Formation-Containment Control of Euler–Lagrange Systems of Leaders With Bounded Unknown Inputs

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Abstract—Motivated by the promising applications of multiple Euler–Lagrange (EL) systems, we study, in this article, the formation-containment (FC) control problem for multiple EL systems of leaders with bounded unknown control inputs and with communication among each other over directed topologies, which can cooperatively generate safe trajectories to avoid obstacles. Given the FC shapes, an algorithm is first proposed to obtain the stress matrix while satisfying certain conditions, based on which a novel adaptive distributed observer to the convex hull is proposed for every follower. An adaptive updating gain is applied to make the observer fully distributed without using the global information of the graph, and a continuous function is designed to restrain the influence of the inputs of the leaders. Then, a local control law using the adaptive distributed observer is presented to accomplish the FC control of EL systems. Based on the Lyapunov stability theory, it is proved that the FC error can be designed as small as possible by adjusting some parameters in the observer.

Index Terms—Adaptive distributed observer, Euler–Lagrange systems, formation-containment (FC) control, multiagent systems, nonautonomous leaders.

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$f_i(t)$ | Eq. (9) | Auxiliary variable. |
$\psi_i(t)$ | Eq. (9) | Auxiliary variable. |
$P$ | Eq. (3) | Solution of the ARE. |
$\mu$ | Eqs. (4) and (5) | Positive constant. |
$\rho$ | Eqs. (4) and (5) | $1/2 < \rho < 1$ and $\rho \in R_{odd}$. |
$\sigma_i$ | Eq. (8) | Positive constant. |
$\nu$ | Eq. (14) | Constant $\nu(t)$. |
$\phi$ | $\gamma + \rho$ | Auxiliary parameter. |

I. INTRODUCTION

COORDINATED control of multiagent systems, whose engineering applications include consensus (see [1]–[3]), event-triggered control (see, e.g., [4]), flocking (e.g., [5]), attitude synchronization of rigid bodies (e.g., [6]–[8]), formation control (e.g., [9]–[11]), and containment control (e.g., [12] and [13]), has aroused considerable research interest in the past few years. Containment control of multiagent systems, inspired by the natural phenomena that male silkworm moths will end up in the safety space spanned by the female silkworm moths by detecting pheromone released by females, aims to enforce the followers to converge to the convex hull formed by the leaders. The containment control problem was solved for continuous-time systems (e.g., [14] and [15]), discrete-time general linear systems (see [16]), and Euler–Lagrange (EL) systems (see [17] and [18]), other nonlinear systems (see [19] and [20]) with only neighboring communication (see [21] and [22]).

In many applications, a group of agents as a whole are intended to perform specific tasks, for example, moving target enclosing and surveillance, while maintaining a desired shape (e.g., Yang et al. [11] and Wang et al. [23]) or time-varying shapes (e.g., [24]–[28]). Some formation control problems are formulated as consensus problem (e.g., [29] and [30]). Affine formation maneuver control (see [31] and [32]) provides a new direction for solving the problem, where the agents as a whole can perform translation, rotation, scaling shear, and/or their combinations.

Formation-containment (FC) control, as the name implies, combines the formation control and containment control. In practice, the autonomous vehicles are likely to perform missions in complex and cluttered environments. The vehicles can
be divided into two classes: 1) those (the leaders) are equipped with sensors to detect the environment and 2) those (the followers) are not. The leaders can generate safe trajectories by utilizing the sensors and then overall agents can reach their destination safely if the followers converge to the safe space the leaders formed. Thus, it is significant to consider the FC control of EL systems. In most research results of FC control problem (see [33]–[35]), the configuration of the agents greatly depends on the Laplacian matrix of the graph topology, and the configuration is fixed once given the graph (e.g., [33]–[36]). Though the containment control problem of multiagent systems under switching topologies was realized in [37], it is a waste of energy for the agents to jump between different formations because of the existence of a row-stochastic matrix. In our result, we consider the opposite problem for the FC control, that is, given the desired formation shapes, an algorithm is given to obtain the stress matrix over the directed communication graph. In such a case, the agents can form a variety of configurations, while in [31] gives a similar algorithm over undirected communication graph.

From some point of view, the FC control is a special case of affine formation control. In affine formation control, the edge weights can be positive or negative, but the assumption that the edge weights are all positive or all negative plays a key role in the containment control. Even so, our control law is applicable for multiagent systems among which the leaders have bounded unknown inputs. In practice, the exosystems are required to have control inputs to generate more complex signals cooperatively such that their trajectories are able to avoid hazardous areas and/or obstacles. To take this situation into consideration, [38] and [39] considered the containment or FC control problem of multiple linear systems with leaders having nonzero inputs. In both results, the system matrices of the leaders are global information that may be unavailable in practice. For the control of EL systems, an important category of nonlinear systems that describes autonomous vehicles, under-actuated surface vessels, robot manipulators, and spacecrafts, there have been some interesting works on the performance constraint control (e.g., [40] and [41]). Similar to the coordination control problem of EL systems in [42] and [43], the leader agents in our work are represented by linear systems, which can be viewed as virtual leaders in real applications. Their leader agents, however, have no control input. To the best of our knowledge, the FC control of EL systems, with leaders having unknown inputs, is still open.

Motivated by the above facts, we aim, in this article, to design a distributed control law that utilizes an adaptive distributed observer to solve the FC control of multiple EL systems with their leaders subject to unknown inputs. The main contributions are four-fold. First, under the assumption that the leaders have bounded control inputs with the boundary and system matrices being unknown to all the followers, we design an adaptive distributed observer for the system matrices of the leaders and then a control law to compensate for their inputs. We note that in [42], it is assumed that the leaders do not have any control input. Different from the works in [36] and [44], which studied the formation tracking problem of multiple EL systems via bounded input, the leaders in our problem are described by linear systems and we removed the assumptions on the systems matrix of the leader agents. Second, given the formation configuration, an algorithm for obtaining the corresponding stress matrix is given. In general, the Laplacian matrix associated with the communication graph is defined, and the element of the Laplacian matrix associated with each edge is usually set as 1, such as works in [33] and [35], where the formation shape is fixed. However, formation shapes are first given in our work, and then an algorithm is proposed to determine the stress matrix and the directed graph. Moreover, the algorithm is proposed for the directed digraph, while the algorithm in [31] is only applicable to undirected digraph. Third, the observers and control protocol we proposed are fully distributed without using the global information of the communication graph. In our problem, the FC problem is solved based on the stress matrix of a directed graph, and the stress matrix is asymmetric. In [36] and [45], the graph topology is assumed to be undirected and its adaptive updating gain cannot be adopted to solve our problem. Finally, in our formulation, the leaders can communicate with each other to generate a safe region for the followers and to form required formations. It is critical for the leaders to interact. On the one hand, they can cooperatively form the expected configuration given positions of other leaders. On the other hand, there is a direct relationship between the position of the leaders and the size of the convex hull formed by them. To summarize, it is challenging to design a distributed control law that employs the stress matrix and estimation of the system matrix of the leaders to avoid using the global information of the graph and to compensate for the bounded unknown inputs of the leaders under a directed graph.

The outline of the rest of this article is as follows. Preliminaries and problem formulation are given in Section II. In Section III, we designed the adaptive distributed observer to the convex hull spanned by the leaders, after which the distributed control protocol using the adaptive distributed observer is proposed for every EL system in Section IV. Next, an illustrative example is shown in Section V to verify our control protocol and we conclude this article by Section VI with some remarks.

Throughout this article, $1_n \in \mathbb{R}^n$ denotes a vector with all elements being 1. $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix. Kronecker product is denoted by $\otimes$. $\|x\|$ denotes the Euclidean norm of a vector $x$. $\lambda(A)$, $\lambda_{\min}(A)$, and $\lambda_{\max}(A)$ represent the eigenvalues, the minimum eigenvalue, and the maximum eigenvalue of matrix $A$, respectively. $X^T$ stands for the transpose of the vector or matrix $X$. $R_{odd}^n = \{x \in \mathbb{R} : x > 0 \text{ and } x \text{ is a ratio of odd integers}\}$

II. PRELIMINARIES AND PROBLEM FORMULATION

We consider a set of $N_f$ EL systems whose dynamic equations are as follows:

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i$$

$i = N_i + 1, \ldots, N_f$, where $N_i$ is the number of leaders; $q_i, \dot{q}_i \in \mathbb{R}^n$ denote the generalized position and velocity, respectively; $M_i(q_i) \in \mathbb{R}^{n \times n}$ is the symmetric and positive definite inertia matrix; $C_i(q_i, \dot{q}_i)\dot{q}_i \in \mathbb{R}^n$ is the vector representing
the Coriolis and centripetal forces; \(G_t(q_i) \in \mathbb{R}^n\) is the vector of gravitational force; and \(\tau_i \in \mathbb{R}^n\) is the generalized force vector.

According to [46], the EL systems in (1) have the following properties.

**Property 1:** \(\dot{M}_i(q_i) = 2C_i(q_i, \dot{q}_i)\) is skew symmetric for \(\forall q_i, \dot{q}_i\).

**Property 2:** For any \(x, y \in \mathbb{R}^n\)

\[
M_i(q_i)x + C_i(q_i, \dot{q}_i)y + G_i(q_i) = Y_i(q_i, \dot{q}_i, x, y)\Theta_i
\]

where \(Y_i(q_i, \dot{q}_i, x, y) \in \mathbb{R}^{n \times p}\) is a known regression matrix and \(\Theta_i \in \mathbb{R}^p\) is a constant vector consisting of the uncertain parameters of the system given in (1).

In our problem, there are \(N_l\) leaders whose desired generalized position vectors \(q_j\), for \(j = 1, \ldots, N_l\), are assumed to be generated by the following exosystem:

\[
\dot{y}_j(t) = S y_j(t) + R y_j(t), \quad q_j(t) = C y_j(t) \tag{2}
\]

where \(y_j \in \mathbb{R}^m\), and \(S \in \mathbb{R}^{m \times m}, R \in \mathbb{R}^{n \times k}, C \in \mathbb{R}^{n \times m}\) are constant matrices.

**Assumption 1:** The inputs of the leader systems \(r_j(t)\) are bounded, which means there exists a positive constant \(\gamma\) such that \(\|r_j(t)\| \leq \gamma\).

**Remark 1:** In our problem, the control input \(r_j(t)\) introduces flexibility to generate more general reference signals and safe trajectories to respond to changes of the environment, such as to avoid obstacles. In addition, we assume that \(\gamma\) is unknown to all the followers.

We view the system composed of (1) and (2) as a multiagent system of \((N_l+N_f)\) agents with (2) being the leaders and (1) the followers. Assume that the network topology of the multiagent system is represented by \(\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}\) with \(\mathcal{V} = \{1, \ldots, N_l, N_l+1, \ldots, N_l+N_f\}\) and \(\mathcal{E} = \mathcal{V} \times \mathcal{V}\). We use \(\mathcal{L} = \{1, \ldots, N_l\}\) to denote the leaders, and \(\mathcal{F} = \{N_l+1, \ldots, N_l+N_f\}\) to denote the followers. For \(i, j \in \mathcal{V}, (j, i) \in \mathcal{E}\) if and only if information can flow from node \(j\) to \(i\). Let \(N_j = \{(j, i) \in \mathcal{E}\}\) denote the neighbor set of agent \(i\). If the graph contains a sequence of edges \((i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)\), then we say there is a directed path from node \(i_1\) to \(i_k\), or \(i_k\) is reachable from \(i_1\).

A formation formed by the outputs of the leaders and the followers is represented by \(q = [q_1^T, \ldots, q_{N_l+N_f}^T]^T\), and every node in the graph \(\mathcal{G}\) is mapped to the position \(q_i\). The compact form of positions of the followers and the leaders is denoted as \(q_f = [q_{N_l+1}^T, \ldots, q_{N_l+N_f}^T]^T\) and \(q_l = [q_1^T, \ldots, q_{N_l}^T]^T\), respectively. For formation \((\mathcal{G}, q)\), a stress is a set of scalars \([w_{ij}]_{(i,j)\in \mathcal{E}}\), where \(w_{ij} \in \mathbb{R}\) is associated with the edge \((j, i)\). If the stress satisfies \(\sum_{j \in N_i} w_{ij} (q_j - q_i) = 0\), it is called an equilibrium stress [31]. The compact form of this equation is \((\Omega \otimes I_n)q = 0\), where \(q = [q_1^T, q_f^T]^T\), \(\Omega \in \mathbb{R}^{(N_l+N_f) \times (N_l+N_f)}\) is called the stress matrix satisfying

\[
[\Omega]_{ij} = \begin{cases} 0, & i \neq j, (i, j) \notin \mathcal{E} \\ -w_{ij}, & i \neq j, (i, j) \in \mathcal{E} \\ \sum_{k \in N_i} w_{ik}, & i = j \end{cases}
\]

and

\[
\Omega = \begin{bmatrix} \Omega_{lf} & \Omega_{lf} \\ \Omega_{lf} & \Omega_{lf} \end{bmatrix}
\]

where \(\Omega_{lf} \in \mathbb{R}^{N_f \times N_l}\) and \(\Omega_{lf} \in \mathbb{R}^{N_f \times N_f}\), is another expression of \(\Omega\) according to the categorization of the leaders and the followers. We denote \(\Omega = \Omega \otimes I_n\).

**Definition 1** [47]: A set \(C \subseteq \mathbb{R}^n\) is convex if \((1 - \lambda)x + \lambda y \in C\), for any \(x, y \in C\) and any \(\lambda \in [0, 1]\). The convex hull \(Co(X)\) of a finite set of points \(X = \{x_1, x_2, \ldots, x_N\}\) is the minimal convex set containing all points in \(X\). That is

\[
Co(X) = \{\sum_{i=1}^{N} \alpha_i x_i | x_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1\}.
\]

By Definition 1, the set composed of the outputs of the leaders is denoted as \(Q = \{q_1, \ldots, q_{N_l}\}\). The convex hull spanned by the outputs of the leaders is represented by \(Co(Q)\).

Similar to the definition of the FC control in [48], we define our problem as follows.

**Problem 1:** Given the multiagent system in (1) and (2), and a digraph \(\mathcal{G}\), find a distributed control law such that when it is applied to the given system:

1) the resulting overall closed-loop system converges to a desired formation, with a preset degree of accuracy, that is, given a scalar \(\varepsilon_1 > 0\) and given a formation vector \(h(t) = [h_1^T(t), h_2^T(t), \ldots, h_{N_l+N_f}^T(t)]^T\), the overall system is said to achieve formation if

\[
\lim_{t \to \infty} \|q_i(t) - h_i(t)\| \leq \varepsilon_1, \quad i \in \mathcal{F} \cup \mathcal{L}
\]

2) all trajectories of the closed-loop systems of the followers asymptotically converge to the convex hull spanned by the outputs of the leaders (2), with a preset degree of accuracy, that is

\[
\lim_{t \to \infty} \|q_i(t) - \sum_{j=1}^{N_l} m_{ij} q_j(t)\| \leq \varepsilon_2
\]

for any initial condition of \(q_i(0)\) and \(v_j(0), i \in \mathcal{F}, j \in \mathcal{L}\), where \(0 \leq m_{ij} \leq 1, \sum_{j=1}^{N_l} m_{ij} = 1\), and \(\varepsilon_2 > 0\).

**Remark 2:** In our problem, due to the existence of the control inputs of the leader systems, the formation error cannot asymptotically converge to zero. Instead, the Euclidean norm of the formation error is bounded by a sufficiently small constant as \(t\) tends to infinity, which is reasonable and acceptable in reality.

To solve Problem 1, the following assumptions are made.

**Assumption 2:** For each follower in (1), there exists at least one leader in (2) that has a directed path to the follower in the digraph \(\mathcal{G}\), and \(\mathcal{G}\) is directed.

**Assumption 3:** For all \(j \in N_l\), the elements of the stress \(w_{ij} > 0\) or all \(w_{ij} < 0\).

**Assumption 4:** All the states and state derivatives of the leader agents are bounded, that is, for \(\forall j \in \mathcal{L}, v_j\) and \(\dot{v}_j\) are bounded.

**Assumption 5:** There exist positive constants \(k_m, k_m, k_e, k_g\) such that for all \(i \in \mathcal{F}, k_m l_{nf} \leq \dot{M}_i(q_i) \leq k_m l_{nf}, \|C_i(q_i, \dot{q}_i)\| \leq K_c\|\dot{q}_i\|, \|G_i(q_i)\| \leq K_g\).

**Remark 3:** Assumptions 2 and 5 have been widely used in the study of coordinated control of EL systems (see [36]). Assumption 2 is a sufficient condition for the solvability of the FC problem of the multiagent system, and Assumption 5 is...
also a property of the EL systems [49]. Assumption 3, a sufficient condition, as will be illustrated in Section III, is important for the creation of the convex hull formed by the leaders. The following Algorithm 1 will show that the FC shape may be formed even if \( w_{ij} > 0 \) and \( w_{ij} < 0 \) exist simultaneously in a graph. Assumption 4 is a sufficient and necessary condition for the stability and convergence of the closed-loop system, according to Barbalat’s Lemma. It is noted that all the eigenvalues of the system matrix \( S \) are assumed to have nonpositive real parts in the works [42], [43]. In our work, the assumption is relaxed to the boundness of the states and its derivative.

III. ADAPTIVE DISTRIBUTED OBSERVER OF THE CONVEX HULL SPANNED BY THE LEADERS

In [31], the value of \( w_{ij} \) can be positive or negative, but there is no control input in the leader systems. In this section, we show that Assumption 3 plays an important role in designing the adaptive distributed observer for leaders with bounded unknown inputs. We have the following lemma.

**Lemma 1:** Under Assumptions 2 and 3, if the stress matrix satisfies (\( \Omega \otimes I_n \)q = 0), then \( q_i, i \in \mathcal{F} \) belongs to the convex hull spanned by the leader systems, that is, \( q_i \in \text{Co}(Q) \).

**Proof:** We use \( \Omega_1 \) and \( \Omega_2 \) to denote the stress matrices satisfying for \( i \neq j \), \( (i, j) \in \mathcal{E} \), all \( w_{ij} > 0 \) and all \( w_{ij} < 0 \), respectively. It is obvious that \( \Omega_1 = -\Omega_2 \). By [50], under Assumptions 2 and 3, we have \(-\Omega_1^{-1} \Omega_2 \) is non-negative and each row of \(-\Omega_1^{-1} \Omega_2 \) has a sum equal to 1. Because \( \Omega_2 = -\Omega_1^{-1} \Omega_1 \) and \( \Omega_2 = -\Omega_1^{-1} \Omega_1 \), \(-\Omega_2^{-1} \Omega_1 \) is non-negative and each row of \(-\Omega_2^{-1} \Omega_2 \) has a sum equal to 1. Thus, \( q_i, i \in \mathcal{F} \), belongs to the convex hull spanned by the leader systems.

**Remark 4:** \( \Omega \otimes I_n \)q = 0 is equivalent to \( \sum_{j \in \mathcal{N}_i} w_{ij}(q_i - q_j) = 0 \) for \( \forall i \in \mathcal{F} \). The fact that \( w_{ij} > 0 \) \((w_{ij} < 0)\) represents an attracting force (repelling force) in edge \((j, i)\), and \( w_{ij}(q_i - q_j) \) stands for the force applied on agent \( i \) by agent \( j \). The mechanical explanation of the result that \( q_i \) belongs to the convex hull formed by its neighbors is under such circumstances all attracting forces (repelling forces) applied to agent \( i \) are balanced. Without loss of generality, we use \( w_{ij} \) to represent \( |w_{ij}| \) in the following section. Since \( \Omega_2 \) is nonsingular, we can obtain \( \Omega_1^{-1} \Omega_2 \)q can be positive or negative, but that the solution is nonunique if \( N_f \geq 2 \). Algorithm 1 provides a method to find appropriate solutions.

**Algorithm 1 Calculation of Equilibrium Stress**

1. Given \( \{q_i\}_{i=1}^{N_l+N_f} \)
2. for \( i = N_l + 1, \ldots, N_l + N_f \)
3. find parameters \( \{m_{i,k}\} \) such that
   \[
   q_i = m_{i,1}q_1 + \cdots + m_{i,N_f}q_{N_f} + m_{i,N_l+N_f}q_{N_l+N_f}
   \]
   4. update \( \{m_{i,k}\} \)
5. end for
6. for \( j = 1, \ldots, N_l + N_f - 1 \)
7. if there exists a set of coefficients \( \{n_{j,k}\} \) such that
   \[
   q_j = n_{j,j+1}q_{j+1} + \cdots + n_{j,N_l+N_f}q_{N_l+N_f}
   \]
   8. save \( \{n_{j,k}\} \)
9. end for
10. for \( i = N_l + 1, \ldots, N_l + N_f \)
11. for \( j = 1, \ldots, N_l + N_f - 1 \)
12. substitute \( q_j \) in step 7 to step 3
13. get the coefficients \( \{l_{i,k}\}_{k=1}^{N_l+N_f-1} \)
14. if every element \( 0 \leq l_{i,k} \leq 1 \) and \( \sum_{k=1}^{N_l+N_f-1} l_{i,k} = 1 \)
15. update \( \{m_{i,k}\} \)
16. end if
17. end for
18. end for
19. if Assumption 2 is not satisfied
20. change the orders of the base vectors for some follower
21. return to step 10

The fact that the followers lie in the convex hull spanned by the leaders guarantees the existence of the solution in steps 2–5, according to the definition of convex hull. However, the neighbors of the followers are all the leaders, and it may cause great communication pressure for the leaders. After steps 6–18, the communication is dispersed among the followers. Since the solution in steps 11–17 depends on the order of the base vectors in \( \{q_i\} \), and Assumption 2 may not be satisfied, we can change the base vector orders to obtain more solutions, as steps 19–21 show. Thus, the algorithm is quite flexible.

Let \( P \) be the unique positive-definite solution of the following algebraic Riccati equation (ARE):

\[
PS + S^TP - 2P^2 + I_m = 0.
\]

Since \((S, I_m)\) is controllable, \( P \) always exists. Because the system matrix \( S \) is unknown to the followers, we design a distributed observer of \( P \) for the followers.

Consider the following dynamic compensator, for \( i \in \mathcal{F} \):

\[
\dot{\gamma}_i(t) = -\mu \left( \sum_{k \in \mathcal{F}} w_{ik} (\gamma_l(t) - \gamma_k(t)) + \sum_{j \in \mathcal{L}} w_{ij} (\gamma_j(t) - \gamma) \right)
\]

\[
\dot{X}_i(t) = -\mu \left( \sum_{k \in \mathcal{F}} w_{ik} (X_l(t) - X_k(t)) + \sum_{j \in \mathcal{L}} w_{ij} (X_j(t) - X) \right)
\]

\[
\dot{s}_i(t) = S_i(t) \eta_i(t) - (a_i(t) + \xi_i^T(t) P_i(t) \xi_i(t)) P_i(t) \xi_i(t)
\]

\[
- (\gamma_i(t) + \rho) R_i(t) f_i(t)
\]
where $X_i$ and $X$ in (5) are auxiliary variables representing $S_i$ and $S; R_i$ and $R; C_i$ and $C; P_i$ and $P$; respectively, and where $\mu$ and $\varphi$ are constants with $\mu > 0, 1/2 < \varphi < 1$ and $\varphi \in R^{+};$

\[
\zeta(t) = \sum_{k \in \mathcal{N}_i} \omega_{k}(\eta_i(t) - \eta_i(t_0)), \quad \text{if } k = j \in \mathcal{L}, \quad \zeta_i(t) = \gamma, \quad X_i(t) = X, \quad \eta_i(t) = v_i(t)
\]

and $\kappa_i$ is a sufficiently small positive constant.

\[
\hat{\alpha}_i(t) = -\alpha_i\alpha_i(t) + 2\zeta_i^T(t)P_i\zeta_i(t)
\]

the compact form of (10) can then be expressed as

\[
\dot{\zeta}_i(t) = (I_{N_i} \otimes S)\eta_i(t) - \Psi_i(t)(I_{N_i} \otimes P)\zeta_i(t) - \phi(I_{N_i} \otimes R)f_i(t) + \tilde{w}(t).
\]

We note that the compact form of $\zeta_i(t)$ is $\zeta(t) = \tilde{\Omega}_i \eta_i(t) + \tilde{\Omega}_i \eta_i(t)$, which stands for convex hull estimation error. Since

\[
\dot{\zeta}_i(t) = (I_{N_i} \otimes S)v_i(t) + (I_{N_i} \otimes R)\tilde{r}(t), \quad \text{where } \tilde{r}(t) = [P_i^T(t), \ldots, P_i^T(t)]^T,
\]

it follows that:

\[
\dot{\zeta}(t) = (I_{N_i} \otimes S - \Omega_i \Psi(t) \otimes P)\zeta(t) - \phi(\Omega_i \otimes R)f_i(t) + (\Omega_i \otimes R)\tilde{r}(t) + (\Omega_i \otimes I_m)\tilde{w}(t).
\]

To show $\zeta(t)$ is ultimately bounded as $t$ tends to infinity, we first construct the Lyapunov functional candidate

\[
V(x) = \frac{1}{2} \sum_{i = N_i+1}^{N_i+N_f} (\psi_i(t) + \alpha_i(t))\zeta_i(t) + \frac{1}{2} \sum_{i = N_i+1}^{N_i+N_f} \alpha_i^2(t)
\]

where $\tilde{\alpha}_i(t) = \alpha_i(t) - \nu$, and $\nu$ is a positive constant to be determined. It is obvious that $V(x)$ is positive definite. If $V(x)$ is ultimately bounded, so is $\zeta(t)$. Next, we will show that $V(x)$ is ultimately bounded by the following steps.

Taking the derivative of $V(x)$ gives

\[
\dot{V}_i(t) = \frac{1}{2} \sum_{i = N_i+1}^{N_i+N_f} [(\dot{\psi}_i(t) + \tilde{\alpha}_i(t))\zeta_i(t)]^T(t)P\zeta_i(t)
\]

\[
+ (\psi_i(t) + \alpha_i(t))\zeta_i(t)
\]

\[
= \sum_{i = N_i+1}^{N_i+N_f} (\tilde{\alpha}_i(t)\zeta_i(t)P\zeta_i(t)
\]

\[
+ (\psi_i(t) + \alpha_i(t))\zeta_i(t)
\]

\[
= \sum_{i = N_i+1}^{N_i+N_f} \alpha_i(t)\zeta_i(t)P\zeta_i(t)
\]

\[
+ \sum_{i = N_i+1}^{N_i+N_f} \tilde{\alpha}_i(t)\zeta_i(t)
\]

\[
\text{where } \tilde{\alpha}_i(t) = \alpha_i(t) - \nu, \quad \text{and } \nu \text{ is a positive constant to be determined.}
\]

It can be shown that $\tilde{w}_i(t) = 0$ when $t \geq T$.

Denoting

\[
\Psi_i(t) = \text{diag} \left[ \psi_{N_i+1}(t), \ldots, \psi_{N_i+N_f}(t) \right]
\]

\[
\tilde{\zeta}(t) = \left[ \tilde{\zeta}_{N_i+1}(t), \ldots, \tilde{\zeta}_{N_i+N_f}(t) \right]^T
\]

\[
\tilde{w}(t) = \left[ \tilde{w}_{N_i+1}(t), \ldots, \tilde{w}_{N_i+N_f}(t) \right]^T
\]

\[
f(t) = \left[ f_{N_i+1}(t), \ldots, f_{N_i+N_f}(t) \right]^T
\]

\[
\text{the compact form of (10) can then be expressed as}
\]

\[
\dot{\zeta}_i(t) = (I_{N_i} \otimes S)\eta_i(t) - \Psi_i(t)(I_{N_i} \otimes P)\zeta_i(t) - \phi(I_{N_i} \otimes R)f_i(t) + \tilde{w}(t).
\]

We note that the compact form of $\zeta_i(t)$ is $\zeta(t) = \tilde{\Omega}_i \eta_i(t) + \tilde{\Omega}_i \eta_i(t)$, which stands for convex hull estimation error. Since

\[
\dot{\zeta}_i(t) = (I_{N_i} \otimes S)v_i(t) + (I_{N_i} \otimes R)\tilde{r}(t), \quad \text{where } \tilde{r}(t) = [P_i^T(t), \ldots, P_i^T(t)]^T,
\]

it follows that:

\[
\dot{\zeta}(t) = (I_{N_i} \otimes S - \Omega_i \Psi(t) \otimes P)\zeta(t) - \phi(\Omega_i \otimes R)f_i(t) + (\Omega_i \otimes R)\tilde{r}(t) + (\Omega_i \otimes I_m)\tilde{w}(t).
\]

To show $\zeta(t)$ is ultimately bounded as $t$ tends to infinity, we first construct the Lyapunov functional candidate

\[
V(x) = \frac{1}{2} \sum_{i = N_i+1}^{N_i+N_f} (\psi_i(t) + \alpha_i(t))\zeta_i(t) + \frac{1}{2} \sum_{i = N_i+1}^{N_i+N_f} \alpha_i^2(t)
\]

where $\tilde{\alpha}_i(t) = \alpha_i(t) - \nu$, and $\nu$ is a positive constant to be determined. It is obvious that $V(x)$ is positive definite. If $V(x)$ is ultimately bounded, so is $\zeta(t)$. Next, we will show that $V(x)$ is ultimately bounded by the following steps.

Taking the derivative of $V(x)$ gives

\[
\dot{V}_i(t) = \frac{1}{2} \sum_{i = N_i+1}^{N_i+N_f} [(\dot{\psi}_i(t) + \tilde{\alpha}_i(t))\zeta_i(t)]^T(t)P\zeta_i(t)
\]

\[
+ (\psi_i(t) + \alpha_i(t))\zeta_i(t)
\]

\[
= \sum_{i = N_i+1}^{N_i+N_f} (\tilde{\alpha}_i(t)\zeta_i(t)P\zeta_i(t)
\]

\[
+ (\psi_i(t) + \alpha_i(t))\zeta_i(t)
\]

\[
= \sum_{i = N_i+1}^{N_i+N_f} \alpha_i(t)\zeta_i(t)P\zeta_i(t)
\]

\[
+ \sum_{i = N_i+1}^{N_i+N_f} \tilde{\alpha}_i(t)\zeta_i(t)
\]

\[
\text{where } \tilde{\alpha}_i(t) = \alpha_i(t) - \nu, \quad \text{and } \nu \text{ is a positive constant to be determined.}
\]

It can be shown that $\tilde{w}_i(t) = 0$ when $t \geq T$.

Denoting

\[
\Psi_i(t) = \text{diag} \left[ \psi_{N_i+1}(t), \ldots, \psi_{N_i+N_f}(t) \right]
\]

\[
\tilde{\zeta}(t) = \left[ \tilde{\zeta}_{N_i+1}(t), \ldots, \tilde{\zeta}_{N_i+N_f}(t) \right]^T
\]

\[
\tilde{w}(t) = \left[ \tilde{w}_{N_i+1}(t), \ldots, \tilde{w}_{N_i+N_f}(t) \right]^T
\]

\[
f(t) = \left[ f_{N_i+1}(t), \ldots, f_{N_i+N_f}(t) \right]^T
\]

\[
\text{the compact form of (10) can then be expressed as}
\]

\[
\dot{\zeta}_i(t) = (I_{N_i} \otimes S)\eta_i(t) - \Psi_i(t)(I_{N_i} \otimes P)\zeta_i(t) - \phi(I_{N_i} \otimes R)f_i(t) + \tilde{w}(t).
\]
We let

\[
\chi_1(t) = 2 \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t) \sigma_i(t) P \tilde{\sigma}_i(t) + \sum_{i=N_l+1}^{N_l+N_f} [\psi_i(t) - \nu] \tilde{\sigma}_i(t).
\]

Substituting (13) into \( \chi_1(t) \) gives

\[
\chi_1(t) = 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes P] \tilde{\sigma}(t) + 2 \sigma^T(t) [\Psi(t) \otimes PS] \tilde{\sigma}(t)
- 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{\sigma}(t)
- 2 \phi \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] f(t)
+ 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t)
\]

\[
\leq \sigma^T(t) [\Psi(t) \otimes (PS + S^T P)] \tilde{\sigma}(t)
+ 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t)
- \lambda_{\min}(\Omega_{\gamma}) \sigma^T(t) [\Psi^2(t) \otimes 2P^2] \tilde{\sigma}(t)
- 2 \phi \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] f(t)
+ 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t).
\]

(16)

By the definition of \( f_i(t) \) in (9), we have

\[
\sigma_i^T(t) PR f_i(t) = \frac{\|R^TP \sigma_i(t)\|^2}{\|R^TP \sigma_i(t)\|^2 + \kappa_i}
\]

and \( \sigma_i^T(t) PR f_i(t) \leq \|R^TP \sigma_i(t)\|, \) for \( k \in F, \ k \neq i. \) Then, it follows that:

\[
-2 \phi \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] f(t) = 2 \phi \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t) \sigma_i(t) T PR
\]

\[
\times \left[ \sum_{k=N_l+1}^{N_l+N_f} w_{ik} [f_k(t) - f_i(t)] - \sum_{j=1}^{N_l} w_{ij} f_j(t) \right]
\]

\[
\leq 2 \phi \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t)
\]

\[
\times \left[ \sum_{k=N_l+1}^{N_l+N_f} w_{ik} \left( \|R^TP \sigma_i(t)\| - \frac{\|R^TP \sigma_i(t)\|^2}{\|R^TP \sigma_i(t)\|^2 + \kappa_i} \right) \right]
\]

\[
- \sum_{j=1}^{N_l} w_{ij} \left( \|R^TP \sigma_i(t)\| - \frac{\|R^TP \sigma_i(t)\|^2}{\|R^TP \sigma_i(t)\|^2 + \kappa_i} \right) \right]
\]

\[
\leq 2 \phi \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t)
\]

\[
\times \left( \sum_{k=N_l+1}^{N_l+N_f} \kappa_i w_{ik} - \sum_{j=1}^{N_l} w_{ij} \frac{\|R^TP \sigma_i(t)\|^2}{\|R^TP \sigma_i(t)\|^2 + \kappa_i} \right).
\]

(17)

From Assumption 1, we have

\[
2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t)
= - \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t) \sigma_i(t) T PR \sum_{j=1}^{N_l} \kappa_i w_{ij} f_j(t)
\]

\[
\leq 2 \gamma \psi_i(t) \sum_{j=1}^{N_l} w_{ij} \|R^TP \sigma_i(t)\|
\]

\[
\leq 2 \phi \psi_i(t) \sum_{j=1}^{N_l} w_{ij} \|R^TP \sigma_i(t)\|. \] (18)

The last inequality is held because \( \phi = \gamma + \rho > \gamma. \) Noting that

\[
-2 \phi \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] f(t) + 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t)
\]

\[
\leq 2 \phi \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t) \left( \sum_{k=N_l+1}^{N_l+N_f} \kappa_i w_{ik} + \sum_{j=1}^{N_l} \kappa_i w_{ij} \right)
\]

\[
= 2 \phi \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t) \sum_{k=1}^{N_l+N_f} \kappa_i w_{ik} + 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t). \] (19)

and substituting (17) and (18) into (16) gives

\[
\chi_1(t) \leq \sigma^T(t) [\Psi(t) \otimes (PS + S^T P)] \tilde{\sigma}(t)
- \lambda_{\min}(\Omega_{\gamma}) \sigma^T(t) [\Psi^2(t) \otimes 2P^2] \tilde{\sigma}(t)
- 2 \phi \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] f(t)
+ 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t)
\times \sum_{k=1}^{N_l+N_f} \kappa_i w_{ik} + 2 \sigma^T(t) [\Psi(t) \Omega_{\gamma} \otimes PR] \tilde{r}(t). \] (20)

By the definition of \( \alpha_i(t) \) in (8), \( \chi_2(t) \) can be written as

\[
\chi_2(t) = \sum_{i=N_l+1}^{N_l+N_f} [\psi_i(t) - \nu] [\Psi_i(t) - \nu] \sigma_i(t) + 2 \sigma_i^T(t) P \tilde{\sigma}_i(t)
\]

\[
= \sigma^T(t) \left( \left[\Psi(t) - \nu I_{N_l} \right] \otimes 2P^2 \right) \tilde{\sigma}(t)
- \sum_{i=N_l+1}^{N_l+N_f} \sigma_i [\alpha_i(t) - \nu] \psi_i(t)
\]

\[
\leq \sigma^T(t) \left( \left[\Psi(t) - \nu I_{N_l} \right] \otimes 2P^2 \right) \tilde{\sigma}(t)
- \frac{1}{4} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i [\alpha_i^2(t) + \tilde{\alpha}_i^2(t)] + \frac{1}{2} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i \nu^2. \] (21)

Here, we note that the last inequality holds because

\[
\psi_i(t) \geq \alpha_i(t) \geq 0
- \frac{1}{2} [\alpha_i(t) - \nu] \psi_i(t) \leq - \frac{1}{4} \alpha_i^2(t) + \frac{1}{4} \nu^2
\]

and

\[
- \frac{1}{2} \tilde{\alpha}_i(t) \left[\tilde{\alpha}_i(t) + \nu\right] \leq - \frac{1}{4} \tilde{\alpha}_i^2(t) + \frac{1}{4} \nu^2.
\]
Next, substituting (20) and (21) into (15) gives
\[
\dot{V}_c(t) \leq -\sigma^T(t)(\Psi(t) \otimes I_m)\zeta(t) \\
+ 2\phi \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t) \sum_{k=1}^{N_l+N_f} \kappa_i w_{ik} \\
+ 2\sigma^T(t)[\Psi(t)\Omega_{\text{eff}} \otimes P]\tilde{\omega}(t) \\
- \frac{1}{4} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i [\alpha_i^2(t) + \tilde{\alpha}_i^2(t)] + \frac{1}{2} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i v_i^2.
\]

Let
\[
u_i = \frac{N_l + N_f}{2\phi} \sum_{k=1}^{N_l+N_f} \kappa_i w_{ik}.
\]

It can be verified that
\[
2\phi \sum_{i=N_l+1}^{N_l+N_f} \psi_i(t) \sum_{k=1}^{N_l+N_f} \kappa_i w_{ik} \\
= \sum_{i=N_l+1}^{N_l+N_f} \alpha_i(t) \nu_i + \sum_{i=N_l+1}^{N_l+N_f} \frac{1}{2} \nu_i^2 \\
\leq \left( \sum_{i=N_l+1}^{N_l+N_f} \frac{1}{4} \sigma_i \alpha_i^2(t) + \sum_{i=N_l+1}^{N_l+N_f} \frac{1}{2} \nu_i^2 \right) \\
+ \left( \frac{1}{2\lambda_{\text{max}}(P)} \sum_{i=N_l+1}^{N_l+N_f} \left[ \sigma_i \alpha_i^2(t) \right] \right) \\
+ \frac{1}{2} \sum_{i=N_l+1}^{N_l+N_f} \nu_i^2 \\
\leq \frac{1}{4} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i \alpha_i^2(t) + \frac{1}{2\lambda_{\text{max}}(P)} \sum_{i=N_l+1}^{N_l+N_f} \left[ \sigma_i \alpha_i^2(t) \right] \\
+ \frac{1}{2} \sum_{i=N_l+1}^{N_l+N_f} \nu_i^2.
\]

Noting that
\[
\frac{1}{2\lambda_{\text{max}}(P)} \sum_{i=N_l+1}^{N_l+N_f} \left[ \sigma_i \alpha_i^2(t) \right] \leq \frac{1}{2} \sigma^T(t)(\Psi(t) \otimes I_m)\zeta(t)
\]
it follows from (22) that:
\[
\dot{V}_c(t) \leq -\frac{1}{2} \sigma^T(t)(\Psi(t) \otimes I_m)\zeta(t) - \frac{1}{4} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i \tilde{\alpha}_i^2(t) \\
+ 2\sigma^T(t)[\Psi(t)\Omega_{\text{eff}} \otimes P]\tilde{\omega}(t) + \tau
\]
where
\[
\tau = \sum_{i=N_l+1}^{N_l+N_f} \left[ \frac{1}{2} \nu_i^2 + \frac{1}{2} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i v_i^2 \right].
\]

Because \(\psi_i(t) \geq \alpha_i(t) \geq 0\), we obtain from (14) that
\[
V_c(t) \leq \sigma^T(t)[\Psi(t) \otimes P]\zeta(t) + \frac{1}{2} \sum_{i=N_l+1}^{N_l+N_f} \tilde{\alpha}_i^2(t) \\
\leq \lambda_{\text{max}}(P) \sigma^T(t)[\Psi(t) \otimes I_m] \zeta(t) + \frac{1}{2} \sum_{i=N_l+1}^{N_l+N_f} \tilde{\alpha}_i^2(t).
\]

Then, it follows from (23) that:
\[
\dot{V}_c(t) \leq -\frac{1}{2} \sigma^T(t)(\Psi(t) \otimes I_m)\zeta(t) - \frac{1}{4} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i \tilde{\alpha}_i^2(t) \\
+ 2\sigma^T(t)[\Psi(t)\Omega_{\text{eff}} \otimes P]\tilde{\omega}(t) + \tau + \beta \sum_{i=N_l+1}^{N_l+N_f} \tilde{\alpha}_i^2(t) \\
+ \beta \left[ -V_c + \lambda_{\text{max}}(P) \sigma^T(t)(\Psi(t) \otimes I_m)\zeta(t) \right] \\
\leq -\beta V_c - \left( \frac{1}{2} - \beta \lambda_{\text{max}}(P) \right) \sigma^T(t)(\Psi(t) \otimes I_m)\zeta(t) + \tau \\
- \frac{1}{4} \sum_{i=N_l+1}^{N_l+N_f} \sigma_i (\tilde{\alpha}_i - \tilde{\alpha}_i^2(t)) \\
+ 2\sigma^T(t)[\Psi(t)\Omega_{\text{eff}} \otimes P]\tilde{\omega}(t)
\]
where \(\beta\) is a positive constant. Choosing \(\beta\) that satisfies
\[
\beta < \min \left\{ \frac{1}{2\lambda_{\text{max}}(P)} \sigma_i \right\}
\]
for all \(i \in F\), we have
\[
\dot{V}_c(t) \leq -\beta V_c(t) + \tau + 2\sigma^T(t)[\Psi(t)\Omega_{\text{eff}} \otimes P]\tilde{\omega}(t).
\]

Note that \(\tilde{\omega}(t)\) is bounded in \([0, T_e]\), which implies \(V_c(t)\) is bounded in \([0, T_e]\), so are \(\zeta(t)\) and \(\tilde{\alpha}_i(t)\). Because \(\tilde{\omega}(t) = 0\) when \(t \geq T_e\), then according to the comparison lemma, i.e., [52, Lemma 3.4], it can be verified that
\[
V_c(t) \leq \left[ V_c(T_e) - \frac{\tau}{\beta} \right] e^{-\beta(t-T_e)} + \frac{1}{\beta} \beta \lambda_{\text{max}}(P) \sigma^T(t)(\Psi(t) \otimes I_m)\zeta(t)
\]
As such, \(V_c(t)\) is ultimately bounded with the upper bound \(\frac{\tau}{\beta}\), which means \(\zeta(t)\) and \(\alpha_i(t)\) are both ultimately bounded, according to the definition of \(V_c(t)\) in (14). Since \(\zeta(t) = \Omega_{\text{eff}} \eta(t) + \Omega_{\text{eff}} \nu(t)\), there exists a positive constant \(\varepsilon_3\) such that
\[
\lim_{t \to \infty} \| \eta(t) + \left( \frac{\Omega_{\text{eff}}^{-1} \Omega_{\text{eff}}}{\epsilon_3} \right) \nu(t) \| \leq \varepsilon_3.
\]

This completes the proof.

Remark 5: From (24), we see that the convex hull estimation error converges to a small bounded set if \(\kappa_i\) and \(\sigma_i\) are selected to be sufficiently small. Note that \(\tilde{\omega}(t)\) is distributed and it is applied to the adaptive distributed observer in (6) to avoid using the global information of the directed digraph. In addition, the dynamic compensator composed by (4)–(6) is all designed in a distributed way with no global information being used. It is thus easy to be implemented. Note that the consensus approach in [45] is applicable to undirected digraph. The stress matrix of directed digraph is asymmetric, and the coupling gain in [45] cannot be directly applied to solve our problem.
Remark 6: The FC control is actually a special class of the affine formation control. Compared with the work in [31], there are advantages for using the adaptive distributed observer of (6). First, the assumptions that \( \{q_i\}_{i=1}^{N_l} \) affinely span \( \mathbb{R}^n \) and the formation \((\tilde{G}, q)\) is universally rigid are not required. These assumptions are needed in [31] to ensure that \( \Omega_{gf} \) is nonsingular. By Lemma 1, the condition that \( \Omega_{gf} \) is nonsingular is satisfied under Assumptions 2 and 3. By Lemma 1 in [31], \( N_l \geq n+1 \) is necessary to ensure that \( \{q_i\}_{i=1}^{N_l} \) affinely span \( \mathbb{R}^n \). In our problem, however, the number of leader systems \( N_l \) can be less than \( n+1 \). In addition, our digraph is directed while the digraph in [31] is undirected. Second, the adaptive distributed observer in (6) takes the control inputs of the leaders into consideration, while the works in [31] does not.

Remark 7: By Lemma 2, since the state estimation error \( \xi(t) = \Omega_{gf}^{1/2}(\hat{\eta}(t) + \tilde{\Omega}_{gf}^{1/2}\Omega_{gf}^{1/2}v(t)) \) is ultimately bounded, then

\[
C_i(t)\left[\eta_i(t) + \left(\Omega_{gf}^{1/2}\Omega_{gf}^{1/2}\right)_{rowi} \otimes I_m \right]v(t) = C_i(t)\eta_i(t) + \left(\Omega_{gf}^{1/2}\Omega_{gf}^{1/2}\right)_{rowi} \otimes I_m q_i(t) + C_i(t)\eta_i(t)
\]

in which \( (\Omega_{gf}^{1/2}\Omega_{gf}^{1/2})_{rowi} \) stands for the \( i \)-th row of \( \Omega_{gf}^{1/2}\Omega_{gf}^{1/2} \) and \(-[(\Omega_{gf}^{1/2}\Omega_{gf}^{1/2})_{rowi} \otimes I_m]q_i(t)\) belongs to \( \text{Co}(Q) \), is also ultimately bounded since \( C_i(t) = 0 \) when \( t \geq T_e \).

IV. FORMATION-CONTAINMENT CONTROL OF MULTIPLE EUCLIDEAN-LAGRANGE SYSTEMS

In this section, we design a distributed control law for the multiple EL systems in (1) to solve Problem 1 by utilizing the adaptive distributed observer given in (4)–(6).

Let

\[
\dot{q}_i(t) = C_i(t)\dot{\eta}_i(t) - \alpha [q_i(t) - C_i(t)\eta_i(t)] \tag{25}
\]

\[
s_i(t) = \dot{q}_i(t) - \tilde{q}_i(t) \tag{26}
\]

where \( \alpha \) is a positive constant. Then

\[
\tilde{q}_i(t) = \tilde{C}_i(t)\dot{\eta}_i(t) + C_i(t)\dot{\eta}_i(t) - \alpha [\dot{q}_i(t) - \tilde{C}_i(t)\eta_i(t) - C_i(t)\dot{\eta}_i(t)] \tag{27}
\]

where \( \dot{\eta}_i(t) \) can be easily obtained from (6). Considering the continuous function \( \tilde{f}_i(t) \), the continuity of \( \tilde{R}_i(t), \tilde{S}_i(t), \tilde{P}_i(t), \) and \( \tilde{\zeta}_i(t) \) ensures the existence of \( \tilde{f}_i(t) \).

By Property 2, there exists a known matrix \( Y_i(t) := Y_i(q_i, q_i, \dot{q}_i, \ddot{q}_i) \)

and an unknown constant vector \( \Theta_j \) such that

\[
M_i(q_i)\ddot{q}_i(t) + C_i(q_i, \dot{q}_i)\dot{q}_i(t) + C_i(t)\eta_i(t) = Y_i(q_i, \dot{q}_i, \ddot{q}_i, \dot{\eta}_i(t))\Theta_j. \tag{28}
\]

Next, we define the following control law:

\[
\tau_i(t) = -K_i s_i(t) + Y_i(t)\tilde{\Theta}_i(t), \quad i \in \mathcal{F} \tag{29}
\]

\[
\tilde{\Theta}_i(t) = -\Lambda_i^{-1}Y_i^T(t)s_i(t) \tag{30}
\]

where \( K_i \in \mathbb{R}^{n \times n} \) and \( \Lambda_i \in \mathbb{R}^{n \times n} \) are two symmetric and positive-definite matrices, \( \tilde{\Theta}_i(t) \) is the estimation of \( \Theta_i \).

We have the following theorem.

Theorem 1: Given the followers in (1) and the leaders in (2), and a digraph \( \tilde{G} \), with Assumptions 1–5, then Problem 1 can be solved by the control law composed of (29), (30).

Proof: Substituting (29) into (1) gives

\[
M_i(q_i)\ddot{q}_i(t) + C_i(q_i, \dot{q}_i)\dot{q}_i(t) + G_i(q_i) = -K_i s_i(t) + Y_i(t)\tilde{\Theta}_i(t). \tag{31}
\]

Let \( \tilde{\Theta}_i(t) = \tilde{\Theta}_i(t) - \Theta_i \). Subtracting \( Y_i(t)\Theta_i(t) \) on both sides of (31) gives

\[
M_i(q_i)s_i(t) + C_i(q_i, \dot{q}_i)s_i(t) = -K_i s_i(t) + Y_i(t)\tilde{\Theta}_i(t),
\]

\[
\tilde{\Theta}_i(t) = -\Lambda_i^{-1}Y_i^T(t)s_i(t). \tag{32}
\]

Next, define a Lyapunov function candidate for the \( i \)-th follower as

\[
V_i(t) = \frac{1}{2} [s_i^T(t)M_i(q_i)s_i(t) + \tilde{\Theta}_i^T(t)\Lambda_i\tilde{\Theta}_i(t)].
\]

By Property 1 of the EL systems

\[
\dot{V}_i(t) = s_i^T(t)M_i(q_i)s_i(t) + \frac{1}{2} s_i^T(t)\dot{M}_i(q_i)s_i(t) + \tilde{\Theta}_i^T(t)\Lambda_i\dot{\tilde{\Theta}}_i(t)
\]

\[
= -s_i^T(t)K_i s_i(t) \leq 0. \tag{33}
\]

We note that (32) implies both \( s_i(t) \) and \( \tilde{\Theta}_i(t) \) are bounded. By Barbalat’s Lemma [46], if \( \dot{V}_i(t) \) is uniformly continuous for all \( t \geq 0 \), then \( \lim_{t \to \infty} s_i(t) = 0 \). To this end, we need to further show that \( \dot{V}_i(t) \) is bounded, that is, to show \( \dot{s}_i(t) \) is bounded. From (26), it holds that

\[
\dot{s}_i(t) = \tilde{q}_i(t) - \tilde{q}_i(t), \tag{34}
\]

where \( \tilde{q}_i(t) \) is given in (27). \( \tilde{q}_i(t) \) is shown to be bounded by the following steps.

1) Substituting (25) into (26) gives

\[
[\dot{q}_i(t) - C_i(t)\dot{\eta}_i(t)] + \alpha [q_i(t) - C_i(t)\eta_i(t)] = s_i(t). \tag{33}
\]

Since \( \alpha > 0 \), (33) is a stable first-order linear system with \( q_i(t) - C_i(t)\eta_i(t) \) being its state variable and \( s_i(t) \) being its input. Since \( s_i(t) \) is already shown to be bounded, both \( q_i(t) - C_i(t)\eta_i(t) \) and \( \dot{q}_i(t) - C_i(t)\dot{\eta}_i(t) \) are thus bounded. By Lemma 2 and Assumption 4, \( \eta_i(t) \) is bounded. According to the definition of \( \dot{\eta}_i(t) \) in (6), \( \dot{\eta}_i(t) \) is also bounded. Thus, both \( q_i(t) \) and \( \dot{q}_i(t) \) are bounded.

2) By (5), \( C_i(t) \) and \( \tilde{C}_i(t) \) are bounded since \( C_i(t) \to C \) in finite time.

3) Considering (6), because \( \eta_i(t) \) and \( \dot{\eta}_i(t) \) are bounded, \( \zeta_i(t) \) and \( \tilde{\zeta}_i(t) \) are bounded. Thus, \( \tilde{f}_i(t) \) is bounded, which implies that \( \dot{\eta}_i(t) \) is bounded.

Thus, by Barbalat’s lemma, it follows that for \( i \in \mathcal{F} \), \( \lim_{t \to \infty} \dot{V}_i(t) = 0 \), which implies \( \lim_{t \to \infty} s_i(t) = 0 \). By (33), we have

\[
\lim_{t \to \infty} (q_i(t) - C_i(t)\eta_i(t)) = 0
\]

and

\[
\lim_{t \to \infty} \dot{q}_i(t) - C_i(t)\dot{\eta}_i(t) = 0.
\]
Fig. 1. (a) Trajectories. (b) Formation error of agents. (c) Estimation error of $\gamma$ for the followers 4 and 5 (the blue circles). (d) Formation error between 139.6 s ≤ $t$ ≤ 139.9 s.

Furthermore, by Remark 7, since

$$C_i(t)\eta_i(t) + \begin{bmatrix} (\Omega_{\text{eff}}^{-1}\Omega_{\beta})_{\text{row}} \otimes I_n \end{bmatrix} q_i(t)$$

is ultimately bounded, if for $i \in F$, we choose

$$h_i(t) = -\begin{bmatrix} (\Omega_{\text{eff}}^{-1}\Omega_{\beta})_{\text{row}} \otimes I_n \end{bmatrix} q_i(t)$$

then $q_i(t) - h_i(t)$ is ultimately bounded. As such, there exist a small constant $\varepsilon_1$ such that

$$\lim_{t \to \infty} \|q_i(t) - h_i(t)\| \leq \varepsilon_1.$$
for some $\varepsilon_1 > 0$ provided that $\kappa_i$ and $\sigma_i$ are sufficiently small. This completes our proof.

Remark 8: As (27) shows, $\tilde{f}_i(t)$ is required, and according to (6), $\tilde{f}_i(t)$ is a function of $\tilde{f}_i(t)$ that exists because $\tilde{f}_i(t)$ is continuous. If $\kappa_i$ is chosen as 0, $\tilde{f}_i(t)$ would be defined as

$$\tilde{f}_i(t) = \begin{cases} \frac{R_i^T P_i(t) \xi_i(t)}{\|R_i^T P_i(t) \xi_i(t)\|}, & \|R_i^T P_i(t) \xi_i(t)\| \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

and it is nondifferentiable, then the control law (29), (30) is no longer valid. Therefore, $\kappa_i$ can not be chosen as 0. Besides, the continuous function $\tilde{f}_i(t)$ can avoid chattering in the control input.

Finally, we note the leader systems, which are described by linear systems, and the followers (EL systems) can be viewed as heterogeneous systems, as Dong et al. in [42] did. In [42], a dynamic gain is also designed to avoid using the global information of the digraph. However, the dynamic gain cannot be applied to solve our problem due to the existence of the control inputs of the leaders. Different methods in [44] and [53] are used to solve the FC control problem of EL systems, and the control law is applied to obstacles-free environments successfully.

V. ILLUSTRATIVE EXAMPLE

Consider a group of two robot manipulators, whose dynamic models are described by the EL systems as follows [54]:

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) = \tau_i, \quad i = 4, 5$$

where $q_i(t)$, $\dot{q}_i(t) \in \mathbb{R}^2$ represent their generalized position and velocity, respectively, and

$$M_i(q_i) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3\cos q_2 \\ \theta_2 + \theta_3\cos q_2 \\ \theta_2 \end{bmatrix}, \quad C_i(q_i, \dot{q}_i) = \begin{bmatrix} -\theta_3(\sin q_2)q_2 \\ -\theta_3(\sin q_2)\dot{q}_1 \\ \theta_3(\sin q_2)\dot{q}_1 \end{bmatrix}, \quad G_i(q_i) = \begin{bmatrix} \theta_4\cos q_1 + \theta_5\cos(q_1 + q_2) \\ \theta_5\cos(q_1 + q_2) \end{bmatrix}.$$ 

The unknown vector is $\Theta_i = \text{col}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, whose actual values are

$$\Theta_4 = \text{col}(0.91, 1.26, 0.22, 1.27, 0.58), \quad \Theta_5 = \text{col}(1.10, 1.36, 0.32, 1.67, 0.74).$$

We noted that the EL systems satisfy Assumption 5. The three leaders are first-order systems

$$\dot{p}_j(t) = v_j(t), \quad j = 1, 2, 3$$

with position $p_j \in \mathbb{R}^2$ as state and velocity $v_j \in \mathbb{R}^2$ as control input. We assume $\|v_j(t)\| \leq 2$, which satisfies Assumption 1.

The graph topology for the system is shown in Fig. 1(a), and it is obvious that Assumptions 2 and 3 are satisfied. The designed parameters in the control law (29), (30) are listed in Table I.

<table>
<thead>
<tr>
<th>$\varphi$</th>
<th>$\mu$</th>
<th>$\rho$</th>
<th>$\alpha$</th>
<th>$K_{4.5}$</th>
<th>$\Lambda_{4.5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>577</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>30I5</td>
<td>0.02I5</td>
</tr>
</tbody>
</table>

For the dynamic compensator in (6), we set $\rho = 2$, and we verify our result by selecting different values of $\kappa_i$ and $\sigma_i$, which have direct relationship with the convex hull estimation error and the formation error.

In Fig. 1(a), $\kappa_{4.5} = 0.001$ and $\sigma_{4.5} = 0.001$ are used for illustration. As illustrated in Fig. 1(a), for the FC control problem, the leaders (represented by the red circles) that are equipped with necessary sensors can communicate with each other to generate safe trajectories to avoid obstacles but it is beyond the scope of our research. Fig. 1(b) shows the formation error for the followers (represented by the blue circles) along the whole process. Fig. 1(d) shows that the accuracy of the formation error converges to $10^{-3}$ when $\kappa_{4.5} = 0.001$, $\sigma_{4.5} = 0.001$, and $i = 4, 5$. It is obvious that the trajectories of all agents can avoid obstacles and the agents thus reach their destination safely under the FC control law. Moreover, there is no variables or parameters using the global information of communication graph, so the distributed observer and control law are fully distributed. Fig. 1(b) and (d) both shows that the formation error is ultimately bounded. The result that $\gamma_i(t) \to \gamma$, $i = 4, 5$ in finite time can be concluded from Fig. 1(c).

Fig. 2 shows the formation errors of the followers with $\kappa_{4.5} = 0.1$ and $\sigma_{4.5} = 0.1$ (left) and $\kappa_{4.5} = 0.01$ and $\sigma_{4.5} = 0.01$ (right) during $139.5 \leq t \leq 140.0$ s are $10^{-1}$ and $10^{-2}$, respectively. Figs. 1(d) and 2 together illustrate that the formation error decreases with decreasing $\kappa_i$ and $\sigma_i$. In practical applications, the formation error is within acceptable limits.
The simulation animation for the example can be viewed at https://youtu.be/U9tBjugI0ec.

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