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Graph-theoretic characterisations of structural controllability for multi-agent system with switching topology†

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This article considers the controllability problem for multi-agent systems. In particular, the structural controllability of multi-agent systems under switching topologies is investigated. The structural controllability of multi-agent systems is a generalisation of the traditional controllability concept for dynamical systems, and purely based on the communication topologies among agents. The main contributions of this article are graph-theoretic characterisations of the structural controllability for multi-agent systems. It turns out that the multi-agent system with switching topology is structurally controllable if and only if the union graph $\mathcal{G}$ of the underlying communication topologies is connected (single leader) or leader–follower connected (multi-leader). Finally, this article concludes with several illustrative examples and discussions of the results and future work.

Keywords: structural controllability; multi-agent system; cooperative control; graphic interpretation

1. Introduction

Due to the latest advances in communication and computation, the distributed control and coordination of the networked dynamic agents has rapidly emerged as a hot multi-disciplinary research area (Lawson and Beard 2002; Dunbar and Murray 2006); Bliman and Ferrari-Trecate 2008), which lies at the intersection of systems control theory, communication and mathematics. In addition, the advances of the research in multi-agent systems are strongly supported by their promising civilian and military applications, such as cooperative control of unmanned air vehicles, autonomous underwater vehicles, space exploration, congestion control in communication networks, air traffic control and so on (Tomlin, Pappas, and Sastry 1998; How, King, and Kuwata 2004). Much work has been done on the formation stabilisation and consensus seeking. Approaches like graph Laplacians for the associated neighbourhood graphs, artificial potential functions, and navigation functions for distributed formation stabilisation with collision avoidance constraints have been developed. Furthermore, inspired by the cooperative behaviour of natural swarms, such as bee flocking, ant colonies and fish schooling, people try to obtain experiences from how the group units make their whole group motions under control just through limited and local interactions among them.

The control of such large-scale complex systems poses several new challenges that fall beyond the traditional methods. Part of the difficulty comes from the fact that the global behaviour of the whole group combined by multiple agents is not a simple summation of the individual agent's behaviour. Actually, the group behaviour can be greatly impacted by the communication protocols or interconnection topology between the agents, which makes the global behaviour display high complexities. Hence, the cooperative control of multi-agent systems is still in its infancy and attracts more and more researchers' attention. One basic question in multi-agent systems that attracts control engineers' interest is what is the necessary information exchanging among agents to make the whole group well-behaved, e.g. controllable. This can be formulated as a controllability problem for multi-agent systems under the leader–follower framework. Roughly speaking, a multi-agent system is controllable if and only if we can drive the whole group of agents to any desirable configurations only based on local interactions between agents and possibly some limited commands to a few agents that serve as leaders. The basic issue is the interplay between control and communication. In particular, we would like to investigate what is the necessary and/or sufficient condition on the graph of communication topologies

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among agents for the controllability of multi-agent systems.

This multi-agent controllability problem was first proposed in Tanner (2004), who formulated it as the controllability of a linear system and proposed a necessary and sufficient algebraic condition based on the eigenvectors of the graph Laplacian. Tanner (2004) focused on fixed topology situation with a particular member which acted as the single leader. The problem was then developed in Ji, Muhammad, and Egerstedt (2006), Ji and Egerstedt (2007), Rahmani and Mesbahi (2006), Rahmani, Ji, Mesbahi, and Egerstedt (2009), Ji, Lin, and Lee (2008), and got some interesting results. For example, in Ji and Egerstedt (2007), it was shown that a necessary and sufficient condition for controllability is not sharing any common eigenvalues between the Laplacian matrix of the follower set and the Laplacian matrix of the whole topology. However, it remains elusive on what exactly the graphical meaning of these algebraic conditions related to the Laplacian matrix. This motivates several research activities on illuminating the controllability of multi-agent systems from a graph theoretical point of view. For example, a notion of anchored systems was introduced in Rahmani and Mesbahi (2006) and it was shown that symmetry with respect to the anchored vertices makes the system uncontrollable. However, so far the research progress using graph theory is quite limited and a satisfactory graphical interpretation of these algebraic controllability conditions turns out to be very challenging. Besides, the weights of communication links among agents have been demonstrated to have a great influence on the behaviour of whole multi-agent group (see, e.g. Moreau 2005). However, in the previous multi-agent controllability literature (Tanner 2004; Ji, Lin, and Lee 2008b), the communication weighting factor is usually ignored. One classical result under this no weighting assumption is that a multi-agent system with complete graphical communication topology is uncontrollable (Tanner 2004). This is counter-intuitive since it means each agent can get direct information from each other but this leads to a bad global behaviour as a team. This shows that too much information exchange may damage the controllability of multi-agent system. In contrast, if we set weights of unnecessary links to be zero and impose appropriate weights to other links so as to use the communication information in a selective way, then it is possible to make the system controllable (Zamani and Lin 2009).

In this article, motivated by the above observation, the weighting factor is taken into account for multi-agent controllability problem. In particular, a new notion for the controllability of multi-agent systems, called structural controllability, which was proposed by us in Zamani and Lin (2009), is investigated directly through the graph–theoretic approach for control systems. The communication topology of whole multi-agent system is described by a weighted graph and the system is called structurally controllable if one may find a set of weights such that the corresponding multi-agent system is controllable in a classical sense. The structural controllability reveals under certain topology whether it is possible to make the whole multi-agent system well-behaved, i.e. controllable here through suitable choice of the communication weights. From another angle, it helps to bring to light the effects of the communication topology on the controllability property of multi-agent systems without worrying about the influence of weighting factors. It turns out that this controllability notation only depends on the topology of the communication scheme in the case of a single leader under a fixed topology (Zamani and Lin 2009).

Another novelty in this article is the successful investigation of impacts of switching topologies on the multi-agent controllability property, for which there is barely graphical interpretation to the best of our knowledge. Note that the results in Tanner (2004), Ji et al. (2006), Ji and Egerstedt (2007), Rahmani and Mesbahi (2006), Rahmani et al. (2009), Ji et al. (2008) and Zamani and Lin (2009) are all focused on multi-agent systems under fixed communication topologies which may restrict their impacts on real applications. In many applications, it may become impossible to keep the communication topology fixed for the whole period. Therefore, it is of practical importance to consider time-varying communication topologies. A natural framework to study the time variance of communication topology is through switched systems (see e.g. Sun, Ge, and Lee 2002; Lin and Antsakis 2007, 2009). In this article, we will focus on multi-agent systems under switching topologies in the framework of switched systems. Some early efforts have been observed in the literature. Necessary and sufficient algebraic conditions for the controllability of multi-agent systems under switching topology were derived in Ji et al. (2008b) and Liu, Chu, Wang, and Xie (2008) based on the developments of controllability study in switched systems. However, these algebraic results lacks graphically intuitive interpretations, which are important since they can provide us significant guidelines for the communication protocol design for multi-agent systems. Therefore, this article aims to fill this gap and propose graphic interpretations of these algebraic conditions for the controllability of multi-agent systems under switching topology. In particular, we follow the setup in Zamani and Lin (2009) and investigate the structural controllability of multi-agent systems with switching communication topologies.

Preliminary results on single-leader case were reported in Liu et al. (2009). Here a more general case: multi-agent system with multi-leader is studied, which is actually an extension of single-leader case. It is assumed that the leaders act as the external or control signal and will not be affected by any other group members. Based on this structural controllability, we propose necessary and sufficient graph theoretic conditions for the structural controllability of multi-agent system with switching topologies. It turns out that the multi-agent system with switching topology is structurally controllable if and only if the union graph $\mathcal{G}$ of the underlying communication topologies is connected (single leader) or leader–follower connected (multi-leader). Some examples are given to underscore our theoretical analysis.

The outline of this article is as follows: In Section 2, we introduce some basic preliminaries and the problem formulation, followed by structural controllability study in Section 3, where a graphic necessary and sufficient condition for the structural controllability under single-leader case is given. In Section 4, graphical interpretation of structural controllability of multi-leader multi-agent system is proposed. In Section 5, some examples are presented to give the readers deeper understanding of our theoretical results. Finally, some concluding remarks are drawn in this article.

2. Preliminaries and problem formulation

2.1 *Graph theory preliminaries*

A weighted graph is an appropriate representation for the communication or sensing links among agents because it can represent both existence and strength of these links among agents. The weighted graph $\mathcal{G}$ with N vertices consists of a vertex set $\mathcal{V}=\{v_1, v_2, \dots, v_N\}$ and an edge set $\mathcal{I}=\{e_1, e_2, \dots, e_{N'}\}$, which is the interconnection links among the vertices. Each edge in the weighted graph represents a bidirectional communication or sensing media. Two vertices are known to be *neighbours* if $(i,j)\in\mathcal{I}$, and the number of neighbours for each vertex is its *valency*. An alternating sequence of distinct vertices and edges in the weighted graph is called a *path*. The weighted graph is said to be *connected* if there exists at least one path between any distinct vertices, and *complete* if all vertices are neighbours to each other.

The *adjacency matrix*, $\mathcal{A}$ is defined as

$$\mathcal{A}_{(i,j)} = \begin{cases} \mathcal{W}_{ij} & (i,j) \in \mathcal{I}, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $\mathcal{W}_{ij} \neq 0$ stands for the weight of edge (j, i). Here, the adjacency matrix $\mathcal{A}$ is $|\mathcal{V}| \times |\mathcal{V}|$ and $|\cdot|$ is the cardinality of a set.

The *Laplacian matrix* of a graph $\mathcal{G}$, denoted as $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ or $\mathcal{L}$ for simplicity, is defined as

$$\mathcal{L}_{(i,j)} = \begin{cases} \Sigma_{j\in\mathcal{N}_i} \mathcal{W}_{ij} & i=j, \\ -\mathcal{W}_{ij} & i\neq j \text{ and } (i,j)\in\mathcal{I}, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

2.2 *Multi-agent structural controllability with switching topology*

Specifically, controllability problem usually cares about how to control N agents based on the leader–follower framework. Take the case of single leader as example. Without loss of generality, assume that the Nth agent serves as the leader and takes commands and controls from outside operators directly, while the rest $N-1$ agents are followers and take controls as the nearest neighbour law.

Mathematically, each agent's dynamics can be seen as a point mass and follows

$$\dot{x}_i = u_i. \tag{3}$$

The control strategy for driving all follower agents is

$$u_i = -\sum_{j\in\mathcal{N}_i} w_{ij}(x_i - x_j) + w_{ii}x_i, \tag{4}$$

where $\mathcal{N}_i$ is the neighbour set of the agent i (could contain i itself), and w_{ij} is weight of the edge from agent j to agent i. On the other hand, the leader's control signal is not influenced by the followers and needs to be designed, which can be represented by

$$\dot{x}_N = u_N. \tag{5}$$

In other words, the leader affects its nearby agents, but it does not get directly affected by the followers since it only accepts the control input from an outside operator. For simplicity, we will use z to stand for x_N in the sequel. It is known that the whole multi-agent system under fixed communication topology can be written as a linear system:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u_N \end{bmatrix}, \tag{6}$$

where $A \in \mathbb{R}^{(N-1)\times(N-1)}$ and $B \in \mathbb{R}^{(N-1)\times 1}$ are both sub-matrices of the corresponding graph Laplacian matrix $-\mathcal{L}$

The communication network of dynamic agents with directed information flow under link failure and creation can usually described by switching topology.

Under m switching topologies, it is clear that the whole system equipped with m subsystems can be written in a compact form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u_N \end{bmatrix}, \tag{7}$$

or, equivalently,

$$\begin{cases} \dot{x} = A_i x + B_i z, \\ \dot{z} = u_N, \end{cases} \tag{8}$$

where $i \in \{1, \ldots, m\}$. $A_i \in \mathbb{R}^{(N-1)\times(N-1)}$ and $B_i \in \mathbb{R}^{(N-1)\times 1}$ are both sub-matrices of the corresponding graph Laplacian matrix $-\mathcal{L}$. The matrix A_i reflects the interconnection among followers, and the column vector B_i represents the relation between followers and the leader under corresponding subsystems. Since the communication topologies among agents are time-varying, so the matrices A_i and B_i are also varying as a function of time. Therefore, the dynamical system described in (7) can be naturally modelled as a switched system (definition can be found latter).

Considering the structural controllability of multi-agent system, system matrices A_i and $\boldsymbol{B}_i$, $i \in \{1, \ldots, m\}$ are structured matrices, which means that their elements are either fixed zeros or free parameters. Fixed zeros imply that there is no communication link between the corresponding agents and the free parameters stand for the weights of the communication links. Our main task here is to find out under what kinds of communication topologies, it is possible to make the group motions under control and steer the agents to the specific geometric positions or formation as a whole group. Now this controllability problem reduces to whether we can find a set of weights w_{ij} such that the multi-agent system (7) is controllable. Then the controllability problem of multi-agent system can now be formulated as the structural controllability problem of switched linear system (7):

Definition 2.1: The multi-agent system (7) with switching topology, whose matrix elements are zeros or free parameters, is said to be structurally controllable if and only if there exist a set of communication weights w_{ij} that can make the system (7) controllable in the classical sense.

2.3 *Switched linear system and controllability matrix*

In general, a switched linear system is composed of a family of subsystems and a rule that governs the switching among them, and is mathematically described by

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t), \tag{9}$$

where $x(t) \in \mathbb{R}^n$ are the states, $u(t) \in \mathbb{R}^r$ are piecewise continuous input and σ: $[t_0, \infty) \rightarrow M \triangleq \{1, \ldots, m\}$ is the switching signal. System (9) contains m subsystems (A_i, B_i), $i \in \{1, \ldots, m\}$ and $\sigma(t) = i$ implies that the ith subsystem (A_i, B_i) is activated at time instance t.

For the controllability problem of switched linear systems, a well-known matrix rank condition was given in Sun and Ge (2004):

Lemma 2.2: (Sun and Ge 2004) *If matrix*:

$$\begin{aligned} &[B_1, \ldots, B_m, A_1 B_1, \ldots, A_m B_1, \ldots, A_m B_m, A_1^2 B_1, \ldots, \\ &\quad A_m A_1 B_1, \ldots, A_1^2 B_m, \ldots, A_m A_1 B_m, \ldots, A_1^{n-1} B_1, \ldots, \\ &\quad A_m A_1^{n-2} B_1, \ldots, A_1 A_m^{n-2} B_m, \ldots, A_m^{n-1} B_m] \end{aligned} \tag{10}$$

has full row rank n, then switched linear system (9) *is controllable, and vice versa.*

This matrix is called the controllability matrix of the corresponding switched linear system (9).

3. Structural controllability of multi-agent system with single leader

The multi-agent system with a single leader under switching topology has been modelled as switched linear system (7). Before proceeding to the structural controllability study, we first discuss the controllability of multi-agent system (7) when all the communication weights are fixed.

After simple calculation, the controllability matrix of switched linear system (7) can be shown to have the following form:

$$\begin{bmatrix} 0, \ldots, 0, B_1, \ldots, B_m, A_1 B_1 \ldots, A_1 A_m^{N-3} B_m, \ldots, A_m^{N-2} B_m \\ 1, \ldots, 1, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0 \end{bmatrix}.$$

This implies that the controllability of the system (7) coincides with the controllability of the following system:

$$\dot{x} = A_i x + B_i z \qquad i \in \{1, \ldots, m\}, \tag{11}$$

which is the extracted dynamics of the followers that correspond to the x component of the equation. Therefore we have following definition:

Definition 3.1: The multi-agent system (7) is said to be structurally controllable under leader z if system (11) is structurally controllable under control input z.

For simplicity, we use (A_i, B_i) $i \in \{1, \ldots, m\}$ to represent switched linear system (11) in the sequel. In (11), each subsystem (A_i, B_i) can be described by a directed graph (Lin 1974):

Definition 3.2: The representation graph of structured system (A_i, B_i) is a directed graph $\mathcal{G}_i$, with vertex set $\mathcal{V}_i = \mathcal{X}_i \cup \mathcal{U}_i$, where $\mathcal{X}_i = \{x_1, x_2, \ldots, x_n\}$, which is called state vertex set and $\mathcal{U}_i = \{u_1, u_2, \ldots, u_r\}$, which is called input vertex set, and edge set $\mathcal{I}_i = \mathcal{I}_{U_iX_i} \cup \mathcal{I}_{X_iX_i}$, where $\mathcal{I}_{U_iX_i} = \{(u_p, x_q) | B_{qp} \neq 0, 1 \leq p \leq r, 1 \leq q \leq n\}$ and $\mathcal{I}_{X_iX_i} = \{(x_p, x_q) | A_{qp} \neq 0, 1 \leq p \leq n, 1 \leq q \leq n\}$ are the oriented edges between inputs and states and between states defined by the interconnection matrices A_i and B_i above. This directed graph (for notational simplicity, we will use digraph to refer to directed graph) $\mathcal{G}_i$ is also called the graph of matrix pair (A_i, B_i) and denoted by $\mathcal{G}_i(A_i, B_i)$.

For each subsystem, we have got a graph G_i with vertex set $\mathcal{V}_i$ and edge set $\mathcal{I}_i$ to represent the underlying communication topologies. As to the whole switched system, the representing graph, which is called *union graph*, is defined as follows:

Definition 3.3: The switched linear system (11) can be represented by a union digraph, defined as a flow structure $\mathcal{G}$. Mathematically, $\mathcal{G}$ is defined as

$$\mathcal{G}_1 \cup \mathcal{G}_2 \cup \ldots \cup \mathcal{G}_m = \{\mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_m; \mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_m\}. \quad (12)$$

Remark 1: It turns out that union graph $\mathcal{G}$ is the representation of linear structured system: $(A_1 + A_2 + \cdots + A_m, B_1 + B_2 + \cdots + B_m)$.

Remark 2 : In many literature works about controllability of multi-agent systems (Tanner 2004; Liu et al. 2008), the underlying communication topology among the agents is represented by undirected graph, which means that the communication among the agents is bidirectional. Here we still adopt this kind of communication topology. Then w_{ij} and w_{ji} are free parameters or zero simultaneously (in numerical realisation, the values of w_{ij} and w_{ji} can be chosen to be different). Besides, one edge in undirected graph can be treated as two oriented edges. Consequently, even though all the analysis and proofs for structural controllability of multi-agent systems are based on the directed graph (the natural graphic representation of matrix pair (A_i, B_i) is digraph), the final result will be expressed in undirected graph form.

Before proceeding further, we need to introduce two definitions which were proposed in Lin (1974) for linear structured system $\dot{x} = Ax + Bu$ first:

Definition 3.4 (Lin 1974)**:** The matrix pair (A, B) is said to be reducible or of form *I* if there exist permutation matrix P such that they can be written in the following form:

$$PAP^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \quad (13)$$

where $A_{11} \in \mathbb{R}^{p \times p}$, $A_{21} \in \mathbb{R}^{(n-p) \times p}$, $A_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$ and $B_{22} \in \mathbb{R}^{(n-p) \times r}$.

Remark 3: Whenever the matrix pair (A, B) is of form I, the system is structurally uncontrollable (Lin 1974) and meanwhile, the controllability matrix $Q = [B, AB, \ldots, A^{n-1}B]$ will have at least one row which is identically zero for all parameter values (Glover and Silverman 1976). If there is no such permutation matrix P, we say that the matrix pair (A, B) is irreducible.

Definition 3.5 (Lin 1974)**:** The matrix pair (A, B) is said to be of form II if there exist permutation matrix P such that they can be written in the following form:

$$\left[PAP^{-1}, PB\right] = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad (14)$$

where $P_2 \in \mathbb{R}^{(n-k) \times (n+r)}$, $P_1 \in \mathbb{R}^{k \times (n+r)}$ with no more than $k-1$ non-zero columns (all the other columns of P_1 have only fixed zero entries).

Here we need to recall a known result in literature for structural controllability of multi-agent system with fixed topology (Zamani and Lin 2009):

Lemma 3.6 (Zamani and Lin 2009)**:** *The multi-agent system with fixed topology under the communication topology $\mathcal{G}$ is structurally controllable if and only if graph $\mathcal{G}$ is connected.*

This lemma proposed an interesting graphic condition for structural controllability in fixed topology situation and revealed that the controllability is totally determined by the communication topology. However, how about in the switching topology situation? According to Lemma 2.2, once we impose proper scalars for the parameters of the system matrix (A_i, B_i) to satisfy the full rank condition, the multi-agent system (11) is structurally controllable. However, this only proposed an algebraic condition. Do we still have very good graphic interpretation for the relationship between the structural controllability and switching interconnection topologies? The following theorem answers this question and gives a graphic necessary and sufficient condition for structural controllability under switching topologies.

Theorem 3.7: *The multi-agent system* (11) *with the communication topologies $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$ is structurally controllable if and only if the union graph $\mathcal{G}$ is connected.*

Proof:

Necessity: Assume that the multi-agent switched system is structurally controllable, we want to prove that the union graph $\mathcal{G}$ is connected, which is equivalent to that the system has no isolated agents in the union graph $\mathcal{G}$ (Zamani and Lin 2009).

Suppose that the union graph $\mathcal{G}$ is disconnected and for simplicity, we will prove by contradiction for the case that there exits only one disconnected agent. The proof can be straightforwardly extended to more general cases with more than one disconnected agents. If there is one isolated agent in the union graph, there are two possible situations: the isolated agent is the leader or one of the followers. On the one hand, if the isolated agent is the leader, it follows that $B_1 + B_2 + \cdots + B_m$ is identically a null vector. So every B_i is a null vector. Easily, we can conclude that the controllability matrix for the switched system is never of full row rank $N-1$, which means that the multi-agent system is not structurally controllable. On the other hand, if the isolated agent is one follower, we get that the matrix pair $(A_1 + A_2 + \cdots + A_m, B_1 + B_2 + \cdots + B_m)$ is reducible. By Definition 3.4, the controllability matrix

$$[B_1 + B_2 + \cdots + B_m,\\ (A_1 + A_2 + \cdots + A_m)(B_1 + B_2 + \cdots + B_m),\\ ,\ldots,\\ (A_1 + A_2 + \cdots + A_m)^{N-2}(B_1 + B_2 + \cdots + B_m)]$$

always has at least one row that is identically zero. Expanding the matrix yields

$$[B_1 + B_2 + \cdots + B_m,\\ A_1B_1 + A_2B_1 + \cdots + A_mB_1 + A_1B_2 + A_2B_2\\ + \cdots + A_mB_2 + \cdots + A_1B_m + A_2B_m \cdots + A_mB_m\\ ,\ldots,\\ A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \cdots + A_m^{N-2}B_m].$$

The zero row is identically zero for every parameter. This implies that every component in this matrix, such as B_i, A_iB_j *and* $A_i^pA_j^qB_r$, has the same row always to be zero. As a result, the controllability matrix

$$[B_1,\ldots,B_m,A_1B_1,\ldots,A_mB_1,\ldots,A_mB_m,A_1^2B_1,\ldots,A_mA_1B_1,\\ \ldots,A_1^2B_m,\ldots,A_mA_1B_m,\ldots,A_1^{n-1}B_1,\ldots,A_mA_1^{n-2}B_1,\ldots,\\ A_1A_m^{n-2}B_m,\ldots,A_m^{n-1}B_m]$$

always has one zero row. Therefore, the multi-agent system (11) is not structurally controllable. Until now, we have got the necessity proved.

Sufficiency: If the union graph $\mathcal{G}$ is connected, we want to prove that the multi-agent system (11) is structurally controllable.

According to Lemma 3.6, the connectedness of the union graph $\mathcal{G}$ implies that the corresponding system $(A_1 + A_2 + A_3 + \cdots + A_m,\quad B_1 + B_2 + B_3 + \cdots + B_m)$ is structurally controllable. Then there exist some scalars for the parameters in system matrices that make the controllability matrix

$$[B_1 + B_2 + \cdots + B_m,\\ (A_1 + A_2 + \cdots + A_m)(B_1 + B_2 + \cdots + B_m),\\ ,\ldots,\\ (A_1 + A_2 + \cdots + A_m)^{N-2}(B_1 + B_2 + \cdots + B_m)]$$

has full row rank $N-1$. Expanding the matrix, it follows that the matrix

$$[B_1 + B_2 + \cdots + B_m,\\ A_1B_1 + A_2B_1 + \cdots + A_mB_1 + A_1B_2 + A_2B_2\\ + \cdots + A_mB_2 + \ldots + A_1B_m + A_2B_m \ldots + A_mB_m\\ ,\ldots,\\ A_1^{N-2}B_1 + A_2A_1^{N-3}B_1 + \cdots + A_m^{N-2}B_m],$$

has full rank $N-1$. Next, we add some column vectors to the above matrix and get

$$[B_1+B_2+\cdots+B_m, B_2,\ldots,B_m,\\ A_1B_1+A_2B_1+\cdots+A_mB_1+A_1B_2+A_2B_2+\cdots+A_mB_2\\ +\cdots+A_1B_m+A_2B_m+\cdots+A_mB_m, A_2B_1, A_3B_1,\ldots,A_mB_m\\ ,\ldots,\\ A_1^{N-2}B_1+A_2A_1^{N-3}B_1+\cdots+A_m^{N-2}B_m, A_2A_1^{N-3}B_1,\ldots,A_m^{N-2}B_m].$$

This matrix still has $N-1$ linear independent column vectors, so it has full row rank. Next, subtract $B_2,\ldots,B_m$ from $B_1+B_2+\cdots+B_m$; subtract $A_2B_1, \ldots, A_mB_m$ from $A_1B_1+A_2B_1+\cdots+A_mB_m$ and subtract $A_2A_1^{N-3}B_1,\ldots,A_m^{N-2}B_m$ from $A_1^{N-2}B_1+A_2A_1^{N-3}B_1+\cdots+A_m^{N-2}B_m$. Since this column fundamental transformation will not change matrix rank, the matrix still has full row rank. Now the matrix becomes

$$[B_1,\ldots,B_m,A_1B_1,\ldots,A_mB_1,\ldots,A_mB_m,A_1^2B_1,\ldots,A_mA_1B_1,\\ \ldots,A_1^2B_m,\ldots,A_mA_1B_m,\ldots,A_1^{n-1}B_1,\ldots,A_mA_1^{n-2}B_1,\ldots,\\ A_1A_m^{n-2}B_m,\ldots,A_m^{n-1}B_m],$$

which is the controllability matrix of system (11) and has full row rank $N-1$. Therefore, the multi-agent system is structurally controllable. □

4. Structural controllability of multi-agent system with multi-leader

In the above discussion, we assume the multi-agent system has totally N agents and the Nth agent serves as the leader and takes commands and controls from outside operators directly, while the rest $N-1$ agents are followers and take controls as the nearest neighbour law. In the following part, we will discuss the situation that several agents are chosen as the leaders of the whole multi-agent systems, which is actually an extension of single-leader case.

Similar to the single-leader case, the multi-agent system with multiple leaders is given by

$$\begin{cases} \dot{x}_i = u_i, & i = 1, \ldots, N, \\ \dot{x}_{N+j} = u_{N+j}, & j = 1, \ldots, n_l, \end{cases} \tag{15}$$

where N and n_l represent the number of followers and leaders, respectively. x_i indicates the state of the ith agent, $i = 1, \ldots, N + n_l$.

The control strategy u_i, $i = 1, \ldots, N$ for driving all follower agents is the same as the single-leader case. The leaders' control signal is still not influenced by the followers and we are allowed to pick u_{N+j}, $j = 1, \ldots, n_l$ arbitrarily. For simplicity, we use vector x to stand for the followers' states and z to stand for the leaders' states.

Then the whole multi-agent system equipped with m communication topologies can be written in a compact form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \tag{16}$$

or, equivalently,

$$\begin{cases} \dot{x} = A_i x + B_i z, \\ \dot{z} = u, \end{cases} \tag{17}$$

where $i \in \{1, \ldots, m\}$. $A_i \in \mathbb{R}^{N\times N}$ and $B_i \in \mathbb{R}^{N\times n_l}$ are both sub-matrices of the corresponding graph Laplacian matrix $-\mathcal{L}$.

The dynamics of the followers can be extracted as

$$\dot{x} = A_i x + B_i z, \quad i \in \{1, \ldots, m\}. \tag{18}$$

Remark 4: Similar to the single-leader case, the structural controllability of system (16) coincides with the structural controllability of system (18). And we say that the multi-agent system (15) with switching topology and multi-leader is structurally controllable if and only if the switched linear system (18) is structurally controllable with z as the control inputs.

Before proceeding further, we first discuss the structural controllability of multi-agent systems with multi-leader under fixed topology with the following dynamics:

$$\dot{x} \quad = Ax + Bz, \tag{19}$$

where $A \in \mathbb{R}^{N\times N}$ and $B \in \mathbb{R}^{N\times n_l}$ are both sub-matrices of the graph Laplacian matrix $-\mathcal{L}$.

In Ji et al. (2008a) and Ji, Wang, Lin, and Wang (2009), a new graph topology, leader–follower connected topology was proposed:

Definition 4.1 (Ji et al. 2008a)**:** A follower subgraph $\mathcal{G}_f$ of the interconnection graph $\mathcal{G}$ is the subgraph induced by the follower set $\mathcal{V}_f$ (here is x). Similarly, a leader subgraph $\mathcal{G}_l$ is the subgraph induced by the leader set $\mathcal{V}_l$ (here is z).

Denote by $\mathcal{G}_{c_1}, \ldots, \mathcal{G}_{c_\gamma}$, the connected components in the follower $\mathcal{G}_f$. The definition of leader-follower connected topology is as follows:

Definition 4.2 (Ji et al. 2009)**:** The interconnection graph $\mathcal{G}$ of multi-agent system (19) is said to be leader–follower connected if for each connected component $\mathcal{G}_{c_i}$ of $\mathcal{G}_f$, there exists a leader in the leader subgraph $\mathcal{G}_l$, so that there is an edge between this leader and a node in $\mathcal{G}_{c_i}, i = 1, \ldots, \gamma$.

Based on this new graph topology, we can derive the criterion for structural controllability for multi-agent system (19) under fixed topology:

Theorem 4.3: *The multi-agent system* (19) *with multi-leader and fixed topology under the communication topology $\mathcal{G}$ is structurally controllable if and only if graph $\mathcal{G}$ is leader–follower connected.*

Proof:
Necessity: The idea of necessity proof is similar to the proof of lemma 3 in Ji et al. (2008a). We assume that there exists one connected component $\mathcal{G}_{c_p}$ not connected to the leader subgraph $\mathcal{G}_l$. Define A_i and B_i matrices as sub-matrices of A and B, the same as the F_i and R_i matrices in lemma 3 of Ji et al. (2008a). Following the analysis in lemma 3 of Ji et al. (2008b), can be easity obtained as the controllability matrix of multi-agent system (19)

$$\mathcal{C} = \begin{bmatrix} B_1 & A_1 B_1 & A_1^2 B_1 & \cdots & A_1^{N-1} B_1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ B_\gamma & A_\gamma B_\gamma & A_\gamma^2 B_\gamma & \cdots & A_\gamma^{N-1} B_\gamma \end{bmatrix}. \tag{20}$$

Consequently, rank $\mathcal{C} =$ row rank $\mathcal{C} < N$. The maximum rank of $\mathcal{C}$ is less than N, which implies that the corresponding multi-agent system (19) is not structurally controllable.

Sufficiency: We adopt the Proof of theorem 1 in Ji et al. (2009) to help us prove the sufficiency. The communication graph $\mathcal{G}$ consists of several connected components $\mathcal{G}^{(i)}$, $i = 1, \ldots, \kappa$, which can be partitioned into two subgraphs: induced leader subgraph $\mathcal{G}_l^{(i)}$ and induced follower subgraph $\mathcal{G}_f^{(i)}$. For each connected components $\mathcal{G}^{(i)}$, $i = 1, \ldots, \kappa$, it can be modelled as a linear system with its system matrices being sub-matrices of A and B matrices. Following the analysis in Theorem 1 in Ji et al. 2009), the following equation can be deduced:

$$rank\ \mathcal{C} = rank\ \mathcal{C}_1 + rank\ \mathcal{C}_2 + \cdots + rank\ \mathcal{C}_\kappa, \tag{21}$$

where $\mathcal{C}$ is the controllability matrix of multi-agent system (19) and $\mathcal{C}_i$ is the controllability matrix of connected component $\mathcal{G}^{(i)}$. The independence of these connected components guarantees the independence of free parameters in the corresponding matrices, which correspond to the communication weights of the links. Consequently, we have that

$$g\text{-}rank\ \mathcal{C} = g\text{-}rank\ \mathcal{C}_1 + g\text{-}rank\ \mathcal{C}_2 + \cdots + g\text{-}rank\ \mathcal{C}_\kappa$$

where g-rank of a structured matrix M is defined to be the maximal rank that M achieves as a function of its free parameters. Besides, if in some connected component $\mathcal{G}^{(i)}$, there is more than one leaders, we can split it into several connected components with single leader or choose one as leader and set all weights of the communication links between the followers and other leaders to be zero. After doing this, connected component $\mathcal{G}^{(i)}$ is a connected topology with single leader. According to Lemma 3.6, $\mathcal{C}_i$ has full g-rank, which equals to the number of follower agents in $\mathcal{G}^{(i)}$. Moreover, there is no common follower agent among the connected components. Consequently, g-rank $\mathcal{C}=N$ and multi-agent system (19) is structurally controllable. □

With the above definitions and theorems, we are in the position to present the graphical interpretation of structural controllability of multi-agent systems under switching topology with multi-leader:

Theorem 4.4: *The multi-agent system* (16) *or* (18) *with the communication topologies* $\mathcal{G}_i$, $i \in \{1, \ldots, m\}$ *and multi-leader is structurally controllable if and only if the union graph* $\mathcal{G}$ *is leader–follower connected.*

Proof: As stated in Remark 1, the union graph $\mathcal{G}$ is the representation of the linear system: $(A_1 + A_2 + A_3 + \cdots + A_m,\ B_1 + B_2 + B_3 + \cdots + B_m)$. Therefore, the condition that the union graph $\mathcal{G}$ is leader–follower connected is equivalent to the condition that linear system $(A_1 + A_2 + A_3 + \cdots + A_m,\ B_1 + B_2 + B_3 + \cdots + B_m)$ is structurally controllable. Following the proof procedure in Theorem 3.7, this result can be proved. □

5. Numerical examples

Next we will give two examples to illustrate the results in this paper and for simplicity, we take single leader case as examples.

We consider here a four-agent network with agent 0 as the leader and with switching topology described by the graphs in Figure 1(a) and (b) (the self-loops are not depicted, because it will not influence the connectivity). Overlaying the subgraphs together can get the union graph $\mathcal{G}$ of this example as shown in Figure 1(c).

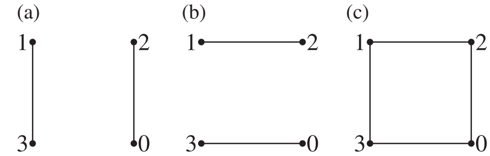

Figure 1. Switched network with two subsystems.

It turns out that the union graph of the switched system is connected. By theorem 3.7, it is clear that the multi-agent system is structurally controllable.

Next, the rank condition of this multi-agent system will be checked.

From Figure 1, calculating the Laplacian matrix for each subgraph topology, it can be obtained that the system matrices of each subsystem are (one thing we should mention here with the control strategy that each agent can use its own state information, the diagonal elements always have free parameters, so we can get the following form of sub-matrix of Laplacian matrix):

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_2 & 0 \\ \lambda_5 & 0 & \lambda_3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \lambda_6 \\ 0 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} \lambda_7 & \lambda_{10} & 0 \\ \lambda_{11} & \lambda_8 & 0 \\ 0 & 0 & \lambda_9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \lambda_{12} \end{bmatrix}.$$

According to Lemma 2.2, the controllability matrix for this switched linear system is: $[B_1, B_2, A_1B_1, A_2B_1, A_1B_2, A_2B_2, A_1^2B_1, A_2A_1B_1, A_1A_2B_1, A_2^2B_1, A_1^2B_2, A_2A_1B_2, A_1A_2B_2, A_2^2B_2]$. After simple calculation, we can find three column vectors in the controllability matrix:

$$\begin{bmatrix} 0 \\ \lambda_6 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ \lambda_{12} \end{bmatrix}, \quad \begin{bmatrix} \lambda_4\lambda_{12} \\ 0 \\ \lambda_3\lambda_{12} \end{bmatrix}.$$

Imposing all the parameters scalar 1, it follows that these three column vectors are linearly independent and this controllability matrix has full row rank. Therefore, the multi-agent system is structurally controllable.

In the second example, we still consider a four-agent network with agent 0 as the leader and with switching topology described by the graphs in Figure 2(a) and (b). Overlaying the subgraphs together can get the union graph $\mathcal{G}$ of this example shown in Figure 2(c). It turns out that the union graph of the switched system is disconnected, because agent 2 is isolated. According to theorem 3.7, it is clear that the multi-agent system is not structurally controllable.

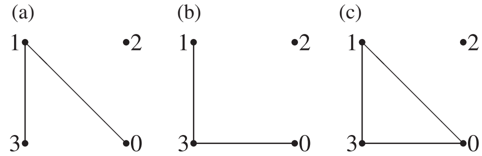

Figure 2. Another switched network with two subsystems.

Similarly, the rank condition of this switched linear system needs to be checked to see whether it is structurally controllable or not.

From Figure 2, calculating the Laplacian matrix for each graphic topology, it is clear that the system matrices of each subsystem are :

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_2 & 0 \\ \lambda_5 & 0 & \lambda_3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \lambda_6 \\ 0 \\ 0 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} \lambda_7 & 0 & \lambda_{10} \\ 0 & \lambda_8 & 0 \\ \lambda_{11} & 0 & \lambda_9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \lambda_{12} \end{bmatrix}.$$

Computing the controllability matrix of this example yields the controllability matrix:

$$\begin{bmatrix} \lambda_6 & 0 & \lambda_1\lambda_6 \ldots \lambda_7\lambda_{10}\lambda_{12} + \lambda_9\lambda_{10}\lambda_{12} \\ 0 & 0 & 0 \quad \ldots \quad 0 \\ 0 & \lambda_{12} & \lambda_5\lambda_6 \ldots \lambda_{10}\lambda_{11}\lambda_{12} + \lambda_9^2\lambda_{12} \end{bmatrix}.$$

This matrix has the second row always to be zero for all the parameter values, which makes the maximum rank of this matrix less than 3. Therefore, this multi-agent system is not structurally controllable.

6. Conclusions and future work

In this article, the structural controllability problem of the multi-agent systems interconnected via a switching weighted topology has been considered. Based on known results in the literature of switched systems and graph theory, graphic necessary and sufficient conditions for the structural controllability of multi-agent systems under switching communication topologies were derived. It was shown that the multi-agent system is structurally controllable if and only if the union graph $\mathcal{G}$ is connected (single leader) or leader–follower connected (multi-leader). The graphic characterisations show a clear relationship between the controllability and interconnection topologies and give us a foundation to design the optimal control effect for the switched multi-agent system.

Some interesting remarks can be made on this result. First, it gives us a clear understanding on what are the necessary information exchanges among agents to make the group of agents behaviour in a desirable way. Second, it provides us a guideline to design communication protocols among dynamical agents. It is required that the resulted communication topology among agents should somehow remain connected as time goes on, which is quite intuitive and reasonable. Third, it is possible to reduce communication load by disable certain links or make them on and off as long as the union graph is connected. Several interesting research questions arise from this scenario. For example, what is the optimal switching sequence of topologies in the sense of minimum communication cost? How to co-design the switching topology path and control signals to achieve desirable configuration in an optimal way? We will investigate these questions in our future research.

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