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Systems & Control Letters 57 (2008) 28–33
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Interconnection of Kronecker canonical form and special coordinate basis of multivariable linear systems

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Received 21 August 2006; received in revised form 2 April 2007; accepted 8 June 2007
Available online 20 July 2007

Abstract

This paper establishes a straightforward interconnection between the Kronecker canonical form and the special coordinate basis of linear systems. Such an interconnection yields an alternative approach for computing the Kronecker canonical form, and as a by-product, the Smith form, of the system matrix of general multivariable time-invariant linear systems. The overall procedure involves the transformation of a given system in the state-space description into the special coordinate basis, which is capable of explicitly displaying all the system structural properties, such as finite and infinite zero structures, as well as system invertibility structures. The computation of the Kronecker canonical form and Smith form of the system matrix is rather simple and straightforward once the given system is put under the special coordinate basis. The procedure is applicable to proper systems and singular systems.

Keywords: Linear systems theory; Kronecker canonical form; Smith form; System structural indices; Singular systems

1. Introduction

The Kronecker canonical form has been extensively used in the literature to capture the invariant indices and structural properties of linear systems. It is now well understood that the system structural properties play a crucial role in the design of control systems. In this paper, we consider a multivariable linear time-invariant system characterized by

$$\Sigma : \begin{cases} E\dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are, respectively, the state, input and output of the given system, and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ are constant matrices. The system Σ is said to be singular if $\text{rank}(E) < n$. Otherwise, it is said to be a proper system. It is well understood in the literature that the structural properties of Σ, such as the finite and infinite zero structures, as well as the system invertibility structures, can be fully captured by its (Rosenbrock) system matrix defined as follows (see, [13]):

$$P_\Sigma(s) = \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} = s\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}. \tag{2}$$

The computational problem of the Kronecker canonical form of $P_\Sigma(s)$ is potentially ill-posed. It involves finding generalized eigenvalues and eigenvectors for singular matrix pencils. Traditionally, the computation of the Kronecker canonical form was carried out through certain iterative reduction schemes (see, for example, [1,8,9,12,16]). Among these approaches, some were based on the reduction of the system matrix to a generalized Schur form (see, for example, [6,7]), and others the generalization of Kublanovskaya's algorithm for the determination of the Jordan structure of a constant matrix (see, for example, [1]).

Instead of focusing on the computational issues of the Kronecker canonical form of matrix pencils, the main objective of this paper is to establish a straightforward interconnection between the Kronecker canonical form and the special coordinate

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doi:10.1016/j.sysconle.2007.06.009

basis of linear systems of Sannuti and Saberi [15]. We will show that it is simple to derive a constructive procedure for computing the Kronecker canonical form, and as a by-product, the Smith form, of $P_\Sigma(s)$ by utilizing the special coordinate basis technique. The special coordinate basis was originally proposed by Sannuti and Saberi [15], and was recently completed by Chen [3], in which all the system structural properties of the special coordinate basis were rigorously justified. The issues on the computation of the special coordinate basis of linear systems have recently been studied in detail in Chu et al. [5], which shows that a raw form of the special coordinate basis can be obtained using some almost orthogonal transformations. The software realization of the special coordinate basis and other related decomposition techniques required is readily available in Lin et al. [10]. Thus, the additional cost for computing the canonical forms mentioned above is very minimal.

To be more specific, we recall that two pencils $sM_1 - N_1$ and $sM_2 - N_2$ of dimensions $m \times n$ are said to be equivalent if there exist constant nonsingular matrices $\tilde{P}$ and $\tilde{Q}$ of appropriate dimensions such that

$$\tilde{Q}(sM_1 - N_1)\tilde{P} = sM_2 - N_2. \tag{3}$$

It was shown in Gantmacher [8] that any pencil $sM - N$ can be reduced to a canonical quasi-diagonal form, which is given by

$$\tilde{Q}(sM - N)\tilde{P} = \begin{bmatrix} \begin{matrix}\text{blkdiag}\{sI - J, L_{l_1}, \ldots, L_{l_{p_b}}, \\ R_{r_1}, \ldots, R_{r_{m_c}}, I - sH\}\end{matrix} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

In (4), the last term, i.e., 0, corresponds to the case when there are redundant columns or rows associated with the input matrices and measurement matrices. J is in Jordan canonical form, and $sI - J$ has the following $\sum_{i=1}^{\delta} d_i$ pencils as its diagonal blocks,

$$sI_{m_{i,j}} - J_{m_{i,j}}(\beta_i) := \begin{bmatrix} s-\beta_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\beta_i & -1 \\ & & & s-\beta_i \end{bmatrix}, \tag{5}$$

$j = 1, 2, \ldots, d_i$, $i = 1, 2, \ldots, \delta$. L_{l_i}, $i = 1, 2, \ldots, p_b$, is an $(l_i + 1) \times l_i$ bidiagonal pencil, i.e.,

$$L_{l_i} := \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}. \tag{6}$$

R_{r_i}, $i = 1, 2, \ldots, m_c$, is an $r_i \times (r_i + 1)$ bidiagonal pencil, i.e.,

$$R_{r_i} := \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}. \tag{7}$$

Finally, H is nilpotent and in Jordan canonical form, and $I - sH$ has the following d pencils as its diagonal blocks,

$$I_{n_j+1} - sJ_{n_j+1}(0) := \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \tag{8}$$

$j = 1, 2, \ldots, d$. Then, $\{(s - \beta_i)^{m_{i,j}}, j = 1, 2, \ldots, d_i\}$ are finite elementary divisors at β_i, $i = 1, 2, \ldots, \delta$. The index sets $\{r_1, r_2, \ldots, r_{m_c}\}$ and $\{l_1, l_2, \ldots, l_{p_b}\}$ are right and left minimal indices, respectively. Lastly, $\{(1/s)^{n_j+1}, j = 1, 2, \ldots, d\}$ are the infinite elementary divisors.

In the context of this paper, we will focus on

$$sM - N = s\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} = P_\Sigma(s), \tag{9}$$

the (Rosenbrock) system matrix pencil associated with Σ. The definition of structural invariants of Σ is based on the invariant indices of its system pencil. In particular, the right and left invertibility indices are, respectively, the right and left minimal indices of the system pencil, the finite and infinite zero structures of the given system are related to the finite and infinite elementary divisors of the system pencil.

The Smith form of the system matrix is another way to capture the invariant zero structure of the given system Σ. We recall the definition of the Smith form from the classical text of Rosenbrock and Storey [14]. Given a polynomial matrix $A(s)$, it was shown in [14] that there exist unimodular transformations $M(s)$ and $N(s)$ such that

$$S(s) = M(s)A(s)N(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{10}$$

where

$$D(s) = \text{diag}\{p_1(s), p_2(s), \ldots, p_r(s)\}, \tag{11}$$

and where each $p_i(s)$, $i = 1, 2, \ldots, r$, is a monic polynomial and $p_i(s)$ is a factor of $p_{i+1}(s)$, $i = 1, 2, \ldots, r - 1$. Note that a unimodular matrix is a square polynomial matrix whose determinant is a nonzero constant. $S(s)$ of (10) is called the Smith canonical form or Smith form of $A(s)$. We will show in this paper that it is straightforward to obtain the Smith form of $P_\Sigma(s)$ once it is transformed into the Kronecker canonical form.

The rest of the paper is organized as follows: in Section 2, we present the main results of this paper, i.e., the interconnection of the Kronecker canonical form and Smith form of the system matrix, $P_\Sigma(s)$, and the special coordinate basis of Σ. The results will be illustrated by a numerical example. Finally, some concluding remarks will be drawn in Section 3.

2. Kronecker and Smith forms of the system matrix

Before proceeding to present our main results, we first show that the Kronecker canonical form and Smith form of the system pencil of singular systems can be captured by converting the singular system into an auxiliary proper system. This can be done as follows. Without loss of generality, we assume that E is in the form of

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{12}$$

and thus A, B and C can be partitioned accordingly as

$$A = \begin{bmatrix} A_{nn} & A_{ns} \\ A_{sn} & A_{ss} \end{bmatrix}, \quad B = \begin{bmatrix} B_n \\ B_s \end{bmatrix}, \quad C = [\, C_n \quad C_s \,]. \tag{13}$$

Rewriting the system pencil of (9) as

$$P_\Sigma(s) = \left[\begin{array}{c|cc} sI - A_{nn} & -A_{ns} & -B_n \\ \hline -A_{sn} & -A_{ss} & -B_s \\ C_n & C_s & D \end{array}\right] = \begin{bmatrix} sI - A_x & -B_x \\ C_x & D_x \end{bmatrix} \tag{14}$$

it is simple to see that the invariant indices of Σ are equivalent to those of a proper system characterized by (A_x, B_x, C_x, D_x). Thus, without loss of generality, we focus on the computation of the Kronecker canonical form and Smith form of the system matrix of a proper system characterized by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{15}$$

i.e., the following matrix pencil,

$$P_\Sigma(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}, \tag{16}$$

throughout the reminder of this paper. We next recall that the Kronecker canonical form of the system matrix of Σ, i.e., $P_\Sigma(s)$, is invariant under nonsingular state, input and output transformations, Γ_s, Γ_i and Γ_o, and is invariant under any state feedback and output injection. Such a fact follows directly from the following manipulation:

$$\begin{aligned} UP_\Sigma(s)V &= \begin{bmatrix} \Gamma_s^{-1} & -\tilde{K}\Gamma_o^{-1} \\ 0 & \Gamma_o^{-1} \end{bmatrix} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \Gamma_s & 0 \\ \Gamma_i \tilde{F} & \Gamma_i \end{bmatrix} \\ &= \begin{bmatrix} sI - (\tilde{A} + \tilde{B}\tilde{F} + \tilde{K}\tilde{C} + \tilde{K}\tilde{D}\tilde{F}) & -(\tilde{B} + \tilde{K}\tilde{D}) \\ \tilde{C} + \tilde{D}\tilde{F} & \tilde{D} \end{bmatrix} \\ &= \begin{bmatrix} sI - A_{KF} & -B_K \\ C_F & \tilde{D} \end{bmatrix}, \end{aligned} \tag{17}$$

where $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is the transformed system and is given by

$$\begin{aligned} \tilde{A} &= \Gamma_s^{-1} A \Gamma_s, \quad \tilde{B} = \Gamma_s^{-1} B \Gamma_i, \\ \tilde{C} &= \Gamma_o^{-1} C \Gamma_s, \quad \tilde{D} = \Gamma_o^{-1} D \Gamma_i, \end{aligned} \tag{18}$$

$\tilde{F}$ and $\tilde{K}$ are, respectively, the state feedback and output injection gain matrices under the coordinate of the transformed system, and finally, Σ_{KF} characterized by the matrix quadruple $(A_{KF}, B_K, C_F, \tilde{D})$ with $A_{KF} = \tilde{A} + \tilde{B}\tilde{F} + \tilde{K}\tilde{C} + \tilde{K}\tilde{D}\tilde{F}$, $B_K = \tilde{B} + \tilde{K}\tilde{D}$ and $C_F = \tilde{C} + \tilde{D}\tilde{F}$ is the resulting transformed system under the state feedback and output injection laws.

We are now ready to show that the Kronecker canonical form of $P_\Sigma(s)$ can be obtained neatly through the special coordinate basis of Σ. The following is a step-by-step algorithm that generates the required nonsingular transformations U and V for the canonical form:

Step KCF 1: Computation of the special coordinate basis of Σ.

Apply the result of Chu et al. [5] (see also [4,15]) to find nonsingular state, input and output transformations, $\Gamma_s \in \mathbb{C}^{n\times n}$, $\Gamma_i \in \mathbb{R}^{m\times m}$ and $\Gamma_o \in \mathbb{R}^{p\times p}$, such that the given system Σ of (15) is transformed into the special coordinate basis as given in Theorem 2.4.1 of Chen [4] or in the following compact form:

$$\begin{aligned} \tilde{A} &= \Gamma_s^{-1} A \Gamma_s \\ &= \begin{bmatrix} A_{aa} & L_{ab}C_b & 0 & L_{ad}C_d \\ 0 & A_{bb} & 0 & L_{bd}C_d \\ B_c E_{ca} & L_{cb}C_b & A_{cc} & L_{cd}C_d \\ B_d E_{da} & B_d E_{db} & B_d E_{dc} & A_{dd}^* + B_d E_{dd} + L_{dd}C_d \end{bmatrix} \\ &\quad + B_0 C_0, \end{aligned}$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = [\, B_0 \quad B_1 \,] = \begin{bmatrix} B_{0a} & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix},$$

and

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} C_{0a} & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix},$$

$$\tilde{D} = \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $A_{dd}^* \in \mathbb{R}^{n_d\times n_d}$, $B_d \in \mathbb{R}^{n_d\times m_d}$ and $C_d \in \mathbb{R}^{m_d\times n_d}$ have the following forms:

$$A_{dd}^* = \text{blkdiag}\{A_{q_1}, \ldots, A_{q_{m_d}}\}, \tag{19}$$

and

$$\begin{aligned} B_d &= \text{blkdiag}\{B_{q_1}, \ldots, B_{q_{m_d}}\}, \\ C_d &= \text{blkdiag}\{C_{q_1}, \ldots, C_{q_{m_d}}\}, \end{aligned} \tag{20}$$

with A_{q_i}, B_{q_i} and C_{q_i}, $i = 1, 2, \ldots, m_d$, being given as follows:

$$\begin{aligned} A_{q_i} &= \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_{q_i} &= [1, 0, \ldots, 0]. \end{aligned} \tag{21}$$

Also, we assume that $A_{aa} \in \mathbb{C}^{n_a\times n_a}$ is already in the Jordan canonical form, i.e.,

$$A_{aa} = \text{blkdiag}\{J_{a,1}, J_{a,2}, \ldots, J_{a,k}\}, \tag{22}$$

where $J_{a,i}$, $i = 1, 2, \ldots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_{a,i} = \text{diag}\{\alpha_i, \alpha_i, \ldots, \alpha_i\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}, \tag{23}$$

and (A_{bb}, C_b), with $A_{bb} \in \mathbb{R}^{n_b\times n_b}$ and $C_b \in \mathbb{R}^{p_b\times n_b}$, is in the form of the observability structural decomposition (see, for example, [2,4,11, Theorem 2.3.1] for its dual version), i.e.,

$$A_{\rm bb} = A_{\rm bb}^* + L_{\rm bb}C_{\rm b} = \text{blkdiag}\{A_{{\rm bb},1}, \ldots, A_{{\rm bb},p_{\rm b}}\} + L_{\rm bb}C_{\rm b}, \tag{24}$$

and

$$C_{\rm b} = \text{blkdiag}\{C_{{\rm b},1}, \ldots, C_{{\rm b},p_{\rm b}}\}, \tag{25}$$

with

$$A_{{\rm bb},i} = \begin{bmatrix} 0 & I_{l_i-1} \\ 0 & 0 \end{bmatrix}, \quad C_{{\rm b},i} = [\,1 \quad 0\,], \quad i = 1, 2, \ldots, p_{\rm b}. \tag{26}$$

Finally, $(A_{\rm cc}, B_{\rm c})$, with $A_{\rm cc} \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$ and $B_{\rm c} \in \mathbb{R}^{n_{\rm c}\times m_{\rm c}}$, is assumed to be in the form of the controllability structural decomposition of Theorem 2.3.1 of [4] (see also [2,11]), i.e.,

$$A_{\rm cc} = A_{\rm cc}^* + B_{\rm c}E_{\rm cc} = \text{blkdiag}\{A_{{\rm cc},1}, \ldots, A_{{\rm cc},m_{\rm c}}\} + B_{\rm c}E_{\rm cc}, \tag{27}$$

and

$$B_{\rm c} = \text{blkdiag}\{B_{{\rm c},1}, \ldots, B_{{\rm c},m_{\rm c}}\}, \tag{28}$$

with

$$A_{{\rm cc},i} = \begin{bmatrix} 0 & I_{l_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{{\rm c},i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad i = 1, 2, \ldots, m_{\rm c}. \tag{29}$$

Step KCF 2: Determination of state feedback and output injection laws.

Let

$$\tilde F = -\begin{bmatrix} C_{0\rm a} & C_{0\rm b} & C_{0\rm c} & C_{0\rm d} \\ E_{\rm da} & E_{\rm db} & E_{\rm dc} & E_{\rm dd} \\ E_{\rm ca} & 0 & E_{\rm cc} & 0 \end{bmatrix}, \tag{30}$$

and

$$\tilde K = -\begin{bmatrix} B_{0\rm a} & L_{\rm ad} & L_{\rm ab} \\ B_{0\rm b} & L_{\rm bd} & L_{\rm bb} \\ B_{0\rm c} & L_{\rm cd} & L_{\rm cb} \\ B_{0\rm d} & L_{\rm dd} & 0 \end{bmatrix}. \tag{31}$$

It is straightforward to verify that the resulting $\Sigma_{\rm KF}$ is characterized by

$$A_{\rm KF} = \begin{bmatrix} A_{\rm aa} & 0 & 0 & 0 \\ 0 & A_{\rm bb}^* & 0 & 0 \\ 0 & 0 & A_{\rm cc}^* & 0 \\ 0 & 0 & 0 & A_{\rm dd}^* \end{bmatrix}, \quad B_{\rm K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{\rm c} \\ 0 & B_{\rm d} & 0 \end{bmatrix}, \tag{32}$$

and

$$C_{\rm F} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{\rm d} \\ 0 & C_{\rm b} & 0 & 0 \end{bmatrix}, \quad \tilde D = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

Step KCF 3: Finishing touches.

It is now simple to verify that the (Rosenbrock) system matrix associated with $\Sigma_{\rm KF}$ has the following form:

1. The corresponding term associated with $J_{{\rm a},i}$ is given by

$$sI - J_{{\rm a},i} = \begin{bmatrix} s-\alpha_i & -1 & & \\ & \ddots & \ddots & \\ & & s-\alpha_i & -1 \\ & & & s-\alpha_i \end{bmatrix}, \tag{34}$$

which is already in the format of (5).

2. The corresponding term associated with $(A_{{\rm bb},i}, C_{{\rm b},i})$ is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & I_{l_i} \end{bmatrix}\begin{bmatrix} C_{{\rm b},i} \\ sI - A_{{\rm bb},i} \end{bmatrix} = \begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}, \tag{35}$$

which is in the format of (6).

3. The corresponding term associated with $(A_{{\rm cc},i}, B_{{\rm c},i})$ is given by

$$[\,sI - A_{{\rm cc},i} \quad -B_{{\rm c},i}\,] = \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}, \tag{36}$$

which is in the format of (7).

4. Lastly, the corresponding term associated with $(A_{q_i}, B_{q_i}, C_{q_i})$ is given by

$$\begin{bmatrix} sI - A_{q_i} & -B_{q_i} \\ C_{q_i} & 0 \end{bmatrix} = \begin{bmatrix} s & -1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & s & -1 & 0 \\ & & & s & -1 \\ 1 & \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

Let

$$U_{q_i} = \begin{bmatrix} 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix}, \quad V_{q_i} = -\begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{38}$$

Then, we have

$$U_{q_i}\begin{bmatrix} sI - A_{q_i} & -B_{q_i} \\ C_{q_i} & 0 \end{bmatrix} V_{q_i} = \begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & 1 & -s \\ & & & 1 \end{bmatrix}, \tag{39}$$

which is now in the format of (8).

The Kronecker canonical form of the system matrix of $\Sigma_{\rm KF}$, or equivalently the system matrix of Σ, i.e., (16), can then be obtained by taking into account the additional transformations given in (35) and (38) together with some appropriate permutation transformations. This completes the algorithm.

Next, we proceed to show the Smith form of the system matrix, $P_\Sigma(s)$, can be explicitly displayed under special coordinate basis as well.

Step Smith 1: Determination of the Kronecker form of $P_\Sigma(s)$.

Utilize the special coordinate basis of Σ to determine the Kronecker canonical form of $P_\Sigma(s)$ as given in the previous algorithm. However, for the Smith form of $P_\Sigma(s)$, we need not to decompose A_{aa} into the Jordan canonical form, which might involve complex transformations. Instead, we leave A_{aa} as a real-valued matrix. Note that the transformations involved in the Kronecker canonical form are constant and nonsingular, and thus unimodular.

Step Smith 2: Determination of unimodular transformations.

1. Using the procedure given in the proof of Theorem 7.4 in Chapter 3 of Rosenbrock and Storey [14], it is straightforward to show that the term $sI - J_{a,i}$ in (34) can be deduced to the following Smith form:

$$(sI - J_{a,i}) \rightleftharpoons \mathrm{diag}\{\overbrace{1, \ldots, 1}^{n_i-1}, (s-\alpha_i)^{n_i}\}. \tag{40}$$

In general, following the procedure given in [14], we can compute two unimodular transformations $M_a(s)$ and $N_a(s)$ such that $sI - A_{aa}$ is transformed into the Smith form, i.e.,

$$M_a(s)(sI - A_{aa})N_a(s) = \{p_{a,1}(s), p_{a,2}(s), \ldots, p_{a,n_a}(s)\}. \tag{41}$$

Clearly, these polynomials are related to the invariant zero structures of the given system Σ.

2. The term corresponding to $(A_{bb,i}, C_{b,i})$ given in (35) has a constant Smith form:

$$\begin{bmatrix} I_{l_i} \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 & & & \\ s & \ddots & & \\ \vdots & \ddots & \ddots & \\ s^{l_i} & \cdots & s & 1 \end{bmatrix} \times \left(\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix} \right) I_{l_i}. \tag{42}$$

Note that the first term on the right-hand side of the above equation is a unimodular matrix.

3. Similarly, the Smith form for the term corresponding to $(A_{cc,i}, B_{c,i})$ given in (36) is also a constant matrix:

$$[I_{r_i} \quad 0] = I_{r_i} \left(\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix} \right) N_{r_i}(s), \tag{43}$$

where

$$N_{r_i}(s) = -\begin{bmatrix} 1 & & & \\ s & \ddots & & \\ \vdots & \ddots & \ddots & \\ s^{r_i} & \cdots & s & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ I_{r_i} & 0 \end{bmatrix} \tag{44}$$

is a unimodular matrix.

4. Lastly, the Smith form for the term corresponding to $(A_{q_i}, B_{q_i}, C_{q_i})$ given in (38) is an identity matrix:

$$I_{q_i+1} = I_{q_i+1} \left(\begin{bmatrix} 1 & -s & & \\ & \ddots & \ddots & \\ & & \ddots & -s \\ & & & 1 \end{bmatrix} \right) \times \begin{bmatrix} 1 & s & \cdots & s^{q_i} \\ & \ddots & \ddots & \vdots \\ & & \ddots & s \\ & & & 1 \end{bmatrix}. \tag{45}$$

Once again, the last term of the equation above is a unimodular matrix.

Finally, in view of (41)–(45) together with some appropriate permutation transformations, it is now straightforward to obtain unimodular transformations $M(s)$ and $N(s)$ such that

$$M(s)P_\Sigma(s)N(s) = \begin{bmatrix} D_\Sigma(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{46}$$

where

$$D_\Sigma(s) = \mathrm{diag}\{\overbrace{1, \ldots, 1}^{n_{bcd}}, p_{a,1}(s), p_{a,2}(s), \ldots, p_{a,n_a}(s)\}, \tag{47}$$

and where $n_{bcd} = n_b + n_c + n_d + m_0 + m_d$.

We illustrate the above results with the following example.

Example 2.1. Consider system characterized by (15) with

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 1 \\ -1 & 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{48}$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{49}$$

which is already in the form of the special coordinate basis with an invariant zero at 1, and $n_a = n_b = n_c = n_d = 1$. Following the algorithm given in Steps KCF 1–3, we obtain

$$\tilde{F} = \begin{bmatrix} 1 & -3 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \\ -1 & -2 \\ 0 & 0 \end{bmatrix},$$

$$A_{KF} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $B_K = B$, $C_F = C$, $\tilde{D} = D$, and the required two nonsingular transformations,

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -3 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix},$$

which transform $P_\Sigma(s)$ into the Kronecker canonical form, i.e.,

$$UP_\Sigma(s)V = \left[\begin{array}{c|c|cc|cc} s-1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & s & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Next, following the algorithm given in Steps SMITH 1–2, we obtain two unimodular matrices,

$$M(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & s-1 \end{bmatrix},$$

and

$$N(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & s-1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 1-s \end{bmatrix},$$

with $\det[M(s)] = -1$ and $\det[N(s)] = 1$, which convert $P_\Sigma(s)$ into the Smith form, i.e.,

$$M(s)P_\Sigma(s)N(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, the polynomial in the entry (4,4) of the above Smith form of $P_\Sigma(s)$, i.e., $s-1$, results from the invariant zero of Σ.

3. Conclusions

In this paper, we have demonstrated that the well known Kronecker canonical form and Smith form of the system matrix of a general multivariable linear system, either proper or singular, can be captured using the special coordinate basis technique in a straightforward manner. The results have been implemented in an m-function in MATLAB and have been reported recently in a technical report [10]. Interested readers might directly contact the authors for a beta version of the tool kit.

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