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Semi-global leader–following output consensus of heterogeneous systems with all agents subject to input saturation

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Abstract

In this paper, we consider the semi-global leader–following output consensus of heterogeneous multi-agent linear systems over a directed communication graph with both the leader agent and the follower agents subject to input saturation. Via the low gain feedback design technique, both the state feedback and output feedback consensus protocols are constructed. In the state feedback case, a fully distributed observer is given to estimate the state of the leader agent. In the output feedback case, for both the informed follower agents and the uninformed follower agents, a state observer and a distributed leader state observer for each follower are designed to estimate the state of the follower itself and the state of the leader agent, respectively. In the distributed leader state observer, additional discontinuous functions are designed to suppress the influence of the bounded input of the leader agent. In both consensus protocols, a term under a condition similar to that used in the robust control is used to compensate for the control input of the leader agent.

KEYWORDS

heterogeneous multi-agent systems, input saturation, low gain feedback, output consensus

1 | INTRODUCTION

In recent years, cooperative control of multi-agent systems is gaining incredible attention due to their extensive applications in the formation control of robots,[1-4] containment control[5-7] and output regulation of multi-agent systems,[8-10] attitude synchronization of spacecraft,[11,12] to name just a few. Among them, the consensus problem is to drive the states/outputs of the agents to reach a same value with local information transmitted from others. Especially, the leader-following consensus or output consensus problem aims to design a distributed control protocol such that the states/outputs of the followers asymptotically converge to that of the leader. For heterogeneous systems, which may have different state dimensions, it would not make sense and also impossible to reach state consensus. Take unmanned ground vehicles (UGVs) and unmanned aerial vehicles (UAVs) for example. It is impossible for them to reach state consensus in three-dimensional space, while they can reach position consensus in the horizontal two-dimensional space to perform tasks.

As an extension of the leader-following output regulation problem, the leader-following output consensus is to track the reference and/or to reject the disturbances that are both generated by the leader system (exosystem).[13-15] In the work of Su and Huang,[15] the leader-following consensus problem is formulated into the output regulation problem. According to whether the states of the agents can be measured or not, the consensus methods are classified into state feedback control protocol and output feedback control protocol. Due to the limitation of communication, the information of the leader is usually unreachable for every follower. In such a case, a great quantity of distributed observer-based output regulation and output consensus problems with local information exchange were investigated.[8,13] It has been verified that the leader-following output consensus problem will be addressed by designing appropriate control laws with the distributed observer, if the communication graph contains a spanning tree rooted from the leader.[16]

It is a fact that in the above works, there is no control input of the leader system, and no input saturation of the follower systems. The semi-global output consensus problem with input saturation of the followers was investigated.[17-19] By semi-global output consensus, we mean the output consensus problem is addressed locally under any a priori given bounded set that acts as a subset of a domain of attraction. In the work of Lin,[20] the low gain feedback design technique was originally designed to solve the semi-global stabilization problem of an individual linear system with saturating actuators, and it is now applied to the cooperative control of multi-agent systems. Via the low gain feedback design technique, Su et al.[17] solved the semi-global leader-following consensus of linear multi-agent systems with input saturation, and Shi et el.[18] considered the same problem of multiple heterogeneous linear systems with the state feedback consensus protocol and the output feedback consensus protocol, and the followers are classified into the informed ones and the uninformed ones. The output consensus protocol designed by Zhao et al.[19] divided the followers into several groups according to the length of the longest directed path between the follower and the leader.

It is interesting to note that the leader has zero control input in the results mentioned above. In fact, the systems with control inputs are able to generate more complex and rich signals for the followers to track, and it is of more practical significance. Moreover, the control input allows the leader to generate trajectories to avoid obstacles so that the whole system can reach its destination safely. Thus, we take the control input of the leader into consideration, as works in References 21,22 did. In this paper, we solve the semi-global leader-following output consensus problem of multiple heterogeneous systems with both the leader and the followers having input saturation over a directed communication graph. Motivated by the work of Shi et al.,[18] both a state feedback-based consensus protocol and an output feedback-based consensus protocol are given. In the case of state feedback, we firstly give a fully distributed observer for each follower to estimate the state of the leader without using the global information of the graph. In the case of output feedback, the followers are divided into two subgroups, the informed ones and the uninformed ones, which have and do not have access to the information of the leader, respectively. In both two subgroups, a distributed leader state observer and a state observer are constructed to estimate the state of the leader and the state of the follower itself, respectively. The state in the distributed leader state observer for the informed followers is used by the uninformed ones to estimate the state of the leader. Since the leader has bounded input, a term is designed in the distributed leader state observers for all the followers. For similar reasons, we modify the control law constructed by Shi et al.[18] to eliminate the influence of the control input of the leader. It has been shown that the consensus protocols can solve the semi-global leader-following output consensus problem with both the leader and the followers subject to input saturation if there is a directed spanning tree in the communication graph.

The remainder of this paper is organized as follows. In Section 2, the definitions of the semi-global state feedback leader-following output consensus problem and semi-global output feedback leader-following output consensus problem, and necessary assumptions are given. Two consensus protocols are, respectively, constructed in Section 3.1 and Section 3.2 to solve the two problems defined in Section 2. We give illustrative examples in Section 4 to verify the effectiveness of our control laws. Finally, we conclude our work in Section 5 with some remarks.

Notation: $1_N \in \mathbb{R}^N$ represents an N-dimensional column vector with all entries being 1. I_N denotes an N-dimensional identity matrix. 0_n is a zero matrix with $n \times n$ dimension. $\otimes$ represents the Kronecker product. X^{T} denotes the transpose of the matrix or vector X. For a symmetric matrix P, $P > 0$ means P is positive definite.

2 | PROBLEM FORMULATION AND ASSUMPTIONS

Consider a group of heterogeneous systems consisting of a leader and N followers. The leader is described as

$$\begin{cases} \dot{w} = Sw + R\sigma_{\gamma}(u_0) \\ y_0 = -Qw, \end{cases} \tag{1}$$

where $w \in \mathbb{R}^s$, $u_0 \in \mathbb{R}^r$ and $y_0 \in \mathbb{R}^q$ are the state, control input and output of the leader system, respectively, $\sigma_\gamma : \mathbb{R}^r \to \mathbb{R}^r$ denotes a vector-valued saturation function, that is, for $s = [s_1, s_2, \dots, s_r]^{\mathrm{T}}$, $\sigma_\gamma(s) = [\sigma_\gamma(s_1), \sigma_\gamma(s_2), \dots, \sigma_\gamma(s_r)]^{\mathrm{T}}$, and for each $j = 1, 2, \dots, r$, $\sigma_\gamma(s_j) = \mathrm{sgn}(s_j)\min\{|s_j|, \gamma\}$, where γ is a known constant.

The dynamics of each follower system is given as

$$\begin{cases} \dot{x}_i = A_i x_i + B_i \sigma_{\Delta_i}(u_i) + d_i(t), \\ y_i = C_i x_i, \qquad\qquad\qquad i = 1, 2, \dots, N, \end{cases} \tag{2}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^q$ are, respectively, the state, control input, and output of the ith follower, respectively, and $d_i(t) \in \mathbb{R}^{n_i}$ represents external disturbance. Assume that the disturbance is caused by the leader and $d_i(t) = W_i w$, then the dynamics of each follower can be rewritten as

$$\begin{cases} \dot{x}_i = A_i x_i + B_i \sigma_{\Delta_i}(u_i) + W_i w, \\ y_i = C_i x_i, \\ e_i = C_i x_i + Q w, \qquad\qquad i = 1, 2, \dots, N. \end{cases} \tag{3}$$

Similar to the output regulation problem, the leader system generates both the reference $-Qw$ to be tracked and the disturbance $W_i w$ to be rejected. It is noted that e_i also represents the output tracking error between the ith follower system and the leader system. Without loss of generality, we assume $\Delta = \Delta_1 = \Delta_2 = \cdots = \Delta_N$. In this paper, we aim to design the control law u_i for each follower such that the output tracking error e_i satisfies $\lim_{t\to\infty} e_i = 0$.

The communication among the leader and the followers is represented by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with $\mathcal{V} = \{0, 1, 2, \dots, N\}$ being the node set and $\mathcal{E} = \mathcal{V} \times \mathcal{V}$ being the edge set. For $i, j \in \mathcal{V}$, $(j, i) \in \mathcal{E}$ if and only if information can flow from node j to i. Then node j is called a neighbor of node i and node i is called a child of node j. We use $\mathcal{F} = \{1, \dots, N\}$ to denote the set of followers, and use $\mathcal{N}_i := \{j : (j, i) \in \mathcal{E}\}$ to represent the set of neighbors of node i. Depending on whether or not the followers have access to the information of the leader, the followers are divided into two classes. The informed ones can obtain the information of the leader, while the uninformed ones can not, and we use $\mathcal{F}_{\mathrm{in}}$ and $\mathcal{F}_{\mathrm{un}}$ to represent the informed followers and the uninformed followers, respectively, that is, $\mathcal{F}_{\mathrm{in}} := \{i : i \in \mathcal{F}, (0, i) \in \mathcal{E}\}$ and $\mathcal{F}_{\mathrm{un}} := \mathcal{F} \setminus \mathcal{F}_{\mathrm{in}}$. Without loss of generality, we assume the first l followers are the informed ones and the left $N - l$ followers are the uninformed ones. If the graph contains a sequence of edges (i_1, i_2), (i_2, i_3), $\dots$, (i_{k-1}, i_k), then we say there is a *directed path* from node i_1 to i_k. For a directed graph $\mathcal{G}$, the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{(N+1)\times(N+1)}$ is defined as $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{(N+1)\times(N+1)}$ is defined as $l_{ij} = -a_{ij}$ if $i \neq j$, and $l_{ii} = \sum_{j=0}^{N} a_{ij}$. According to the classification of the leader, the informed followers and the uninformed followers, $\mathcal{L}$ can be partitioned as

$$\mathcal{L} = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 \\ \mathbf{0} & \mathcal{L}_4 & \mathcal{L}_5 \end{bmatrix},$$

where $\mathcal{L}_1 \in \mathbb{R}^{l\times 1}$, $\mathcal{L}_2 \in \mathbb{R}^{l\times l}$, $\mathcal{L}_3 \in \mathbb{R}^{l\times(N-l)}$, $\mathcal{L}_4 \in \mathbb{R}^{(N-l)\times l}$, and $\mathcal{L}_5 \in \mathbb{R}^{(N-l)\times(N-l)}$.

Assumption 1. There is a directed path from the leader node 0 to each follower node i.

Lemma 1 (Li et al.[23]). *Under Assumption 1, all eigenvalues of matrices* $\begin{bmatrix} \mathcal{L}_2 & \mathcal{L}_3 \\ \mathcal{L}_4 & \mathcal{L}_5 \end{bmatrix}$ *and* $\mathcal{L}_5$ *have positive real parts.*

Assumption 2. The following regulator equations[24]

$$\begin{aligned} \Pi_i S &= A_i \Pi_i + B_i \Gamma_i + W_i \\ C_i \Pi_i + Q &= 0, \quad i = 1, 2 \dots, N, \end{aligned}$$

have a pair of solutions $\Pi_i \in \mathbb{R}^{n_i\times s}$ and $\Gamma_i \in \mathbb{R}^{m_i\times s}$.

Assumption 3. The pair (S, Q) is detectable.

Assumption 4. For each follower $i \in \mathcal{F}$, the pair (A_i, B_i) is stabilizable, (A_i, C_i) is detectable, and all eigenvalues of A_i have nonpositive real parts.

Lemma 2 (Lin[20]). *Suppose Assumption 4 holds. For each $\epsilon \in (0, 1]$, there exists a unique positive definite matrix $P_i(\epsilon) \in \mathbb{R}^{n_i \times n_i}$, $i \in \mathcal{F}$ of the following parametric algebraic Riccati equation (ARE):*

$$A_i^{\mathrm{T}} P_i(\epsilon) + P_i(\epsilon) A_i - 2P_i(\epsilon) B_i B_i^{\mathrm{T}} P_i(\epsilon) = -\epsilon I_{n_i}. \tag{4}$$

In addition, $\lim_{t \to \infty} P_i(\epsilon) = 0$.

The low gain feedback design technique is originally designed for linear systems with input saturation to solve the semi-global stabilization problem. With the solution $P_i(\epsilon)$ used in the control law, the parametric ARE in (4) not only guarantees the stability of the closed-loop system, but also allows the control input to be unsaturated for any a priori given and bounded set of initial conditions by choosing ϵ small enough.

Assumption 5. The linear matrix equations

$$B_i E_i - \Pi_i R = 0, \tag{5}$$

have solutions E_i, for each $i \in \mathcal{F}$.

Assumption 6. For each $i \in \mathcal{F}$, there exist positive constants $\delta_i < \Delta - \|\gamma E_i\|_\infty$ and a time $T \geq 0$ such that $\|\Gamma_i w\|_{\infty,T} \leq \Delta - \|\gamma E_i\|_\infty - \delta_i, i \in \mathcal{F}$, for all w with $w(0) \in \mathcal{W}_0$, where $\mathcal{W}_0$ is a priori given bounded set, and $\|\Gamma_i w\|_{\infty,T} := \sup_{t \geq T} \|\Gamma_i w\|_\infty$.

Remark 1. Assumption 1 is widely used in the coordinated control of multiple agents.[8,13] Assumptions 2–4 impose necessary conditions for the stability of our output consensus problem of heterogeneous multi-agent systems. The function of Assumption 5 is similar to the condition used by Graham et al.[25] for robust control with disturbances, which implies that the influence of the unknown control input of the leader u_0 can be compensated by the control protocol u_i of every follower. As shown later, Assumption 6 plays an important role in guaranteeing the control protocols unsaturated.

For the case that the states of the leader and the followers can be measured, a state feedback control law, using a distributed observer to be designed to estimate the state of the leader, will be proposed to solve the semi-global leader-following output consensus problem.

Firstly, consider the following fully distributed observer:

$$\dot{\hat{w}}_i = S\hat{w}_i - \left(\alpha_i(t) + \zeta_i^{\mathrm{T}}(t) P_S \zeta_i(t)\right) P_S \zeta_i(t) - \gamma R f_{i,1}(t), \quad i \in \mathcal{F}, \tag{6}$$

where $\zeta_i(t) = a_{i0}(\hat{w}_i(t) - w(t)) + \sum_{j \in \mathcal{F}} a_{ij}(\hat{w}_i(t) - \hat{w}_j(t))$, $P_S > 0$ is a solution of the following ARE

$$P_S S + S^{\mathrm{T}} P_S - P_S^2 + I_s = 0, \tag{7}$$

the gain $\alpha_i(t)$ is updated as the following law

$$\dot{\alpha}_i(t) = \zeta_i^{\mathrm{T}} P_S^2 \zeta_i(t)$$

with $\alpha_i(0) > 0$, and the function $f_{i,1}$ is defined as

$$f_{i,1}(t) = \begin{cases} \frac{R^{\mathrm{T}} P_S \zeta_i(t)}{\|R^{\mathrm{T}} P_S \zeta_i(t)\|}, & \|R^{\mathrm{T}} P_S \zeta_i(t)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

As shown by Hua et al.,[22] $\hat{w}_i$ is an estimation of the state of the leader. In the fully distributed observer, $\zeta_i(t)$ denotes estimation errors of $\hat{w}_i(t)$ for followers i relative to its neighbors. $\alpha_i(t)$ is the updating gain to avoid using the global information of the graph, and $f_{i,1}(t)$ is to suppress the influence of the control input of the leader.

Problem 1 (State feedback-based semi-global leader-following output consensus problem). Consider the group of heterogeneous systems composed of (1) and (3). Assume that Assumptions 1–6 hold. Let $x = [x_1^{\mathrm{T}}, \dots, x_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$, where $n = n_1 + n_2 + \cdots + n_N$, and $\hat{w} = [\hat{w}_1^{\mathrm{T}}, \dots, \hat{w}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Ns}$. For any a priori bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$, $\mathcal{W}_0 \subset \mathbb{R}^s$ and $\hat{\mathcal{W}}_0 \subset \mathbb{R}^{Ns}$, design a state feedback consensus control protocol u_i of the form

$$u_i = f_i(x_i, \hat{w}_i),$$

such that for $[x^{\mathrm{T}}(0), w^{\mathrm{T}}(0), \hat{w}^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_0 \times \mathcal{W}_0 \times \hat{\mathcal{W}}_0$, the leader-following output consensus is achieved, that is, for $i \in \mathcal{F}$, the output error e_i satisfies

$$\lim_{t\to\infty} e_i(t) = 0.$$

However, in practice, the state information of the leader and the followers may not be available, and only the outputs can be measured. In such a case, we will design a set of output feedback-based distributed observers and control protocols. Since the informed followers have access to the output of the leader, while the uninformed followers do not, the observers that estimate the state of the leader for the informed and uninformed followers will be different. Moreover, the output-based observer of each follower that estimates the state of the follower itself is needed.

Consider the following dynamic compensator:

$$\dot{\hat{v}}_0 = S\hat{v}_0 + L_S(y_0 + Q\hat{v}_0) + R\sigma_\gamma(u_0), \tag{8}$$

where $\hat{v}_0$ is the state of the dynamic compensator, $L_S \in \mathbb{R}^{s\times q}$ is a constant matrix such that $S + L_S Q$ is Hurwitz. The existence of L_S is guaranteed by Assumption 3. It will be shown in the following section that $\lim_{t\to\infty}(w - \hat{v}_0) = 0$.

Since $S + L_S Q$ is Hurwitz, then for any positive definite matrix $F \in \mathbb{R}^{s\times s}$, there exists a positive definite matrix $G \in \mathbb{R}^{s\times s}$ that solves the following equation:[26]

$$G(S + L_S Q) + (S + L_S Q)^{\mathrm{T}} G = -F. \tag{9}$$

For the ith informed followers, consider the following dynamic compensator,

$$\dot{\hat{x}}_i = A_i\hat{x}_i + W_i\hat{v}_0 + B_i\sigma_\Delta(u_i) - L_{A,i}(y_i - C_i\hat{x}_i), \tag{10a}$$

$$\dot{\hat{v}}_i = S\hat{v}_i + L_S(y_0 + Q\hat{v}_i) + \gamma R f_{i,2}(t), \quad i \in \mathcal{F}_{\mathrm{in}}, \tag{10b}$$

where $f_{i,2}(t)$ is designed to make up for the influence of the leader, and it is defined as

$$f_{i,2}(t) = \begin{cases} \frac{R^{\mathrm{T}}G^{\mathrm{T}}(\hat{v}_0 - \hat{v}_i)}{\|(\hat{v}_0 - \hat{v}_i)^{\mathrm{T}}GR\|}, & \|(\hat{v}_0 - \hat{v}_i)^{\mathrm{T}}GR\| \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

$\hat{v}_i$ and $\hat{x}_i$ can be viewed as the estimations of the state of the leader and the state of the ith informed follower agent. $L_{A,i}$ is a constant matrix such that $A_i + L_{A,i}C_i$ is Hurwitz. The existence of $L_{A,i}$ is guaranteed by Assumption 4.

For the ith uninformed followers, the distributed observer is constructed as

$$\dot{\hat{v}}_i = S\hat{v}_i - \mu\left(\sum_{j=1}^{l} a_{ij}(\hat{v}_i - \hat{v}_j) + \sum_{j=l+1}^{N} a_{ij}(\hat{v}_i - \hat{v}_j)\right) + \gamma R f_{i,3}(t), \quad i \in \mathcal{F}_{\mathrm{un}}, \tag{11a}$$

$$\dot{\hat{x}}_i = A_i\hat{x}_i + W_i\hat{v}_i + B_i\sigma_\Delta(u_i) - L_{A,i}(y_i - C_i\hat{x}_i), \tag{11b}$$

where $\mu > 0$ is a sufficiently large constant such that $I_{N-l} \otimes S - \mu\mathcal{L}_5 \otimes I_s$ is Hurwitz. Let $H = [h_i] \in \mathbb{R}^{(N-l)s\times(N-l)s}$ with $h_i \in \mathbb{R}^{s\times s}$, be a positive definite and block diagonal matrix such that $K \in \mathbb{R}^{(N-l)s\times(N-l)s} > 0$ is obtained by the following equation:

$$H(I_{N-l} \otimes S - \mu\mathcal{L}_5 \otimes I_s) + (I_{N-l} \otimes S - \mu\mathcal{L}_5 \otimes I_s)^{\mathrm{T}} H = -K. \tag{12}$$

Note that $\hat{v}_j, j = 1, 2, \dots, l$, is the state of the observer in (10b). The meaning of $f_{i,3}$ is the same as that of $f_{i,2}$. Denote $\bar{v}_i = \frac{1}{\sum_{k=1}^{l} a_{ik}} \sum_{k=1}^{l} a_{ik}(\hat{v}_k - \hat{v}_i)$, then $f_{i,3}$ is defined as

$$f_{i,3}(t) = \begin{cases} \frac{R^{\mathrm{T}} h_i \bar{v}_i}{\|\bar{v}_i^{\mathrm{T}} h_i R\|}, & \|\bar{v}_i^{\mathrm{T}} h_i R\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Then, the definition of the output feedback-based semi-global output consensus problem is given.

Problem 2 (Output feedback-based semi-global leader-following output consensus problem). Consider the group of heterogeneous systems composed of (1) and (3). Assume that Assumptions 1–6 hold. Let $x = [x_1^{\mathrm{T}}, \dots, x_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$, where $n = n_1 + n_2 + \cdots + n_N$, $\hat{x} = [\hat{x}_1^{\mathrm{T}}, \hat{x}_2^{\mathrm{T}}, \dots, \hat{x}_N^{\mathrm{T}}]^{\mathrm{T}}$ and $\hat{v} = [\hat{v}_1^{\mathrm{T}}, \hat{v}_2^{\mathrm{T}}, \dots, \hat{v}_N^{\mathrm{T}}]^{\mathrm{T}}$. For any given bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$, $\mathcal{W}_0 \subset \mathbb{R}^s$, $\hat{\mathcal{X}}_0 \subset \mathbb{R}^n$ and $\hat{\mathcal{V}}_0 \subset \mathbb{R}^{Ns}$, design an output feedback-based distributed control law u_i of the form

$$u_i = f_i(\hat{x}_i, \hat{v}_i),$$

such that for $[x^{\mathrm{T}}(0), w^{\mathrm{T}}(0), \hat{v}^{\mathrm{T}}(0), \hat{x}^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_0 \times \mathcal{W}_0 \times \hat{\mathcal{V}}_0 \times \hat{\mathcal{X}}_0$, the leader-following output consensus is achieved, that is, for $i \in \mathcal{F}$, the output error e_i satisfies

$$\lim_{t\to\infty} e_i(t) = 0.$$

3 | OUTPUT CONSENSUS OVER DIRECTED NETWORKS

In this section, the semi-global leader-following output consensus problems in Problem 1 and Problem 2 are, respectively, solved in Section 3.1 and Section 3.2. Let $P_i(\epsilon)$ be the solution to the parametric ARE in (4). For convenience, we denote $P_i = P_i(\epsilon)$ hereafter.

To begin with, the following lemma is given.

Lemma 3 (Cai et al.[27]). *Consider the following system:*

$$\dot{x} = \epsilon F x + F_1(t)x + F_2(t),$$

where $x \in \mathbb{R}^n$, $F \in \mathbb{R}^{n\times n}$ is Hurwitz, $\epsilon > 0$ is a constant, $F_1(t) \in \mathbb{R}^{n\times n}$ and $F_2(t) \in \mathbb{R}^n$ are bounded and continuous for all $t \geq t_0$. We have if $F_1(t),\ F_2(t) \to 0$ as $t \to \infty$ (exponentially), then for any $x(t_0)$ and any $\epsilon > 0$, $x \to 0$ as $t \to \infty$ (exponentially).

3.1 | Semi-global output consensus via state feedback

Let E_i be the solution of the equation in (5), and we denote $H_i := P_i B_i E_i$. Then, we construct the following state feedback consensus control law for every follower:

$$u_i = -B_i^{\mathrm{T}} P_i (x_i - \Pi_i \hat{w}_i) + \Gamma_i \hat{w}_i - \gamma E_i g_{i,1}(t), \quad i \in \mathcal{F}, \tag{14}$$

where $\hat{w}_i$ is the state of the fully distributed observer in (6), $g_{i,1}(t)$ is defined as

$$g_{i,1}(t) = \begin{cases} \frac{H_i^{\mathrm{T}} \hat{\xi}_i(t)}{\|H_i^{\mathrm{T}} \hat{\xi}_i(t)\|}, & \|H_i^{\mathrm{T}} \hat{\xi}_i(t)\| \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

and $\hat{\xi}_i := x_i - \Pi_i \hat{w}_i$. We note that $\|g_{i,1}(t)\| = 1$ or 0.

Theorem 1. *Consider the group of heterogeneous systems composed of (1) and (3). Assume that Assumptions 1–6 hold. Then, Problem 1 is solved by the low gain state feedback control law (14). That is, for any a priori bounded sets $\mathcal{X}_0 \subset \mathbb{R}^n$,*

$\mathcal{W}_0 \subset \mathbb{R}^s$ and $\hat{\mathcal{W}}_0 \subset \mathbb{R}^{Ns}$, there exists an $\epsilon^* \in (0, 1]$, such that for $[x^{\mathrm{T}}(0), w^{\mathrm{T}}(0), \hat{w}^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_0 \times \mathcal{W}_0 \times \hat{\mathcal{W}}_0$ and each $\epsilon \in (0, \epsilon^*]$, the output consensus error e_i for each follower agent satisfies $\lim_{t\to\infty} e_i = 0$.

Proof. It has been proved by Hua et al.[22] that $\lim_{t\to\infty}(\hat{w}_i(t) - w(t)) = 0, i \in \mathcal{F}$ globally, without using the global information of the graph.

Denote $\xi_i = x_i - \Pi_i w$, $\xi = [\xi_1^{\mathrm{T}}, \xi_2^{\mathrm{T}}, \dots, \xi_N^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{w} = [w_1^{\mathrm{T}}, w_2^{\mathrm{T}}, \dots, w_N^{\mathrm{T}}]^{\mathrm{T}}$, $A = \mathrm{diag}\{A_1, A_2, \dots, A_N\}$, $B = \mathrm{diag}\{B_1, B_2, \dots, B_N\}$, $C = \mathrm{diag}\{C_1, C_2, \dots, C_N\}$, $W = \mathrm{diag}\{W_1, W_2, \dots, W_N\}$, $E = \mathrm{diag}\{E_1, E_2, \dots, E_N\}$, $\Pi = \mathrm{diag}\{\Pi_1, \Pi_2, \dots, \Pi_N\}$, $\Gamma = \mathrm{diag}\{\Gamma_1, \Gamma_2, \dots, \Gamma_N\}$, $P = \mathrm{diag}\{P_1, P_2, \dots, P_N\}$. Let $u = [u_1^{\mathrm{T}}, u_2^{\mathrm{T}}, \dots, u_N^{\mathrm{T}}]^{\mathrm{T}}$ and $\overline{w} = 1_N \otimes w$, then, we have

$$\begin{cases} \dot{x} = Ax + B\sigma_\Delta(u) + W\overline{w}, \\ y = Cx, \\ e = Cx + (I_N \otimes Q)\overline{w}, \qquad i = 1, 2, \dots, N. \end{cases}$$

The compact form of the control protocol in (14) satisfies

$$u = -B^{\mathrm{T}} P(x - \Pi\hat{w}) + \Gamma\hat{w} - \gamma E g_1(t), \tag{16}$$

where $g_1(t) = [g_{1,1}^{\mathrm{T}}, g_{2,1}^{\mathrm{T}}, \dots, g_{N,1}^{\mathrm{T}}]^{\mathrm{T}}$. Denote $\bar{u}_0 = 1_N \otimes u_0$, under Assumption 2, the dynamics of ξ is

$$\dot{\xi} = A\xi - B\Gamma\overline{w} + B\sigma_\Delta(u) - \Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0). \tag{17}$$

Define the Lyapunov function

$$V_\xi = \xi^{\mathrm{T}} P \xi, \tag{18}$$

where $P = P(\epsilon)$ is the solution of the parametric ARE (4). Since $\lim_{t\to\infty}(w - \hat{w}) = 0$, there exists a time $T_1 > T$ such that

$$\|\Gamma(w - \hat{w})\|_{\infty, T_1} \leq \frac{1}{2}\delta.$$

Then, under Assumption 6, we have

$$\|\Gamma\hat{w}\|_{\infty, T_1} \leq \|\Gamma w\|_{\infty, T_1} + \|\Gamma(w - \hat{w})\|_{\infty, T_1} \leq \Delta - \|\gamma E\| - \frac{1}{2}\delta.$$

For any $[x^{\mathrm{T}}(0), w^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_0 \times \mathcal{W}_0$, $\xi(T_1)$ belongs to a compact set $\mathcal{X}_1$, independent of ϵ, since ξ is determined by a linear differential equation with bounded inputs $\sigma_\Delta(u)$ and $\sigma_\gamma(\bar{u}_0)$. Let $c_1 > 0$ be such that

$$\sup_{\xi(T_1) \in \mathcal{X}_1, \epsilon \in (0,1]} V_\xi \leq c_1.$$

Such a c_1 exists because $\mathcal{X}_1$ is a bounded set. Define $L_{V_\xi}(c_1) := \{\xi : V_\xi \leq c_1\}$. Then, there exists an $\epsilon^* \in (0, 1]$ such that for any $\epsilon \in (0, \epsilon^*]$,

$$\| - B^{\mathrm{T}} P(x - \Pi\hat{w})\|_{\infty, T_1} \leq \frac{1}{2}\delta.$$

Such an ϵ^* exists because $\lim_{\epsilon\to 0} P(\epsilon) = 0$ and $(x - \Pi\hat{w})$ is within a bounded set. Considering the control protocol in (16), we have

$$\|u\|_{\infty, T_1} \leq \| - B^{\mathrm{T}} P(x - \Pi\hat{w})\|_{\infty, T_1} + \|\Gamma\hat{w}\|_{\infty, T_1} + \|\gamma E g_1(t)\| \leq \Delta,$$

thus, $\sigma_\Delta(u) = u$. Replace the control input u in (17) by the control protocol (16) gives

$$\dot{\xi} = (A - BB^{\mathrm{T}} P)\xi - \gamma B E g_1(t) - \Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0) - (B\Gamma + BB^{\mathrm{T}} P\Pi)\tilde{w},$$

where $\tilde{w} = \overline{w} - \hat{w}$. Then, the derivative of the Lyapunov function in (18) satisfies

$$\begin{aligned}\dot{V}_\xi &= \xi^{\mathrm{T}}(A^{\mathrm{T}}P + PA - 2PBB^{\mathrm{T}}P)\xi - 2\gamma\xi^{\mathrm{T}}PBEg_1(t) - 2\xi^{\mathrm{T}}P\Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0) - 2\xi^{\mathrm{T}}P(B\Gamma + BB^{\mathrm{T}}P\Pi)\tilde{w} \\ &= -\epsilon\xi^{\mathrm{T}}\xi - 2\gamma\xi^{\mathrm{T}}PBEg_1(t) - 2\xi^{\mathrm{T}}P\Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0) - 2\xi^{\mathrm{T}}P(B\Gamma + BB^{\mathrm{T}}P\Pi)\tilde{w},\end{aligned} \tag{19}$$

Define a auxiliary variable

$$\overline{g}_{i,1} = \begin{cases} \frac{H_i^{\mathrm{T}}\xi_i(t)}{\|H_i^{\mathrm{T}}\xi_i(t)\|}, & \|H_i^{\mathrm{T}}\xi_i(t)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\lim_{t\to\infty}(\xi_i - \hat{\xi}_i) = 0$, it is easy to show that $\lim_{t\to\infty}(\overline{g}_{i,1}(t) - g_{i,1}(t)) = 0$.

By the definition of $g_{i,1}$ in (15),

$$\begin{aligned}-2\gamma\xi^{\mathrm{T}}PBEg_1(t) &= \sum_{i=1}^{N} -2\gamma\xi_i^{\mathrm{T}}P_iB_iE_ig_{i,1}(t) = \sum_{i=1}^{N} - 2\gamma\xi_i^{\mathrm{T}}H_ig_{i,1}(t) \\ &= \sum_{i=1}^{N} -2\gamma\xi_iH_i\overline{g}_{i,1}(t) + \sum_{i=1}^{N}2\gamma\xi_iH_i(\overline{g}_{i,1}(t) - g_{i,1}(t)) \\ &= \sum_{i=1}^{N} -2\gamma\|\xi_i^{\mathrm{T}}H_i\| + \sum_{i=1}^{N}2\gamma\xi_iH_i(\overline{g}_{i,1}(t) - g_{i,1}(t)).\end{aligned} \tag{20}$$

Since u_0 is subject to input saturation, it follows that

$$-2\xi^{\mathrm{T}}P\Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0) = \sum_{i=1}^{N} -2\xi_i^{\mathrm{T}}P_i\Pi_iR\sigma_\gamma(u_0) = \sum_{i=1}^{N} -2\xi_i^{\mathrm{T}}P_iB_iE_i\sigma_\gamma(u_0) = \sum_{i=1}^{N} -2\xi_i^{\mathrm{T}}H_i\sigma_\gamma(u_0) \leq \sum_{i=1}^{N}2\gamma\|\xi_i^{\mathrm{T}}H_i\|, \tag{21}$$

where the second equality holds because of Assumption 5. Combining (20), (21), and (19) gives

$$\dot{V}_\xi \leq -\epsilon\xi^{\mathrm{T}}\xi + \sum_{i=1}^{N}2\gamma\xi_iH_i(\overline{g}_{i,1}(t) - g_{i,1}(t)) - 2\xi^{\mathrm{T}}P(B\Gamma + BB^{\mathrm{T}}P\Pi)\tilde{w}.$$

Since $\lim_{t\to\infty}(\overline{g}_{i,1}(t) - g_{i,1}(t)) = 0$ and $\lim_{t\to\infty}\tilde{w} = 0$, then according to the comparison lemma (lemma 3.4 of Khalil[28]) and Lemma 3, it can be verified that $\lim_{t\to\infty} V_\xi = 0$, which implies that $\lim_{t\to\infty}\xi_i = 0$, that is $\lim_{t\to\infty}(x_i - \Pi_iw) = 0$. Then, the output error e_i satisfies

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty}(C_ix_i + Qw) = \lim_{t\to\infty}(C_i(x_i - \Pi_iw) + C_i\Pi_iw + Qw) = \lim_{t\to\infty} C_i(x_i - \Pi_iw) = 0.$$

The above facts complete the proof. ∎

3.2 | Semi-global output consensus via output feedback

Based on the distributed observers (8)–(11), we construct the following output-based control protocol

$$u_i = -B_i^{\mathrm{T}}P_i(\hat{x}_i - \Pi_i\hat{v}_i) + \Gamma_i\hat{v}_i - \gamma E_ig_{i,2}(t), \quad i \in \mathcal{F}, \tag{22}$$

where $\hat{v}_i$ and $\hat{x}_i$ are the states in the distributed observers (10) and (11), and $g_{i,2}(t)$ is defined as

$$g_{i,2}(t) = \begin{cases} \frac{H_i^{\mathrm{T}}\hat{\psi}_i(t)}{\|H_i^{\mathrm{T}}\hat{\psi}_i(t)\|}, & \|H_i^{\mathrm{T}}\hat{\psi}_i(t)\| \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

and $\hat{\psi}_i := \hat{x}_i - \Pi_i\hat{v}_i$.

Theorem 2. *Consider the group of heterogeneous systems composed of (1) and (3). Assume that Assumptions 1–6 hold. Then, Problem 2 is solved by the low gain output feedback control law (22). That is, for any a priori bounded sets* $\mathcal{X}_0 \subset \mathbb{R}^n$, $\mathcal{W}_0 \subset \mathbb{R}^s$, $\hat{\mathcal{X}}_0 \subset \mathbb{R}^n$ *and* $\hat{\mathcal{V}}_0 \subset \mathbb{R}^{Ns}$, *there exists an* $\epsilon^* \in (0, 1]$, *such that for* $[x^{\mathrm{T}}(0), w^{\mathrm{T}}(0), \hat{x}^{\mathrm{T}}(0), \hat{v}^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_0 \times \mathcal{W}_0 \times \hat{\mathcal{X}}_0 \times \hat{\mathcal{V}}_0$ *and each* $\epsilon \in (0, \epsilon^*]$, *the output error* e_i *for each follower agent satisfies* $\lim_{t\to\infty} e_i = 0$.

Proof. Firstly, consider the output-based observer in (8). Let $\tilde{v}_0 = w - \hat{v}_0$. Then, the dynamics of $\tilde{v}_0$ satisfies

$$\dot{\tilde{v}}_0 = (S + L_S Q)\tilde{v}_0.$$

Because $L_S \in \mathbb{R}^{s\times q}$ is such that $S + L_S Q$ is Hurwitz, we have $\lim_{t\to\infty} \tilde{v}_0 = 0$.

Denote the distributed error $\tilde{x}_i = x_i - \hat{x}_i$, and $\tilde{v}_i = w - \hat{v}_i$ $i \in \mathcal{F}$. For the i-th informed followers, under the distributed observer (10a), the dynamics of $\tilde{x}_i$ is

$$\dot{\tilde{x}}_i = \dot{x}_i - \dot{\hat{x}}_i = (A_i + L_{A,i}C_i)\tilde{x}_i + W_i\tilde{v}_0, \quad i \in \mathcal{F}_{\mathrm{in}}.$$

Since $A_i + L_{A,i}C_i$ is Hurwitz, according to Lemma 3, we have $\lim_{t\to\infty} \tilde{x}_i = 0$. The dynamics of $\tilde{v}_i$ is

$$\dot{\tilde{v}}_i = \dot{w} - \dot{\hat{v}}_i = (S + L_S Q)\tilde{v}_i + R\sigma_\gamma(u_0) - \gamma R f_{i,2}, \quad i \in \mathcal{F}_{\mathrm{in}}. \tag{24}$$

Define the following Lyapunov function

$$V_{\tilde{v}_i} = \tilde{v}_i^{\mathrm{T}} G \tilde{v}_i,$$

where G is the solution of (9). Its derivative along the system (24) is

$$\begin{aligned}
\dot{V}_{\tilde{v}_i} &= -F\tilde{v}_i^{\mathrm{T}}\tilde{v}_i + 2\tilde{v}_i^{\mathrm{T}} G R\sigma_\gamma(u_0) - 2\gamma\tilde{v}_i^{\mathrm{T}} G R f_{i,2} \\
&= -F\tilde{v}_i^{\mathrm{T}}\tilde{v}_i + 2\tilde{v}_i^{\mathrm{T}} G R\sigma_\gamma(u_0) - 2\gamma\tilde{v}_i^{\mathrm{T}} G R\bar{f}_{i,2} + 2\gamma\tilde{v}_i^{\mathrm{T}} G R(\bar{f}_{i,1} - f_{i,2}) \\
&\leq -F\tilde{v}_i^{\mathrm{T}}\tilde{v}_i + 2\gamma\|\tilde{v}_i^{\mathrm{T}} G R\| - 2\gamma\|\tilde{v}_i^{\mathrm{T}} G R\| + 2\gamma\tilde{v}_i^{\mathrm{T}} G R(\bar{f}_{i,1} - f_{i,2}) \\
&= -F\tilde{v}_i^{\mathrm{T}}\tilde{v}_i + 2\gamma\tilde{v}_i^{\mathrm{T}} G R(\bar{f}_{i,2} - f_{i,2}),
\end{aligned}$$

where $\bar{f}_{i,2}$ is a auxiliary variable defined as

$$\bar{f}_{i,2}(t) = \begin{cases} \frac{R^{\mathrm{T}}G^{\mathrm{T}}(w-\hat{v}_i)}{\|(w-\hat{v}_i)^{\mathrm{T}}GR\|}, & \|(w-\hat{v}_i)^{\mathrm{T}}GR\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It has been shown that $\lim_{t\to\infty}(w - \hat{v}_0) = 0$, thus, $\lim_{t\to\infty}(\bar{f}_{i,2} - f_{i,2}) = 0$. Similar to the proof of state feedback-based output consensus problem, it can be concluded that $\lim_{t\to\infty} V_{\tilde{v}_i} = 0$, that is, $\lim_{t\to\infty}(w - v_i) = 0$, $i \in \mathcal{F}_{\mathrm{in}}$. Therefore, for $i \in \mathcal{F}_{\mathrm{in}}$, $\hat{x}_i$ and $\hat{v}_i$ are estimations of the state of the ith informed follower and the state of the leader, respectively. Since the uninformed followers do not have access to the information of the leader, $\hat{x}_i$ and $\hat{v}_i$, $i \in \mathcal{F}_{\mathrm{in}}$, will be transmitted to the uninformed followers by the communication graph.

For the ith uninformed follower, $\tilde{v}_i$ is determined by

$$\dot{\tilde{v}}_i = \dot{w} - \dot{\hat{v}}_i = S\tilde{v}_i - \mu\left(\sum_{j=1}^{l} a_{ij}(\hat{v}_i - \hat{v}_j) + \sum_{j=l+1}^{N} a_{ij}(\hat{v}_i - \hat{v}_j)\right) - \gamma R f_{i,3} + R\sigma_\gamma(u_0), \quad i \in \mathcal{F}_{\mathrm{un}}. \tag{25}$$

Denote $\tilde{v}_b = [\tilde{v}_{l+1}^{\mathrm{T}}, \tilde{v}_{l+2}^{\mathrm{T}}, \ldots, \tilde{v}_N^{\mathrm{T}}]^{\mathrm{T}}$ be the compact form of state error $\tilde{v}_i$ for $i \in \mathcal{F}_{\mathrm{un}}$. Then, the compact form of (25) is

$$\dot{\tilde{v}}_b = (I_{N-l} \otimes S - \mu(\mathcal{L}_5 \otimes I_s))\tilde{v}_b - \mu(\mathcal{L}_4 \otimes I_s)\tilde{v}_a - \gamma(I_{N-l} \otimes R)\bar{f}_3(t) + I_{N-l} \otimes R\sigma_\gamma(\bar{u}_0), \tag{26}$$

where $\tilde{v}_a = [\tilde{v}_1^{\mathrm{T}}, \tilde{v}_2^{\mathrm{T}}, \ldots, \tilde{v}_l^{\mathrm{T}}]^{\mathrm{T}}, \bar{f}_3 = [f_{l+1,3}^{\mathrm{T}}, f_{l+2,3}^{\mathrm{T}}, \ldots, f_{N,3}^{\mathrm{T}}]^{\mathrm{T}}$. Since all eigenvalues of $\mathcal{L}_5$ have positive real parts, there exists a positive constant μ such that $(I_{N-l} \otimes S - \mu\mathcal{L}_5 \otimes I_s)$ is Hurwitz.

Define the Lyapunov function

$$V_{\tilde{v}_b} = \tilde{v}_b^{\mathrm{T}} H \tilde{v}_b,$$

where H is the solution of (12). Its derivative along the system (26) satisfies

$$\dot{V}_{\tilde{v}_b} = -K\tilde{v}_b^{\mathrm{T}}\tilde{v}_b - 2\gamma\tilde{v}_b^{\mathrm{T}} H(I_{N-l} \otimes R)\overline{f}_3(t) + 2\tilde{v}_b^{\mathrm{T}} H\left(I_{N-l} \otimes R\sigma_\gamma(\bar{u}_0)\right) - 2\mu\tilde{v}_b^{\mathrm{T}} H(\mathcal{L}_4 \otimes I_s)\tilde{v}_a. \tag{27}$$

Since H is a block diagonal matrix, we have

$$2\tilde{v}_b^{\mathrm{T}} H\left(I_{N-l} \otimes R\sigma_\gamma(\bar{u}_0)\right) = \sum_{i=l+1}^{N} 2\tilde{v}_i^{\mathrm{T}} h_i R\sigma_\gamma(u_0) \leq \sum_{i=l+1}^{N} 2\gamma\|\tilde{v}_i^{\mathrm{T}} h_i R\|. \tag{28}$$

Firstly, we define a auxiliary variable

$$\overline{f}_{i,3}(t) = \begin{cases} \frac{R^{\mathrm{T}} h_i(w-\hat{v}_i)}{\|(w-\hat{v}_i)^{\mathrm{T}} h_i R\|}, & \|(w-\hat{v}_i)^{\mathrm{T}} h_i R\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

From the definition of $f_{i,3}(t)$ in (13) and $\overline{f}_{i,3}(t)$, it satisfies

$$\begin{aligned} -2\gamma\tilde{v}_b^{\mathrm{T}} H(I_{N-l} \otimes R)\overline{f}_3(t) &= \sum_{i=l+1}^{N} -2\gamma\tilde{v}_i^{\mathrm{T}} h_i R f_{i,3}(t) \\ &= \sum_{i=l+1}^{N} -2\gamma\tilde{v}_i^{\mathrm{T}} h_i R(f_{i,3}(t) - \overline{f}_{i,3}(t) + \overline{f}_{i,3}(t)) \\ &= \sum_{i=l+1}^{N} -2\gamma\tilde{v}_i^{\mathrm{T}} h_i R\overline{f}_{i,3}(t) - \sum_{i=l+1}^{N} 2\gamma\tilde{v}_i^{\mathrm{T}} h_i R(f_{i,3}(t) - \overline{f}_{i,3}(t)) \\ &= \sum_{i=l+1}^{N} -2\gamma\|\tilde{v}_i^{\mathrm{T}} h_i R\| - \sum_{i=l+1}^{N} 2\gamma\tilde{v}_i^{\mathrm{T}} h_i R(f_{i,3}(t) - \overline{f}_{i,3}(t)), \end{aligned} \tag{29}$$

Taking (29) and (28) into (27) gives

$$\dot{V}_{\tilde{v}_b} \leq -K\tilde{v}_b^{\mathrm{T}}\tilde{v}_b - 2\mu\tilde{v}_b^{\mathrm{T}} H(\mathcal{L}_4 \otimes I_s)\tilde{v}_a - \sum_{i=l+1}^{N} 2\gamma\tilde{v}_i^{\mathrm{T}} h_i R(f_{i,3}(t) - \overline{f}_{i,3}(t)).$$

Consider the definition of $f_{i,3}(t)$ in (13) and $\overline{f}_{i,3}(t)$ defined above. Since $\lim_{t\to\infty}(w - \hat{v}_i) = 0$ for $i \in [1, 2, \ldots, l]$, we have $\lim_{t\to\infty}\left(w - \frac{1}{\sum_{k=1}^{l} a_{ik}}\sum_{k=1}^{l} a_{ik}\hat{v}_k\right) = 0$. Therefore, we have $\lim_{t\to\infty}(f_{i,3}(t) - \overline{f}_{i,3}(t)) = 0$. Moreover, because $\lim_{t\to\infty}\tilde{v}_a = 0$, then similar to the proof of state feedback-based output consensus problem, it can be concluded that $\lim_{t\to\infty} V_{\tilde{v}_b} = 0$, that is, $\lim_{t\to\infty}(w - v_i) = 0$, $i \in \mathcal{F}_{\mathrm{un}}$. Under the distributed observer (11b), the dynamics of $\tilde{x}_i$ satisfies

$$\dot{\tilde{x}}_i = \dot{w} - \dot{\hat{x}}_i = (A_i + L_{A,i}C_i)\tilde{x}_i + W_i(w - \hat{v}_i), \quad i \in \mathcal{F}_{\mathrm{un}}.$$

Since $A_i + L_{A,i}C_i$ is Hurwitz and $\lim_{t\to\infty}(w - \hat{v}_i) = 0$ for $i \in \mathcal{F}_{\mathrm{un}}$, we have $\lim_{t\to\infty}(x_i - \hat{x}_i) = 0$.

Recall that $\xi = x - \Pi\overline{w}$, the dynamics of ξ in (17) is also satisfied. Denote $\tilde{x} = [\tilde{x}_1^{\mathrm{T}}, \tilde{x}_2^{\mathrm{T}}, \ldots, \tilde{x}_N^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{v} = [\tilde{v}_1^{\mathrm{T}}, \tilde{v}_2^{\mathrm{T}}, \ldots, \tilde{v}_N^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{v} = [\hat{v}_1^{\mathrm{T}}, \hat{v}_2^{\mathrm{T}}, \ldots, \hat{v}_N^{\mathrm{T}}]^{\mathrm{T}}$, $g_2(t) = [g_{1,2}(t)^{\mathrm{T}}, g_{2,2}(t)^{\mathrm{T}}, \ldots, g_{N,2}(t)^{\mathrm{T}}]^{\mathrm{T}}$, then the controller (22) can be rewritten as the following compact form:

$$u = -B^{\mathrm{T}} P(\hat{x} - \Pi\hat{v}) - \gamma E g_2(t) + \Gamma\hat{v} = -B^{\mathrm{T}} P(x - \Pi\overline{w}) - \gamma E g_2(t) + \Gamma\hat{v} + B^{\mathrm{T}} P\tilde{x} - B^{\mathrm{T}} P\Pi\tilde{v}.$$

Define the Lyapunov function

$$V_\xi = \xi^{\mathrm{T}} P \xi, \tag{30}$$

where $P = P(\epsilon)$. Since $\lim_{t\to\infty} \tilde{v} = 0$ and $\lim_{t\to\infty} \tilde{x} = 0$, there exists a time $T_2 > T$ such that

$$\|\Gamma(w - \hat{v})\|_{\infty,T_2} \le \frac{1}{2}\delta, \quad \|B^{\mathrm{T}} P \tilde{x}\| \le \rho_1 < \frac{1}{4}\delta, \quad \|B^{\mathrm{T}} P \Pi \tilde{v}\| \le \rho_2 < \frac{1}{4}\delta,$$

then under Assumption 6, the following inequality holds.

$$\|\Gamma\hat{v}\|_{\infty,T_2} \le \|\Gamma(\hat{v} - w)\|_{\infty,T_2} + \|\Gamma w\|_{\infty,T_2} \le \Delta - \|\gamma E\| - \frac{1}{2}\delta.$$

For any $[x^{\mathrm{T}}(0), w^{\mathrm{T}}(0), \hat{x}^{\mathrm{T}}(0), \hat{v}^{\mathrm{T}}(0)]^{\mathrm{T}} \in \mathcal{X}_0 \times \mathcal{W}_0 \times \hat{\mathcal{X}}_0 \times \hat{\mathcal{V}}_0$, $\xi(T_2)$ is bounded by a compact set $\mathcal{X}_2$, independent of ϵ, since ξ is determined by a linear differential equation with bounded inputs $\sigma_\Delta(u)$ and $\sigma_\gamma(\bar{u}_0)$. Let $c_2 > 0$ be a constant such that

$$\sup_{\xi(T_2)\in\mathcal{X}_2, \epsilon\in(0,1]} V_\xi \le c_2.$$

c_2 exists because $\mathcal{X}_2$ is bounded. Define $L_{V_\xi}(c_2) := \{\xi : V_\xi \le c_2\}$. Then, there exists an $\epsilon^* \in (0,1]$ such that for any $\epsilon \in (0, \epsilon^*]$, the inequality

$$\| - B^{\mathrm{T}} P(x - \Pi\overline{w})\|_{\infty,T_2} \le \frac{1}{2}\delta - \rho_1 - \rho_2.$$

Such an ϵ^* exists because $\lim_{\epsilon\to 0} P = 0$ and $\xi = x - \Pi\overline{w}$ is bounded. Thus, by Assumption 6, we have

$$\|u\|_{\infty,T_2} \le \| - B^{\mathrm{T}} P(x - \Pi\overline{w})\|_{\infty,T_2} + \|\gamma E g_2(t)\| + \|\Gamma\hat{v}\| + \|B^{\mathrm{T}} P\tilde{x}\| + \|B^{\mathrm{T}} P\Pi\tilde{v}\| \le \Delta.$$

Therefore, we have $\sigma_\Delta(u) = u$.

Substituting the control protocol (22) into (17), we have

$$\dot{\xi} = (A - BB^{\mathrm{T}} P)\xi - \gamma B E g_2(t) - \Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0) - B\Gamma\tilde{v} + B^{\mathrm{T}} P\tilde{x} - B^{\mathrm{T}} P\Pi\tilde{v}.$$

The derivative of the Lyapunov function (30) satisfies

$$\dot{V}_\xi = -\epsilon\xi^{\mathrm{T}}\xi - 2\gamma\xi^{\mathrm{T}} PBE g_2(t) - 2\xi^{\mathrm{T}} P\Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0) - 2\xi^{\mathrm{T}} PB\Gamma\tilde{v} + 2\xi^{\mathrm{T}} PB^{\mathrm{T}} P\tilde{x} - B^{\mathrm{T}} P\Pi\tilde{v}. \tag{31}$$

According to the definition of $g_{i,2}(t)$ in (23), it gives

$$\begin{aligned}
-2\gamma\xi^{\mathrm{T}} PBE g_2(t) &= \sum_{i=1}^{N} -2\gamma\xi_i^{\mathrm{T}} P_i B_i E_i g_{i,2} \\
&= \sum_{i=1}^{N} -2\gamma\xi_i^{\mathrm{T}} H_i \overline{g}_{i,2} + \sum_{i=1}^{N} -2\gamma\xi_i^{\mathrm{T}} H_i (g_{i,2}(t) - \overline{g}_{i,2}(t)) \\
&= \sum_{i=1}^{N} -2\gamma\|\xi_i^{\mathrm{T}} H_i\| - 2\gamma\xi_i^{\mathrm{T}} H_i (g_{i,2}(t) - \overline{g}_{i,2}(t)),
\end{aligned} \tag{32}$$

where $\overline{g}_{i,2}(t)$ is defined as

$$\overline{g}_{i,2}(t) = \begin{cases} \frac{H_i^{\mathrm{T}}\xi_i(t)}{\|H_i^{\mathrm{T}}\xi_i(t)\|}, & \|H_i^{\mathrm{T}}\xi_i(t)\| \ne 0, \\ 0, & \text{otherwise}. \end{cases}$$

Since $\lim_{t\to\infty}(x_i - \hat{x}_i) = 0$ and $\lim_{t\to\infty}(w - \hat{v}) = 0$, we have $\lim_{t\to\infty}(\xi_i(t) - \psi_i(t)) = 0$, which implies $\lim_{t\to\infty}(g_{i,2}(t) - \overline{g}_{i,2}(t)) = 0$. Since u_0 is subject to the standard saturation function, we have

$$-2\xi^{\mathrm{T}} P\Pi(I_N \otimes R)\sigma_\gamma(\bar{u}_0) = \sum_{i=1}^{N} -2\xi_i^{\mathrm{T}} P_i \Pi_i R \sigma_\gamma(u_0) = \sum_{i=1}^{N} -2\xi_i^{\mathrm{T}} P_i B_i E_i \sigma_\gamma(u_0) \leq \sum_{i=1}^{N} 2\gamma \|\xi_i^{\mathrm{T}} H_i\|. \tag{33}$$

Taking (32) and (33) into (31) gives

$$\dot{V}_\xi \leq -\epsilon \xi^{\mathrm{T}} \xi - 2\gamma \xi_i^{\mathrm{T}} H_i (g_{i,2} - \overline{g}_{i,2}(t)) - 2\xi^{\mathrm{T}} PB\Gamma\tilde{v} + 2\xi^{\mathrm{T}} PB^{\mathrm{T}} P\tilde{x} - B^{\mathrm{T}} P\Pi\tilde{v}.$$

Since $\lim_{t\to\infty}(g_{i,2} - \overline{g}_{i,2}(t)) = 0$, $\lim_{t\to\infty} \tilde{x} = 0$, and $\lim_{t\to\infty} \tilde{v} = 0$, it can be verified that $\lim_{t\to\infty} \xi = 0$, that is, $\lim_{t\to\infty}(x_i - \Pi_i w) = 0$. Then, it follows that

$$\lim_{t\to\infty} e_i = \lim_{t\to\infty} C_i \xi_i = 0.$$

This completes the proof. ∎

Remark 2. It is noted that the two consensus protocols (14) and (22) solve the semi-global output consensus of multi-agent systems, which implies that they rely on the initial state conditions. Different from the results that the initial errors need to be sufficiently small, the semi-global results allow the initial values to be within any bounded sets.

4 | ILLUSTRATIVE EXAMPLES

In this section, we aim to achieve the output consensus of a nonholonomic robot and four quadrotors using the control laws (14) and (22), respectively, where the nonholonomic robot acts as the leader, and the quadrotors are the followers. According to Young and Beard,[29] the nonholonomic robot can be feedback linearized to a double-integrator system

$$\begin{aligned} \dot{w} &= \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix} w + \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix} \sigma_\gamma(u_0) \\ y_0 &= [I_2 \;\; 0_2]w, \end{aligned} \tag{34}$$

where $w = [p_0^{\mathrm{T}}, \dot{p}_0^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^4$. By comparing the systems (1) and (34), one can easy to get the matrices S, R, and Q. Following Hehn and D'Andrea,[30] the quadrotors can be modeled by a triple integrator. In the multi-agent system, we assume the nonholonomic robot generates a disturbance $W_i w$ to the quadrotors. Thus, the dynamics of the quadrotors can be expressed as

$$\begin{aligned} \dot{x}_i &= \begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & 0_2 \end{bmatrix} x_i + \begin{bmatrix} 0_2 \\ 0_2 \\ I_2 \end{bmatrix} \sigma_\gamma(u_0) + \begin{bmatrix} 0_2 & I_2 \\ 0_2 & -I_2 \\ 0_2 & -I_2 \end{bmatrix} w \\ y_i &= [I_2 \;\; 0_2 \;\; 0_2]x_i, \quad i = 1,2,3,4, \end{aligned} \tag{35}$$

where $x_i = [p_i^{\mathrm{T}}, \dot{p}_i^{\mathrm{T}}, \ddot{p}_i^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^6$. In this example, we assume that quadrotors fly at a fixed altitude. Similarly, one can easy to obtain the matrices A_i, B_i, C_i, and W_i by comparing systems (3) and (35). It is easy to verify that Assumptions 3 and 4 are satisfied. We can note that the outputs of agents are $y_i = p_i \in \mathbb{R}^2$, $i = 0,1,2,3,4$. So the leader-following output consensus problem can be viewed as the nonholonomic robot's position tracking for quadrotors.

Solving the equations in Assumptions 2 and 4 gives

$$\Pi_i = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix}, \quad \Gamma_i = [0_2 \;\; I_2], \quad E_i = I_2, \quad i = 1,2,3,4.$$

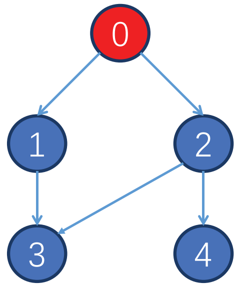

FIGURE 1 The communication graph

The communication graph $\mathcal{G}$ of the agents is shown in Figure 1. It is obvious that Assumption 1 is satisfied. Among the followers, the first two are informed ones and the left two are uninformed ones. The corresponding matrix $\mathcal{L}$ and $\mathcal{L}_5$ are

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathcal{L}_5 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that all eigenvalues of $\mathcal{L}$ and $\mathcal{L}_5$ have positive real parts.

The bounds of the control input of the leader γ and the followers Δ_i are set as 2 and 5, respectively, for $i = 1, 2, 3, 4$. We have $\|\gamma E_i\|_\infty = 1$, $i = 1, 2, 3, 4$. During the simulation, the control input of the leader is set as $u_0 = [1 \quad 1]^{\mathrm{T}} \cdot \sin(-4t)$. We have $\|\Gamma_i w\|_\infty < 1$, $i = 1, 2, 3, 4$, $\forall t > 0$, So Assumption 6 is satisfied.

4.1 | Semi-global output consensus via state feedback

In this section, we show the effectiveness of Theorem 1, which holds under the state feedback control law (14). The initial state of the leader, the followers, and the distributed observer in (6) are set as random values between $[-2, 2]$. One can easy to solve the ARE in (7). As shown in Figure 2, the states $\hat{w}_i$, $i = 1, 2, 3, 4$, of the fully distributed observer (6) asymptotically converge to the state of the leader.

We set the low gain parameter $\epsilon = 0.001$. One can easy to solve the parametric ARE in (4) with the chosen ϵ. The outputs of agents and the output consensus errors are shown in Figure 3. It can be seen that the output tracking error asymptotically converges to 0. Thus, Problem 1 is solved by the state feedback consensus control protocol (14) with $\epsilon = 0.001$.

4.2 | Semi-global output consensus via output feedback

In this section, we verify the effectiveness of Theorem 2, which solves the output consensus problem via the output feedback consensus protocol (22).

In distributed observers 10 and (11), the matrix $L_{A,i} = -[3I_2 \quad 3I_2 \quad I_2]^{\mathrm{T}}$, $i = 1, 2, 3, 4$. We can verify that $A_i + L_{A,i}C_i$ is Hurwitz.

For the informed followers 1 and 2, the matrix in (10b) is set as $L_S = [3I_2 \quad 2I_2]^{\mathrm{T}}$. One can easy to solve the equation (9) with $F = I_4$. For the uninformed followers, we construct the distributed observer in (11b) and (11a) with $\mu = 2$, and

$$h_3 = \begin{bmatrix} 0.125 & 0 & 0.0156 & 0 \\ 0 & 0.125 & 0 & 0.0156 \\ 0.0156 & 0 & 0.1289 & 0 \\ 0 & 0.0156 & 0 & 0.1289 \end{bmatrix}, \quad h_4 = \begin{bmatrix} 0.25 & 0 & 0.0625 & 0 \\ 0 & 0.25 & 0 & 0.0625 \\ 0.0625 & 0 & 0.2813 & 0 \\ 0 & 0.0625 & 0 & 0.2812 \end{bmatrix}.$$

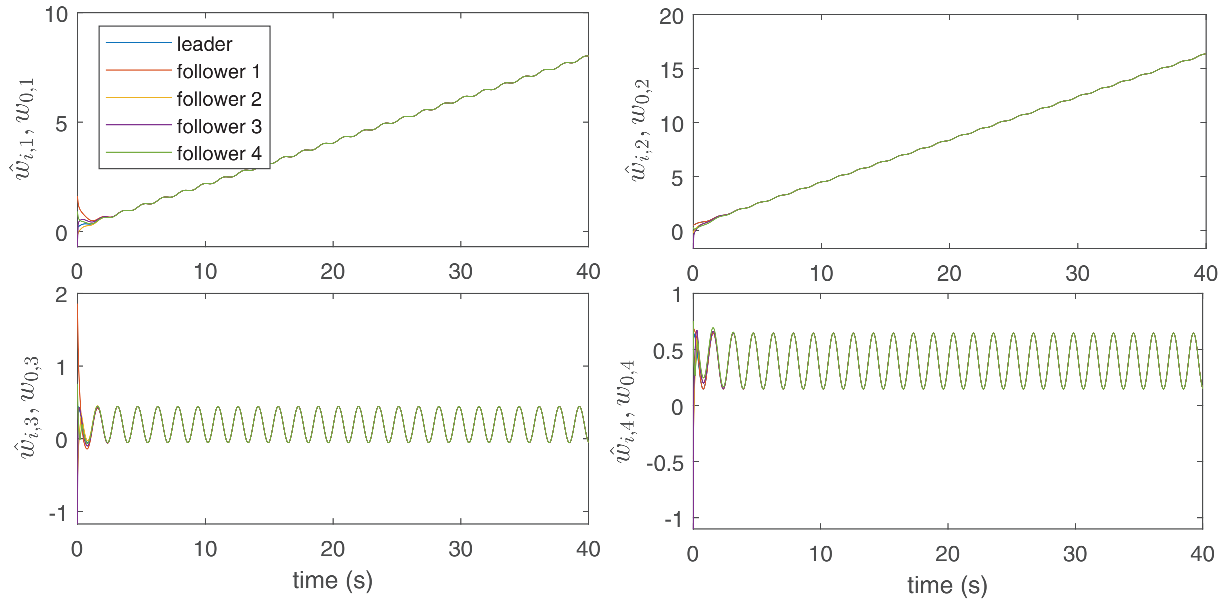

FIGURE 2 State of the leader and its estimations $\hat{w}_i$, $i = 1, 2, 3, 4$ for every follower

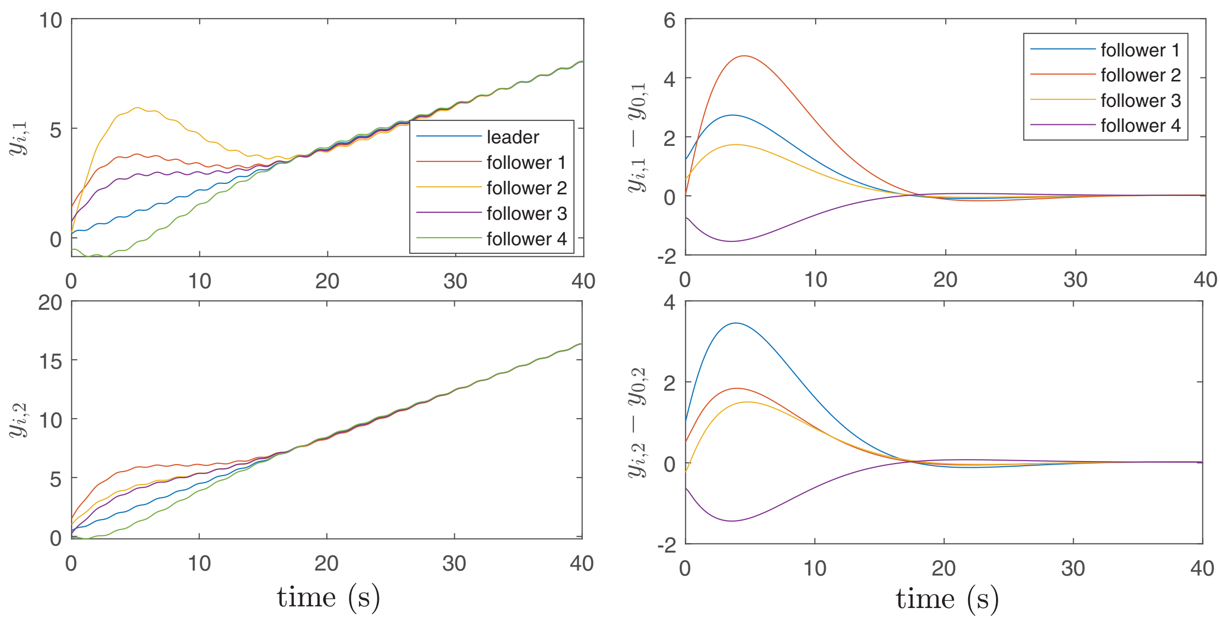

FIGURE 3 Simulation result under the state feedback consensus control law (14) with $\epsilon = 0.001$. (Left) Outputs of the leader and the four followers. (Right) The output consensus errors between the four followers and the leader

In this example, the initial conditions of the states of the leader $w(0)$ in (1), the states of the four followers $x_i(0)$, $i = 1, 2, 3, 4$ in (3), the states of the observers in (8), (10), and in (11) are set as random values between [0,5]. As shown in Figure 4, the estimation errors $\hat{v}_i - w$ and $\hat{x}_i(t) - x_i(t)$, $i = 1, 2, 3, 4$, converge to zero.

The low gain parameter is chosen as $\epsilon = 0.0001$. Figure 5 shows that when $\epsilon = 0.0001$, the outputs of the four followers y_i, $i = 1, 2, 3, 4$, are regulated to the output of the leader y_0.

5 | CONCLUSIONS

In this paper, we have investigated the semi-global leader-following output consensus problem with both the leader and the followers subject to input saturation over a directed graph. According to whether the followers

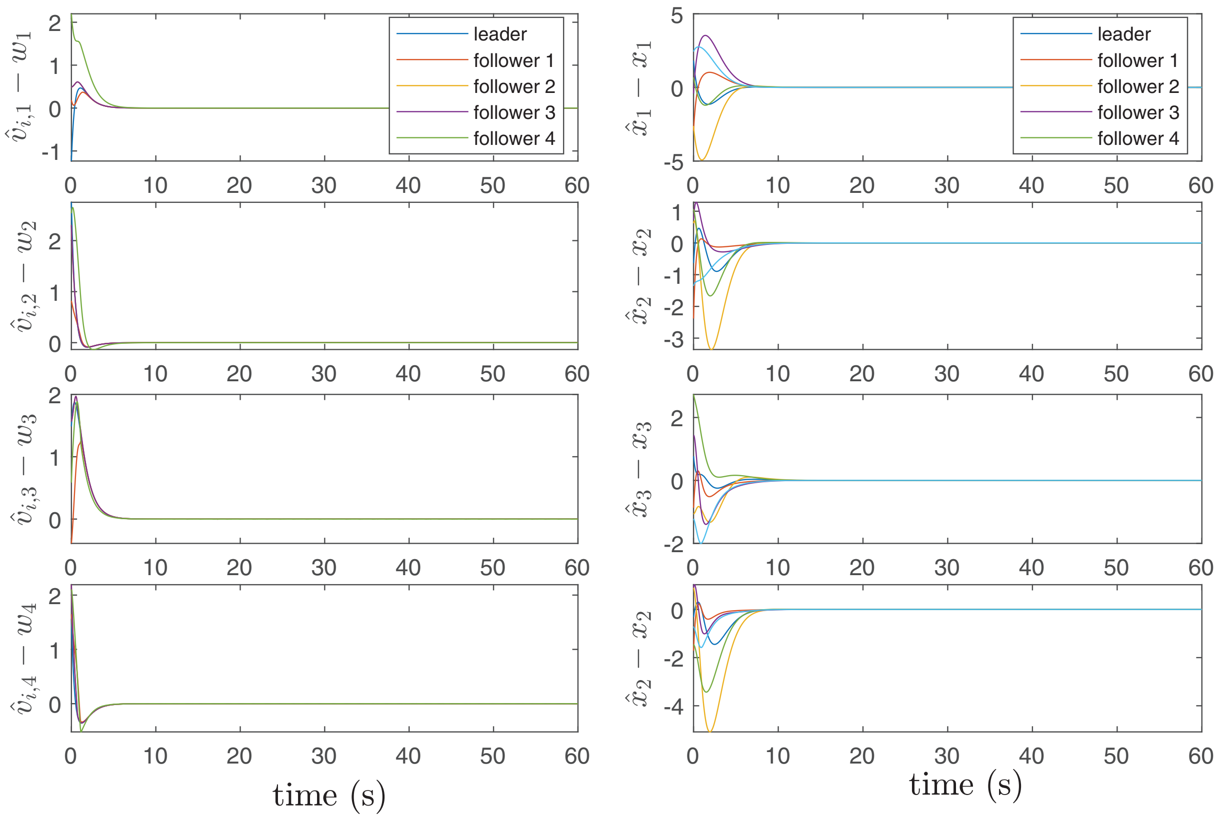

FIGURE 4 (Left) Elements of the estimation error $\hat{v}_i - w$, $i = 1, 2, 3, 4$. (Right) Elements of the estimation error $\hat{x}_i - x_i$, $i = 1, 2, 3, 4$

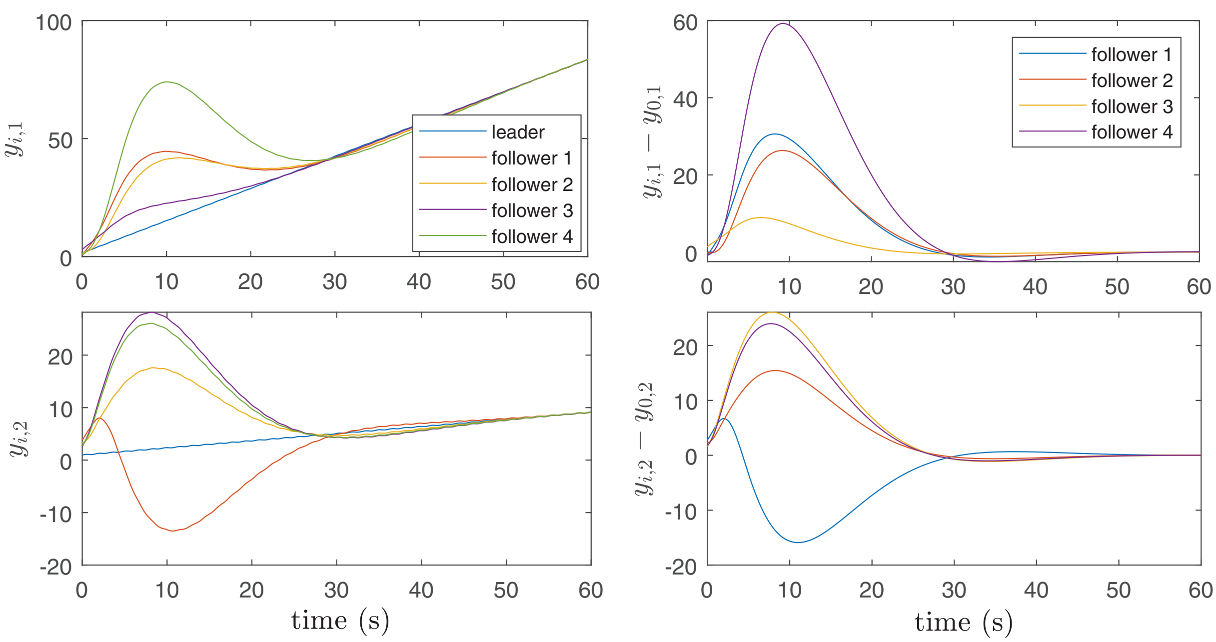

FIGURE 5 Simulation result under the output feedback consensus control law (22) with $\epsilon = 0.0001$. (Left) Outputs of the leader and the four followers. (Right) Elements of the output consensus errors $y_i - y_0$, $i = 1, 2, 3, 4$

have access to the information of the leader, they are divided into the informed ones and the uninformed ones. Via the low gain feedback design technique and the output regulation theory, state feedback-based and output feedback-based consensus protocols were constructed. Especially, the existence of the input of the leader raised difficulties in designing distributed observers and control laws. In view of this, auxiliary terms were designed to make up for the influence of input of the leader. Finally, the constructed control protocols were verified by a practical example.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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