VARIANT FACTOR TECHNIQUE FOR TRACKING CONTROL OF A CLASS OF NONLINEAR SYSTEMS WITH INPUT SATURATION

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Abstract

A variant factor (VF) technique is developed for the control design of a class of single-input and single-output affine nonlinear systems with input saturation. The main ideas of VF are to convert an affine nonlinear system into a linear time-invariant system similar to the dynamic inversion and then to use the ideas of the composite nonlinear feedback (CNF) technique to design a control law for the converted linear system with the input saturation. A VF is introduced to guarantee the resulting closed-loop system asymptotically stable in presence of the input saturation. The VF control laws retain the similar properties of the CNF control laws. With the VF control law, the system output can asymptotically track the step reference and the resulting closed-loop system can achieve better tracking performance than that with the control law designed with the dynamic inversion.

Key Words

Nonlinear systems and control, dynamic inversion, composite nonlinear feedback, input saturation, tracking control

1. Introduction

The nonlinear system control design has been attracting much interest in the academic circle. This is because the structural properties of the nonlinear systems are various. The dynamic inversion [1], [2] symbolizes a breakthrough in the nonlinear system control design. The dynamic inversion is applicable to the control design of a class of the affine nonlinear systems. The main ideas of the dynamic inversion are to convert an affine nonlinear system into a linear time-invariant (LTI) system with a non-singular nonlinear state transformation and the nonlinear state feedback control so-called the exact linearization, and then to apply all kinds of the LTI system control design methods to design a linear control law for the exactly linearized time-invariant system, and finally to form a complete control law for the original nonlinear system by combining the nonlinear state transformation, nonlinear state feedback control and the designed linear control law. The exact linearization is capable of handling the nonlinearities completely in a class of the affine nonlinear systems and the resulting complete control laws are valid to the original nonlinear systems. Thus, the dynamic inversion becomes one of the fundamental methods in the nonlinear system control design.

There were many literatures published on the dynamic inversion [3]–[6]. The main attempt of those publications is to utilize the dynamic inversion to deal with the nonlinearities in the control design to which the dynamic inversion is easily applicable. To handle the nonlinearities to which the dynamic inversion is not easily applicable, various derivatives of the dynamic inversion were developed such as the adaptive dynamic inversion [7]–[9], the hierarchical dynamic inversion [10], [11]. The main ideas of those derivatives are to transform a nonlinear system into a new formulation to which the dynamic inversion is easily applicable, to separate dynamics of a nonlinear system in the time scale and/or to partition a nonlinear system into a number of subsystems to which the dynamic inversion is easily applicable.

However, those methods have to assume that there is no input saturation in the nonlinear systems. The input saturation is one of typical nonlinearities in the control design to which the dynamic inversion is not easily applicable. A technique so-called the composite nonlinear feedback (CNF) [12], [13] is capable of handling the input saturation in the control design for the LTI systems. The main idea of CNF is to design a control law comprising a linear control law and a nonlinear part. The nonlinear part is a multiplication of a nonlinear gain and a linear feedback expression. The nonlinear gain is designed to

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yield a low damping-ratio to achieve quick rising time when there is large tracking error and a high damping-ratio to remove the overshoot caused by the low damping-ratio when the system output approaches the target of the step reference. The resulting closed-loop system can achieve better tracking performance than those with the control laws designed with the generic linear system control design methods.

Many derivatives of CNF were developed to improve the robustness against the disturbances and applicability to the generic references [14]–[17]. An application of CNF to deal with the input saturation in a class of the nonlinear systems was published in [18]. However, there was only considered the control design for an LTI actuator with input saturation incorporated in a stable nonlinear system. Actually, the input saturation in the nonlinear systems is left to be studied. This motivated us to develop a variant factor (VF) technique to deal with the input saturation in a class of the affine nonlinear systems.

VF is a derivative of the dynamic inversion and CNF, in which the exact linearization is applied at first to convert an affine nonlinear system into an LTI system and then the ideas of CNF are adopted to design a linear control law and a VF at a chosen nominal point. The VF control design follows the procedures of the dynamic inversion and the guidelines of CNF. A VF is introduced to the control laws designed with the dynamic inversion for the nonlinear systems without input saturation to form the VF control laws for those with input saturation. Similarly to the CNF control design for the LTI systems with input saturation, the VF can be designed to result in a low damping-ratio to achieve quick rising time when there is large tracking error and a high damping-ratio to remove the overshoot caused by the low damping-ratio when the system output approaches the step reference. With the VF control law, the system output of a class of the affine nonlinear systems with input saturation can asymptotically track the step reference and the resulting closed-loop system can achieve better tracking performance than that with the control law designed with dynamic inversion.

In this paper, the VF technique will be proposed for tracking control design of a class of the nonlinear systems with input saturation. An illustrative example will be given to show the VF control design and the simulation results will be presented to demonstrate the tracking performance improved by VF. The outline of this paper is as follows. The VF technique and the relevant theoretical results are presented in Section 2. In Section 3, an illustrative example and the simulation results are given. Finally, the concluding remarks are drawn.

2. VF Technique

The VF technique will be presented for the tracking control design of a class of the single-input and single-output (SISO) affine nonlinear systems with input saturation. A control law will be designed with VF at first and then the theoretical results are given. Design of the VF will be presented finally.

We consider a class of the SISO affine nonlinear systems with input saturation as follows:

$$\begin{cases} \dot{x} = f(x) + g(x)\mathrm{sat}(u) \\ y = h(x) \end{cases} \tag{1}$$

where $x \in \Re^n$, $y \in \Re$ and $u \in \Re$ are the state, output and control variables, respectively. sat is a function defined as follows,

$$\mathrm{sat}(u) = \begin{cases} \overline{u}, & u > \overline{u} \\ u, & |u| \leq \overline{u} \\ -\overline{u}, & u < -\overline{u} \end{cases} \tag{2}$$

where $\overline{u}$ is the ceiling value of u. We need the assumptions such as (a) $f(x) \in \Re^n$ and $g(x) \in \Re^n$ are smooth vector fields and $h(x) \in \Re$ is a smooth function on a compact and connected set X of $\Re^n$; (b) the relative dynamic degree of (1) is $r \leq n$ on X; (c) the zero dynamics of (1) is stable on X if any.

Based on [1], [2], if the relative dynamic degree of (1) is r, there exists a non-singular transformation $\Phi(x)$ from X to $Z \times Z_0$ which is a compact and connected set of $\Re^n$, as follows,

$$\begin{pmatrix} z \\ z_0 \end{pmatrix} = \Phi(x), \ \ \forall x \in X, \ \begin{pmatrix} z \\ z_0 \end{pmatrix} \in Z \times Z_0 \tag{3}$$

where $z \in \Re^r$, $z_0 \in \Re^{n-r}$ and,

$$\begin{aligned} \Phi'(x) &= (L_f^0 h(x) \quad \cdots \quad L_f^{r-1} h(x) \\ &\qquad \phi_{r+1}(x) \quad \cdots \quad \phi_n(x)) \\ L_f^i h(x) &= f'(x) \frac{\partial L_f^{i-1} h(x)}{\partial x}, \quad \text{for } \ i = 1, \ldots \\ L_f^0 h(x) &= h(x) \\ g'(x) \frac{\partial \phi_i(x)}{\partial x} &= 0, \quad \text{for } \ i = r+1, \ldots, n \end{aligned}$$

where $\phi_i(x)$ is a smooth function on X for $i = r+1, \ldots, n$. With the nonlinear state transformation, $\Phi(x)$, (1) can be converted as follows,

$$\begin{cases} \dot{z} = Az + bv \\ y = Cz \\ \dot{z}_0 = z_{\mathrm{d}}(z, z_0) \end{cases} \tag{4}$$

where $v := s(z, z_0) + p(z, z_0)\mathrm{sat}(u)$, z_0 is the state variable of the zero dynamics and z is the state variable of the LTI system. $z_{\mathrm{d}}(z, z_0) \in \Re^{n-r}$ is a smooth vector field and $s(z,z_0)\in\Re$ and $p(z,z_0)\in\Re$ are smooth functions on $Z\times Z_0$.

$$
\begin{aligned}
z_d'(z,z_0) &= \left(f'(x)\frac{\partial\phi_{r+1}(x)}{\partial x}\cdots f'(x)\frac{\partial\phi_n(x)}{\partial x}\right)\\
s(z,z_0) &= f'(x)\frac{\partial L_f^r h(x)}{\partial x}\\
p(z,z_0) &= g'(x)\frac{\partial L_f^r h(x)}{\partial x}\\
x &= \Phi^{-1}(z,z_0)
\end{aligned}
$$

where Φ^{-1} denotes the inverse of Φ, which is well defined as Φ is non-singular from X to $Z\times Z_0$. The system matrices are:

$$
A=\begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix},\quad b=\begin{bmatrix}0\\ 1\end{bmatrix},\quad C=[1\quad 0]
$$

where I denotes the identity matrix with appropriate dimensions.

Our task is to design a control law with the VF technique for (1) so that the output, y, can asymptotically track a step reference, $r_f\in\Re$. As the nonlinear state transformation, $\Phi(x)$, is non-singular from X to $Z\times Z_0$, (4) is equivalent to (1). Thus, we can design a control law for (4) at first and then combine the control law for (4) and the nonlinear state transformation to form the control law for (1). Since zero dynamics of (1) is assumed to be stable, we just need to design a control law for:

$$
\begin{cases}\dot z = Az+bv\\ y=Cz\end{cases}\tag{5}
$$

where $v:=s(z,z_0)+p(z,z_0)\mathrm{sat}(u)$. A linear control law can be designed for (5) so that y can track a step reference, r_f, in which v is regarded as the control variable, v_c, as follows,

$$
v_c = F(z-G_e r_f)\tag{6}
$$

where $F=-R^{-1}b'P$ and $G_e=(A+bF)^{-1}b[C(A+bF)^{-1}b]^{-1}$. P is the solution of the Riccati equation as follows,

$$
PA+A'P-PbR^{-1}b'P+Q=0\tag{7}
$$

where $R>0$ is a scalar and $Q>0$ is a definitely positive matrix. If $p(z,z_0)$ is invertible at $\forall(z'\quad z_0')'\in Z\times Z_0$, without consideration of input saturation in (1), a control law can be designed with dynamic inversion by letting $v=v_c$ as follows,

$$
\begin{cases}u=p^{-1}(z,z_0)[F(z-G_e r_f)-s(z,z_0)]\\ \begin{pmatrix}z\\ z_0\end{pmatrix}=\Phi(x)\end{cases}\tag{8}
$$

To handle input saturation in (1), a VF is introduced to (8) so that a complete control law for (1) can be presented as follows,

$$
\begin{cases}u=p^{-1}(z,z_0)[\rho F(z-G_e r_f)-s(z,z_0)]\\ \begin{pmatrix}z\\ z_0\end{pmatrix}=\Phi(x)\end{cases}\tag{9}
$$

where $\rho\in\Re$ is the VF introduced, which is a continuous function to be designed to improve tracking performance. Evidently, when $\rho=1$, (9) is the same as (8) designed with dynamic inversion for (1) without input saturation.

Additionally, a nominal point $x_n\in X$ is chosen so that $(z_n'\quad z_{0,n}')'=\Phi(x_n)$, where $(z_n'\quad z_{0,n}')'\in Z\times Z_0$. Next, we need to define a set at the chosen nominal point. For a scalar $\tau\in(0,1)$, there exists a scalar $\mu>0$ so that,

$$
\begin{aligned}
X_{iv} &:= \{x : x'Px\le\mu\}\\
&\Longrightarrow |p^{-1}(z_n,z_{0,n})F\,x|\le(1-\tau)\overline{u}
\end{aligned}\tag{10}
$$

Then, we have the following conclusion.

Theorem 2.1 *Suppose that (a) $f(x)$ and $g(x)$ are smooth vector fields and $h(x)$ is a smooth function on a compact and connected set X of $\Re^n$; (b) the relative dynamic degree of (1) is $r\le n$ on X; (c) the zero dynamics of (1) is stable on X if any; (d) $p(z,z_0)$ is invertible at $\forall(z'\quad z_0')'\in Z\times Z_0$ in which $(z'\quad z_0')'=\Phi(x)$ is non-singular from X to $Z\times Z_0$. Then, with the control law, (9), the system output of (1), y, can asymptotically track a step reference, $r_f\in\Re$, if:*

1. *There exists a scalar $\tau\in(0,1)$ so that,*

$$
|p^{-1}(z,z_0)s(z,z_0)|\le\tau\overline{u}\qquad \forall\begin{pmatrix}z\\ z_0\end{pmatrix}\in Z\times Z_0\tag{11}
$$

2. *ρ is a continuous function and $x_n\in X$ is a chosen nominal point so that*

$$
\begin{aligned}
\rho &\ge 1/2 \text{ and,}\\
r_p &:= p(z,z_0)p^{-1}(z_n,z_{0,n})\ge 1/2
\end{aligned}\tag{12}
$$

$$
\begin{pmatrix}z_n\\ z_{0,n}\end{pmatrix}=\Phi(x_n),\ \forall\begin{pmatrix}z\\ z_0\end{pmatrix}\in Z\times Z_0
$$

3. *Let x_{iv} denote the initial value of x. Then, the step reference, r_f. satisfies:*

$$
z_{iv}-G_e r_f\in X_{iv},\ \begin{pmatrix}z_{iv}\\ z_{0,iv}\end{pmatrix}=\Phi(x_{iv})\tag{13}
$$

Proof: Let $\tilde{z} := z - G_e r_f$. Then,

$$\begin{aligned} u &= p^{-1}(z,z_0)[\rho F(z - G_e r_f) - s(z,z_0)]| \\ &= p^{-1}(z,z_0)\rho F\tilde{z} - p^{-1}(z,z_0)s(z,z_0) \end{aligned} \tag{14}$$

When $\tilde{z} \in X_{iv}$, it implies that,

$$|p^{-1}(z_n, z_{0,n})F\tilde{z}| \leq (1-\tau)\overline{u} \tag{15}$$

Note that $r_p > 0$ on $Z \times Z_0$ as $p(z, z_0)$ is a smooth function and invertible on the compact and connected set $Z \times Z_0$. Otherwise, there should be a point, $(z' \quad z_0')' \in Z \times Z_0$, at which $p(z, z_0) = 0$ is not invertible.

When $0 < \rho \leq r_p$, with (11) and (15),

$$\begin{aligned} |u| &\leq |p^{-1}(z,z_0)\rho F\tilde{z}| + |p^{-1}(z,z_0)s(z,z_0)| \\ &\leq |p^{-1}(z,z_0)r_p F\tilde{z}| + |p^{-1}(z,z_0)s(z,z_0)| \\ &= |p^{-1}(z_n,z_{0,n})F\tilde{z}| + |p^{-1}(z,z_0)s(z,z_0)| \\ &\leq (1-\tau)\overline{u} + \tau\overline{u} = \overline{u} \end{aligned}$$

Thus,

$$\text{sat}(u) = p^{-1}(z,z_0)[\rho F\tilde{z} - s(z,z_0)] \tag{16}$$

When $\rho > r_p > 0$,

$$\begin{aligned} u &= p^{-1}(z,z_0)[\rho F\tilde{z} - s(z,z_0)] \\ &= p^{-1}(z,z_0)(\rho - r_p)F\tilde{z} \\ &\quad + p^{-1}(z,z_0)[r_p F\tilde{z} - s(z,z_0)] \end{aligned} \tag{17}$$

With (11) and (15), the second item in (17) can be bounded as follows,

$$\begin{aligned} &|p^{-1}(z,z_0)[r_p F\tilde{z} - s(z,z_0)]| \\ &\quad \leq |p^{-1}(z,z_0)r_p F\tilde{z}| + |p^{-1}(z,z_0)s(z,z_0)| \\ &\quad = |p^{-1}(z_n,z_{0,n})F\tilde{z}| + |p^{-1}(z,z_0)s(z,z_0)| \\ &\quad \leq (1-\tau)\overline{u} + \tau\overline{u} = \overline{u} \end{aligned}$$

Thus,

$$\begin{aligned} \text{sat}(u) &= qp^{-1}(z,z_0)(\rho - r_p)F\tilde{z} \\ &\quad + p^{-1}(z,z_0)[r_p F\tilde{z} - s(z,z_0)] \end{aligned} \tag{18}$$

where $q \in (0,1]$. So, $\text{sat}(u)$ can be formulated as follows,

$$\text{sat}(u) = p^{-1}(z,z_0)[\overline{\rho} F\tilde{z} - s(z,z_0)] \tag{19}$$

where,

$$\overline{\rho} = \begin{cases} \rho & 0 < \rho \leq r_p \\ q\rho + (1-q)r_p & \rho > r_p, q \in (0,1] \end{cases}$$

Let $[f_0 \quad f_1] := F$ and f_0 is a non-zero scalar. If $f_0 = 0$, $A + bF$ can be checked to be not asymptotically stable. It violates that $A + bF$ is asymptotically stable as with (7),

$$(A + bF)'P + P(A + bF) = -Q - PbR^{-1}b'P$$

Then,

$$\begin{aligned} AG_e &= A(A+bF)^{-1}b[C(A+bF)^{-1}b]^{-1} \\ &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -f_1 f_0^{-1} & f_0^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &\quad \times [C(A+bF)b]^{-1} = 0 \end{aligned}$$

Thus,

$$\begin{aligned} \dot{\tilde{z}} &= \dot{z} = Az + b[s(z,z_0) + p(z,z_0)\text{sat}(u)] \\ &= (A + b\overline{\rho}F)\tilde{z} + AG_e r_f \\ &= (A + b\overline{\rho}F)\tilde{z} \end{aligned}$$

Therefore, the closed-loop system comprising (4) and (9) can be written as follows,

$$\begin{cases} \dot{\tilde{z}} = (A + b\overline{\rho}F)\tilde{z} \\ \dot{z}_0 = z_d(z,z_0) \end{cases} \tag{20}$$

As (20) is divided into two parts and the second part is the zero dynamics which is assumed to be stable, we only need to prove stability of the first part. Choose a Lyapunov function as follows,

$$V = \tilde{z}'P\tilde{z} \tag{21}$$

Then, derivative of V can be computed along the trajectory of (20) and with (7) as follows,

$$\begin{aligned} \dot{V} &= \dot{\tilde{z}}'P\tilde{z} + \tilde{z}'P\dot{\tilde{z}} \\ &= \tilde{z}'[(A + b\overline{\rho}F)'P + P(A + b\overline{\rho}F)]\tilde{z} \\ &= \tilde{z}'[(A - b\overline{\rho}R^{-1}b'P)'P \\ &\quad + P(A - b\overline{\rho}R^{-1}b'P)]\tilde{z} \\ &= \tilde{z}'[-Q + Pb(1 - 2\overline{\rho})R^{-1}b'P]\tilde{z} \end{aligned} \tag{22}$$

Clearly, if $\overline{\rho} \geq 1/2$, that is,

$$\begin{aligned} \rho \geq 1/2 \ \text{ and } \ r_p &\geq 1/2, \quad \forall \begin{pmatrix} z \\ z_0 \end{pmatrix} \in Z \times Z_0 \\ \dot{V} &\leq -\tilde{z}'Q\tilde{z} < 0 \end{aligned} \tag{23}$$

It implies that once $\tilde{z} \in X_{iv}$, $\tilde{z}$ will never be out of X_{iv} and $\dot{V} < 0$ if (11) and (12) are satisfied.

As the initial value of $\tilde{z}$ is $z_{iv} - G_e r_f \in X_{iv}$ provided by (13), $\tilde{z}$ will be inside X_{iv} and $\dot{V} < 0$ if (11) and (12) are satisfied. Therefore, the closed-loop system comprising (4) and (9) is asymptotically stable.

$$\lim_{t\to\infty} \tilde{z} = 0, \Longrightarrow \lim_{t\to\infty} z = G_e r_f$$
$$\text{and} \quad \lim_{t\to\infty} y = \lim_{t\to\infty} Cz = r_f \tag{24}$$

This completes the proof of Theorem 2.1.

Remark 2.1

1. *The VF, ρ, plays an important role in (9) so that the control law designed at the chosen nominal point is applicable to other points in X.*
2. *Similarly to control design with CNF for linear time-invariant systems with input saturation, ρ can be designed to smoothly yield low and high damping-ratios to improve tracking performance.*
3. *If $\rho = r_p$, the resulting closed-loop system is asymptotically stable and the control signal does not violate the input limit.*
4. *If the relative dynamic degree of (1) is $r=1$ on X, $A=0$. Then, (12) becomes $\rho > 0$ without any constraint on r_p.*

2.1 Design of VF

The VF, ρ, can be designed to be a continuous function to result in low and high damping-ratios to improve tracking performance.

Generally, ρ is designed to be a continuous function of the tracking error, $|y - r_f|$, such as,

$$\rho = \begin{cases} \rho_1, & \rho_1 \geq 1/2 \\ 1/2, & \rho_1 < 1/2 \end{cases}$$
$$\rho_1 = \frac{\alpha}{1 + e^{\beta|y - r_f|}} \tag{25}$$

where $\alpha > 0$ and $\beta > 0$ are parameters which can be determined based on the specific application. With the specific VF, low damping-ratio is produced to achieve fast rising time when the tracking error is large and higher damping-ratio is given to remove the overshoot caused by the low damping-ratio when the output approaches the target of a step reference. As the VF is a continuous function, there is no switch happened in tracking process.

As the relationship between the damping-ratio and the tracking performance is well defined in second-order linear time-invariant systems, the above-mentioned guideline is good to design the variant factor for second-order linear time-invariant systems.

However, it is not easy to identify explicitly the relationship between the damping-ratio and the tracking performance in high-order linear time-invariant systems. Nonetheless, the guideline is still applicable in principle.

A VF will be designed in the next section as VF is applied to design a control law for an illustrative example.

3. An Illustrative Example

An illustrative example is given to show the control design with VF for an SISO affine nonlinear system with input saturation. Simulations will be made to compare the tracking performance between the closed-loop system resulting from VF and that from the dynamic inversion.

The SISO affine nonlinear system with input saturation is presented as follows:

$$\begin{cases} \dot{x} = f(x) + g(x)\mathrm{sat}(u) \\ y = h(x) \end{cases} \tag{26}$$

where $x \in \Re^3$, $u \in \Re$ and $y \in \Re$ are state, control and output variables, respectively. sat is a function defined in (2) and $\overline{u} = 1$.

$$x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad f(x) = \begin{pmatrix} -x_1 s_2 s_3 c_3 \\ -c_2 s_3 c_3^{-1} + x_1 s_2^2 s_3 \\ s_2 s_3^2 + x_1 s_2 c_2 c_3 \end{pmatrix}$$

$$g(x) = \begin{pmatrix} -x_1^2 s_3 c_3^2 \\ 0 \\ -x_1 c_3^3 \end{pmatrix}, \quad h(x) = x_1 s_2 c_3$$

$$s_2 := \sin x_2, \quad c_2 := \cos x_2$$
$$s_3 := \sin x_3, \quad c_3 := \cos x_3$$

By following procedures of control design with dynamic inversion, we can easily calculate that the relative dynamic degree of (26) is 2 and construct a transformation as follows,

$$\begin{pmatrix} z \\ z_0 \end{pmatrix} = \phi(x) = x_1 \begin{pmatrix} s_2 c_3 \\ -s_3 \\ c_2 c_3 \end{pmatrix}, \quad x \in X \subset \Re^3$$

$$\begin{pmatrix} z \\ z_0 \end{pmatrix} \in Z \times Z_0 \subset \Re^3, \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := z$$

$$X := \left\{ x \; : \; \begin{array}{l} x_1 \in [5\sqrt{2}, 100], \\ x_2 \in [-2\pi, 2\pi], \\ x_3 \in [-0.49\pi, 0.49\pi] \end{array} \right\} \tag{27}$$

$$Z \times Z_0 := \left\{ \begin{pmatrix} z \\ z_0 \end{pmatrix} : \begin{array}{l} z_1^2 + z_2^2 + z_0^2 \in [50, 100^2], \\ z_1^2 + z_0^2 \in [50 c_{49}^2, 100^2], \\ z_2^2 \leq 100^2 s_{49}^2 \end{array} \right\}$$

where $s_{49} = \sin 0.49\pi$ and $c_{49} = \cos 0.49\pi$. $(z' \quad z_0)' = \phi(x)$ can be checked non-singular from X to $Z \times Z_0$ and (26) can be converted with (27) as follows,

$$\begin{cases} \dot{z} = Az + bv \\ y = Cz \\ \dot{z}_0 = z_1 z_2 \end{cases} \quad (28)$$

where $v := -z_1 z_0 + (z_1^2 + z_0^2)\text{sat}(u)$ and:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b - \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0]$$

$z_0 \in \Re$ is the state variable of the zero dynamics of (28), which is stable as z_2 should be zero in steady states.

A point $x_n = (10 \quad 0.25\pi \quad 0)' \in X$ is chosen as the nominal point for control design with VF so that $z_n = (5\sqrt{2} \quad 0)'$ and $z_{0,n} = 5\sqrt{2}$. A control law is designed with VF for (26) at the chosen nominal point as follows,

Table 1
Tracking Performance

Initial Point x_0	Settling Time (s) VF/DI	Over-Shoot (%) VF/DI
$(10 \quad 0.25\pi \quad 0)'$	0.5/0.69	0/0
$(5\sqrt{2} \quad 0.25\pi \quad 0)'$	0.5/0.7	0/0
$(20 \quad 0.25\pi \quad 0)'$	0.5/0.69	0/0

$$\begin{cases} u_{VF} = (z_1^2 + z_0^2)^{-1} \\ \qquad \times [\rho F_{VF}(z - G_e r_f) + z_1 z_0] \\ \begin{pmatrix} z \\ z_0 \end{pmatrix} = \phi(x) \end{cases} \quad (29)$$

where $F_{VF} = -(47.769 \quad 9.7743)$, $G_e' = (1 \quad 0)$ and r_f is the step reference with a amplitude of 1. ρ is designed as a function of $|y - r_f|$ as follows,

$$\rho = \begin{cases} \rho_1, & \rho_1 \geq 1/2 \\ 1/2, & \rho_1 < 1/2 \end{cases}$$
$$\rho_1 = \frac{4}{1 + e^{5|y - r_f|}} \quad (30)$$

For comparison between dynamic inversion and VF, another control law is designed with dynamic inversion for (26) as follows,

$$\begin{cases} u_{DI} = (z_1^2 + z_0^2)^{-1} \\ \qquad \times [F_{DI}(z - G_e r_f) + z_1 z_0] \\ \begin{pmatrix} z \\ z_0 \end{pmatrix} = \phi(x) \end{cases} \quad (31)$$

where $F_{DI} = -(47.769 \quad 13.823)$.

Simulation is made in three initial points in which the output is controlled to track a step reference with amplitude of 1. The tracking performance is summarized in Table 1 and system responses are shown in Figs. 1–4. The

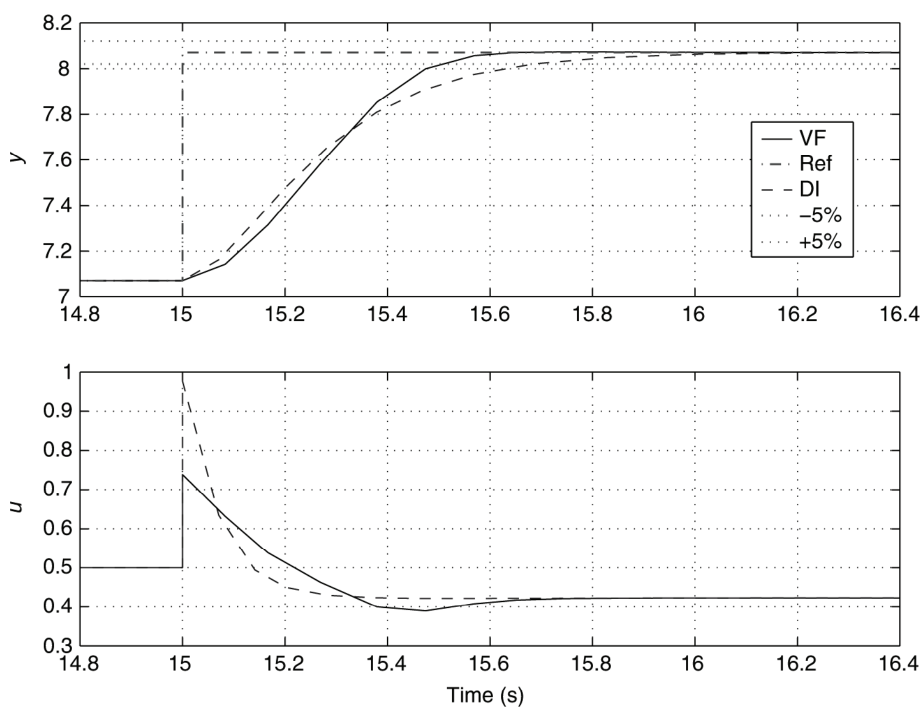

Figure 1. Simulation in the nominal point, $x_0 = (10, 0.25\pi, 0)'$: output response and control signal.

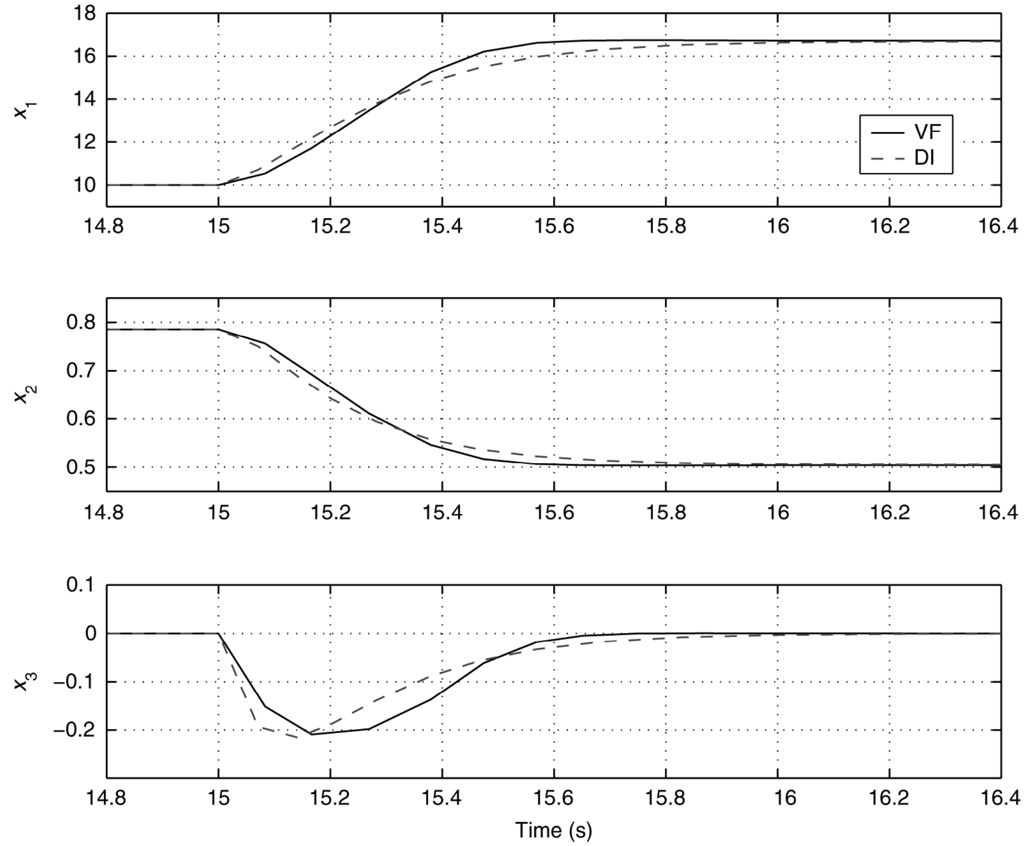

Figure 2. Simulation in the nominal point, $x_0 = (10, 0.25\pi, 0)'$: state responses.

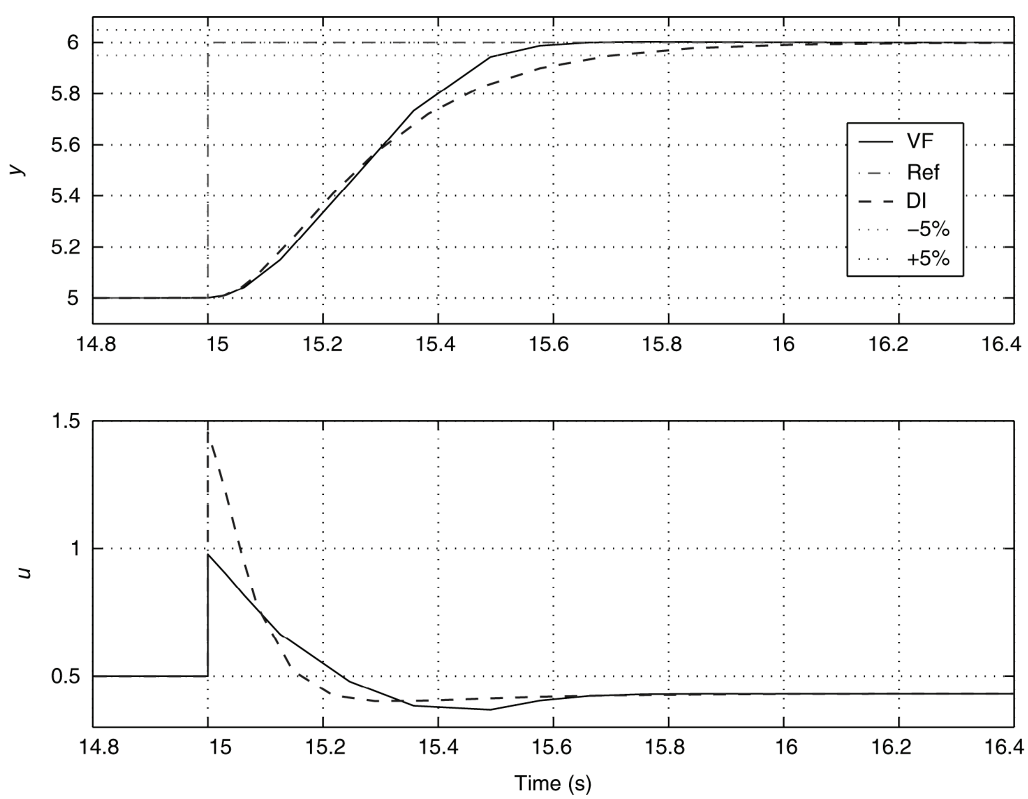

Figure 3. Simulation in non-nominal point, $x_0 = (5\sqrt{2}, 0.25\pi, 0)'$.

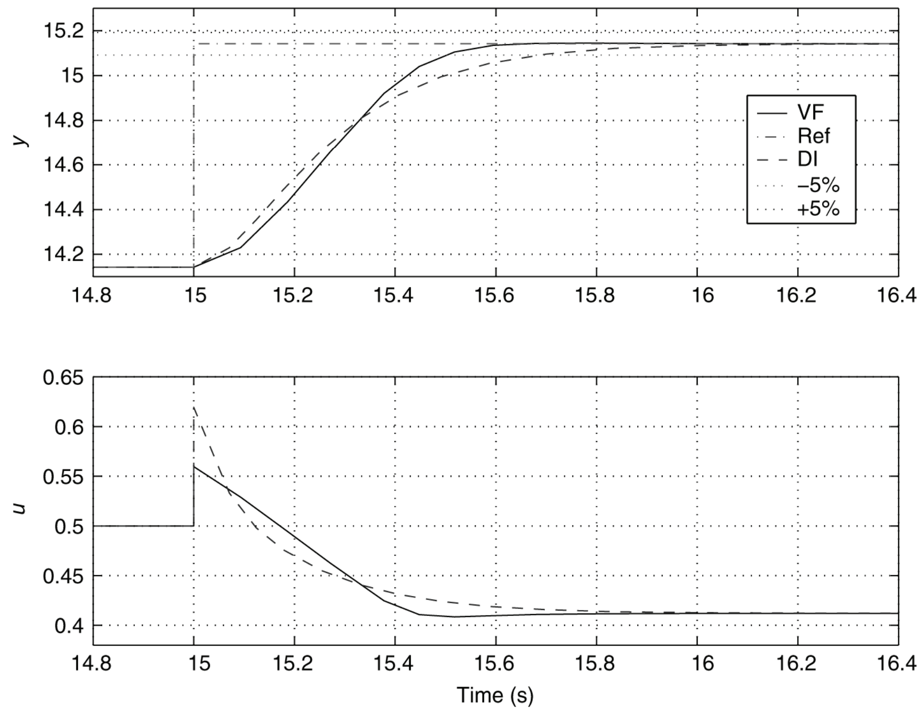

Figure 4. Simulation in non-nominal point, $x_0 = (20, 0.25\pi, 0)'$.

simulation results demonstrate that the tracking performance of the resulting closed-loop system with the control law, (29), designed with VF is better than those with the control law, (31), designed with dynamic inversion.

4. Conclusion

The VF technique has been developed for the tracking control design of a class of the SISO affine nonlinear systems with input saturation. The control law designed with VF at the chosen nominal point is applicable to other defined points and the resulting closed-loop system can achieve better tracking performance than that with the control law designed with the dynamic inversion. Nonetheless, there is still much theoretical research left to be studied such as design of the VF in the high-order LTI systems.

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