0$ be a constant such that

$$
\sup_{[\tilde{x}_i(T_1), v_i(T_1), w(T_1), \tilde{\eta}_i(T_1)]^{\mathrm{T}} \in \tilde{\mathcal{X}}_{i,T_1} \times \mathcal{V}_{i,T_1} \times \mathcal{W}_{T_1} \times \mathcal{Z}_{i,T_1}} V \le c. \qquad (20)
$$

Define $L_V(c) := \big\{[\tilde{x}_i(T_1)^{\mathrm{T}}, v_i(T_1)^{\mathrm{T}}, w(T_1)^{\mathrm{T}}, \tilde{\eta}_i(T_1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n_i+q_i+s} : V \le c\big\}$. Let $\epsilon^* \in (0, 1]$ be such that, for all $\epsilon \in (0, \epsilon^*]$, $[\tilde{x}_i^{\mathrm{T}}(T_1), v_i^{\mathrm{T}}(T_1), w(T_1), \tilde{\eta}_i^{\mathrm{T}}(T_1)]^{\mathrm{T}} \in L_V(c)$ implies that

$$\begin{aligned} \|K_i\tilde{x}_i\| &\le \frac{1}{60}\Delta, & \|K_iA_i\tilde{x}_i\| &\le \frac{1}{60}\Delta,\\ \|K_iB_iv_i\| &\le \frac{1}{60}\Delta, & \|K_iB_i\mathrm{sat}_{\Delta_1}(v_i)\| &\le \frac{1}{60}\Delta,\\ \|K_iB_i\Gamma_iw\| &\le \frac{1}{60}\Delta, & \|B_i^{\mathrm{T}}P_iB_i\| &\le \frac{12}{23}. \end{aligned} \tag{21}$$

For easy references, we let

$$k_i := K_i\tilde{x}_i, \quad h_i := K_iA_i\tilde{x}_i, \quad r_i := K_iB_iv_i, \quad d_i := \Gamma_iS\tilde{\eta}_i, \quad m_i := K_iB_i\Gamma_iw,$$

and

$$t_i := K_iB_i\mathrm{sat}_{\Delta_1}(v_i), \quad n_i := K_i(A_i\Pi_i\tilde{\eta}_i + B_i\Gamma_i\tilde{\eta}_i).$$

Substituting (17) and (18) into (14) gives

$$\begin{aligned} \Delta V \le& -\epsilon\sum_{i=1}^{N}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i\\ &+\sum_{i=1}^{N}\Big(2k_i(\mathrm{sat}_{\Delta_1}(v_i)-\Gamma_iw+k_i)-k_i^2+\frac{12}{23}(\mathrm{sat}_{\Delta_1}(v_i)-\Gamma_iw+k_i)^2\Big)\\ &+\sum_{i=1}^{N}4\Big(\big[h_i+t_i-m_i+v_i-\Gamma_iSw+\mathrm{sat}_{\Delta_2}(-h_i-r_i+m_i+\Gamma_iSw-v_i+n_i+d_i)\big]^2\\ &\quad-(v_i-\Gamma_iw+k_i)^2\Big). \end{aligned} \tag{22}$$

By Assumption 1, Γ_iw and Γ_iSw vary between their extreme values.

1. If $\|v_i - \Gamma_iSw\| \le \frac{11}{12}\Delta$, which implies $\|\mathrm{sat}_{\Delta_1}(v_i) - \Gamma_iw\| \le \|v_i - \Gamma_iw\|$, then we have

$$\begin{aligned} \Delta V \le& -\epsilon\sum_{i=1}^{N}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i\\ &+\sum_{i=1}^{N}\Big(-k_i^2+2k_i(v_i-\Gamma_iw+k_i)+4\big(n_i+d_i\big)^2\\ &\qquad+\frac{12}{23}(v_i-\Gamma_iw+k_i)^2-4(v_i-\Gamma_iw+k_i)^2\Big)\\ \le& -\epsilon\sum_{i=1}^{N}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i \end{aligned}$$

$$
\begin{aligned}
&+4\sum_{i=1}^{N}\tilde{\eta}_i^{\mathrm{T}}(K_iA_i\Pi_i+K_iB_i\Gamma_i+\Gamma_iS)^{\mathrm{T}}(K_iA_i\Pi_i+K_iB_i\Gamma_i+\Gamma_iS)\tilde{\eta}_i\\
&+\sum_{i=1}^{N}\Big(-k_i^2+2k_i(v_i-\Gamma_iw+k_i)-(v_i-\Gamma_iw+k_i)^2+(v_i-\Gamma_iw+k_i)^2\\
&\qquad+\frac{12}{23}(v_i-\Gamma_iw+k_i)^2-4(v_i-\Gamma_iw+k_i)^2\Big)\\
\le&-\epsilon\sum_{i=1}^{N}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i\\
&+4\tilde{\eta}^{\mathrm{T}}\big[KA\Pi+KB\Gamma+\Gamma(I_N\otimes S)\big]^{\mathrm{T}}\big[KA\Pi+KB\Gamma+\Gamma(I_N\otimes S)\big]\tilde{\eta}\\
&+\sum_{i=1}^{N}\Big(-(v_i-\Gamma_iw)^2-\frac{57}{23}(v_i-\Gamma_iw+k_i)^2\Big),
\end{aligned}
\tag{23}
$$

where

$$
K=\text{blockdiag}\{K_1,\ldots,K_N\},\quad \Pi=\text{blockdiag}\{\Pi_1,\ldots,\Pi_N\},
$$

$$
\Gamma=\text{blockdiag}\{\Gamma_1,\ldots,\Gamma_N\},\quad A=\text{blockdiag}\{A_1,\ldots,A_N\},
$$

and

$$
B=\text{blockdiag}\{B_1,\ldots,B_N\}.
$$

Since $\lim_{t\to\infty}\tilde{\eta}_i=0$, one can conclude from (23) that

$$
\Delta V<0,\quad \forall[\tilde{x}_i(T_1),v_i(T_1),w(T_1),\tilde{\eta}(T_1)]^{\mathrm{T}}\in L_V(c)\backslash\{0\}. \tag{24}
$$

2. Otherwise, if $\|v_i-\Gamma_iSw\|\ge\frac{11}{12}\Delta$, which means $\|v_i\|\ge\frac{11}{12}\Delta$, it follows that

$$
\begin{aligned}
\Delta V\le&-\epsilon\sum_{i=1}^{N}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i\\
&+\sum_{i=1}^{N}\Big[2k_i\big(\text{sat}_{\Delta_1}(v_i)-\Gamma_iw\big)+k_i^2+\frac{12}{23}\big(\text{sat}_{\Delta_1}(v_i)-\Gamma_iw+k_i\big)^2\\
&\qquad+4(v_i-\Gamma_iSw+h_i+t_i-m_i-\Delta)^2-4(v_i-\Gamma_iw+k_i)^2\Big]\\
\le&-\epsilon\sum_{i=1}^{N}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i\\
&+\sum_{i=1}^{N}\Big[\frac{1}{60^2}\Delta^2+2\cdot\frac{1}{60}\Delta\cdot1.9\Delta+\frac{12}{23}\left(1+0.9+\frac{1}{60}\right)\Delta^2
\end{aligned}
$$

$$+4\left(2\times\frac{11}{12}+\frac{1}{20}-1+\frac{1}{60}\right)\left(-1+\frac{1}{20}+\frac{1}{60}\right)\Delta^2\Bigg]$$

$$\leq -\epsilon\sum_{i=1}^{N}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i - N\Delta^2, \tag{25}$$

which means (24) holds.

Thus, we conclude from (23) and (25) that

$$\Delta V < 0, \quad \forall[\tilde{x}_i(T_1), v_i(T_1), w(T_1), \tilde{\eta}(T_1)]^{\mathrm{T}} \in L_V(c)\backslash\{0\},$$

which implies

$$\lim_{t\to\infty}\tilde{x}_i = \lim_{t\to\infty}(x_i - \Pi_i w) = 0, \quad \lim_{t\to\infty}(v_i - \Gamma_i w) = 0,$$

and

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}\big(C_i(\tilde{x}_i + \Pi_i w) + Qw\big) = 0.$$

This completes our proof. □

4 An Illustrative Example

We consider a group of multi-agent systems consisting of one leader and four followers, whose switching communication network is as described in Fig. 1.

The leader system is given by (2) with

$$S = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The four follower systems are described by (1). In particular, for $i = 1, 2$,

$$A_i = \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \quad W_i = \begin{bmatrix} -0.2 & -0.7 \\ 0.6 & -0.2 \\ 0.0 & -0.1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and for $i = 3, 4$,

$$A_i = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix}, \quad W_i = \begin{bmatrix} 0.3 & -0.7 \\ -0.7 & 0.2 \\ -0.2 & -0.7 \end{bmatrix}, \quad C_i = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$

The solutions of the regulator equation in (6) are

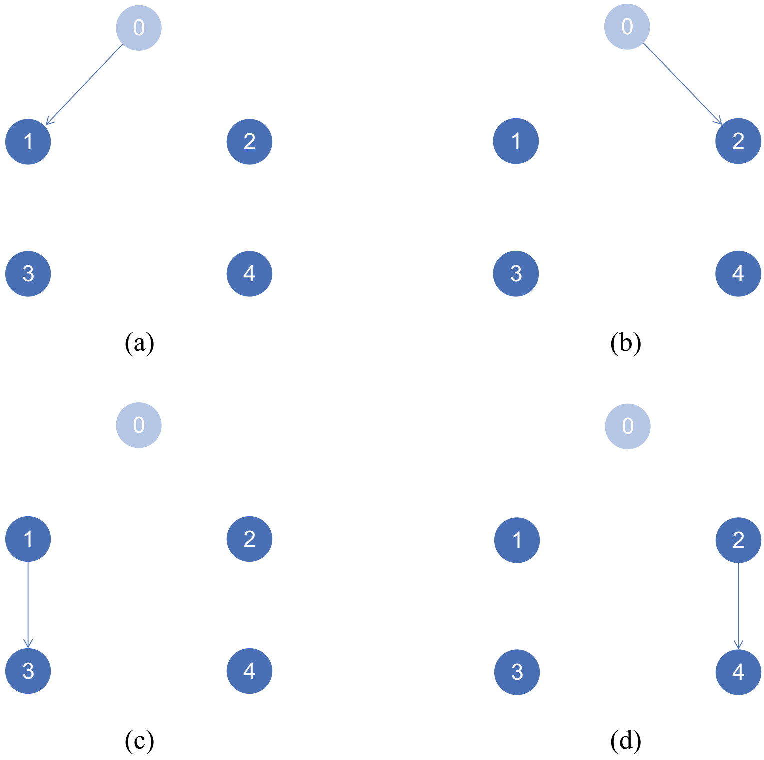

Fig. 1 Switching digraph $\bar{\mathcal{G}}_{\sigma(t)}$ with $\mathcal{P} = \{1, 2, 3, 4\}$. **a** $\bar{\mathcal{G}}_1$. **b** $\bar{\mathcal{G}}_2$. **c** $\bar{\mathcal{G}}_3$. **d** $\bar{\mathcal{G}}_4$

$$\Pi_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad i = 1, 2$$

and

$$\Pi_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad i = 3, 4, 5.$$

We assume in this example that follower systems have the same actuator "time constants", that is, for $i = 1, 2, 3, 4$, $\alpha = 0.5$. We also assume $\Delta = 8$. It is easy to verify that Assumptions 1–5 are satisfied.

The initial system states and actuator positions of the followers are chosen as

$$\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & -2 \\ 3 & 2 & -1 & 0 \\ 4 & 0 & 3 & -1 \end{bmatrix}$$

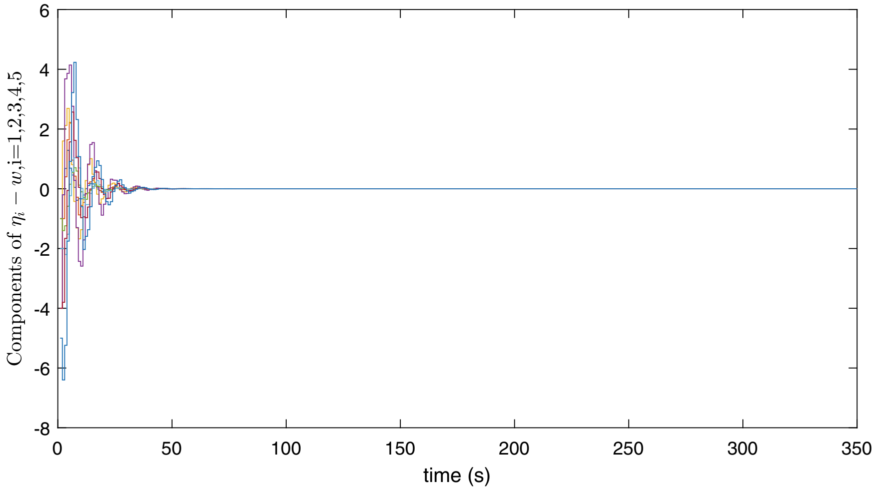

Fig. 2 Components of estimation errors $\eta_i - w$ for $i = 1, 2, 3, 4$

and

$$\begin{bmatrix} v_1(0) & v_2(0) & v_3(0) & v_4(0) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 & -2 \end{bmatrix}.$$

The initial states of the distributed observers (10) are set as random values.

We consider the low gain parameter $\epsilon = 0.01$. The solutions of the discrete time parametric ARE (3) are

$$P_i(0.01) = \begin{bmatrix} 0.4626 & 0.9970 & 1.0487 \\ 0.9970 & 3.4640 & 4.7468 \\ 1.0487 & 4.7468 & 9.9810 \end{bmatrix}, \quad i = 1, 2$$

and

$$P_i(0.01) = \begin{bmatrix} 0.0133 & -0.0001 & -0.0006 \\ -0.0001 & 0.4684 & 1.0234 \\ -0.0006 & 1.0234 & 4.6907 \end{bmatrix}, \quad i = 3, 4.$$

The estimation errors $\eta_i - w$ and output consensus error e_i, $i = 1, 2, 3, 4$, are respectively shown in Figs. 2 and 3. Clearly, the leader-following output consensus is achieved.

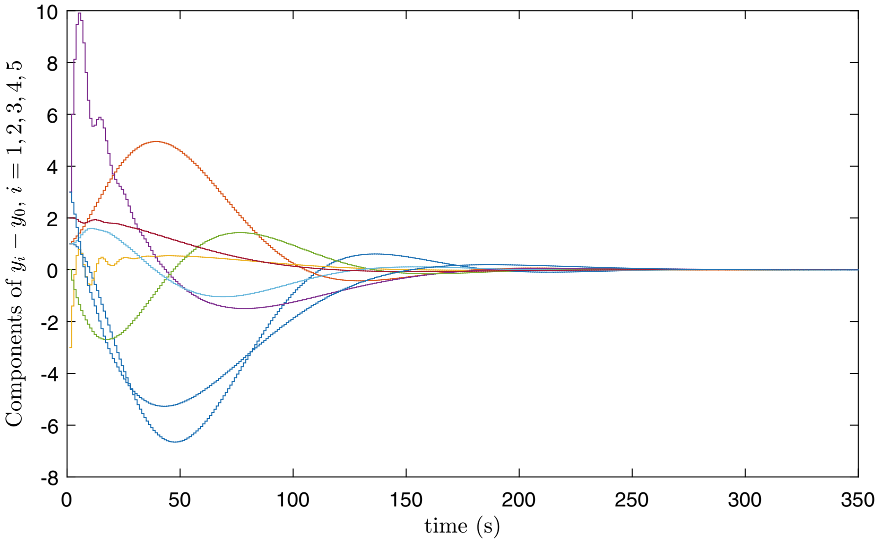

Fig. 3 Output consensus errors e_i, $i = 1, 2, 3, 4$, under the consensus protocol (11) with $\epsilon = 0.01$

5 Conclusion

We have considered in this chapter a group of heterogeneous linear systems subject to actuator position and rate saturation, and switching topologies. A distributed control law has been designed to solve the semi-global leader-following output consensus problem. In the near future, we will investigate a global control protocol for systems with actuator position and rate saturation.

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