Assignment of Complete Structural Properties of Linear Systems via Sensor Selection

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***Abstract*—For $\dot{x} = Ax + Bu$, the problem of structural assignment via sensor selection is to find an output equation, $y = Cx + Du$, such that the resulting system (A, B, C, D) has the pre-specified structural properties, such as the finite and infinite zero structures as well as the invertibility properties. In this paper, we establish a set of necessary and sufficient conditions under which a complete set of system structural properties can be assigned, and an explicit algorithm for constructing the required matrix pair (C, D).**

***Index Terms*—Actuator selection, finite zeros, infinite zeros, Kronecker invariants, linear systems, sensor selection, structural assignment.**

I. Introduction

THE problem of assigning structural properties of a linear system via sensor selection [1] is, for a linear system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{1}$$

to find a system output

$$y = Cx + Du \tag{2}$$

such that the resulting system (A, B, C, D) has all the pre-specified structural properties, such as the finite and infinite zero structures and the invertibility properties [2]. Such a problem is also referred to as the sensor selection problem. Another problem that is dual to the sensor selection problem is the actuator selection problem, which is, for a given matrix pair (A, C), to find a matrix pair (B, D) such that the resulting system (A, B, C, D) has the desired structural properties.

Recall that the system matrix pencil $P_\Sigma(s)$ is defined for the system Σ characterized by (1), (2) or the quadruple (A, B, C, D)

$$P_\Sigma(s) = \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}. \tag{3}$$

The structural assignment problem can thus be viewed as a matrix pencil completion problem. That is, for given A and B, find C and D such that $P_\Sigma(s)$ has the pre-specified Kronecker invariants [3].

Solutions to the sensor and actuator selection problems would build a linkage between achievable control performances and hardware implementation, and provide a foundation upon which trade-offs can be incorporated in an early stage of overall engineering design process. Traditionally, control theory views actuators and sensors as a part of system dynamics and focuses only on analysis and control design for the system under a given set of actuators and sensors located at their fixed locations. It is now widely recognized that achievable control performances hinge on the selection of sensors and actuators along with their locations, which together with the plant dynamics, determine the structural properties of the overall system. Indeed, very often, significant performance improvement can be achieved by simple relocation of some of the sensors and actuators. For example, it is well understood that it is troublesome to deal with systems with nonminimum-phase finite zeros in control system design. However, the designer is fortunately able to remove the troublesome nonminimum-phase finite zeros and obtain better performance by appropriately adding or relocating sensors or actuators.

The selection of sensors and actuators and their locations also arises in a variety of other applications, such as flexible structures [4]–[7], distributed processes [8]–[10], wireless networks [11], fault detection and isolation [12], and maneuvering targets tracking [13].

The complete set of invariants of matrix pencils under nonsingular transformations are captured by Kronecker invariants as finite and infinite elementary divisors, and column and row minimal indices [3]. A numerically stable algorithm for computing Kronecker invariants can be found in [14]. In 1973, by taking a geometric approach, Morse [2] established that the structure of a linear system is completely characterized by a set of invariant factors $\mathcal{I}_1$ and three sets of integers, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$, all of which are invariant under nonsingular state, input and output transformations, state feedback and output injection. In particular, the invariant factors $\mathcal{I}_1$ represents the finite zero structure of the system, $\mathcal{I}_4$ represents its infinite zero structure, and $\mathcal{I}_2$ and $\mathcal{I}_3$ characterize its right and left invertibility properties respectively. We note that Morse index lists $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ coincide with the Kronecker invariants of system matrix pencil (3). In particular, $\mathcal{I}_1$ is the finite elementary divisors, $\mathcal{I}_2$ and $\mathcal{I}_3$ are respectively the column and row minimal indices, and $\mathcal{I}_4$ is related to the infinite elementary divisors.

The problem of structural assignment was first studied by Rosenbrock in [1], in which finite zeros are assigned by choosing the output matrices. Indeed, most results on the

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structural assignment have pertained to the assignment of finite zero (invariant zero or transmission zero) structures (see, e.g., squaring down [15]–[19], feedforward [20], structured additive transformations [21]). The finite zero assignment can be treated as a pole assignment problem with state or output feedback [22]–[24]. Boley and Dooren [25] studied the problem of zero placement for an arbitrary matrix pencil by the addition of new rows or columns and shown how additional rows or columns can be appended to place as many zeros as possible.

In 1995, [26] proposed a technique which is capable of simultaneously assigning finite and infinite zero structures. Recently, we made an attempt to deal with the assignment of a complete set of system structures, including finite and infinite zero structures and invertibility structures [27]. In particular, in [27], we identified a set of sufficient conditions, and under these conditions developed an algorithm that leads to the assignment of a complete set of structural properties.

The structural assignment problem was solved in [28] in terms of homogeneous invariant factors when the underlying field is infinite. However, as pointed out in [29], it is very difficult to extract from the relations provided in [28] a set of necessary and sufficient conditions that ensure the existence of C and D such that the system (A, B, C, D) has the prescribed infinite elementary divisors. Motivated by this observation, [29] adopted a similar strategy as in [1] to present a set of necessary and sufficient conditions under which an infinite zero structure can be assigned. More recently in [30], we established a set of necessary and sufficient conditions for the assignability of a set of structural properties which includes finite zeros, infinite zeros and row minimal indices, and provided an explicit algorithm to construct the required matrices.

We also note that there are many results on a related matrix pencil completion problem [31], [32]. This problem is, for $E, A \in \mathbb{R}^{n\times p}$, to find matrix pencils $H_{12}(s)$, $H_{21}(s)$ and $H_{22}(s)$, such that

$$\begin{bmatrix} sE-A & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

has the pre-specified Kronecker invariants. Dodig in [33], [34] gave a set of simple and explicit necessary and sufficient conditions for the existence of a matrix pencil with prescribed Kronecker invariants and a regular subpencil. These results in general do not apply to the structural property assignment problem considered in this paper, in which only constant matrices C and D, rather than matrix pencils $H_{12}(s)$, $H_{21}(s)$ and $H_{22}(s)$, can be selected.

In this paper, we will establish a set of necessary and sufficient conditions for the assignability of a complete set of structural properties, including the finite and infinite zeros properties and the invertibility properties, and will develop a numerical algorithm for the explicit construction of the required pair (C, D). As a result, we give a complete solution to sensor and actuator selection problems.

The remainder of this paper is organized as follows. Section II includes some background materials. Section III presents some preliminary results which will lead to our main results in Section IV. Section V contains some examples that illustrate various aspects of the results of this paper. Section VI concludes the paper.

Throughout the paper, we use $x = \{x_1, x_2, \cdots, x_n\}$ to denote a set and $x = [x_1, x_2, \cdots, x_n]$ an ordered set. Set minus between two sets x and y is denoted as $x \setminus y$. For any $x = \{x_1, x_2, \cdots, x_n\} \in \mathbb{R}^n$, $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ denotes the elements of x in the non-increasing order. Similarly, $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ denotes the elements of x in the non-decreasing order. For two polynomials α and β, $\alpha|\beta$ denotes "α divides β," and $d(\alpha)$ denotes the degree of α. For an integer k, denote $\varrho_k = [1\ 0_{1\times(k-1)}] \in \mathbb{R}^{1\times k}$

$$\vartheta_k = \begin{bmatrix} 0_{(k-1)\times 1} \\ 1 \end{bmatrix} \in \mathbb{R}^k, \quad \aleph_k = \begin{bmatrix} 0 & I_{k-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}.$$

II. Background Materials

Definition 2.1: [35] For $x, y \in \mathbb{R}^n$, $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]},\ k = 1, 2, \cdots, n-1,\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$$

or, equivalently

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)},\ k = 1, 2, \cdots, n-1,\ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

When $x \prec y$, x is said to be majorized by y (or, y majorizes x). This notation and terminology were originally introduced in [36].

Definition 2.2: [35] For $x, y \in \mathbb{R}^n$, $x \prec^w y$, if

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \quad k = 1, 2, \cdots, n.$$

When $x \prec^w y$, x is said to be weakly supermajorized by y. Equivalently, we write $y \succ^w x$.

Definition 2.3: For $x, y \in \mathbb{R}^n$, $x \leq y$, if

$$x_{(i)} \leq y_{(i)}, \quad i = 1, 2, \cdots, n.$$

Next we recall the equivalence of matrix pencils [3]. For a matrix pencil $sM - N$, there exist nonsingular matrices $\tilde{Q}$ and $\tilde{P}$ such that

$$\tilde{Q}(sM-N)\tilde{P} = \begin{bmatrix} \text{blkdiag}\left\{sI-J, L_{l_1}, \cdots, L_{l_{p_b}}, R_{r_1}, \cdots, R_{r_{m_c}}, I-sH\right\} & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

where J is in the Jordan canonical form, and $sI - J$ has the following $\Sigma_{i=1}^{\delta} d_i$ pencils as its diagonal blocks

$$sI_{m_{i,j}} - J_{m_{i,j}}(\beta_i) := \begin{bmatrix} s-\beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\beta_i & -1 \\ & & & s-\beta_i \end{bmatrix}$$

$j=1,2,\cdots,d_i$, $i=1,2,\cdots,\delta$. L_{l_i}, $i=1,2,\cdots,p_{\rm b}$, is an $(l_i+1)\times l_i$ bidiagonal pencil, and R_{r_i}, $i=1,2,\cdots,m_{\rm c}$, is an $r_i\times(r_i+1)$ bidiagonal pencil, i.e.,

$$L_{l_i}:=\begin{bmatrix}-1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s\end{bmatrix},\quad R_{r_i}:=\begin{bmatrix}s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1\end{bmatrix}.$$

Finally, H is nilpotent and in Jordan canonical form, and $I-sH$ has the following $m_{\rm d}$ pencils as its diagonal blocks

$$I_{n_j+1}-sJ_{n_j+1}(0):=\begin{bmatrix}1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1\end{bmatrix}$$

$j=1,2,\cdots,m_{\rm d}$. Then, invariant factors of J are finite elementary divisors. The sets $\{r_1,r_2,\cdots,r_{m_{\rm c}}\}$ and $\{l_1,l_2,\cdots,l_{p_{\rm b}}\}$ are column and row minimal indices, respectively. Lastly, $\{(1/s)^{n_j+1}, j=1,2,\cdots,m_{\rm d}\}$ are the infinite elementary divisors. The form (4) is called the Kronecker canonical form.

In what follows, we recall the controllability indices and the nice basis indices of a pair (A,B) with $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$. There exists only finite elementary divisors and row minimal indices in the matrix pencil $[sI-A\ \ -B]$. The zeros of finite elementary divisors are the set of uncontrollable modes, while row minimal indices are the controllability indices of the pair (A,B). The controllability indices can also be computed from

$$\Xi(A,B,j)=[b_1b_2\cdots b_m\vdots Ab_1Ab_2\cdots Ab_m\vdots\cdots \cdots\vdots A^{j-1}b_1A^{j-1}b_2\cdots A^{j-1}b_m]$$

where b_i is the ith column of B, and j is a non-negative integer. Search for linearly independent columns of $\Xi(A,B,n)$ from left to right and rearrange them as

$$b_1Ab_1\cdots A^{k_1-1}b_1, b_2Ab_2\cdots A^{k_2-1}b_2,\cdots \cdots, b_mAb_m\cdots A^{k_m-1}b_m.$$

The controllability indices of the pair (A,B) are defined as $k=\{k_1,k_2,\cdots,k_m\}$. If $\Sigma_{i=1}^m k_i=n$, the system (A,B) is controllable.

Consider an ordered set of non-negative integers $\tau=[\tau_1,\tau_2,\cdots,\tau_m]$, we define a function Θ as

$$\Theta(A,B,\tau)=[b_1Ab_1\cdots A^{\tau_1-1}b_1\vdots b_2Ab_2\cdots A^{\tau_2-1}b_2\vdots\cdots \cdots\vdots b_mAb_m\cdots A^{\tau_m-1}b_m].$$

When $\tau_i=0$, items related to b_i are eliminated from $\Theta(A,B,\tau)$.

Definition 2.4: [37] The ordered set of non-negative integers $r=[r_1,r_2,\cdots,r_m]$ is called the indices of a nice basis associated with a controllable pair (A,B) if $\Theta(A,B,r)$ is nonsingular.

It is obvious that if r is the indices of a nice basis, then $\sum_{i=1}^m r_i=n$.

We next recall some properties of the controllability indices k of the pair (A,B). Define

$$\xi_j=\text{rank}\left(\Xi(A,B,j)\right),\quad j=1,\cdots,n$$

and let $\xi_0=0$. We have

$$\begin{aligned}\xi_j-\xi_{j-1}&=\text{card}\left\{i\in\{1,2,\cdots,m\}:k_i\ge j\right\}\\ 2\xi_j-\xi_{j+1}-\xi_{j-1}&=\text{card}\left\{i\in\{1,2,\cdots,m\}:k_i=j\right\}.\end{aligned}$$

It means that the controllability indices can be determined by ξ_j, $j=0,1,\cdots,n$. It can be verified that the following equation:

$$\begin{aligned}\text{rank}\left(\Xi(A,B,j)\right)&=\text{rank}\left(\Xi\left(T_{\rm S}^{-1}AT_{\rm S},T_{\rm S}^{-1}BT_{\rm I},j\right)\right)\\ &=\text{rank}\left(\Xi(A-BK,B,j)\right)\\ &\quad j=1,2,\cdots,n\end{aligned}$$

holds for nonsingular $T_{\rm S}\in\mathbb{R}^{n\times n}$ and $T_{\rm I}\in\mathbb{R}^{m\times m}$ and state feedback $K\in\mathbb{R}^{m\times n}$. Thus, the controllability indices of (A,B) are invariant under state and input transformations and state feedback.

Lemma 2.1: For $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$, there exist nonsingular $T_{\rm S}\in\mathbb{R}^{n\times n}$ and $T_{\rm I}\in\mathbb{R}^{m\times m}$, and state feedback $K\in\mathbb{R}^{m\times n}$ such that

$$T_{\rm S}^{-1}[A|B]\begin{bmatrix}T_{\rm S} & 0\\ K & T_{\rm I}\end{bmatrix}=\left[\begin{array}{cc|cc}A_0 & 0 & 0 & 0\\ 0 & A_\gamma^\star & 0 & B_\gamma\end{array}\right]\qquad(5)$$

with

$$\begin{aligned}A_\gamma^\star&=\text{blkdiag}\left\{\aleph_{k_{h_0+1}},\aleph_{k_{h_0+2}},\cdots,\aleph_{k_m}\right\}\\ B_\gamma&=\text{blkdiag}\left\{\vartheta_{k_{h_0+1}},\vartheta_{k_{h_0+2}},\cdots,\vartheta_{k_m}\right\}\end{aligned}$$

where $\lambda(A_0)$ is the set of uncontrollable modes, and $k=\{k_1,k_2,\cdots,k_m\}$ is the controllability indices with $0=k_1=\cdots=k_{h_0}<k_{h_0+1}\le\cdots\le k_m$. □

The controllability indices and the nice basis indices of a matrix pair have the following majorization relationship.

Lemma 2.2: [38] Consider a controllable pair (A,B) with controllability indices k. Let r be the indices of a nice basis associated with (A,B). Then, $k\prec r$. □

The following lemma gives the relationship between invariant factors and the eigenvalue structure of a matrix.

Lemma 2.3: [3] Let $A\in\mathbb{C}^{n\times n}$ and its eigenvalues be λ_i, $i=1,2,\cdots,\varsigma$, with the sizes of their Jordan blocks being $n_{i,j}$, $j=1,2,\cdots,\tau_i$, $i=1,2,\cdots,\varsigma$, where $n_{i,1}\ge n_{i,2}\ge\cdots\ge n_{i,\tau_i}$. Then, the invariant factors of A are given by

$$\begin{aligned}\alpha_j&=1,\ \ j=1,2,\cdots,n-\max\{\tau_1,\tau_2,\cdots,\tau_\varsigma\};\\ \alpha_{n-j+1}&=\prod_{i=1}^{\varsigma}(s-\lambda_i)^{n_{i,j}},\ \ j=1,2,\cdots,\max\{\tau_1,\tau_2,\cdots,\tau_\varsigma\}\end{aligned}$$

where $n_{i,j}=0$ if $j>\tau_i$. □

For an $A\in\mathbb{R}^{n\times n}$, its eigenvalues are self-conjugated. Thus its invariant factors α_i, $i=1,2,\cdots,n$, have real coefficients, and their factors are $(s+\mu_j)$ or $(s+\mu_j)^2+\omega_j^2$, where $\mu_j,\omega_j\in\mathbb{R}$.

Lemma 2.4: [39] Let $A\in\mathbb{R}^{n\times n}$ and $\alpha_1|\alpha_2|\cdots|\alpha_n$ be its invariant factors. Then, there exists a $B\in\mathbb{R}^{n\times m}$ such that the pair (A,B) is controllable with controllability indices $k=$

$\{0,\cdots,0,k_1,k_2,\cdots,k_r\}$, where $k_1,k_2,\cdots,k_r$ are positive, if and only if

$$\alpha_i=1,\quad i=1,2,\cdots,n-r;$$
$$\{k_1,k_2,\cdots,k_r\}\prec\{d(\alpha_{n-r+1}),d(\alpha_{n-r+2}),\cdots,d(\alpha_n)\}.$$

□

Indeed, the conditions in Lemma 2.4 are equivalent to

$$0_{n-m}\cup k\prec\{d(\alpha_1),d(\alpha_2),\cdots,d(\alpha_n)\}.$$

Next, we recall the structural decomposition of the system (A,B,C,D). Sannuti and Saberi [40], [41] developed an algorithm to construct input, state and output transformations that decompose the system into a normal form, which explicitly displays all the structural properties as identified by Morse [2]. A toolkit [42] in MATLAB environment containing such a normal form is currently available online. The implemented in the toolkit [42] is based on a numerically stable algorithm recently reported in [43], together with an enhanced procedure reported in [44].

By [2], [40], [41], we have the following result.

Lemma 2.5: Consider

$$\Sigma:\quad\begin{cases}\dot{x}=Ax+Bu\\ y=Cx+Du\end{cases}\tag{6}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and $y\in\mathbb{R}^p$. There exist nonsingular transformations $\Gamma_{\rm S}\in\mathbb{R}^{n\times n}$, $\Gamma_{\rm O}\in\mathbb{R}^{p\times p}$ and $\Gamma_{\rm I}\in\mathbb{R}^{m\times m}$, and feedback $K\in\mathbb{R}^{m\times n}$ such that (See equation at bottom of page)

$$\begin{bmatrix}\Gamma_{\rm S}^{-1}&0\\0&\Gamma_{\rm O}^{-1}\end{bmatrix}\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]\begin{bmatrix}\Gamma_{\rm S}&0\\K&\Gamma_{\rm I}\end{bmatrix}=\left[\begin{array}{c|c}\tilde{A}_z&\tilde{B}\\\hline\tilde{C}&\tilde{D}\end{array}\right]=$$
$$\left[\begin{array}{cccc|cccc}A_{\rm a}&L_{\rm ab}C_{\rm b}&0&L_{\rm ad}C_{\rm d}&0&B_{0\rm a}&0&0\\0&A_{\rm bb}^{\star}+L_{\rm bb}C_{\rm b}&0&L_{\rm bd}C_{\rm d}&0&B_{0\rm b}&0&0\\0&0&A_{\rm cc}^{\star}&L_{\rm cd}C_{\rm d}&0&B_{0\rm c}&0&B_{\rm c}\\0&0&0&A_{\rm dd}^{\star}+L_{\rm dd}C_{\rm d}&0&B_{0\rm d}&B_{\rm d}&0\\\hline0&0&0&0&0&0&0&0\\0&0&0&0&0&I_{m_0}&0&0\\0&0&0&C_{\rm d}&0&0&0&0\\0&C_{\rm b}&0&0&0&0&0&0\end{array}\right]\tag{7}$$

where

$$A_{\rm bb}^{\star}={\rm blkdiag}\left\{\aleph_{l_1},\aleph_{l_2},\cdots,\aleph_{l_{p_{\rm b}}}\right\}$$
$$C_{\rm b}={\rm blkdiag}\left\{\varrho_{l_1},\varrho_{l_2},\cdots,\varrho_{l_{p_{\rm b}}}\right\}$$
$$A_{\rm cc}^{\star}={\rm blkdiag}\left\{\aleph_{r_1},\aleph_{r_2},\cdots,\aleph_{r_{m_{\rm c}}}\right\}\tag{8}$$
$$B_{\rm c}={\rm blkdiag}\left\{\vartheta_{r_1},\vartheta_{r_2},\cdots,\vartheta_{r_{m_{\rm c}}}\right\}\tag{9}$$
$$A_{\rm dd}^{\star}={\rm blkdiag}\left\{\aleph_{q_1},\aleph_{q_2},\cdots,\aleph_{q_{m_{\rm d}}}\right\}\tag{10}$$
$$B_{\rm d}={\rm blkdiag}\left\{\vartheta_{q_1},\vartheta_{q_2},\cdots,\vartheta_{q_{m_{\rm d}}}\right\}\tag{11}$$
$$C_{\rm d}={\rm blkdiag}\left\{\varrho_{q_1},\varrho_{q_2},\cdots,\varrho_{q_{m_{\rm d}}}\right\}.\tag{12}$$

□

Remark 2.1: The finite zero structure of Σ (Morse index $\mathcal{I}_1$) is given by the invariant factors of $A_{\rm aa}$. The left invertibility structure ($\mathcal{I}_3$) is given by $\{l_1,l_2,\cdots,l_{p_{\rm b}}\}$, and the right invertibility structure ($\mathcal{I}_2$) is given by $\{r_1,r_2,\cdots,r_{m_{\rm c}}\}$. The system Σ has $m_0={\rm rank}(D)$ infinite zeros of order 0. The infinite zeros of orders greater than 0 ($\mathcal{I}_4$) are given by $\{q_1,q_2,\cdots,q_{m_{\rm d}}\}$. That is, each q_i corresponds to an infinite zero of order q_i. Also, Σ is left invertible if $\mathcal{I}_2$ is empty, right invertible if $\mathcal{I}_3$ is empty, invertible if both $\mathcal{I}_2$ and $\mathcal{I}_3$ are empty, and degenerate if both $\mathcal{I}_2$ and $\mathcal{I}_3$ are present.

Remark 2.2: Based on (7), the Kronecker canonical form of the system matrix pencil can be computed easily. In particularly, We first apply output injection $F\in\mathbb{R}^{n\times p}$ to the system, i.e. left multiple both sides of (7) with

$$\begin{bmatrix}I&F\\0&I\end{bmatrix}$$

to remove the terms $L_{\rm ab}C_{\rm b}$, $L_{\rm bb}C_{\rm b}$, $L_{\rm ad}C_{\rm d}$, $L_{\rm bd}C_{\rm d}$, $L_{\rm cd}C_{\rm d}$, $L_{\rm dd}C_{\rm d}$, $B_{0\rm a}$, $B_{0\rm b}$, $B_{0\rm c}$ and $B_{0\rm d}$ in (7), and then use permute operations in columns and rows. In Kronecker canonical form, finite elementary divisors are given by the invariant factors of $A_{\rm aa}$, row minimal indices are given by $\{l_1,l_2,\cdots,l_{p_{\rm b}}\}$, column minimal indices given by $\{r_1,r_2,\cdots,r_{m_{\rm c}}\}$, and the infinite elementary divisors $\{\overbrace{1/s,\cdots,1/s}^{m_0},(1/s)^{q_1+1},(1/s)^{q_2+1},\cdots,(1/s)^{q_{m_{\rm d}}+1}\}$ [44].

III. PRELIMINARY RESULTS

The problem of assigning controllability indices is, for a given A, to find a B, such that the pair (A,B) has the prescribed controllability indices and the uncontrollable mode structure. Based on the invariant factors of a matrix, Zaballa [39] identified a set of necessary and sufficient conditions under which the controllability indices are assignable. In what follows, we establish a set of necessary and sufficient conditions for (A,B) to have the prescribed controllability indices in term of the eigenstructure of A. Such a new approach to establishing necessary and sufficient conditions will facilitate our development of an explicit algorithm for structural assignment in the next section. By Lemmas 2.3 and 2.4, we have the following lemma.

Lemma 3.1: Let $A\in\mathbb{R}^{n\times n}$ and its eigenvalues be λ_i, $i=1,2,\cdots,\varsigma$, with the sizes of their Jordan blocks being $n_{i,j}$, $j=1,2,\cdots,\tau_i$, $i=1,2,\cdots,\varsigma$, where

$$n_{i,1}\geq n_{i,2}\geq\cdots\geq n_{i,\tau_i}\tag{13}$$

and let k be a set with m non-negative integers and $\sum_{i=1}^m k_i=n$. Then, there exists a $B\in\mathbb{R}^{n\times m}$ such that the pair (A,B) has controllability indices k if and only if

$$k\prec\left\{\sum_{i=1}^{\varsigma}n_{i,1},\sum_{i=1}^{\varsigma}n_{i,2},\cdots,\sum_{i=1}^{\varsigma}n_{i,m}\right\}\tag{14}$$

where undefined $n_{i,j}$'s are set to be zero. □

Note that (14) implies that $\max\{\tau_1,\tau_2,\cdots,\tau_\varsigma\}\leq m$.

Consider an $A\in\mathbb{R}^{n\times n}$ as in Lemma 3.1, the algebraic multiplicity of its eigenvalue λ_i is $m_i=\sum_{t=1}^{\tau_i}n_{i,t}$. Thus, the set of eigenvalues of A, including repeated ones, is given by

$$\lambda(A)=\{\overbrace{\lambda_1,\cdots,\lambda_1}^{m_1},\overbrace{\lambda_2,\cdots,\lambda_2}^{m_2},\cdots,\overbrace{\lambda_\varsigma,\cdots,\lambda_\varsigma}^{m_\varsigma}\}.$$

Let Λ_1 be a self-conjugated subset of $\lambda(A)$

$$\Lambda_1 = \{\overbrace{\lambda_1, \cdots, \lambda_1}^{\hat{m}_1}, \overbrace{\lambda_2, \cdots, \lambda_2}^{\hat{m}_2}, \cdots, \overbrace{\lambda_\varsigma, \cdots, \lambda_\varsigma}^{\hat{m}_\varsigma}\}$$

where $\hat{m}_i \le m_i$. There exists a $\bar{T} \in \mathbb{R}^{n\times n}$ such that

$$\bar{T}^{-1}A\bar{T} = \begin{bmatrix} \bar{A}_1 & * \\ 0 & \bar{A}_2 \end{bmatrix} \tag{15}$$

with $\lambda(\bar{A}_1) = \Lambda_1$.

Now we consider a special decomposition in the form of (15). There exists a $T_1 \in \mathbb{C}^{n\times n}$ such that

$$T_1^{-1}AT_1 = \mathrm{blkdiag}\{J_1, J_2, \cdots, J_\varsigma\}$$

with

$$\begin{aligned} J_i &= \mathrm{blkdiag}\{J_{i,\tau_i}, J_{i,\tau_i-1}, \cdots, J_{i,1}\} \\ J_{i,j} &= \lambda_i I_{n_{i,j}} + \aleph_{n_{i,j}}. \end{aligned}$$

Rewrite

$$J_i = \begin{bmatrix} Z_{i,1} & * \\ 0 & Z_{i,2} \end{bmatrix}, \quad Z_{i,1} \in \mathbb{C}^{\hat{m}_i \times \hat{m}_i}.$$

There exists a permutation matrix T_2 such that

$$(T_1T_2)^{-1}A(T_1T_2) = \begin{bmatrix} Z_1 & * \\ 0 & Z_2 \end{bmatrix}$$

where

$$\begin{aligned} Z_1 &= \mathrm{blkdiag}\{Z_{1,1}, Z_{2,1}, \cdots, Z_{\varsigma,1}\} \\ Z_2 &= \mathrm{blkdiag}\{Z_{1,2}, Z_{2,2}, \cdots, Z_{\varsigma,2}\} \end{aligned}$$

with $\lambda(Z_1) = \Lambda_1$. There exist ι_i, $i = 1, 2, \cdots, \varsigma$, such that

$$\Sigma_{t=1}^{\iota_i-1} n_{i,t} < \hat{m}_i \le \Sigma_{t=1}^{\iota_i} n_{i,t}.$$

Thus, the eigenstructure of Z_2 are given by λ_i, $i = 1, 2, \cdots, \varsigma$, with the sizes of their Jordan blocks being $\mathfrak{y}_{i,j}$, $j = 1, 2, \cdots, \tau_i$, $i = 1, 2, \cdots, \varsigma$, as

$$\begin{aligned} \mathfrak{y}_{i,1} &= n_{i,1}, \cdots, \mathfrak{y}_{i,\iota_i-1} = n_{i,\iota_i-1} \\ \mathfrak{y}_{i,\iota_i} &= \hat{m}_i - \sum_{t=1}^{\iota_i-1} n_{i,t} \\ \mathfrak{y}_{i,\iota_i+1} &= \cdots = \mathfrak{y}_{i,\tau_i} = 0. \end{aligned}$$

It is obvious that $0 < \mathfrak{y}_{i,\iota_i} \le n_{i,\iota_i}$. Since Λ_1 is self-conjugated, there exists a $T \in \mathbb{R}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \tag{16}$$

where the eigenstructure of A_2 is the same as that of Z_2. We denote the eigenstructure of A_2 in (16) as

$$\mathfrak{N}(A, \Lambda_1) = \{\mathfrak{y}_{i,j} | j = 1, 2, \cdots, \tau_i, i = 1, 2, \cdots, \varsigma\}.$$

It is obvious that A_2 in (16) contains Jordan blocks with larger sizes, while A_1 contains Jordan blocks with smaller sizes. Suppose the eigenstructure of $\bar{A}_2$ in (15) are given by λ_i, $i = 1, 2, \cdots, \varsigma$, with the sizes of their Jordan blocks being $\bar{n}_{i,j}$, $j = 1, 2, \cdots, \tau_i$, $i = 1, 2, \cdots, \varsigma$. Then we have the following relationship:

$$\left\{\sum_{i=1}^{\varsigma} \bar{n}_{i,1}, \sum_{i=1}^{\varsigma} \bar{n}_{i,2}, \cdots, \sum_{i=1}^{\varsigma} \bar{n}_{i,\alpha}\right\} \prec \left\{\sum_{i=1}^{\varsigma} \mathfrak{y}_{i,1}, \sum_{i=1}^{\varsigma} \mathfrak{y}_{i,2}, \cdots, \sum_{i=1}^{\varsigma} \mathfrak{y}_{i,\alpha}\right\} \tag{17}$$

where $\alpha = \max\{\tau_1, \tau_2, \cdots, \tau_\varsigma\}$, and the undefined $\mathfrak{y}_{i,j}$'s and $\bar{n}_{i,j}$'s are set to zero.

Theorem 3.1: Let $A \in \mathbb{R}^{n\times n}$ and its eigenvalues be λ_i, $i = 1, 2, \cdots, \varsigma$, with the sizes of their Jordan blocks being $n_{i,j}$, $j = 1, 2, \cdots, \tau_i$, $i = 1, 2, \cdots, \varsigma$. Let Λ_1 be a set of complex scalars and l be a set with m nonnegative integers. Then, there exists a $C \in \mathbb{R}^{m\times n}$ such that the pair (A, C) has the set of unobservable modes Λ_1, and the observability indices l, if and only if $\Lambda_1 \subseteq \lambda(A)$ is self-conjugated, and

$$l \prec \left\{\sum_{i=1}^{\varsigma} \mathfrak{y}_{i,1}, \sum_{i=1}^{\varsigma} \mathfrak{y}_{i,2}, \cdots, \sum_{i=1}^{\varsigma} \mathfrak{y}_{i,m}\right\} \tag{18}$$

where $\{\mathfrak{y}_{i,j}\} = \mathfrak{N}(A, \Lambda_1)$ and the undefined $\mathfrak{y}_{i,j}$'s are set to be zero. □

Proof: Necessity: There exists a $T_{\mathrm{S}} \in \mathbb{R}^{n\times n}$ such that

$$T_{\mathrm{S}}^{-1}AT_{\mathrm{S}} = \begin{bmatrix} A_{11} & * \\ 0 & A_{22} \end{bmatrix}, \quad CT_{\mathrm{S}} = [0 \quad C_2]$$

where (A_{22}, C_2) is observable. It is obvious that the set of unobservable modes $\Lambda_1 = \lambda(A_{11}) \in \lambda(A)$ is self-conjugated.

Denote the eigenvalues of A_{22} by λ_i with the sizes of their Jordan blocks being $\bar{n}_{i,j}$, $j = 1, 2, \cdots, \tau_i$, $i = 1, 2, \cdots, \varsigma$. By Lemma 3.1, we have

$$l \prec \left\{\sum_{i=1}^{\varsigma} \bar{n}_{i,1}, \sum_{i=1}^{\varsigma} \bar{n}_{i,2}, \cdots, \sum_{i=1}^{\varsigma} \bar{n}_{i,m}\right\}.$$

Therefore, (18) is obtained by (17).

Sufficiency: We will give a constructive proof. We decompose A as in (16)

$$T_1^{-1}AT_1 = \begin{bmatrix} A_{11} & * \\ 0 & A_{22} \end{bmatrix}$$

where $\lambda(A_{11}) = \Lambda_1$, and the eigenvalues of A_{22} are given by λ_i with the sizes of their Jordan blocks being $\mathfrak{y}_{i,j}$, $j = 1, 2, \cdots, \tau_i$, $i = 1, 2, \cdots, \varsigma$. By Lemma 3.1, there exists a C_2 such that the pair (A_{22}, C_2) is observable with l as its observability indices. Let $C = T_1[0\ C_2]$. Then, the pair (A, C) has observability indices l and the set of unobservable modes Λ_1. □

Next, we extend the definition of the indices of nice basis to a pair (A, B) that might not be controllable.

Definition 3.1: Consider a pair (A, B) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and controllability indices k. An ordered set of non-negative integers $r = [r_1, r_2, \cdots, r_m]$ is called the indices of a nice basis (of controllable subspace) associated with (A, B), if $\mathrm{rank}(\Theta(A, B, r)) = \sum_{i=1}^{m} r_i = \sum_{i=1}^{m} k_i$.

By decomposing the pair (A, B) into controllable and uncontrollable parts, and using Lemma 2.2, we have the following lemma.

Lemma 3.2: Consider the pair (A,B) with controllability indices k. Let r be the indices of a nice basis (of controllable subspace) associated with (A,B). Then, $k \prec r$. □

In what follows, we show how to extend a full column rank subspace $\Theta(A,B,\tau)$ to a nice basis.

Lemma 3.3: Consider the pair (A,B). Let τ be an ordered set such that $\Theta(A,B,\tau)$ is of full column rank. Then there exist the indices μ of a nice basis of (A,B), such that $\tau \le \mu$. □

Proof: We can extend $\Theta(A,B,\tau)$ to a nice basis of (A,B) in the following way. For b_1, find the smallest μ_1 such that $A^{\mu_1}b_1$ is linearly dependent on $[\Theta(A,B,\tau); A^{\tau_1}b_1, A^{\tau_1+1}b_1, \cdots, A^{\mu_1-1}b_1]$. Then, for b_2, find the smallest μ_2 such that $A^{\mu_2}b_2$ is linearly dependent on $[\Theta(A,B,\tau); A^{\tau_1}b_1, A^{\tau_1+1}b_1, \cdots, A^{\mu_1-1}b_1; A^{\tau_2}b_2, \ A^{\tau_2+1}b_2, \cdots, A^{\mu_2-1}b_2]$. Continue in this way, until we find an m element ordered set μ. Obviously, $\tau \le \mu$. By the construction, $\Theta(A,B,\mu)$ is of full column rank, and $A^{\mu_i}b_i$ is linearly dependent on the columns of $\Theta(A,B,\mu)$, i.e.,

$$A^{\mu_i}b_i = \Theta(A,B,\mu)K_i, \quad K_i \in \mathbb{R}^{\varphi}, \quad i = 1,2,\cdots,m \quad (19)$$

where $\varphi = \sum_{i=1}^{m}\mu_i$. By (19), it can be proven that

$$A^j b_i = \Theta(A,B,\mu)P_{i,j}$$

for $P_{i,j} \in \mathbb{R}^{\varphi}$, $j = \mu_i, \mu_i+1, \cdots, n-1$, $i = 1,2,\cdots,m$. Thus,

$$\sum_{i=1}^{m}\mu_i = \operatorname{rank}(\Theta(A,B,\mu)) = \operatorname{rank}(\Xi(A,B,n)) = \sum_{i=1}^{m}k_i.$$

Therefore, μ is the indices of a nice basis associated with (A,B). □

Next lemma follows from Lemma 3.2 and Lemma 3.3 directly.

Lemma 3.4: Consider the pair (A,B) with controllability indices k. Let τ be an ordered set such that $\Theta(A,B,\tau)$ is of full column rank. Then $k \prec^{w} \tau$. □

The following definition will simplify the description of our main results in the next section.

Definition 3.2: Let $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_{m-s}\}$, $\beta = \{\beta_1, \beta_2, \cdots, \beta_s\}$ and $k = \{k_1, k_2, \cdots, k_m\}$ be three sets of non-decreasing nonnegative integers. Define $\delta = \Delta(\alpha, \beta, k)$ as a reordered set of $\alpha \cup \beta$ relating to k as follows. First, s elements in δ are defined by the elements of β as

$$\sigma_j = \begin{cases} \max\{i : k_i \le \beta_s\}, & j = s \\ \max\{i : k_i \le \beta_j, i < \sigma_{j+1}\}, & j = s-1, s-2, \cdots, 1. \end{cases}$$
$$\delta_{\sigma_j} = \beta_j, \quad j = s, s-1, \cdots, 1.$$

The remaining $m-s$ elements of δ are defined by the elements of α as follows. Let

$$\tau = \{1,2,\cdots,m\} \setminus \{\sigma_1, \sigma_2, \cdots, \sigma_s\}$$

which is in the non-decreasing order, and let

$$\delta_{\tau_j} = \alpha_j, \quad j = 1,2,\cdots,m-s.$$

□

In other words, the ordered set δ are obtained by replacing elements of k with those of α and β. First, starting from the largest to the smallest, replace each element of β for the last original element in k that is not larger than itself. After that, replace the elements of α for the remaining elements of k in the non-decreasing order.

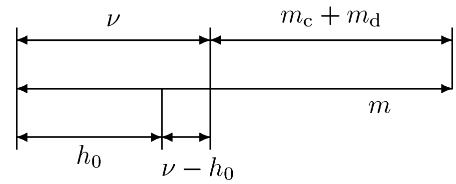

Fig. 1. Relationship among m_c, m_d, h_0, ν and m.

Remark 3.1: Note that $\delta = \Delta(\alpha, \beta, k)$ is well-defined if and only if $\{k_1, k_2, \cdots, k_s\} \le \beta$.

Example 3.1: Let $\alpha = \{1,1,7\}$, $\beta = \{2,4\}$ and $k = \{2,2,3,3,5\}$. We can define $\delta = \Delta(\alpha,\beta,k) = [1,2,1,4,7]$. Similarly, for $\alpha = \{1,2\}$, $\beta = \{4,4\}$ and $k = \{2,2,3,5\}$, $\delta = \Delta(\alpha,\beta,k) = [1,4,4,2]$. On the other hand, if $\alpha = \{1\}$, $\beta = \{2,2\}$ and $k = \{1,3,4\}$, it can be verified that $\delta = \Delta(\alpha,\beta,k)$ is not well-defined.

The following lemma is crucial in developing our main results in the next section.

Lemma 3.5: Consider $\hat{A}_1 \in \mathbb{R}^{n\times n}$ and $\hat{B}_1 \in \mathbb{R}^{n\times m}$

$$\hat{A}_1 = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ E_{\rm ab} & A_{\rm ab} & 0 & L_{\rm abd}C_{\rm d} \\ 0 & 0 & A_{\rm cc}^{\star} & L_{\rm cd}C_{\rm d} \\ 0 & 0 & 0 & A_{\rm dd}^{\star} + L_{\rm dd}C_{\rm d} \end{bmatrix} \quad (20)$$

$$\hat{B}_1 = \begin{bmatrix} 0_{n_0\times h_0} & 0 & 0 & 0 \\ 0 & B_{*0\rm ab} & 0 & 0 \\ 0 & B_{*0\rm c} & 0 & B_{\rm c} \\ 0 & B_{*0\rm d} & B_{\rm d} & 0 \end{bmatrix} \quad (21)$$

where $A_0 \in \mathbb{R}^{n_0\times n_0}$, and $A_{\rm cc}^{\star}$, $B_{\rm c}$, $A_{\rm dd}^{\star}$, $B_{\rm d}$ and $C_{\rm d}$ are in the forms of (8)–(12) with $r_1 \le r_2 \le \cdots \le r_{m_{\rm c}}$, $q_1 \le q_2 \le \cdots \le q_{m_{\rm d}}$. Denote $r = \{r_1, r_2, \cdots, r_{m_{\rm c}}\}$, $q = \{q_1, q_2, \cdots, q_{m_{\rm d}}\}$. Let $k = \{k_1, k_2, \cdots, k_m\}$ be a set of integers with $0 = k_1 = \cdots = k_{h_0} < k_{h_0+1} \le k_{h_0+2} \le \cdots \le k_m$. Let $\nu = m - m_{\rm c} - m_{\rm d}$. Then, there exist $A_{\rm ab}$, $L_{\rm abd}$, $L_{\rm cd}$, $L_{\rm dd}$, $B_{*0\rm ab}$, $B_{*0\rm c}$ and $B_{*0\rm d}$, such that $(\hat{A}_1, \hat{B}_1)$ has controllability indices k and the set of uncontrollable modes $\lambda(A_0)$ if

1) The ordered set $\delta = \Delta(0_\nu \cup q, r, k)$ is well-defined and

$$\sum_{i=1}^{j}\delta_i \le \sum_{i=1}^{j}k_i, \quad j = 1,2,\cdots,m; \quad (22)$$

2) $n_0 + \sum_{i=1}^{m}k_i = n$.

□

Proof: Inequality (22) implies that $m_{\rm c} + m_{\rm d} \le m - h_0$. Thus, $h_0 \le \nu$ (see Fig. 1). The controllability indices of $(\hat{A}_1, \hat{B}_1)$ depend on the choice of $A_{\rm ab}$, $L_{\rm abd}$, $L_{\rm cd}$, $L_{\rm dd}$ $B_{*0\rm ab}$, $B_{*0\rm c}$ and $B_{*0\rm d}$. If all these matrices are equal to zero, then the controllability indices of $(\hat{A}_1, \hat{B}_1)$ are the set $0_\nu \cup q \cup r$.

The m element set δ are defined as follows. The $m_{\rm c}$ elements of δ are first defined from r:

$$\sigma_j = \begin{cases} \max\{i : k_i \le r_{m_{\rm c}}\}, & j = m_{\rm c} \\ \max\{i : k_i \le r_j, i < \sigma_{j+1}\}, & j = m_{\rm c}-1, m_{\rm c}-2, \cdots, 1. \end{cases}$$
$$\delta_{\sigma_j} = r_j, \quad j = m_{\rm c}, m_{\rm c}-1, \cdots, 1.$$

The remaining $m - m_{\rm c}(= \nu + m_{\rm d})$ of δ are defined as follows. Let

$$\tau = \{1, 2, \cdots, m\} \setminus \{\sigma_1, \sigma_2, \cdots, \sigma_{m_{\rm c}}\}.$$

By Fig. 1,

$$\tau_j = j, \quad j = 1, 2, \cdots, \nu.$$

Define

$$\delta_{\tau_j} = \begin{cases} 0, & j = 1, 2, \cdots, \nu \\ q_{j-\nu}, & j = \nu+1, \nu+2, \cdots, \nu+m_{\rm d}. \end{cases}$$

By the definition of σ_j, we have

$$\delta_{\sigma_j} \geq k_{\sigma_j}, \quad j = 1, 2, \cdots, m_{\rm c}. \tag{23}$$

Define

$$\psi_j = \delta_j - k_j, \quad j = 1, 2, \cdots, m.$$

If δ_j is defined from r, ψ_j will be said to be associated with r. Otherwise, if δ_j is defined from q, ψ_j will be said to be associated with q.

By the definition, we have

$$\delta_j = k_j = \psi_j = 0, \quad j = 1, 2, \cdots, h_0$$

and

$$\delta_j = 0, \quad k_j > 0, \quad \psi_j < 0, \quad j = h_0+1, h_0+2, \cdots, \nu.$$

The integers $\psi_j, j = \tau_{\nu+1}, \tau_{\nu+2}, \cdots, \tau_{\nu+m_{\rm d}}$, which are associated with q, can be negative, zero or positive. However, due to (23), the integers $\psi_j, j = \sigma_1, \sigma_2, \cdots, \sigma_{m_{\rm c}}$, which are associated with r, can only be zero or positive.

We partition $B_{*0\rm ab}$, $B_{*0\rm c}$, $B_{*0\rm d}$ $L_{\rm abd}$, $L_{\rm cd}$ and $L_{\rm dd}$ as follows:

$$B_{*0{\rm ab}\{i,j\}} \in \mathbb{R}, \quad i=1,2,\cdots,n_{0\rm ab}, \quad j=1,2,\cdots,\nu-h_0$$
$$B_{*0{\rm c}\{i,j\}} \in \mathbb{R}^{r_i\times 1}, \quad i=1,2,\cdots,m_{\rm c}, \quad j=1,2,\cdots,\nu-h_0$$
$$B_{*0{\rm d}\{i,j\}} \in \mathbb{R}^{q_i\times 1}, \quad i=1,2,\cdots,m_{\rm d}, \quad j=1,2,\cdots,\nu-h_0.$$
$$L_{{\rm abd}\{i,j\}} \in \mathbb{R}, \quad i=1,2,\cdots,n_{0\rm ab}, \quad j=1,2,\cdots,m_{\rm d}$$
$$L_{{\rm cd}\{i,j\}} \in \mathbb{R}^{r_i\times 1}, \quad i=1,2,\cdots,m_{\rm c}, \quad j=1,2,\cdots,m_{\rm d}$$
$$L_{{\rm dd}\{i,j\}} \in \mathbb{R}^{q_i\times 1}, \quad i=1,2,\cdots,m_{\rm d}, \quad j=1,2,\cdots,m_{\rm d}$$

where

$$n_{0\rm ab} = \sum_{i=1}^{m} k_i - \sum_{i=1}^{m_{\rm c}} r_i - \sum_{i=1}^{m_{\rm d}} q_i.$$

Under the conditions of Lemma 3.5, we assign $B_{*0\rm ab}$, $B_{*0\rm c}$, $B_{*0\rm d}$, $L_{\rm abd}$, $L_{\rm cd}$ and $L_{\rm dd}$ by the following steps.

Algorithm 1

Initial Step: Let $\ell = 0$; $s = 0$; $\beta = 0$; $B_{*0\rm ab} = 0$; $B_{*0\rm c} = 0$; $B_{*0\rm d} = 0$; $L_{\rm abd} = 0$; $L_{\rm cd} = 0$; $L_{\rm dd} = 0$; $p_w = 0$, $w = 1, 2, \cdots, m$; $j = h_0$ and $n_+ = \sum_{j=\nu+1,\psi_j>0}^{m} \psi_j$.

Step R: Find the next j such that $\psi_j < 0$. Let $t = -\psi_j$. If $0 < n_+ - \ell < -\psi_j$, let $\beta = -\psi_j - n_+ + \ell$.

Case 1. If $j \leq \nu$,

Sub-case 1.1. If $\ell \geq n_+$, assign $(A_{{\rm ab}\{s+1:s+t,s+1:s+t\}}, B_{*0{\rm ab}\{s+1:s+t,j-h_0\}})$ to be controllable. Let $\ell = \ell + t$ and $s = s + t$.

Sub-case 1.2. If $\ell < n_+$, find the smallest $w > j$ with $\psi_w > \rho_w$. If ψ_w is associated with r, find a z such that $\sigma_z = w$, and let the $(\rho_w - \beta + t)$-th element of $B_{*0{\rm c}\{z,j\}}$ be nonzero. Otherwise, if ψ_w is associated with q, find an e such that $\tau_{\nu+e} = w$, and let the $(\rho_w - \beta + t)$-th element of $B_{*0{\rm d}\{e,j\}}$ be nonzero. If $\psi_w - \rho_w < t$, let $\rho_w = \psi_w$, $\ell = \ell + t$, $t = t - \psi_w + \rho_w$, and go back to Case 1. If $\psi_w - \rho_w \geq t$, let $\rho_w = \rho_w + t$ and $\ell = \ell + t$.

Case 2. If $j > \nu$, find an x such that $\tau_{\nu+x} = j$.

Sub-case 2.1. If $\ell \geq n_+$, assign $(A_{{\rm ab}\{s+1:s+t,s+1:s+t\}}, L_{{\rm abd}\{s+1:s+t,x\}})$ to be controllable. Let $\ell = \ell + t$ and $s = s + t$.

Sub-case 2.2. If $\ell < n_+$, find the smallest $w > j$ with $\psi_w > \rho_w$. If ψ_w is associated with r, find a z such that $\sigma_z = w$, and let the $(\rho_w - \beta + t)$-th element of $L_{{\rm cd}\{z,x\}}$ be nonzero. Otherwise, if ψ_w is associated with q, find an e such that $\tau_{\nu+e} = w$, and let the $(\rho_w - \beta + t)$-th element of $L_{{\rm dd}\{e,x\}}$ be nonzero. If $\psi_w - \rho_w < t$, let $\rho_w = \psi_w$, $\ell = \ell + t$, $t = t - \psi_w + \rho_w$, and go back to Case 2. If $\psi_w - \rho_w \geq t$, let $\rho_w = \rho_w + t$ and $\ell = \ell + t$.

If $\ell < n_{0\rm ab} + n_+$, go to Step R.

End.

We assign $E_{\rm ab}$ and the undefined elements in the lower triangular partition of $A_{\rm ab}$ arbitrarily. It can be verified that, with the resulting $A_{\rm ab}$, $L_{\rm abd}$, $L_{\rm cd}$, $L_{\rm dd}$ $B_{*0\rm ab}$, $B_{*0\rm c}$ and $B_{*0\rm d}$, the pair $(\hat{A}_1, \hat{B}_1)$ has the controllability indices k and the set of uncontrollable modes given by A_0.

This completes the proof of Lemma 3.5. □

Example 3.2: Let $n = 13$, $r = \{5\}$ and $q = \{7\}$. We would like to search for a pair $(\hat{A}_1, \hat{B}_1)$ in the form of (20) with the controllability indices $k = \{1, 3, 3, 6\}$. The conditions in Lemma 3.5 are satisfied. By Algorithm 1, we have

$$n_0 = 0, \quad n_{0\rm ab} = 1, \quad A_{\rm ab} = 1$$
$$L_{\rm abd} = 0, \quad L_{\rm cd} = 0, \quad L_{\rm dd} = 0$$
$$B_{*0\rm ab} = [0 \quad 1], \quad B_{*0\rm c} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}'$$
$$B_{*0\rm d} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'.$$

Similarly, let $n = 12$, $r = \{4\}$ and $q = \{1, 6\}$. By Algorithm 1, we have

$$n_0 = 0, \quad n_{0\rm ab} = 1, \quad A_{\rm ab} = 1, \quad L_{\rm abd} = [1 \quad 0]$$
$$L_{\rm cd} = 0, \quad L_{\rm dd} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$$
$$B_{*0\rm ab} = 0, \quad B_{*0\rm c} = [1 \quad 0 \quad 0 \quad 0]', \quad B_{*0\rm d} = 0$$

such that the pair $(\hat{A}_1, \hat{B}_1)$ has the controllability indices $k = \{1, 3, 3, 5\}$.

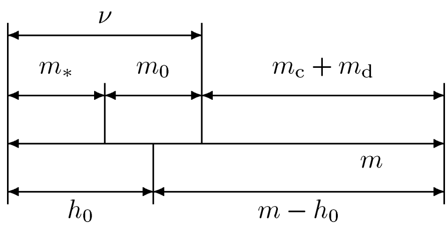

Fig. 2. Relationship among $m_*, m_0, m_c, m_d, h_0, \nu$ and m.

IV. Main Results

The following theorem deals with the assignment of infinite zeros and the column minimal indices. It generalizes the result of [29], where only the infinite zeros is considered.

Theorem 4.1: Consider a pair (A, B) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and its controllability indices k, $0 = k_1 = \cdots = k_{h_0} < k_{h_0+1} \leq \cdots \leq k_m$. Let m_0 be a nonnegative integer, and $r = \{r_1, r_2, \cdots, r_{m_c}\}$, $q = \{q_1, q_2, \cdots, q_{m_d}\}$ be two sets of non-decreasing positive integers. Let $\nu = m - m_c - m_d$. Then, there exist C and D such that the system (A, B, C, D) has m_0 infinite zeros of order 0, and Morse index lists $\mathcal{I}_2 = r$ and $\mathcal{I}_4 = q$ if and only if,

1) $m_c + m_d \leq m - h_0 \leq m_0 + m_c + m_d \leq m$;
2) The ordered set $\delta = \Delta(0_\nu \cup q, r, k)$ is well-defined and

$$\sum_{i=1}^{j} \delta_i \leq \sum_{i=1}^{j} k_i, \quad j = 1, 2, \cdots, m. \tag{24}$$

Proof: Necessity: By Lemma 2.5, there exist nonsingular Γ_S, Γ_O and Γ_I, and feedback gain K such that (7) holds. It is obvious that

$$m_0 + m_c + m_d \leq m, \quad m_c + m_d \leq \operatorname{rank}(B) = m - h_0.$$

Also, we have

$$m - h_0 = \operatorname{rank}\left(\begin{bmatrix} B_{0a} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0a} & B_d & 0 \end{bmatrix}\right) \leq m_0 + m_c + m_d.$$

Thus, the necessity of Condition 1) is proven. The necessity of Condition 1) is depicted in Fig. 2 with $m_* = \nu - m_0$. It is obvious that $\operatorname{rank}([B' \ D']) = m - m_*$.

We will next show the necessity of Condition 2). Define $n_d = \sum_{i=1}^{m_d} q_i$. Let $\tilde{b}_i$ be the ith column of $\tilde{B}$, $i = 1, 2, \cdots, m$. Thus, $\tilde{b}_{\nu+i}$, $i = 1, 2, \cdots, m_d$, are related to B_d, and $\tilde{b}_{m-m_c+i}$, $i = 1, 2, \cdots, m_c$, are related to B_c. It can be verified that

$$\begin{aligned}(\tilde{A}_z)^j \tilde{b}_{m-m_c+i} &= 0, \quad j \geq r_i, i = 1, 2, \cdots, m_c \\ (\tilde{A}_z)^j \tilde{b}_{\nu+i} &= \begin{bmatrix} * \\ 0_{n_d \times 1} \end{bmatrix}, \quad j \geq q_i, i = 1, 2, \cdots, m_d. \end{aligned} \tag{25}$$

Let

$$\xi = [0_\nu, q], \quad \zeta = [0_\nu, q, r].$$

It can be verified that

$$\Theta(\tilde{A}_z, \tilde{B}, \zeta) = \begin{bmatrix} 0 & 0 \\ 0 & Y_c \\ Y_d & 0 \end{bmatrix} \tag{26}$$

where

$$\begin{aligned} Y_c &= \operatorname{blkdiag}\left\{\varsigma_{r_1}, \varsigma_{r_2}, \cdots, \varsigma_{r_{m_c}}\right\} \\ Y_d &= \operatorname{blkdiag}\left\{\varsigma_{q_1}, \varsigma_{q_2}, \cdots, \varsigma_{q_{m_d}}\right\} \end{aligned}$$

and ς_i is an $i \times i$ matrix with the elements in the inverse diagonal being 1 s, and all the other elements being 0 s. Clearly, $\Theta(\tilde{A}_z, \tilde{B}, \zeta)$ is of full column rank. Thus, by Lemma 3.4

$$k \prec^w \zeta. \tag{27}$$

Consequently, we have

$$\sum_{i=1}^{m} \delta_i \leq \sum_{i=1}^{m} k_i. \tag{28}$$

From (25), we obtain $k_1 \leq r_1$. Similarly, we have $\{k_1, k_2, \cdots, k_{m_c}\} \leq r$. Thus, $\delta = \Delta(0_\nu \cup q, r, k)$ is well-defined.

Delete the repeated elements of k and rearrange the remaining elements as $\kappa = \{\kappa_1, \kappa_2, \cdots, \kappa_s\}$ with $\kappa_1 < \kappa_2 < \cdots < \kappa_s$. Define

$$\begin{aligned} \eta_j &= \operatorname{card}\{i \in \{1, 2, \cdots, m\} : k_i \leq \kappa_j\} \\ \varphi_j &= \operatorname{card}\{i \in \{1, 2, \cdots, m\} : r_i < \kappa_{j+1}\} \\ \pi_j &= \eta_j - \varphi_j \end{aligned}$$

for $1 \leq j \leq s - 1$. Obviously

$$k_i = \kappa_j, \quad \eta_{j-1} + 1 \leq i \leq \eta_j. \tag{29}$$

If we can show

$$\sum_{i=1}^{\eta_j} \delta_i \leq \sum_{i=1}^{\eta_j} k_i, \quad j = 1, 2, \cdots, s - 1 \tag{30}$$

then we can prove (24) in the following way. Consider the subset $\{\delta_i : \eta_{j-1} + 1 \leq i \leq \eta_j\}$. Suppose that b elements in this subset come from r, then $\delta_i \geq \kappa_j$, for $\eta_j - b + 1 \leq i \leq \eta_j$. The remaining part of this subset, i.e., δ_i, $\eta_{j-1} + 1 \leq i \leq \eta_j - b$ come from ξ, and are in the non-decreasing order. As a result, there exists a c, $\eta_{j-1} + 1 \leq c \leq \eta_j$, such that

$$\begin{cases} \delta_i < \kappa_j, & \eta_{j-1} + 1 \leq i \leq \eta_j - c \\ \delta_i \geq \kappa_j, & \eta_j - c + 1 \leq i \leq \eta_j. \end{cases} \tag{31}$$

By (28), (29), (30) and (31), we obtain

$$\sum_{i=1}^{t} \delta_i \leq \sum_{i=1}^{t} k_i, \quad \eta_{j-1} + 1 \leq t \leq \eta_j.$$

And thus, we have (24).

Now we only need to prove (30). We divide the proof into two cases: $\xi_{\pi_j} \geq r_{\varphi_j}$ and $\xi_{\pi_j} < r_{\varphi_j}$.

In the case that $\xi_{\pi_j} \geq r_{\varphi_j}$

$$\sum_{i=1}^{\eta_j} \delta_i = \sum_{i=1}^{\pi_j} \xi_i + \sum_{i=1}^{\varphi_j} r_i = \sum_{i=1}^{\eta_j} \zeta_{(i)}.$$

Thus, by (27), we have

$$\sum_{i=1}^{\eta_j} \delta_i \leq \sum_{i=1}^{\eta_j} k_i.$$

Next, we consider the case of $\xi_{\pi_j} < r_{\varphi_j}$. The proof of (30) for this case is a little involved. We define an m element ordered set β^0 as follows:

$$\beta_i^0 = \begin{cases} \zeta_i, & \zeta_i < \kappa_{j+1} - 1 \\ \kappa_{j+1} - 1, & \zeta_i \geq \kappa_{j+1} - 1, \end{cases} \quad i = 1, 2, \cdots, m.$$

Let $\beta^1 = \beta^0$, and if $[\Theta(\tilde{A}_z, \tilde{B}, \beta^0), \tilde{A}_z^{\beta_1^0} \tilde{b}_1, \tilde{A}_z^{\beta_1^0+1} \tilde{b}_1, \cdots, \tilde{A}_z^{\kappa_{j+1}-1} \tilde{b}_1]$ is of full column rank, let $\beta_1^1 = \kappa_{j+1}$. Let $\beta^2 = \beta^1$, and if $[\Theta(\tilde{A}_z, \tilde{B}, \beta^1), \tilde{A}_z^{\beta_2^1} \tilde{b}_2, A_z^{\beta_2^1+1} \tilde{b}_2, \cdots, \tilde{A}_z^{\kappa_{j+1}-1} \tilde{b}_2]$ is of full column rank, let $\beta_2^2 = \kappa_{j+1}$. Continue in this way for all $\tilde{b}_i$ for i from 1 through m, we define β^i, $i = 1, 2, \cdots, m$. Let μ_j be the number of elements in β^m which are not bigger than $\kappa_{j+1} - 1$. Denote all rows of $\tilde{B}$ with $\beta_t^m \leq \kappa_{j+1} - 1$ as $\tilde{B}_{\mathrm{I}}$. Because of (25), all rows of $\tilde{B}$ associated with $r_t \leq \kappa_{j+1} - 1$ are included in $\tilde{B}_{\mathrm{I}}$.

Similarly, we define an m element ordered set α^0 as follows: search for linearly independent columns of $\Xi(\tilde{A}_z, \tilde{B}, \kappa_{j+1} - 1)$ from left to right and rearrange them as

$$\tilde{b}_1 \tilde{A}_z \tilde{b}_1 \, \cdots \, \tilde{A}_z^{\alpha_1^0 - 1} \tilde{b}_1, \tilde{b}_2 \tilde{A}_z \tilde{b}_2 \, \cdots \, \tilde{A}_z^{\alpha_2^0 - 1} \tilde{b}_2, \cdots$$
$$\cdots, \tilde{b}_m \tilde{A}_z \tilde{b}_m \, \cdots \, \tilde{A}_z^{\alpha_m^0 - 1} \tilde{b}_m.$$

Denote

$$\alpha^0 = \left\{\alpha_1^0, \alpha_2^0, \cdots, \alpha_m^0\right\}.$$

Let $\alpha^1 = \alpha^0$, and if $[\Theta(\tilde{A}_z, \tilde{B}, \alpha^0), \tilde{A}_z^{\alpha_1^0} \tilde{b}_1, \tilde{A}_z^{\alpha_1^0+1} \tilde{b}_1, \cdots, \tilde{A}_z^{\kappa_{j+1}-1} \tilde{b}_1]$ is of full column rank, let $\alpha_1^1 = \kappa_{j+1}$. Let $\alpha^2 = \alpha^1$, and if $[\Theta(\tilde{A}_z, \tilde{B}, \alpha^1), \tilde{A}_z^{\alpha_2^1} \tilde{b}_2, A_z^{\alpha_2^1+1} \tilde{b}_2, \cdots, \tilde{A}_z^{\kappa_{j+1}-1} \tilde{b}_2]$ is of full column rank, let $\alpha_2^2 = \kappa_{j+1}$. Continue in this way for all $\tilde{b}_i$ for i from 1 through m, we define α^i, $i = 1, 2, \cdots, m$. It is obviously that there are η_j elements in α^m which are not bigger than $\kappa_{j+1} - 1$. By the above construction, we have the following relation among subspaces:

$$\Theta(\tilde{A}_z, \tilde{B}, \beta^0) \subseteq \Theta(\tilde{A}_z, \tilde{B}, \alpha^0) = \Xi(\tilde{A}_z, \tilde{B}, \kappa_{j+1} - 1) \tag{32}$$

$$\begin{aligned} &\left[\Theta(\tilde{A}_z, \tilde{B}, \beta^i), \tilde{A}_z^{\beta_{i+1}^i} \tilde{b}_{i+1}, \tilde{A}_z^{\beta_{i+1}^i+1} \tilde{b}_{i+1}, \cdots, \tilde{A}_z^{\kappa_{j+1}-1} \tilde{b}_{i+1}\right] \\ &\subseteq \left[\Theta(\tilde{A}_z, \tilde{B}, \alpha^i), \tilde{A}_z^{\alpha_{i+1}^i} \tilde{b}_{i+1}, \tilde{A}_z^{\alpha_{i+1}^i+1} \tilde{b}_{i+1}, \cdots, \tilde{A}_z^{\kappa_{j+1}-1} \tilde{b}_{i+1}\right] \\ &= \Theta(\tilde{A}_z, \tilde{B}, \chi^i), \quad i = 0, 1, \cdots, m-1 \end{aligned} \tag{33}$$

where χ^i is an m element ordered set with the first $i+1$ elements being κ_{j+1} and the remaining elements being $\kappa_{j+1} - 1$. Therefore, by (32) and (33),

$$\eta_j \geq \mu_j.$$

Therefore, $\beta_{(i)}^m$, $i = 1, 2, \cdots, \eta_j$, includes three parts:

1) $\mu_j - \varphi_j$ elements contained in the first π_j elements of $0_\nu \cup q$;
2) $\eta_j - \mu_j$ elements of κ_{j+1};
3) r_i, $i = 1, 2, \cdots, \varphi_j$.

Thus

$$\sum_{i=1}^{\eta_j} \beta_{(i)}^m \geq (\eta_j - \mu_j)\kappa_{j+1} + \sum_{i=1}^{\mu_j - \varphi_j} \xi_i + \sum_{i=1}^{\varphi_j} r_i.$$

Consider $\xi_{\pi_j} < r_{\varphi_j} < \kappa_{j+1}$, we obtain

$$\sum_{i=1}^{\eta_j} \beta_{(i)}^m \geq \sum_{i=1}^{\pi_j} \xi_i + \sum_{i=1}^{\varphi_j} r_i = \sum_{i=1}^{\eta_j} \delta_i.$$

We also have

$$\beta_{(i)}^m = \kappa_{j+1}, \quad i = \eta_j + 1, \cdots, m. \tag{34}$$

Since $\Theta(\tilde{A}_z, \tilde{B}, \beta^m)$ is of full column rank and consider (34), by Lemma 3.4, we obtain

$$\sum_{i=1}^{\eta_j} \beta_{(i)}^m \leq \sum_{i=1}^{\eta_j} k_i.$$

And thus

$$\sum_{i=1}^{\eta_j} \delta_i \leq \sum_{i=1}^{\eta_j} k_i.$$

This complete the proof of (30).

Sufficiency: We will give an algorithm that would yield the desired matrices C and D.

Algorithm 2

1) By Lemma 2.1, find nonsingular state and input transformations T_{S1} and T_{I1}, and feedback K_1 such that (5) holds.
2) By Lemma 3.5, find $(\hat{A}_1, \hat{B}_1)$ in the form of (20) with the controllability indices k and the set of uncontrollable modes given by $\lambda(A_0)$. Assign $\hat{C}_1 \in \mathbb{R}^{(m_0+m_{\mathrm{d}})\times n}$ and $\hat{D}_1 \in \mathbb{R}^{(m_0+m_{\mathrm{d}})\times m}$ as follows:

$$\hat{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\mathrm{d}} \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} 0_{m_0 \times m_*} & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

where C_{d} is in the form of (12). The system $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ has the desired structural properties.
3) By Lemma 2.1, find nonsingular state and input transformations T_{S2} and T_{I2}, and feedback K_2 such that $(\hat{A}_1, \hat{B}_1)$ is transformed into (5). Let

$$C = \left[\hat{C}_1 T_{\mathrm{S2}} - \hat{D}_1 T_{\mathrm{I2}}(K_2 - K_1)\right] T_{\mathrm{S1}}^{-1} \tag{36}$$

$$D = \hat{D}_1 T_{\rm I2} T_{\rm I1}^{-1}. \tag{37}$$

End.

By Algorithm 2, we have

$$A = T_{\rm S1} T_{\rm S2}^{-1} \left[\hat{A}_1 T_{\rm S2} - \hat{B}_1 T_{\rm I2} (K_2 - K_1) \right] T_{\rm S1}^{-1}$$
$$B = T_{\rm S1} T_{\rm S2}^{-1} \hat{B}_1 T_{\rm I2} T_{\rm I1}^{-1}.$$

Thus, the system (A, B, C, D) can be transformed into $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ by using state and input transformations and state feedback. We finally obtain a set of the desired (C, D) as

$$\Omega = \left\{ (\Gamma_{\rm o} C, \Gamma_{\rm o} D) | \Gamma_{\rm o} \in \mathbb{R}^{(m_{\rm d}+m_0)\times(m_{\rm d}+m_0)} \text{ is nonsingular} \right\}.$$

This completes the proof of Theorem 4.1. □

Remark 4.1: If r is an empty set, the conditions of Theorem 4.1 can be reduced to the condition of [29], in which only the assignment of infinite zeros is considered.

The following theorem deals with the assignment of a complete set of structural properties.

Theorem 4.2: Consider the pair (A, B) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, its controllability indices k, $0 = k_1 = \cdots = k_{h_0} < k_{h_0+1} \le \cdots \le k_m$, and the set uncontrollable modes given by $A_0 \in \mathbb{R}^{n_0\times n_0}$. Let the eigenvalues of A_0 be λ_i, $i = 1, 2, \cdots, \varsigma$, with the sizes of their Jordan blocks being $n_{i,j}$, $j = 1, 2, \cdots, \tau_i$, $i = 1, 2, \cdots, \varsigma$. Let $n_{\rm a}$, $p_{\rm b}$, $m_{\rm c}$, $m_{\rm d}$ and m_0 be nonnegative integers, Λ_1 be a set with $n_{\rm a}$ self-conjugated complex scalars, and $r = \{r_1, r_2, \cdots, r_{m_{\rm c}}\}$, $l = \{l_1, l_2, \cdots, l_{p_{\rm b}}\}$ and $q = \{q_1, q_2, \cdots, q_{m_{\rm d}}\}$ be three sets of non-decreasing positive integers. Let $\nu = m - m_{\rm c} - m_{\rm d}$. Then, there exist C and D such that the system (A, B, C, D) has finite zeros Λ_1, m_0 infinite zeros of order 0, and the Morse index lists $\mathcal{I}_2 = r$, $\mathcal{I}_3 = l$ and $\mathcal{I}_4 = q$ if and only if

1) $m_{\rm c} + m_{\rm d} \le m - h_0 \le m_0 + m_{\rm c} + m_{\rm d} \le m$;
2) $l \prec \{\theta + \sum_{i=1}^{\varsigma} \mathfrak{y}_{i,1}, \sum_{i=1}^{\varsigma} \mathfrak{y}_{i,2}, \cdots, \sum_{i=1}^{\varsigma} \mathfrak{y}_{i,p_{\rm b}}\}$, where $\theta = \sum_{j=1}^{p_{\rm b}} l_j + n_{\rm f} - n_0$, $n_{\rm f}$ is the number of elements in $\Lambda_1 \cap \lambda(A_0)$, $\{\mathfrak{y}_{i,j} | j = 1, 2, \cdots, \tau_i, i = 1, 2, \cdots, \varsigma\} = \mathfrak{N}(A_0, \Lambda_1 \cap \lambda(A_0))$ and the undefined $\mathfrak{y}_{i,j}$'s are set to be zero;
3) The ordered set $\delta = \Delta(0_\nu \cup q, r, k)$ is well-defined and
$$\sum_{i=1}^{j} \delta_i \le \sum_{i=1}^{j} k_i, \quad j = 1, 2, \cdots, m; \tag{38}$$
4) $n_{\rm a} + \sum_{i=1}^{p_{\rm b}} l_i + \sum_{i=1}^{m_{\rm c}} r_i + \sum_{i=1}^{m_{\rm d}} q_i = n$.

Proof: Necessity: The necessity of Conditions 1) and 3) follows directly from Theorem 4.1. Condition 4) is necessary for dimensional compatibility.

The necessity of Condition 2) can be proven by using Theorem 3.1. Consider $(\tilde{A}_z, \tilde{B})$ in the form of (7). The eigenvalues of

$$\begin{bmatrix} A_{\rm cc}^\star & L_{\rm cd} C_{\rm d} \\ 0 & A_{\rm dd}^\star + L_{\rm dd} C_{\rm d} \end{bmatrix}$$

can be changed by using state feedback. This means that the eigenstructure of uncontrollable modes A_0 is entirely contained in $A_{\rm con}$ with

$$A_{\rm con} = \begin{bmatrix} A_{\rm a} & L_{\rm ab} C_{\rm b} \\ 0 & A_{\rm bb}^\star + L_{\rm bb} C_{\rm b} \end{bmatrix}.$$

Suppose that the eigenvalues of $A_{\rm bb}^\star + L_{\rm bb} C_{\rm b}$ are given by λ_i, $i = 1, 2, \cdots, \tau$, with the sizes of their Jordan blocks being $\bar{n}_{i,j}$, $j = 1, 2, \cdots, p_{\rm b}$, $i = 1, 2, \cdots, \tau$. Then, by Lemma 3.1 and (17)

$$l \prec \left\{ \sum_{i=1}^{\tau} \bar{n}_{i,1}, \sum_{i=1}^{\tau} \bar{n}_{i,2}, \cdots, \sum_{i=1}^{\tau} \bar{n}_{i,p_{\rm b}} \right\}$$
$$\prec \left\{ \theta + \sum_{i=1}^{\tau} \mathfrak{y}_{i,1}, \sum_{i=1}^{\tau} \mathfrak{y}_{i,2}, \cdots, \sum_{i=1}^{\tau} \mathfrak{y}_{i,p_{\rm b}} \right\}.$$

Sufficiency: We will establish the sufficiency by construction. We first consider the pair $(\hat{A}_1, \hat{B}_1)$ in the form of (20). Denote

$$A_\beta = \begin{bmatrix} A_0 & 0 \\ E_{\rm ab} & A_{\rm ab} \end{bmatrix}. \tag{39}$$

Following Algorithm 1, we assign $A_{\rm ab}$ such that $n_{0\rm ab} - n_{\rm f}$ eigenvalues of $A_{\rm ab}$ are in $\Lambda_1 \setminus (\Lambda_1 \cap \lambda(A_0))$, and the remaining $n_{\rm f}$ eigenvalues are distinct and not in Λ_1. By Theorem 3.1, we can assign $C_{*\rm b}$ such that $(A_\beta, C_{*\rm b})$ has its the set of unobservable modes contained in Λ_1, and its observability indices is l, which satisfies Condition 2).

Similar to Algorithm 2, instead of assigning $\hat{C}_1$ and $\hat{D}_1$ in the forms of (35), we now assign

$$\hat{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\rm d} \\ C_{*\rm b} & 0 & 0 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} 0_{m_*\times m_0} & I_{m_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{40}$$

To show that the system $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ has the desired structural properties, we find nonsingular T_* such that

$$T_*^{-1} A_\beta T_* = \begin{bmatrix} A_{\rm a} & W_{\rm ab} \\ 0 & A_{\rm bb} \end{bmatrix}, \quad C_{*\rm b} T_* = [0 \quad C_{\rm b}]$$

where $\lambda(A_{\rm aa}) = \Lambda_1$. Since $(A_{\rm bb}, C_{\rm b})$ is observable, there exists an $L \in \mathbb{R}^{n_{\rm b}\times p_{\rm b}}$, $n_{\rm b} = \sum_{i=1}^{p_{\rm b}} l_i$, such that

$$\lambda(A_{\rm bb} - LC_{\rm b}) \cap \lambda(A_{\rm a}) = \emptyset.$$

Therefore, the Sylvester equation

$$-A_{\rm a} Y_1 + Y_1 (A_{\rm bb} - LC_{\rm b}) = W_{\rm ab}$$

has a unique solution Y_1. Let

$$\hat{T}_* = \begin{bmatrix} I & Y_1 \\ 0 & I \end{bmatrix}.$$

We obtain

$$(T_* \hat{T}_*)^{-1} A_\beta (T_* \hat{T}_*) = \begin{bmatrix} A_{\rm a} & (Y_1 L) C_{\rm b} \\ 0 & A_{\rm bb} \end{bmatrix}$$
$$C_{*\rm b} (T_* \hat{T}_*) = [0 \quad C_{\rm b}].$$

Thus, the system $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ has finite zeros Λ_1 and m_0 infinite zeros of order 0. Its infinite zeros of order greater than 0 are q, the right invertibility indices are r, and the left invertibility indices are l.

We thus obtain a set of the desired (C, D) as

$$\Omega = \Big\{(\Gamma_o C, \Gamma_o D) | \Gamma_o \in \mathbb{R}^{(m_0+p_b+m_d)\times(m_0+p_b+m_d)} \text{ is nonsingular}\Big\}$$

where C and D are in the forms of (36) and (37). This completes the proof of Theorem 4.2. □

Remark 4.2: The most important step in the constructive algorithm in the proof of Theorems 4.1 and 4.2 is the construction of $(\hat{A}_1, \hat{B}_1)$. The pair $(\hat{A}_1, \hat{B}_1)$ has a pre-specified set of uncontrollable modes and controllability indices, and it also has a form similar to $(\tilde{A}_z, \tilde{B})$ as in (7). Thus, it is easy to assign $(\hat{C}_1, \hat{D}_1)$ such that the system $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ has the desired Morse index lists. In other words, $(\hat{A}_1, \hat{B}_1)$ plays a role that links a matrix pair with certain controllability indices and a matrix quadruple with a certain set of Morse index lists. The construction of $(\hat{A}_1, \hat{B}_1)$ in Lemma 3.5 is not unique, and any matrix pair provides such a linkage can serve as $(\hat{A}_1, \hat{B}_1)$. As will be seen in Example 5.5, in some situations, $(\hat{A}_1, \hat{B}_1)$ could be constructed by only small adjustments in the model of system dynamics.

Remark 4.3: If the uncontrollable mode matrix A_0 is cyclic, which means that the Jordan form of A_0 has one Jordan block associated with each distinct eigenvalue, then Condition 2) in Theorem 4.2 can be simplified. More specially, we can assign $A_{\rm ab}$ such that A_β is cyclic, thus the majorization constraint with respect to l can be removed. Therefore, Condition 2) in Theorem 4.2 simplifies to

2)* $\Lambda_1 = \Theta \cup \Delta_1$, where $\Delta_1 \subseteq \lambda(A_0)$ and Θ is self-conjugated.

Remark 4.4: In Theorem 4.2, we only consider the algebraic multiplicity of finite zeros. We can also take into account the geometric multiplicity of finite zeros. Suppose that the desired eigenstructure of finite zeros is given by $\bar{A}_{\rm aa} \in \mathbb{R}^{n_{\rm a}\times n_{\rm a}}$, then the assignment of this finite zero structure and the left invertibility structure is a little more involved, as $A_{\rm ab}$ and $E_{\rm ab}$ in the constructive algorithm in Lemma 3.5 can no longer be chosen freely. In particular, for $\bar{A}_{\rm aa}$ to be assignable, it is required that there exist $E_{\rm ab}$ and $A_{\rm ab}$ such that A_β in (39) can be transformed into

$$T_*^{-1}\begin{bmatrix} A_0 & 0 \\ E_{\rm ab} & A_{\rm ab}\end{bmatrix} T_* = \begin{bmatrix} \bar{A}_{\rm aa} & W_{\rm ab} \\ 0 & \bar{A}_{\rm bb}\end{bmatrix}$$

and $l \prec \{\sum_{i=1}^{\varsigma} n_{i,1}, \sum_{i=1}^{\varsigma} n_{i,2}, \cdots, \sum_{i=1}^{\varsigma} n_{i,p_{\rm b}}\}$, where $n_{i,j}$, $j = 1, 2, \cdots, \tau_i$, $i = 1, 2, \cdots, \varsigma$, are the sizes of the Jordan blocks associated with $\lambda_i = \lambda(\bar{A}_{\rm bb})$ and $n_{i,1} \geq n_{i,2} \geq \cdots \geq n_{i,\tau_i}$, $i = 1, 2, \cdots, \varsigma$. In this case, we can assign $C_{*\rm b}$ in (40) as $C_{*\rm b} = [0 \ \bar{C}_{\rm b}]T_*^{-1}$, where $(\bar{A}_{\rm bb}, \bar{C}_{\rm b})$ is observable with observability indices l. Following the algorithm in Theorem 4.2, we obtain the desired (C, D).

Remark 4.5: In our earlier algorithm [27], in order to be assignable, the desired orders of infinite zeros must be equal to or less than the elements in the controllability indices of (A, B), and the desired right invertible indices must be equal to the elements in controllability indices of (A, B). In our current algorithm, no such constraints are imposed.

Note that in Theorems 4.1 and 4.2, the necessary and sufficient conditions are given only in terms of the controllability indices k and uncontrollable modes A_0. Fig. 3 summarizes in a graphical form our assignment of a complete set of structural properties. In Theorem 4.1, we focus only on the structural assignment of m_0, $\mathcal{I}_2$ and $\mathcal{I}_4$, while in Theorem 4.2, we consider the assignment of a complete set of structural properties by assigning the additional properties $\mathcal{I}_1$ and $\mathcal{I}_3$.

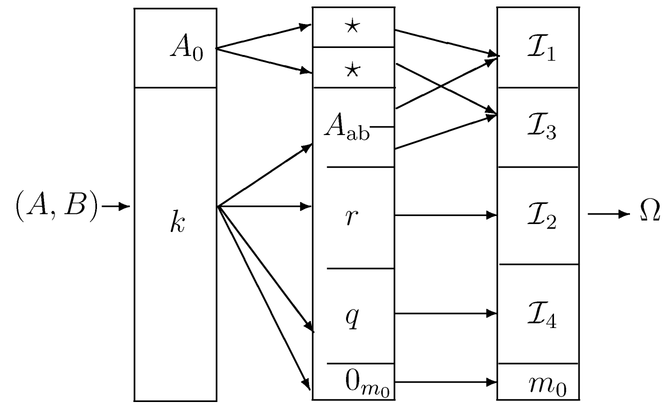

Fig. 3. Graphical summary of the structural assignment.

V. Examples

In this section, we will present several examples to illustrate various scenarios of the structural assignment problem. These examples also show how our results generalize the existing results in the literature.

We first consider an example where the required structural properties are determined to be not assignable.

Example 5.1: Consider a pair (A, B) with controllability indices $k = \{2, 4\}$. Let $r = \{3\}$ and $q = \{1\}$, and define $\delta = \Delta(\alpha, \beta, k) = [3, 1]$. Condition 2) in Theorem 4.1 is not satisfied, Thus, there do not exist (C, D) such that the resulting (A, B, C, D) has $\mathcal{I}_2 = r$ and $\mathcal{I}_4 = q$.

The following example considers the assignment of all four structural properties.

Example 5.2: Consider the linear system (1) with

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We would like to choose C and D such that the resulting system (A, B, C, D) has a finite zero at -1, $\mathcal{I}_2 = \{1\}$, $\mathcal{I}_3 = \{1\}$, and infinite zeros structure $\mathcal{I}_4 = \{1\}$.

Following the constructive algorithm in the proof of Theorem 4.2, we proceed as follows:

1) By Lemma 2.1, the pair (A, B) has an uncontrollable mode 0 and controllability indices $\{1, 2\}$, and can be transformed into (5) by

$$T_{\mathrm{S1}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ \frac{\sqrt{2}}{2} & 1 & 0 & 0 \end{bmatrix}$$

$$T_{\mathrm{I1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

2) By Lemma 3.5, find

$$\hat{A}_1 = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \quad \hat{B}_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 1 \\ \hline 1 & 0 \end{array}\right].$$

Assign

$$\hat{C}_1 = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right], \quad \hat{D}_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right].$$

The resulting system $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ has the desired structural properties.

3) By Lemma 2.1, find

$$T_{\mathrm{S2}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$T_{\mathrm{I2}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to transform $(\hat{A}_1, \hat{B}_1)$ into the form of (5). By (36), (37),

$$C = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the desired output matrices are given by

$$\Omega = \left\{ \left(\Gamma_{\mathrm{o}} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \middle| \Gamma_{\mathrm{o}} \in \mathbb{R}^{2\times 2} \text{ is nonsingular.} \right\}.$$

The following example considers the assignment of $\mathcal{I}_2$ and $\mathcal{I}_4$, whose elements are bigger than the elements in k. The explicit algorithm in [27] cannot deal with this situation.

Example 5.3: Consider

$$A = \begin{bmatrix} \aleph_2 & 0 & 0 \\ 0 & \aleph_2 & 0 \\ 0 & 0 & \aleph_2 \end{bmatrix}, \quad B = \begin{bmatrix} \vartheta_2 & 0 & 0 \\ 0 & \vartheta_2 & 0 \\ 0 & 0 & \vartheta_2 \end{bmatrix}.$$

It is controllable with controllability indices $k = \{2,2,2\}$. We would like to assign output matrices C and D such that the resulting system (A,B,C,D) has $\mathcal{I}_2 = \{3\}$ and $\mathcal{I}_4 = \{3\}$, but no finite zeros.

It is easy to verify that conditions in Theorem 4.2 are satisfied. Assign $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$

$$\hat{A}_1 = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \hat{B}_1 = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\hat{C}_1 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad \hat{D}_1 = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right].$$

It can be verified that the pair $(\hat{A}_1, \hat{B}_1)$ has controllability indices $k = \{2,2,2\}$ and the system $(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ has the desired structural properties. Following Algorithm 2, we obtain the desired (C,D) as

$$C = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following example considers the assignment of finite zeros with or without pre-specified eigenstructure.

Example 5.4: Consider a pair (A,B) with

$$A = \begin{bmatrix} I_3 & 0 \\ 0 & \aleph_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vartheta_3 \end{bmatrix}.$$

Obviously, the following $(\hat{A}_1, \hat{B}_1)$ has the same uncontrollable eigenstructure and controllability indices as those of (A,B)

$$\hat{A}_1 = \left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \hat{B}_1 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 1 \end{array}\right].$$

We can assign C and D such that (A,B,C,D) has finite zeros $\Lambda_1 = \{1,1,1\}, \mathcal{I}_3 = l = \{2\}$ and $\mathcal{I}_4 = q = \{1\}$ by letting

$$\hat{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The resulting C and D are given by

$$C = \begin{bmatrix} 0 & -1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But there do not exist C and D such that the system (A,B,C,D) has the specific structure of the finite zeros

$$\bar{A}_{\mathrm{aa}} = I_3 + \aleph_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{I}_3 = l = \{2\}$ and $\mathcal{I}_4 = q = \{1\}$. Indeed, after the assignment of finite zeros, only identity matrix I_2 in A_β is left to be assigned as $\mathcal{I}_3$. But, as observed in Remark 4.4, such an assignment requires that $l \prec \{1,1\}$, which obviously cannot be satisfied here.

We can however assign C and D such that (A, B, C, D) has finite zeros $\bar{A}_{\rm aa} = I_3 + \aleph_3$, and $\mathcal{I}_3 = l = \{1, 1\}$, $\mathcal{I}_4 = q = \{1\}$ by letting

$$\hat{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The desired C and D are given by

$$C = \begin{bmatrix} 0 & -1 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Yet there do not exist C and D, such that (A, B, C, D) has the finite zeros $\bar{A}_{\rm aa} = I_5$ and $\mathcal{I}_4 = q = \{1\}$. Indeed, to assign finite zeros I_5, $A_{\rm ab}$ can only be chosen as I_2. And for this fixed $A_{\rm ab}$, there does not exist $L_{\rm abd} \in \mathbb{R}^{2\times 1}$ such that the pair $(A_{\rm ab}, L_{\rm abd})$ is controllable. Therefore, there do not exist the required $(\hat{A}_1, \hat{B}_1)$.

Finally, we consider the problem of sensor selection for a mechanical system.

Example 5.5: Consider a benchmark problem for robust control of a flexible mechanical system (see Fig. 4). The problem is to control the displacement of the third mass by applying a force to the first mass. The dynamic model of the system is given by

$$\begin{cases} m_1 \ddot{x}_1 = k_1(x_2 - x_1) + u \\ m_2 \ddot{x}_2 = k_1(x_1 - x_2) + k_2(x_3 - x_2) + w_2 \\ m_3 \ddot{x}_3 = k_2(x_2 - x_3) + w_3 \end{cases}$$

where x_1, x_2 and x_3 are respectively the positions of Mass 1 (with a mass of m_1), Mass 2 (with a mass of m_2) and Mass 3 (with a mass of m_3), k_1 and k_2 are spring constants, u is the input force, and w_2 and w_3 are the disturbances, such as friction forces and unmeasured external forces. The output z to be controlled is the position of the third mass. For simplicity, we choose $m_1 = m_2 = m_3 = 1$ and $k_1 = k_2 = 1$. Thus, the system is represented by

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{pmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w_2 \\ w_3 \end{pmatrix} \\ z &= C_2 x = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0] x. \end{aligned} \tag{41}$$

Although simple in nature, this problem provides an interesting example on how sensor selection can affect the performance of the resulting control system. It is simple to verify that the subsystem (A, B, C_2) is of minimum-phase and invertible. Hence, the disturbance w can be decoupled from the output to be controlled, i.e., z, to an arbitrarily small degree by state feedback [44]. Our objective is to identify a measurement output, or the sensor locations, such that a feedback of the measurement output would yield the same performance as the state feedback. This can be made possible by choosing a measurement output $y = C_1 x$ such that the subsystem (A, E, C_1) is left invertible and of minimum-phase [44]. Thus, at least two measurements are needed.

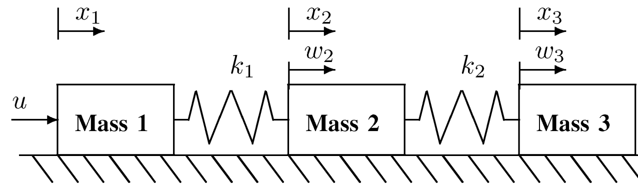

Fig. 4. Three-mass-two-spring flexible mechanical system.

The pair (A, E) is in (41) is controllable with controllability indices $k = \{2, 4\}$.

Suppose we are to assign C_1 such that (A, E, C_1) is invertible with infinite zeros {2, 4}. The (A, E) is already in the form of the required $(\hat{A}_1, \hat{B}_1)$ in Theorem 4.2. C_1 is simply given by

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

which means that the positions of Mass 1 and Mass 3 (x_1, x_3) are measured. It can be verified that the almost disturbance decoupling is achievable by measurement feedback.

Next, we assign C_1 such that (A, E, C_1) is invertible with infinite zeros {2, 2}. Similarly, we assign

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In this case, the positions of Mass 2 and Mass 3 are measured, and the finite zeros of the resulting system are $\pm j$. The subsystem (A, E, C_1) is of weakly minimum phase. The almost disturbance decoupling is achievable, but the controller is more complicated, as the system is only weakly minimum phase. For this reason, we would like to assign C_1 such that the system (A, E, C_1) has finite zeros with negative real parts. We assign

$$\hat{A}_1 = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & \alpha & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right], \quad \hat{B}_1 = E.$$

Note that $\hat{A}_1$ is the same as A, except that the (2,2) entry is now α. The pair $(\hat{A}_1, \hat{B}_1)$ has controllability indices {2, 4} and

$$A_{\rm ab} = \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix}$$

has eigenvalues at $-(1/2)\alpha \pm j(\sqrt{4 - \alpha^2}/2)$ for $\alpha \in (0, 2)$. Assign

$$\hat{C}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

By Algorithm 2, we obtain

$$C_1 = \begin{bmatrix} 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The resulting system (A, E, C_1) is now invertible with infinite zeros $\{2, 2\}$ and stable finite zeros $-(\alpha/2)\alpha \pm j(\sqrt{4-\alpha^2}/2)$ for any $\alpha \in (0, 2)$.

On the other hand, if the positions of Mass 1 and Mass 2 are measured, i.e.,

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

the subsystem (A, E, C_1) is not invertible, and thus, the almost disturbance decoupling cannot be achieved.

VI. Conclusion

In this paper, we have revisited and provided a complete solution to the classical problem of structural assignment for linear systems. We considered a complete set of structural properties, including the finite and infinite zero structures and the invertibility structure. We established a set of necessary and sufficient conditions under which these structural properties are assignable. An algorithm to construct the desired output matrices that result in the prescribed structural properties was also given. Several numerical examples were worked out in detail to illustrate various scenarios in the assignment of structural properties.

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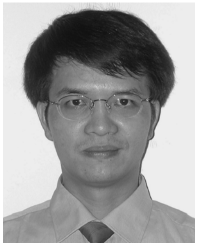

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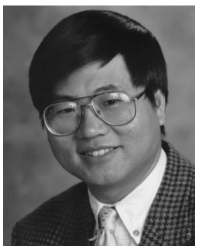

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