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Explicit constructions of global stabilization and nonlinear H_∞ control laws for a class of nonminimum phase nonlinear multivariable systems

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SUMMARY

This paper investigates a global stabilization problem and a nonlinear H_∞ control problem for a class of nonminimum phase nonlinear multivariable systems. To avoid the complicated recursive design procedure, an asymptotic time-scale and eigenstructure assignment method is adopted to construct the control laws for the stabilization problem and the nonlinear H_∞ control problem. A sufficient solvability condition is established on the unstable zero dynamics of the system for global stabilization problem and nonlinear H_∞ control problem, respectively. Moreover, based on the sufficient solvability condition, an upper bound of the achievable L_2-gain is estimated for the nonlinear H_∞ control problem. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

We consider a class of uncertain nonlinear systems whose nonlinearities are unknown but depended only on the output. By transforming the system into a so-called special coordinate basis (SCB) form [1, 2], the system is exactly a nonlinear system in the output feedback form which has been extensively studied in the literature. The geometric conditions for transforming an affine nonlinear system into the output feedback form are given in [3]. In the past two decades, various control problems have been investigated for the nonlinear system in output feedback form, such as global

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stabilization [4], nonlinear output regulation [5–7], unknown disturbance rejection [8, 9], just to name a few. However, most of these works are based on a minimum phase assumption. That is, the zero dynamics of the nonlinear system is assumed to be stable. Only a few results are for the nonminimum phase systems (see, e.g. [10–13]). Nevertheless, the systematic design method for the global stabilization problem of the nonminimum phase systems is limited to the systems with one-dimensional unstable zero dynamics [10, 13]. Recently, in [14], we developed a global stabilization technique for the nonminimum phase nonlinear systems with high-order unstable zero dynamics for single-input and single-output (SISO) systems. To construct the control law, a recursive algorithm developed by Tsinias [15] is used. However, the recursive algorithm leads to tedious and complex calculation for high-order systems. Especially, for the multi-input and multi-output (MIMO) systems, the recursive algorithm needs to be performed separately for each input channel. As a special case of the MIMO system, the SISO system is invertible. The stabilization method proposed in [14] for SISO systems is not directly applicable to the MIMO systems if the MIMO systems are not invertible. In the literature, the time-scale method is a familiar tool to solve the control problems for the systems in various special structural forms. For example, Marino *et al.* [16] used a time-scale method to solve almost disturbance and almost input–output decoupling problems for linear systems in a pseudo-canonical form. This pseudo-canonical form is slightly different from the Morse pseudo-canonical form, but the Morse (pseudo-) canonical form can be easily deduced from it [16]. It is well known that the Morse canonical form gives information on zero structure, observability, controllability and invertibility of the system. However, the outputs of the system are coupled with the inputs in the Morse canonical form and the pseudo-canonical form developed in [16]. As shown in Section 2, the SCB form not only gives a more clear structure on zero structure, observability, controllability and invertibility of the system, but also gives a clearly decoupled structure of inputs and outputs (see, e.g. [1, 2, 17] for details on the SCB form). With the virtues of the SCB form, in this paper, we also adopt a time-scale method called the asymptotic time-scale and eigenstructure assignment (ATEA) method, originated in [18, 19] for solving linear control problems, to construct the control laws to avoid the complicated calculations for the MIMO systems. Moreover, the extended method can tackle right invertible MIMO systems.

The nonlinear H_∞ control problem has attracted much research effort since the works of Van der Schaft [20, 21], and many interesting results are available in the literature, see [22–28] and references therein. The solvability of the nonlinear H_∞ control problem involves in the solvability of a γ-related Hamilton–Jacobi (HJ) equation, where γ is a desired L_2-gain from the disturbance input to the system output. If $\gamma>0$ is arbitrary, the nonlinear H_∞ control problem is known as an almost disturbance decoupling problem. It was shown that the almost disturbance decoupling problem is solvable if the disturbance input does not affect the unstable part of zero dynamics of the system, [29–31], or if the zero dynamics contains only a special chain of integrators [32]. However, for more general situations, disturbance decoupling is generally not feasible. One has to seek to design a controller that achieves a pre-specified L_2-gain $\gamma>\gamma^*$, where γ^* is the best achievable performance for the problem, i.e. the problem is solvable for $\gamma>\gamma^*$ and not for $\gamma<\gamma^*$. The optimal value γ^* can be nicely calculated for linear H_∞ control problem. For more details, see [33–35]. However, how to calculate γ^* exactly and directly is still open for nonlinear H_∞ control problem. But the problem of estimating the optimal γ^* was investigated in [26, 36]. The estimation of optimal L_2-gain for nonlinear H_∞ control problem is obtained in [36] under the assumption that the zero input system is stable. In [26], an upper bound of the optimal value γ^* is computed for a class of nonlinear systems with a second-order zero dynamics. In this paper, we make another effort to obtain an upper estimate of the optimal L_2-gain γ^* based on a sufficient

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solvability condition of the nonlinear H_∞ control problem. The sufficient condition is established by using the global stabilization technique proposed in Section 3 during which an H_∞ control law is constructed explicitly without solving any HJ equations.

The paper is organized as follows. Section 2 gives the problem formulation and a simple introduction on the SCB form. In Section 3, the ATEA method is applied to construct a linear state feedback control law for the global stabilization problem. Section 4 solves a nonlinear H_∞ control problem by using this stabilization technique. Section 5 extends the results to the systems that have zeros on the imaginary axis. In Section 6, an illustrative example is given for solving an H_∞ control problem. Finally, we draw some concluding remarks in Section 7.

2. PROBLEM FORMULATION AND SYSTEM TRANSFORMATION

Consider the nonlinear system of the form

$$\begin{aligned} \dot{\xi} &= A\xi + Bv + \Psi(y) + \mathscr{G}(\xi)w \\ y &= C\xi \end{aligned} \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^s$ the disturbance input, $v \in \mathbb{R}^m$ the control input, $y \in \mathbb{R}^p$ the system output and

$$\Psi(y) = \begin{bmatrix} \psi_1(y) \\ \vdots \\ \psi_n(y) \end{bmatrix}, \quad \mathscr{G}(\xi) = \begin{bmatrix} g_{11}(\xi) & \cdots & g_{1s}(\xi) \\ \vdots & \vdots & \vdots \\ g_{n1}(\xi) & \cdots & g_{ns}(\xi) \end{bmatrix}$$

where $\psi_i(y) : \mathbb{R}^p \to \mathbb{R}$, $i = 1, \ldots, n$, $g_{ij}(\xi) : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, $j = 1, \ldots, s$ are some smooth nonlinear functions, and $\psi_i(0) = 0$.

The global stabilization problem by linear feedback: Consider system (1) with $w = 0$ and find a linear state feedback control law of the form

$$v = K\xi \tag{2}$$

such that the equilibrium at $\xi = 0$ of the closed-loop system consisting of (1) and (2) is globally asymptotically stable.

The nonlinear H_∞ control problem by linear feedback: Given $\gamma > 0$, find, if possible, a linear state feedback control law of form (2) such that the equilibrium at $\xi = 0$ of the closed-loop system consisting of (1) and (2) is globally asymptotically stable and has an L_2-gain, from the exogenous disturbance input w to the output y, that is less than or equal to γ, i.e.

$$\int_o^T \|y(t)\|^2 \,\mathrm{d}t \leqslant \gamma^2 \int_0^T \|w(t)\|^2 \,\mathrm{d}t \tag{3}$$

for all $T \geqslant 0$ and zero initial state $\xi(0) = 0$.

The following assumptions are made in this paper.

Assumption A1: (A, B) is stabilizable, and (A, B, C) is right invertible.

Assumption A2: The linear system (A, B, C) has no invariant zeros on the imaginary axis.

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Assumption A3: There exist n positive real numbers k_i, $i=1,\ldots,n$, such that

$$|\psi_i(y)|\leqslant k_i\|y\| \quad \forall y\in\mathbb{R}^p \tag{4}$$

Assumption A4: There exist positive real numbers l_{ij}, $i=1,\ldots,n$, $j=1,\ldots,s$, such that

$$|g_{ij}(\xi)|\leqslant l_{ij} \quad \forall \xi\in\mathbb{R}^n \tag{5}$$

To establish the solvability of the stabilization problem and the nonlinear H_∞ control problem, we transform system (1) into the SCB form. Specifically, using the result of SCB (see, e.g. [1, 2]), if (A,B,C) is right invertible, there exist nonsingular matrices Γ_s, Γ_i and Γ_o such that

$$\bar{A}=\Gamma_s^{-1}A\Gamma_s=\begin{bmatrix} A_{aa}^- & 0 & 0 & 0 & L_{ad}^-C_d \\ 0 & A_{aa}^0 & 0 & 0 & L_{ad}^0C_d \\ 0 & 0 & A_{aa}^+ & 0 & L_{ad}^+C_d \\ B_cE_{ca}^- & B_cE_{ca}^0 & B_cE_{ca}^+ & A_{cc} & L_{cd}C_d \\ B_dE_{da}^- & B_dE_{da}^0 & B_dE_{da}^+ & B_dE_{dc} & A_{dd} \end{bmatrix}$$

$$\bar{B}=\Gamma_s^{-1}B\Gamma_i=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}$$

$$\bar{C}=\Gamma_o^{-1}C\Gamma_s=[0\ \ 0\ \ 0\ \ 0\ \ C_d]$$

where in particular,

$$A_{dd}=A_{dd}^*+B_dE_{dd}+L_{dd}C_d$$

with

$$A_{dd}^*=\text{blkdiag}\{A_{q_1},A_{q_2},\ldots,A_{q_{m_d}}\}$$
$$B_d=\text{blkdiag}\{B_{q_1},B_{q_2},\ldots,B_{q_{m_d}}\}$$
$$C_d=\text{blkdiag}\{C_{q_1},C_{q_2},\ldots,C_{q_{m_d}}\}$$

The matrices A_{q_i}, B_{q_i} and C_{q_i}, $i=1,2,\ldots,m_d$, have the form

$$A_{q_i}=\begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix},\quad B_{q_i}=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad C_{q_i}=[1,0,\ldots,0]$$

and $A_{aa}^-\in\mathbb{R}^{n_a^-\times n_a^-}$, $A_{aa}^0\in\mathbb{R}^{n_a^0\times n_a^0}$, $A_{aa}^+\in\mathbb{R}^{n_a^+\times n_a^+}$, $A_{cc}\in\mathbb{R}^{n_c\times n_c}$ with $n_a^-+n_a^0+n_a^++n_c+\sum_{i=1}^{m_d}q_i=n$. It should be noted that $m_d=p$ in this case. Moreover, all the eigenvalues of A_{aa}^- are strictly in the

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left-half plane, all those of A_{aa}^0 are on the imaginary axis and all those of A_{aa}^+ are strictly in the right-half plane.

Define the state, output and input transformations

$$x=\Gamma_s^{-1}\xi, \quad y_d=\Gamma_o^{-1}y, \quad u=\Gamma_i^{-1}v \tag{6}$$

and partition x and u as follows:

$$x=\begin{bmatrix} x_a^- \\ x_a^0 \\ x_a^+ \\ x_c \\ x_d \end{bmatrix}, \quad u=\begin{bmatrix} u_d \\ u_c \end{bmatrix}$$

where $x_a^+\in\mathbb{R}^{n_a^+}$, $x_a^0\in\mathbb{R}^{n_a^0}$ and $x_a^-\in\mathbb{R}^{n_a^-}$, $x_c\in\mathbb{R}^{n_c}$, $u_c\in\mathbb{R}^{m_c}$ with $m_c+m_d=m$, and

$$x_d=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{bmatrix}, \quad y_d=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_d} \end{bmatrix}, \quad u_d=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{bmatrix}$$

where $x_i\in\mathbb{R}^{q_i}$, $y_i\in\mathbb{R}$, $u_i\in\mathbb{R}$ for $i=1,2,\ldots,m_d$. Respectively, denote

$$\Phi(y_d)=\Gamma_s^{-1}\Psi(\Gamma_o y_d)=\begin{bmatrix} \phi_a^-(y_d) \\ \phi_a^0(y_d) \\ \phi_a^+(y_d) \\ \phi_c(y_d) \\ \phi_d(y_d) \end{bmatrix}, \quad \mathscr{H}(x)=\Gamma_s^{-1}\mathscr{G}(\Gamma_s x)=\begin{bmatrix} \mathscr{H}_a^-(x) \\ \mathscr{H}_a^0(x) \\ \mathscr{H}_a^+(x) \\ \mathscr{H}_c(x) \\ \mathscr{H}_d(x) \end{bmatrix}$$

Then, system (1) is transformed into the SCB form

$$\begin{aligned}
\dot{x}_a^- &= A_{aa}^- x_a^- + L_{ad}^- y_d + \phi_a^-(y_d) + \mathscr{H}_a^-(x)w \\
\dot{x}_a^0 &= A_{aa}^0 x_a^0 + L_{ad}^- y_d + \phi_a^0(y_d) + \mathscr{H}_a^0(x)w \\
\dot{x}_a^+ &= A_{aa}^+ x_a^+ + L_{ad}^+ y_d + \phi_a^+(y_d) + \mathscr{H}_a^+(x)w \\
\dot{x}_c &= A_{cc}x_c + L_{cd}y_d + \phi_c(y_d) + \mathscr{H}_c(x)w + B_c(u_c + E_{ca}^- x_a^- + E_{ca}^0 x_a^0 + E_{ca}^+ x_a^+) \\
\dot{x}_d &= A_{dd}^* x_d + L_{dd}y_d + \phi_d(y_d) + \mathscr{H}_d(x)w \\
&\quad + B_d(u_d + E_{da}^- x_a^- + E_{da}^0 x_a^0 + E_{da}^+ x_a^+ + E_{dc}x_c + E_{dd}x_d) \\
y_d &= C_d x_d
\end{aligned} \tag{7}$$

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Remark 2.1
Since $\Phi(y_d)=\Gamma_s^{-1}\Psi(\Gamma_o y_d)$, $\Phi(0)=\Gamma_s^{-1}\Psi(0)=0$. Assumption A3 implies that there exists a positive real k_0 such that

$$\|\Phi(y_d)\| \leqslant k_0 \|y_d\| \tag{8}$$

Moreover, there exist constant matrices $D_a^+ \in \mathbb{R}^{n_a^+ \times q_1}$, $D_a^0 \in \mathbb{R}^{n_a^0 \times r}$ and a Lebesgue measurable function matrix $G(y_d) \in \mathbb{R}^{r \times p}$ such that

$$\begin{bmatrix} \phi_a^0(y_d) \\ \phi_a^+(y_d) \end{bmatrix} = \begin{bmatrix} D_a^0 \\ D_a^+ \end{bmatrix} G(y_d) y_d \tag{9}$$

where $(G(y_d))^{\mathrm{T}} G(y_d) \leqslant I$ for all $y_d \in \mathbb{R}^p$, and r is an appropriate positive integer.

Remark 2.2
Under Assumption A4, it is clear that $\|\mathscr{H}_a^-(x)\|$, $\|\mathscr{H}_a^0(x)\|$, $\|\mathscr{H}_a^+(x)\|$, $\|\mathscr{H}_c(x)\|$ and $\|\mathscr{H}_d(x)\|$ are bounded for all $x \in \mathbb{R}^n$. Moreover, there exist two constant matrices $H_a^0 \in \mathbb{R}^{n_a^0 \times s}$ and $H_a^+ \in \mathbb{R}^{n_a^+ \times s}$ such that

$$\begin{bmatrix} \mathscr{H}_a^0(x) \\ \mathscr{H}_a^+(x) \end{bmatrix} \begin{bmatrix} \mathscr{H}_a^0(x) \\ \mathscr{H}_a^+(x) \end{bmatrix}^{\mathrm{T}} \leqslant \begin{bmatrix} H_a^0 \\ H_a^+ \end{bmatrix} \begin{bmatrix} H_a^0 \\ H_a^+ \end{bmatrix}^{\mathrm{T}} \tag{10}$$

3. STABILIZATION BY ASYMPTOTIC TIME-SCALE AND EIGENSTRUCTURE ASSIGNMENT

In this section, we use the ATEA method to solve the global stabilization problem for systems (1) with $w=0$ under Assumptions A1–A3. Assumption A2 implies that $n_a^0=0$, that is, the dynamic equation of x_a^0 does not appear in (7). As will be seen in Section 5, this assumption can be removed. The concept of the ATEA method was originally proposed in [19] and developed fully in Chen [18, 33]. It is decentralized in nature and is in fact rooted in the concept of singular perturbation methods of Kokotovic *et al.* [37]. Such a design method has been utilized intensively to solve many linear control problems, such as H_∞ control, H_2 optimal control, loop transfer recovery and the disturbance decoupling problem.

Theorem 3.1
Under Assumptions A1–A3, let $P_L>0$ and $P_D \geqslant 0$ be the solution of

$$P_L (A_{aa}^+)^{\mathrm{T}} + A_{aa}^+ P_L = L_{ad}^+ (L_{ad}^+)^{\mathrm{T}} \tag{11}$$

$$P_D (A_{aa}^+)^{\mathrm{T}} + A_{aa}^+ P_D = D_a^+ (D_a^+)^{\mathrm{T}} \tag{12}$$

respectively. If $P_L > P_D$, then the global stabilization problem is solvable by a linear state feedback.

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Proof
First, let us construct a linear state feedback control law by the algorithm of ATEA [1]. Since $P_L > P_D$, we have

$$P = (P_L - P_D)^{-1} > 0 \tag{13}$$

Define and partition F_s as follows:

$$F_s = (L_{ad}^+)^{\mathrm{T}} P = \begin{bmatrix} F_{s_1} \\ F_{s_2} \\ \vdots \\ F_{s_{m_d}} \end{bmatrix}$$

where F_{s_i} are of dimensions $1 \times n_a^+$.

By the property of the SCB form [1], (A_{cc}, B_c) is stabilizable. Thus, there exists a matrix $F_c \in \mathbb{R}^{m_c \times n_c}$ such that

$$A_{cc}^c = A_{cc} - B_c F_c \tag{14}$$

is stable.

Now, let

$$\Lambda_i = \{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iq_i}\}, \quad i = 1, 2, \ldots, m_d$$

be the sets of q_i elements, all in the strict left-half plane, which are closed under complex conjugation. For $i = 1, 2, \ldots, m_d$, we define

$$p_i(s) = \prod_{j=1}^{q_i} (s - \lambda_{ij}) = s^{q_i} + F_{i1} s^{q_i - 1} + \cdots + F_{iq_i - 1} s + F_{iq_i}$$

and a sub-gain matrix parameterized by a tuning parameter, ε,

$$\bar{F}_i(\varepsilon) = \frac{1}{\varepsilon^{q_i}} \left[F_{iq_i}, \varepsilon F_{iq_i - 1}, \ldots, \varepsilon^{q_i - 1} F_{i1} \right] \tag{15}$$

and let

$$\bar{F}_s(\varepsilon) = \begin{bmatrix} F_{s_1} F_{1q_1} / \varepsilon^{q_1} \\ F_{s_2} F_{2q_2} / \varepsilon^{q_2} \\ \vdots \\ F_{s_{m_d}} F_{m_d q_{m_d}} / \varepsilon^{q_{m_d}} \end{bmatrix} \tag{16}$$

Then, the ATEA state feedback gain is given by

$$F(\varepsilon) = -\Gamma_i (\bar{F}(\varepsilon) + \bar{F}_0) \Gamma_s^{-1} \tag{17}$$

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where

$$\bar{F}(\varepsilon)=\begin{bmatrix}0 & \bar{F}_s(\varepsilon) & 0 & \bar{F}_d(\varepsilon)\\ 0 & 0 & F_c & 0\end{bmatrix} \tag{18}$$

$$\bar{F}_0=\begin{bmatrix}E_{da}^- & E_{da}^+ & E_{dc} & E_{dd}\\ E_{ca}^- & E_{ca}^+ & 0 & 0\end{bmatrix} \tag{19}$$

and

$$\bar{F}_d(\varepsilon)=\text{blkdiag}\{\bar{F}_1(\varepsilon),\bar{F}_2(\varepsilon),\ldots,\bar{F}_{m_d}(\varepsilon)\} \tag{20}$$

We claim that there exists an $\varepsilon^*>0$ such that

$$v=F(\varepsilon)\xi$$

solves the global stabilization problem of system (1) for all $0<\varepsilon\leqslant\varepsilon^*$. In fact, denote $x_s=x_a^+$ and

$$A_{ss}=A_{aa}^+,\quad B_s=L_{ad}^+$$

It is clear that the closed-loop system in the SCB form is given by

$$\begin{aligned}
\dot{x}_a^- &= A_{aa}^- x_a^- + L_{ad}^- y_d + \phi_a^-(y_d)\\
\dot{x}_s &= A_{ss}x_s + B_s y_d + \phi_a^+(y_d)\\
\dot{x}_c &= (A_{cc}-B_cF_c)x_c + L_{cd}y_d + \phi_c(y_d)\\
\dot{x}_d &= (A_{dd}^* - B_d\bar{F}_d(\varepsilon))x_d - B_d\bar{F}_s(\varepsilon)x_s + L_{dd}y_d + \phi_d(y_d)\\
y_d &= C_d x_d
\end{aligned} \tag{21}$$

Noting that A_{aa}^- and $A_{cc}-B_cF_c$ are stable matrices, and $\Phi(y_d)$ satisfies the linear growth condition (8), to show the stability of (21), it suffice to show that

$$\begin{aligned}
\dot{x}_s &= A_{ss}x_s + B_s y_d + \phi_a^+(y_d)\\
\dot{x}_d &= (A_{dd}^* - B_d\bar{F}_d(\varepsilon))x_d - B_d\bar{F}_s(\varepsilon)x_s + L_{dd}y_d + \phi_d(y_d)\\
y_d &= C_d x_d
\end{aligned} \tag{22}$$

is asymptotically stable. To this end, we define a state transformation

$$\bar{x}_s=x_s,\quad \bar{x}_i=x_i+\begin{bmatrix}F_{s_i}\\ 0\\ \vdots\\ 0\end{bmatrix}x_s,\quad i=1,2,\ldots,m_d,\quad \bar{x}_d=\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\\ \vdots\\ \bar{x}_{m_d}\end{pmatrix} \tag{23}$$

Then, using the special structure of A^*_{dd}, B_d and C_d, we have

$$\begin{aligned}\dot{x}_s &= (A_{ss}-B_sF_s)x_s+L^+_{ad}\bar{y}_d+\phi^+_a(\bar{y}_d-F_sx_s)\\ \dot{\bar{x}}_i &= \left[A_{q_i}-\frac{1}{\varepsilon^{q_i}}B_{q_i}F_iS_i(\varepsilon)\right]\bar{x}_i+\bar{L}_{is}x_s+\bar{L}_{id}\bar{y}_d+\bar{\phi}_{id}(\bar{y}_d-F_sx_s)\\ \bar{y}_d &= y_d+F_sx_s=C_d\bar{x}_d\end{aligned}\tag{24}$$

where

$$F_i=[F_{iq_i},F_{iq_{i-1}},\ldots,F_{i1}],\quad S_i(\varepsilon)=\mathrm{diag}\{1,\varepsilon,\ldots,\varepsilon^{q_i-1}\}\tag{25}$$

and

$$\bar{L}_{is}=\begin{bmatrix}F_{s_i}\\0\\\vdots\\0\end{bmatrix}(A_{ss}-B_sF_s)+L_{is}-L_{id}F_s,\quad \bar{L}_{id}=\begin{bmatrix}F_{s_i}\\0\\\vdots\\0\end{bmatrix}B_s+L_{id}\tag{26}$$

$$\bar{\phi}_{id}(y_d)=\begin{bmatrix}F_{s_i}\\0\\\vdots\\0\end{bmatrix}\phi^+_a(y_d)+\phi_{id}(y_d)\tag{27}$$

with

$$\begin{bmatrix}L_{1d}\\L_{2d}\\\vdots\\L_{m_dd}\end{bmatrix}=L_{dd},\quad \begin{bmatrix}\phi_{1d}(y_d)\\\phi_{2d}(y_d)\\\vdots\\\phi_{m_dd}(y_d)\end{bmatrix}=\phi_d(y_d)$$

It should be noted that $\bar{L}_{is}$ and $\bar{L}_{id}$ are independent on ε. Moreover, by the linear growth condition (8), there exist constants $\kappa_1,\kappa_2,\ldots,\kappa_{m_d}$ such that

$$\|\bar{\phi}_{id}(y_d)\|\leqslant\kappa_i\|y_d\|\tag{28}$$

for $i=1,2,\ldots,m_d$.

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Now, define another state transformation for system (24),

$$\tilde{x}_s = \bar{x}_s, \quad \tilde{x}_i = S_i(\varepsilon)\bar{x}_i, \quad i = 1, 2, \ldots, m_d, \quad \tilde{x}_d = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{m_d} \end{pmatrix} \tag{29}$$

We have

$$\begin{aligned} \dot{\tilde{x}}_s &= (A_{ss} - B_s F_s)\tilde{x}_s + B_s \tilde{y}_d + \phi_a^+(\tilde{y}_d - F_s \tilde{x}_s) \\ \dot{\tilde{x}}_i &= \frac{1}{\varepsilon}[A_{q_i} - B_{q_i} F_i]\tilde{x}_i + S_i(\varepsilon)\bar{L}_{is}\tilde{x}_s + S_i(\varepsilon)\bar{L}_{id}\tilde{y}_d + S_i(\varepsilon)\bar{\phi}_{id}(\tilde{y}_d - F_s\tilde{x}_s) \\ \tilde{y}_d &= C_d \tilde{x}_d \end{aligned} \tag{30}$$

Let P_i, $i = 1, 2, \ldots, m_d$, be positive-definite solutions of

$$P_i(A_{q_i} - B_{q_i} F_i) + (A_{q_i} - B_{q_i} F_i)^{\mathrm{T}} P_i = -I \tag{31}$$

and define a Lyapunov function

$$V(\tilde{x}_s, \tilde{x}_d) = (\tilde{x}_s)^{\mathrm{T}} P \tilde{x}_s + \sum_{i=1}^{m_d} \tilde{x}_i^{\mathrm{T}} P_i \tilde{x}_i \tag{32}$$

Then the derivation of (32) along the trajectory of (30) is given by

$$\begin{aligned} \dot{V} = {} & (\tilde{x}_s)^{\mathrm{T}}((A_{ss} - B_s F_s)^{\mathrm{T}} P + P(A_{ss} - B_s F_s) - F_s^{\mathrm{T}}(G(\Delta))^{\mathrm{T}}(D_a^+)^{\mathrm{T}} P - P D_a^+ G(\Delta) F_s)\tilde{x}_s \\ & + 2(\tilde{x}_s)^{\mathrm{T}} P L_{ad}^+ \tilde{y}_d + 2(\tilde{x}_s)^{\mathrm{T}} P D_a^+ G(\Delta)\tilde{y}_d \\ & + \sum_{i=1}^{m_d} \left(\frac{1}{\varepsilon} \tilde{x}_i^{\mathrm{T}}((A_{q_i} - B_{q_i} F_i)^{\mathrm{T}} P_i + P_i(A_{q_i} - B_{q_i} F_i))\tilde{x}_i + 2\tilde{x}_i^{\mathrm{T}} P_i S_i(\varepsilon)\bar{L}_{is}\tilde{x}_s \right) \\ & + \sum_{i=1}^{m_d} (2\tilde{x}_i P_i S_i(\varepsilon)\bar{L}_{id}\tilde{y}_d + \tilde{x}_i^{\mathrm{T}} P_i S_i(\varepsilon)\bar{\phi}_{id}(\Delta)) \end{aligned}$$

where $\Delta = \tilde{y}_d - F_s \tilde{x}_s$.

Using (11)–(13), we have

$$(A_{aa}^+)^{\mathrm{T}} P + P A_{aa}^+ + P(D_a^+(D_a^+)^{\mathrm{T}} - L_{ad}^+(L_{ad}^+)^{\mathrm{T}})P = 0$$

Noting that $F_s = B_s^{\mathrm{T}} P = (L_{ad}^+)^{\mathrm{T}} P$ and $(G(\Delta))^{\mathrm{T}} G(\Delta) \leqslant I$

$$\begin{aligned} & (A_{ss} - B_s F_s)^{\mathrm{T}} P + P(A_{ss} - B_s F_s) - F_s^{\mathrm{T}}(G(\Delta))^{\mathrm{T}}(D_a^+)^{\mathrm{T}} P - P D_a^+ G(\Delta) F_s \\ & \quad = -P(L_{ad}^+(L_{ad}^+)^{\mathrm{T}} + D_a^+(D_a^+)^{\mathrm{T}} + L_{ad}^+(G(\Delta))^{\mathrm{T}}(D_a^+)^{\mathrm{T}} + D_a^+ G(\Delta)(L_{ad}^+)^{\mathrm{T}})P \\ & \quad \leqslant -\varepsilon_0 I \end{aligned}$$

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for some $\varepsilon_0>0$. Thus

$$\dot{V}(\tilde{x}_s,\tilde{x}_d)\leqslant -\varepsilon_0(\tilde{x}_s)^{\mathrm{T}}\tilde{x}_s+2(\tilde{x}_s)^{\mathrm{T}}PL_{ad}^{+}\tilde{y}_d+2(\tilde{x}_s)^{\mathrm{T}}PD_a^{+}G(\Delta)\tilde{y}_d$$
$$+\sum_{i=1}^{m_d}\left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s+2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta)\right)$$

Since $(G(\Delta))^{\mathrm{T}}G(\Delta)\leqslant I$, $\bar{L}_{id}$ and $\bar{L}_{is}$ are independent on ε and $\bar{\phi}_{id}(\Delta)$ satisfies the linear growth condition (28), it is not difficult to show that there exists an $\varepsilon^*>0$ such that

$$\dot{V}(\tilde{x}_s,\tilde{x}_d)\leqslant -\varepsilon_1\left\|\begin{matrix}\tilde{x}_s\\ \tilde{x}_d\end{matrix}\right\|^2$$

for all $0<\varepsilon\leqslant\varepsilon^*$, where ε_1 is some positive real. This completes the proof of Theorem 3.1. □

4. NONLINEAR H_∞ CONTROL

In this section, we show that the ATEA technique can be used to solve the nonlinear H_∞ control problem which yields the following theorem.

Theorem 4.1

Under Assumptions A1–A4, let $P_L>0$, $P_D\geqslant 0$ and $P_H\geqslant 0$ be the solution of

$$A_{aa}^{+}P_L+P_L(A_{aa}^{+})^{\mathrm{T}}=L_{ad}^{+}(L_{ad}^{+})^{\mathrm{T}} \tag{33}$$

$$A_{aa}^{+}P_D+P_D(A_{aa}^{+})^{\mathrm{T}}=D_a^{+}(D_a^{+})^{\mathrm{T}} \tag{34}$$

$$A_{aa}^{+}P_H+P_H(A_{aa}^{+})^{\mathrm{T}}=H_a^{+}(H_a^{+})^{\mathrm{T}} \tag{35}$$

respectively. If there exists a $0<c<1$ such that

$$P_c=P_L-\frac{1}{c}P_D>0 \tag{36}$$

then the global nonlinear H_∞ control problem is solvable for $\gamma>\hat{\gamma}_+$, where

$$\hat{\gamma}_+=\sqrt{\lambda_{\max}(P_c^{-1}P_H)/(1-c)} \tag{37}$$

Proof

As the proof of Theorem 3.1, we first construct a state feedback control law by the ATEA method, and then we show that this state feedback control law solves the nonlinear H_∞ control problem. Specifically, define

$$P=\left(P_L-\frac{1}{c}P_D-\frac{1}{(1-c)\gamma^2}P_H\right)^{-1} \tag{38}$$

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Since $P_L > (1/c)P_D$ and $\gamma > \hat{\gamma}_+$, $P > 0$. Now, let F_s is given by

$$F_s = (L_{ad}^+)^{\mathrm{T}} P = \begin{bmatrix} F_{s_1} \\ F_{s_2} \\ \vdots \\ F_{s_{m_d}} \end{bmatrix}$$

where F_{s_i} are of dimensions $1 \times n_a^+$. Then, following the same lines of proof of Theorem 3.1, the ATEA state feedback gain is given by

$$F(\varepsilon) = -\Gamma_i (\bar{F}(\varepsilon) + \bar{F}_0)\Gamma_s^{-1} \tag{39}$$

where $\bar{F}(\varepsilon)$ and $\bar{F}_0$ are given by (18) and (19).

Next, we show that there exists an $\varepsilon^* > 0$ such that the state feedback control law

$$v = F(\varepsilon)\xi \tag{40}$$

solves the nonlinear H_∞ control problem for any $0 < \varepsilon \leqslant \varepsilon^*$.

Denote $x_s = x_a^+$ and

$$A_{ss} = A_{aa}^+, \quad B_s = L_{ad}^+$$

Transforming the closed-loop system (1) and (40) into the SCB form yields

$$\begin{aligned}
\dot{x}_a^- &= A_{aa}^- x_a^- + L_{ad}^- y_d + \phi_a^-(y_d) + \mathscr{H}_a^-(x)w \\
\dot{x}_s &= A_{ss} x_s + B_s y_d + \phi_a^+(y_d) + \mathscr{H}_a^+(x)w \\
\dot{x}_c &= (A_{cc} - B_c F_c)x_c + L_{cd} y_d + \phi_c(y_d) + \mathscr{H}_c(x)w \\
\dot{x}_d &= (A_{dd}^* - B_d \bar{F}_d(\varepsilon))x_d - B_d \bar{F}_s(\varepsilon)x_s + L_{dd} y_d + \phi_d(y_d) + \mathscr{H}_d(x)w \\
y_d &= C_d x_d
\end{aligned} \tag{41}$$

Making state transformations

$$\tilde{x}_a^- = x_a^-, \quad \tilde{x}_s = x_s, \quad \tilde{x}_c = x_c, \quad \tilde{x}_i = S_i(\varepsilon)\left(x_i + \begin{bmatrix} F_{s_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_s \right), \quad i = 1, 2, \ldots, m_d \tag{42}$$

on (41) and denoting

$$\tilde{x}_d = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{m_d} \end{pmatrix}$$

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we have

$$\begin{aligned}
\dot{\tilde{x}}_a^- &= A_{aa}^-\tilde{x}_a^- + L_{ad}^-(\tilde{y}_d - F_s\tilde{x}_s) + \phi_a^-(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_a^-(x)w \\
\dot{\tilde{x}}_s &= (A_{ss} - B_sF_s)\tilde{x}_s + B_s\tilde{y}_d + \phi_a^+(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_a^+(x)w \\
\dot{\tilde{x}}_c &= (A_{cc} - B_cF_c)\tilde{x}_c + L_{cd}(\tilde{y}_d - F_s\tilde{x}_s) + \phi_c(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_c(x)w \\
\dot{\tilde{x}}_i &= \frac{1}{\varepsilon}[A_{q_i} - B_{q_i}F_i]\tilde{x}_i + S_i(\varepsilon)\bar{L}_{is}\tilde{x}_s + S_i(\varepsilon)\bar{L}_{id}\tilde{y}_d + S_i(\varepsilon)\bar{\phi}_{id}(\tilde{y}_d - F_s\tilde{x}_s) \\
&\quad + S_i(\varepsilon)\bar{\mathscr{H}}_{id}(x)w, \quad i = 1, 2, \ldots, m_d \\
\tilde{y}_d &= C_d\tilde{x}_d
\end{aligned} \tag{43}$$

where $\bar{L}_{is}$, $\bar{L}_{id}$ and $\bar{\phi}_{id}$ are the same as (26) and (27), and $\bar{\mathscr{H}}_{id}(x)$ is given by

$$\bar{\mathscr{H}}_{id}(x) = \begin{bmatrix} F_{s_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \mathscr{H}_a^+(x) + \mathscr{H}_{id}(x) \quad \text{with} \quad \begin{bmatrix} \mathscr{H}_{1d}(x) \\ \mathscr{H}_{2d}(x) \\ \vdots \\ \mathscr{H}_{m_d d}(x) \end{bmatrix} = \mathscr{H}_d(x) \tag{44}$$

Let $P_i > 0$, $i = 1, 2, \ldots, m_d$, be the positive-definite solutions of (31) and define

$$V_1(\tilde{x}_s, \tilde{x}_d) = (\tilde{x}_s)^{\mathrm{T}} P\tilde{x}_s + \sum_{i=1}^{m_d} \tilde{x}_i^{\mathrm{T}} P_i\tilde{x}_i \tag{45}$$

Then, we have

$$\begin{aligned}
\dot{V}_1(\tilde{x}_s, \tilde{x}_d) &\leqslant 2(\tilde{x}_s)^{\mathrm{T}} P((A_{ss} - B_sF_s)\tilde{x}_s + B_s\tilde{y}_d + \phi_a^+(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_a^+(x)w) \\
&\quad + \sum_{i=1}^{m_d} \left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i + 2\tilde{x}_i^{\mathrm{T}} P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s + 2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d \right) \\
&\quad + \sum_{i=1}^{m_d} (2\tilde{x}_i^{\mathrm{T}} P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta) + 2\tilde{x}_i^{\mathrm{T}} P_iS_i(\varepsilon)\bar{H}_{id}w) \\
&\leqslant (x_s)^{\mathrm{T}} \left(PA_{aa}^+ + (A_{aa}^+)^{\mathrm{T}} P + P\left(\frac{1}{(1-c)\gamma^2} H_a^+(H_a^+)^{\mathrm{T}} + \frac{1}{c} D_a^+(D_a^+)^{\mathrm{T}} - L_{ad}^+(L_{ad}^+)^{\mathrm{T}} \right) P \right) x_s \\
&\quad - (x_s)^{\mathrm{T}} P\left(cL_{ad}^+(L_{ad}^+)^{\mathrm{T}} + \frac{1}{c} D_a^+(D_a^+)^{\mathrm{T}} + D_a^+G(\Delta)(L_{ad}^+)^{\mathrm{T}} + L_{ad}^+(G(\Delta))^{\mathrm{T}}(D_a^+)^{\mathrm{T}} \right) Px_s \\
&\quad - (1-c)y_d^{\mathrm{T}} y_d + (1-c)\gamma^2 w^{\mathrm{T}} w + (1-c)\tilde{y}_d^{\mathrm{T}}\tilde{y}_d + 2(\tilde{x}_s)^{\mathrm{T}} PD_a^+G(\Delta)\tilde{y}_d + c(\tilde{x}_s)^{\mathrm{T}} PL_{ad}^+\tilde{y}_d
\end{aligned}$$

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$$+\sum_{i=1}^{m_d}\left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s+2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d\right)$$

$$+\sum_{i=1}^{m_d}(2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta)+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{H}_{id}w)$$

where $\Delta=\tilde{y}_d-F_s\tilde{x}_s$. Using (33)–(35) and (38), we have

$$PA_{aa}^{+}+(A_{aa}^{+})^{\mathrm{T}}P+P\left[\frac{1}{(1-c)\gamma^2}H_a^{+}(H_a^{+})^{\mathrm{T}}+\frac{1}{c}D_a^{+}(D_a^{+})^{\mathrm{T}}-L_{ad}^{+}(L_{ad}^{+})^{\mathrm{T}}\right]P=0$$

Moreover, since $(G_a^{+})^{\mathrm{T}}G_a^{+}\leqslant I$, there exists a positive real $\varepsilon_0>0$ such that

$$P\left(cL_{ad}^{+}(L_{ad}^{+})^{\mathrm{T}}+\frac{1}{c}D_a^{+}(D_a^{+})^{\mathrm{T}}+D_a^{+}G(\Delta)(L_{ad}^{+})^{\mathrm{T}}+L_{ad}^{+}(G(\Delta))^{\mathrm{T}}(D_a^{+})^{\mathrm{T}}\right)P\geqslant\varepsilon_0I$$

Thus, we have

$$\begin{aligned}\dot{V}_1(\tilde{x}_s,\tilde{x}_d)\leqslant{}&-\varepsilon_0(x_s)^{\mathrm{T}}x_s-(1-c)y_d^{\mathrm{T}}y_d+(1-c)\gamma^2w^{\mathrm{T}}w\\&+(1-c)\tilde{y}_d^{\mathrm{T}}\tilde{y}_d+2(\tilde{x}_s)^{\mathrm{T}}PD_a^{+}G(\Delta)\tilde{y}_d+c(\tilde{x}_a^{+})^{\mathrm{T}}PL_{ad}^{+}\tilde{y}_d\\&+\sum_{i=1}^{m_d}\left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s+2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d\right)\\&+\sum_{i=1}^{m_d}(2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta)+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\mathscr{H}}_{id}(x)w)\end{aligned}$$

Noting that $\bar{L}_{id}$ and $\bar{L}_{is}$ are independent on ε, $\|\bar{\mathscr{H}}_{id}(x)\|$ is bounded for all $x\in\mathbb{R}^n$ and $\bar{\phi}_{id}(\cdot)$ satisfies the linear growth condition (28), for any arbitrary small $\delta_0>0$, there exist positive reals $\varepsilon_1>0$ and $\varepsilon^*>0$ such that

$$\dot{V}_1(\tilde{x}_s,\tilde{x}_d)\leqslant-\varepsilon_1\left\|\begin{matrix}\tilde{x}_s\\\tilde{x}_d\end{matrix}\right\|^2-(1-c)\|y_d\|^2+(1-c)(\gamma^2+\delta_0^2)\|w\|^2$$

for all $0<\varepsilon\leqslant\varepsilon^*$.

Now let $P_a>0$ and $P_0>0$ be the positive-definite solutions of

$$P_aA_{aa}^{-}+(A_{aa}^{-})^{\mathrm{T}}P_a=-I \tag{46}$$

$$P_0(A_{cc}-B_cF_c)+(A_{cc}-B_cF_c)^{\mathrm{T}}P_0=-I \tag{47}$$

Define a Lyapunov function

$$V(\tilde{x})=\varepsilon_2(\tilde{x}_a^{-})^{\mathrm{T}}P_a\tilde{x}_a^{-}+\varepsilon_3\tilde{x}_c^{\mathrm{T}}P_0\tilde{x}_c+V_1(\tilde{x}_s,\tilde{x}_d) \tag{48}$$

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where ε_2 and ε_3 are positive real numbers to be defined later. Then calculating the derivation of (48) along the trajectory of (43), we have

$$\begin{aligned}
\dot{V}(\tilde{x}) \leqslant & -\varepsilon_2\|\tilde{x}_a^-\|^2+2\varepsilon_2(\tilde{x}_a^-)^{\mathrm{T}}P_a(L_{ad}^-(C_d\tilde{x}_d-F_s\tilde{x}_s)+\phi_a^-(C_d\tilde{x}_d-F_s\tilde{x}_s)+\mathscr{H}_a^-(x)w) \\
& -\varepsilon_3\|\tilde{x}_c\|^2+2\varepsilon_3\tilde{x}_c^{\mathrm{T}}P_0(L_{cd}(C_d\tilde{x}_d-F_s\tilde{x}_s)+\phi_c(C_d\tilde{x}_d-F_s\tilde{x}_s)+\mathscr{H}_c(x)w) \\
& -\varepsilon_1\left\|\begin{matrix}\tilde{x}_a^+\\ \tilde{x}_d\end{matrix}\right\|^2-(1-c)\|y_d\|^2+(1-c)(\gamma^2+\delta_0^2)\|w\|^2 \\
\leqslant & (-\varepsilon_2+\varepsilon_2^2r_1)\|x_a^-\|^2+\frac{\varepsilon_1}{4}\left\|\begin{matrix}\tilde{x}_a^+\\ \tilde{x}_d\end{matrix}\right\|^2+(1-c)\delta_a^2\|w\|^2 \\
& +(-\varepsilon_3+\varepsilon_3^2r_2)\|x_a^-\|^2+\frac{\varepsilon_1}{4}\left\|\begin{matrix}\tilde{x}_a^+\\ \tilde{x}_d\end{matrix}\right\|^2+(1-c)\delta_c^2\|w\|^2 \\
& -\varepsilon_1\left\|\begin{matrix}\tilde{x}_a^+\\ \tilde{x}_d\end{matrix}\right\|^2-(1-c)\|y_d\|^2+(1-c)(\gamma^2+\delta_0^2)\|w\|^2
\end{aligned}$$

where δ_a and δ_c are arbitrary small real numbers and

$$r_1 \geqslant \frac{8}{\varepsilon_1}(\|P_a\|\|L_{ad}^-[C_d-F_s]\|)^2+\frac{8k_0}{\varepsilon_1}(\|P_a\|\|[C_d-F_s]\|)^2+\left(\frac{1}{\delta_a\sqrt{1-c}}\|P_a\mathscr{H}_a^-(x)\|\right)^2$$

$$r_2 \geqslant \frac{8}{\varepsilon_1}(\|P_0\|\|L_{cd}[C_d-F_s]\|)^2+\frac{8k_0}{\varepsilon_1}(\|P_0\|\|[C_d-F_s]\|)^2+\left(\frac{1}{\delta_c\sqrt{(1-c)}}\|P_0\mathscr{H}_c(x)\|\right)^2$$

Since $\mathscr{H}_a^-(x)$ and $\mathscr{H}_c(x)$ are bounded, so are r_1 and r_2. Now select ε_2 and ε_3 such that

$$\varepsilon_2<\frac{1}{r_1}, \quad \varepsilon_3<\frac{1}{r_2}$$

Then there exists an $\varepsilon_4>0$ such that

$$\dot{V}(\tilde{x})\leqslant-\varepsilon_4\|\tilde{x}\|^2-(1-c)\|y_d\|^2+(1-c)(\gamma^2+\delta_0^2+\delta_a^2+\delta_c^2)\|w\|^2 \tag{49}$$

Integrating two sides of (49) for 0 to T yields

$$\int_0^T \|y_d\|^2\,\mathrm{d}t\leqslant\int_0^T(\gamma^2+\delta^2)\|w\|^2\,\mathrm{d}t$$

where $\delta=\delta_0+\delta_a+\delta_c$. This completes the proof of Theorem 4.1. □

Remark 4.1
In Theorem 4.1, a design parameter c is introduced. Noting that the nonlinear term $\phi_a^+(y)$ is regarded as an input uncertainty in the global stabilization controller design, the design parameter

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c is a compromise between the global stabilization and the H_∞ control. In stabilization controller design, P satisfies

$$(A_{aa}^+)^{\mathrm{T}} P + P_+ A_{aa}^+ + P[D_a^+ (D_a^+)^{\mathrm{T}} - L_{ad}^+ (L_{ad}^+)^{\mathrm{T}}] P = 0$$

while in H_∞ controller design, P satisfies

$$(A_{aa}^+)^{\mathrm{T}} P + P A_{aa}^+ + P \left[\frac{1}{(1-c)\gamma^2} H_a^+ (H_a^+)^{\mathrm{T}} - (1-c) L_{ad}^+ (L_{ad}^+)^{\mathrm{T}} + \frac{1}{c} D_a^+ (D_a^+)^{\mathrm{T}} - c L_{ad}^+ (L_{ad}^+)^{\mathrm{T}} \right] P = 0$$

When $c=0$ (only if $D_a^+=0$, i.e. $\phi_a^+(y)=0$), the nonlinear H_∞ control problem is reduced to the linear H_∞ control problem. On the other hand, when $c=1$ (only if $H_a^+=0$), the nonlinear H_∞ control problem is reduced to the stabilization problem. When L_{ad}^+, D_a^+ and H_a^+ are fixed, we can calculate the achievable L_2-gain estimation by solving the following optimal problem on c:

$$\hat{\gamma}_+^* = \min_{\substack{0<c<1 \\ P_L - P_D/c > 0}} \sqrt{\frac{\lambda_{\max}((P_L - P_D/c)^{-1} P_H)}{1-c}} \tag{50}$$

Remark 4.2

In [26], an upper estimate of the optimal value γ^* is given for a class of nonlinear systems with a second-order zero dynamics of the form

$$\dot{x}_1 = f_{11}(x_1) + f_{12}(x_1, x_2) x_2 + p_1(x_1, x_2) w \tag{51}$$

$$\dot{x}_2 = f_{21}(x_1) + f_{22}(x_1, x_2) x_2 + p_2(x_1, x_2) w + u \tag{52}$$

where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. The necessary condition $L_{f_{11}} V_1(x_1) < 0$ implies that (51) is the stable part of the zero dynamics. That is, the estimation can be calculated only for the systems with one-dimensional unstable zero dynamics. The zero dynamics considered in this paper is of the form

$$x_a^- = A_{aa}^- x_a^- + L_{ad}^- u + \phi_a^-(u) \tag{53}$$

$$x_a^+ = A_{aa}^+ x_a^+ + L_{ad}^+ u + \phi_a^+(u) \tag{54}$$

where $x_a^- \in \mathbb{R}^{n_a^-}$ and $x_a^+ \in \mathbb{R}^{n_a^+}$. Equation (53) is the stable part of zero dynamics, and (54) is the unstable one. Since n_a^+ need not equal one, our method can tackle the systems with high-order unstable zero dynamics. In the special case $\phi_a^+(u)=0$, (54) reduces to a linear system. In this case, it is not difficult to show that the upper estimate $\hat{\gamma}_+^* = \sqrt{\lambda_{\max}(P_L^{-1} P_H)} = \gamma^*$.

5. TACKLING ZEROS ON THE IMAGINARY AXIS

In this section, we extend the results of Sections 3 and Section 4 to the systems which have zeros on the imaginary axis, i.e. remove Assumption A2. Without Assumption A2, n_a^0 may not equal to zero.

Theorem 5.1
Under Assumptions A1 and A3, let $P_L>0$ and $P_D\geqslant 0$ be the unique solution of (11) and (12), respectively. If $P_L>P_D$ and

$$x^\star(D_a^0(D_a^0)^{\mathrm{T}}-L_{ad}^0(L_{ad}^0)^{\mathrm{T}})x<0 \tag{55}$$

for any eigenvector x of $-(A_{aa}^0)^{\mathrm{T}}$, the global stabilization control problem is solvable by a linear state feedback.

Proof
Define

$$P=\begin{bmatrix} Z & Y^{\mathrm{T}} \\ Y & X \end{bmatrix}^{-1} \tag{56}$$

where

$$X=P_L-P_D \tag{57}$$

and Y is the unique solution of

$$A_{aa}^+Y+Y(A_{aa}^0)^{\mathrm{T}}+D_a^+(D_a^0)^{\mathrm{T}}-L_{ad}^+(L_{ad}^0)^{\mathrm{T}}=0 \tag{58}$$

and $Z>0$ is a solution of the following Lyapunov inequality:

$$A_{aa}^0Z+Z(A_{aa}^0)^{\mathrm{T}}+D_a^0(D_a^0)^{\mathrm{T}}-L_{ad}^0(L_{ad}^0)^{\mathrm{T}}<0 \tag{59}$$

Since all the eigenvalues of A_{aa}^0 are on the imaginary axis and (55) is satisfied, by Theorem 4 of [35], for any Z_0, there exists a solution Z of the Lyapunov inequality (59) such that $Z>Z_0$. Since $P_L>P_D$ implies $X>0$, there exists a solution $Z>0$ of (59) such that $P>0$. Now, let

$$F_s=[(L_{ad}^0)^{\mathrm{T}}\ \ (L_{ad}^+)^{\mathrm{T}}]P=\begin{bmatrix} F_{s_1} \\ F_{s_2} \\ \vdots \\ F_{s_{m_d}} \end{bmatrix}$$

where F_{s_i} are of dimensions $1\times(n_a^0+n_a^+)$. Similar to the proof of Theorem 3.1, we can design an ATEA state feedback gain

$$F(\varepsilon)=-\Gamma_i(\bar{F}(\varepsilon)+\bar{F}_0)\Gamma_s^{-1} \tag{60}$$

where $\bar{F}(\varepsilon)$ is given by (18) and

$$\bar{F}_0=\begin{bmatrix} E_{da}^- & E_{da}^0 & E_{da}^+ & E_{dc} & E_{dd} \\ E_{ca}^- & E_{ca}^0 & E_{ca}^+ & 0 & 0 \end{bmatrix} \tag{61}$$

since n_a^0 may not equal to zero.

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Next, we show that there exists an $\varepsilon^*>0$ such that

$$v=F(\varepsilon)\xi$$

solves the global stabilization problem of system (1) for all $0<\varepsilon\leqslant\varepsilon^*$. Denote

$$x_s=\begin{bmatrix} x_a^0 \\ x_a^+ \end{bmatrix},\quad A_{ss}=\begin{bmatrix} A_{aa}^0 & 0 \\ 0 & A_{aa}^+ \end{bmatrix},\quad B_s=\begin{bmatrix} L_{ad}^0 \\ L_{ad}^+ \end{bmatrix},\quad D_s=\begin{bmatrix} D_a^0 \\ D_a^+ \end{bmatrix},\quad \phi_s(y_d)=\begin{bmatrix} \phi_a^0(y_d) \\ \phi_a^+(y_d) \end{bmatrix}$$

Then the closed-loop system in the SCB form is given by

$$\begin{aligned}
\dot{x}_a^- &= A_{aa}^- x_a^- + L_{ad}^- y_d + \phi_a^-(y_d)\\
\dot{x}_s &= A_{ss}x_s + B_s y_d + \phi_s(y_d)\\
\dot{x}_c &= (A_{cc}-B_cF_c)x_c + L_{cd}y_d + \phi_c(y_d)\\
\dot{x}_d &= (A_{dd}^* - B_d\bar{F}_d(\varepsilon))x_d - B_d\bar{F}_s(\varepsilon)x_a^+ + L_{dd}y_d + \phi_d(y_d)\\
y_d &= C_d x_d
\end{aligned} \tag{62}$$

It is clear that (62) has exactly the same form of (21). Noting that A_{aa}^- and $A_{cc}-B_cF_c$ are stable matrices and $\Phi(y_d)$ satisfies the linear growth condition (8), to show the stability of (62), we just need to show

$$\begin{aligned}
\dot{x}_s &= A_{ss}x_s + B_s y_d + \phi_s(y_d)\\
\dot{x}_d &= (A_{dd}^* - B_d\bar{F}_d(\varepsilon))x_d - B_d\bar{F}_s(\varepsilon)x_a^+ + L_{dd}y_d + \phi_d(y_d)\\
y_d &= C_d x_d
\end{aligned} \tag{63}$$

is asymptotically stable. To this end, we define a state transformation

$$\tilde{x}_s = x_s,\ \tilde{x}_i = S_i(\varepsilon)\left(x_i + \begin{bmatrix} F_{s_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_s\right),\quad i=1,2,\ldots,m_d,\quad \tilde{x}_d = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{m_d} \end{pmatrix} \tag{64}$$

We have

$$\begin{aligned}
\dot{\tilde{x}}_s &= (A_{ss}-B_sF_s)\tilde{x}_s + B_s\tilde{y}_d + \phi_s(\tilde{y}_d - F_s\tilde{x}_a^+)\\
\dot{\tilde{x}}_i &= \frac{1}{\varepsilon}[A_{q_i} - B_{q_i}F_i]\tilde{x}_i + S_i(\varepsilon)\bar{L}_{is}\tilde{x}_a^+ + S_i(\varepsilon)\bar{L}_{id}\tilde{y}_d + S_i(\varepsilon)\bar{\phi}_{id}(\tilde{y}_d - F_s\tilde{x}_a^+)\\
\tilde{y}_d &= C_d\tilde{x}_d
\end{aligned} \tag{65}$$

where $\bar{L}_{is}$, $\bar{L}_{id}$ and $\bar{\phi}_{id}(\cdot)$ are defined by (26) and (27). Let P_i, $i=1,2,\ldots,m_d$, be positive-definite solutions of

$$P_i(A_{q_i} - B_{q_i}F_i) + (A_{q_i} - B_{q_i}F_i)^{\mathrm{T}}P_i = -I \tag{66}$$

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and define a Lyapunov function

$$V(\tilde{x}_s,\tilde{x}_d)=(\tilde{x}_s)^{\mathrm{T}}P\tilde{x}_s+\sum_{i=1}^{m_d}\tilde{x}_i^{\mathrm{T}}P_i\tilde{x}_i \tag{67}$$

Then the derivation of (67) along the trajectory of (65) is given by

$$\begin{aligned}\dot{V} &= (\tilde{x}_s)^{\mathrm{T}}((A_{ss}-B_sF_s)^{\mathrm{T}}P+P(A_{ss}-B_sF_s)-F_s^{\mathrm{T}}(G(\Delta))^{\mathrm{T}}(D_s)^{\mathrm{T}}P-PD_sG(\Delta)F_s)\tilde{x}_s\\ &\quad+2(\tilde{x}_s)^{\mathrm{T}}PL_{ad}^{+}\tilde{y}_d+2(\tilde{x}_s)^{\mathrm{T}}PD_sG(\Delta)\tilde{y}_d\\ &\quad+\sum_{i=1}^{m_d}\left(\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}((A_{q_i}-B_{q_i}F_i)^{\mathrm{T}}P_i+P_i(A_{q_i}-B_{q_i}F_i))\tilde{x}_i+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s\right)\\ &\quad+\sum_{i=1}^{m_d}(2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d+\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta))\end{aligned}$$

where $\Delta=\tilde{y}_d-F_s\tilde{x}_s$. Using (11), (12) and (57), we have

$$A_{aa}^{+}X+X(A_{aa}^{+})^{\mathrm{T}}+(D_a^{+}(D_a^{+})^{\mathrm{T}}-L_{ad}^{+}(L_{ad}^{+})^{\mathrm{T}})=0 \tag{68}$$

Furthermore, with (59), (58) and (68) imply that

$$PA_{ss}+(A_{ss})^{\mathrm{T}}P+P(D_s(D_s)^{\mathrm{T}}-B_s(B_s)^{\mathrm{T}})P\leqslant 0$$

Noting that $F_s=B_s^{\mathrm{T}}P$, we have

$$\begin{aligned}&(A_{ss}-B_sF_s)^{\mathrm{T}}P+P(A_s-B_sF_s)-F_s^{\mathrm{T}}(G(\Delta))^{\mathrm{T}}(D_s)^{\mathrm{T}}P-PD_sG(\Delta)F_s\\ &\quad\leqslant -P(B_s(B_s)^{\mathrm{T}}+D_s(D_s)^{\mathrm{T}}+B_s(G(\Delta))^{\mathrm{T}}(D_s)^{\mathrm{T}}+D_sG(\Delta)(B_s)^{\mathrm{T}})P\\ &\quad\leqslant -\varepsilon_0 I\end{aligned}$$

for some $\varepsilon_0>0$.

Thus,

$$\begin{aligned}\dot{V}(\tilde{x}_s,\tilde{x}_d) &\leqslant -\varepsilon_0(\tilde{x}_s)^{\mathrm{T}}\tilde{x}_s+2(\tilde{x}_s)^{\mathrm{T}}PL_{ad}^{+}\tilde{y}_d+2(\tilde{x}_s)^{\mathrm{T}}PD_sG(\Delta)\tilde{y}_d\\ &\quad+\sum_{i=1}^{m_d}\left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s+2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta)\right)\end{aligned}$$

Since $(G_a^{+}(\Delta))^{\mathrm{T}}G_a^{+}(\Delta)\leqslant I$, $\bar{L}_{id}$ and $\bar{L}_{is}$ are independent on ε and $\bar{\phi}_{id}(\Delta)$ satisfies the linear growth condition (28), it is clear that there exists an $\varepsilon^*>0$ such that

$$\dot{V}(\tilde{x}_s,\tilde{x}_d)\leqslant-\varepsilon_1\left\|\begin{matrix}\tilde{x}_s\\ \tilde{x}_d\end{matrix}\right\|^2$$

for all $0<\varepsilon\leqslant\varepsilon^*$, where ε_1 is some positive real. That is, (65), thus (63), is asymptotically stable. □

Similarly, we have the following theorem for the nonlinear H_∞ control problem for the systems which have zeros on the imaginary axis.

Theorem 5.2
Under Assumptions A1, A3 and A4, let $P_L>0$, $P_D\geqslant 0$ and $P_H\geqslant 0$ be the unique solutions of (33), (34) and (35), respectively. Assume that there exists a $0<c<1$ such that

$$P_c = P_L - \frac{1}{c} P_D > 0 \tag{69}$$

and

$$x^\star \left(\frac{1}{c} D_a^0 (D_a^0)^\mathrm{T} - L_{ad}^0 (L_{ad}^0)^\mathrm{T} \right) x < 0 \tag{70}$$

for any eigenvector x of $-(A_{aa}^0)^\mathrm{T}$, then the nonlinear H_∞ control problem is solvable for a given $\gamma > \hat{\gamma} := \max\{\hat{\gamma}_+, \hat{\gamma}_0\}$, where

$$\hat{\gamma}_+ = \sqrt{\lambda_{\max}(P_c^{-1} P_H)/(1-c)} \tag{71}$$

and

$$\hat{\gamma}_0 = \sqrt{\max_{\|x\|=1} \left\{ \frac{x^\star H_a^0 (H_a^0)^\mathrm{T} x}{(1-c) x^\star (L_{ad}^0 (L_{ad}^0)^\mathrm{T} - \frac{1}{c} D_a^0 (D_a^0)^\mathrm{T}) x} \right\}} \tag{72}$$

for any eigenvector x of $-(A_{aa}^0)^\mathrm{T}$.

Proof
Define

$$P = \begin{bmatrix} Z & Y^\mathrm{T} \\ Y & X \end{bmatrix}^{-1} \tag{73}$$

where

$$X = P_L - \frac{1}{c} P_D - \frac{1}{(1-c)\gamma^2} H_a^+ (H_a^+)^\mathrm{T} \tag{74}$$

and Y is the unique solution of

$$A_{aa}^+ Y + Y (A_{aa}^0)^\mathrm{T} + \frac{1}{(1-c)\gamma^2} H_a^+ (H_a^0)^\mathrm{T} + \frac{1}{c} D_a^+ (D_a^0)^\mathrm{T} - L_{ad}^+ (L_{ad}^0)^\mathrm{T} = 0 \tag{75}$$

and Z is a solution of the following Lyapunov inequality:

$$A_{aa}^0 Z + Z (A_{aa}^0)^\mathrm{T} + \frac{1}{(1-c)\gamma^2} H_a^0 (H_a^0)^\mathrm{T} + \frac{1}{c} D_a^0 (D_a^0)^\mathrm{T} - L_{ad}^0 (L_{ad}^0)^\mathrm{T} < 0 \tag{76}$$

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Since $\gamma > \max\{\hat{\gamma}_+, \hat{\gamma}_0\}$, we have $X>0$ and

$$x^\star \left(\frac{1}{(1-c)\gamma^2} H_a^0 (H_a^0)^{\mathrm{T}} + \frac{1}{c} D_a^0 (D_a^0)^{\mathrm{T}} - L_{ad}^0 (L_{ad}^0)^{\mathrm{T}} \right) x < 0 \tag{77}$$

for any eigenvector x of $-(A_a^0)^{\mathrm{T}}$. Then, by Theorem 4 of [35], there exists a $Z>0$ of the Lyapunov inequality (76) such that $P>0$. Let

$$F_s = [(L_{ad}^0)^{\mathrm{T}} \;\; (L_{ad}^+)^{\mathrm{T}}] P = \begin{bmatrix} F_{s_1} \\ F_{s_2} \\ \vdots \\ F_{s_{m_d}} \end{bmatrix}$$

where F_{s_i} are of dimensions $1 \times (n_a^0 + n_a^+)$. Similar to the proof of Theorem 3.1, we can design an ATEA state feedback gain as

$$F(\varepsilon) = -\Gamma_i (\bar{F}(\varepsilon) + \bar{F}_0) \Gamma_s^{-1} \tag{78}$$

where $\bar{F}(\varepsilon)$ and $\bar{F}_0$ are given by (18) and (61), respectively.

Next, we need to show that there exists an $\varepsilon^* > 0$ such that

$$v = F(\varepsilon)\xi \tag{79}$$

solves the global stabilization problem of system (1) for all $0 < \varepsilon \leqslant \varepsilon^*$. Toward this target, denote

$$x_s = \begin{bmatrix} x_a^0 \\ x_a^+ \end{bmatrix}, \quad A_{ss} = \begin{bmatrix} A_{aa}^0 & 0 \\ 0 & A_{aa}^+ \end{bmatrix}, \quad B_s = \begin{bmatrix} L_{ad}^0 \\ L_{ad}^+ \end{bmatrix}$$

$$D_s = \begin{bmatrix} D_a^0 \\ D_a^+ \end{bmatrix}, \quad \phi_s(y_d) = \begin{bmatrix} \phi_a^0(y_d) \\ \phi_a^+(y_d) \end{bmatrix}, \quad \mathscr{H}_s(x) = \begin{bmatrix} \mathscr{H}_a^0(x) \\ \mathscr{H}_a^+(x) \end{bmatrix}$$

and transforming the closed-loop system (1) and (79) into the SCB form yields

$$\begin{aligned}
\dot{x}_a^- &= A_{aa}^- x_a^- + L_{ad}^- y_d + \phi_a^-(y_d) + \mathscr{H}_a^-(x) w \\
\dot{x}_s &= A_{ss} x_s + B_s y_d + \phi_s(y_d) + \mathscr{H}_s(x) w \\
\dot{x}_c &= (A_{cc} - B_c F_c) x_c + L_{cd} y_d + \phi_c(y_d) + \mathscr{H}_c(x) w \\
\dot{x}_d &= (A_{dd}^* - B_d \bar{F}_d(\varepsilon)) x_d - B_d \bar{F}_s(\varepsilon) x_s + L_{dd} y_d + \phi_d(y_d) + \mathscr{H}_d(x) w \\
y_d &= C_d x_d
\end{aligned} \tag{80}$$

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Making state transformations

$$\tilde{x}_a^- = x_a^-, \quad \tilde{x}_s = x_s, \quad \tilde{x}_c = x_c, \quad \tilde{x}_i = S_i(\varepsilon)\left(x_i + \begin{bmatrix} F_{s_i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_s \right), \quad i = 1, 2, \ldots, m_d \tag{81}$$

on (80) and denoting

$$\tilde{x}_d = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_{m_d} \end{pmatrix}$$

we have

$$\begin{aligned}
\dot{\tilde{x}}_a^- &= A_{aa}^- \tilde{x}_a^- + L_{ad}^-(\tilde{y}_d - F_s\tilde{x}_s) + \phi_a^-(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_a^-(x)w \\
\dot{\tilde{x}}_s &= (A_{ss} - B_sF_s)\tilde{x}_s + B_s\tilde{y}_d + \phi_s(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_s(x)w \\
\dot{\tilde{x}}_c &= (A_{cc} - B_cF_c)\tilde{x}_c + L_{cd}(\tilde{y}_d - F_s\tilde{x}_s) + \phi_c(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_c(x)w \\
\dot{\tilde{x}}_i &= \frac{1}{\varepsilon}[A_{q_i} - B_{q_i}F_i]\tilde{x}_i + S_i(\varepsilon)\bar{L}_{is}\tilde{x}_s + S_i(\varepsilon)\bar{L}_{id}\tilde{y}_d + S_i(\varepsilon)\bar{\phi}_{id}(\tilde{y}_d - F_s\tilde{x}_s) \\
&\quad + S_i(\varepsilon)\bar{\mathscr{H}}_{id}(x)w, \quad i = 1, 2, \ldots, m_d \\
\tilde{y}_d &= C_d\tilde{x}_d
\end{aligned} \tag{82}$$

where $\bar{L}_{is}$, $\bar{L}_{id}$, $\bar{\phi}_{id}$ and $\bar{\mathscr{H}}_{id}(x)$ are defined in (26), (27) and (44). Let $P_i > 0$, $i = 1, 2, \ldots, m_d$, be the positive-definite solutions of (31) and define

$$V(\tilde{x}_s, \tilde{x}_d) = (\tilde{x}_s)^{\mathrm{T}} P\tilde{x}_s + \sum_{i=1}^{m_d} \tilde{x}_i^{\mathrm{T}} P_i \tilde{x}_i \tag{83}$$

Then

$$\begin{aligned}
\dot{V}(\tilde{x}_s, \tilde{x}_d) &\leqslant 2(\tilde{x}_s)^{\mathrm{T}} P((A_{ss} - B_sF_s)\tilde{x}_s + B_s\tilde{y}_d + \phi_s(\tilde{y}_d - F_s\tilde{x}_s) + \mathscr{H}_s(x)w) \\
&\quad + \sum_{i=1}^{m_d}\left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i + 2\tilde{x}_i^{\mathrm{T}} P_i S_i(\varepsilon)\bar{L}_{is}\tilde{x}_s + 2\tilde{x}_i P_i S_i(\varepsilon)\bar{L}_{id}\tilde{y}_d \right) \\
&\quad + \sum_{i=1}^{m_d}(2\tilde{x}_i^{\mathrm{T}} P_i S_i(\varepsilon)\bar{\phi}_{id}(\Delta) + 2\tilde{x}_i^{\mathrm{T}} P_i S_i(\varepsilon)\bar{H}_{id}w) \\
&\leqslant (x_s)^{\mathrm{T}}\left(PA_{aa}^+ + (A_{aa}^+)^{\mathrm{T}} P + P\left(\frac{1}{(1-c)\gamma^2} H_s(H_s)^{\mathrm{T}} + \frac{1}{c} D_s(D_s)^{\mathrm{T}} - B_s(B_s)^{\mathrm{T}} \right) P \right) x_s
\end{aligned}$$

$$
\begin{aligned}
&-(x_s)^{\mathrm{T}} P\left(cB_s(B_s)^{\mathrm{T}}+\frac{1}{c}D_s(D_s)^{\mathrm{T}}+D_sG(\Delta)B_s^{\mathrm{T}}+B_s(G(\Delta))^{\mathrm{T}}(D_s)^{\mathrm{T}}\right)Px_s\\
&-(1-c)y_d^{\mathrm{T}}y_d+(1-c)\gamma^2w^{\mathrm{T}}w+(1-c)\tilde{y}_d^{\mathrm{T}}\tilde{y}_d+2(\tilde{x}_s)^{\mathrm{T}}PD_sG(\Delta)\tilde{y}_d+c(\tilde{x}_s)^{\mathrm{T}}PL_{ad}^{+}\tilde{y}_d\\
&+\sum_{i=1}^{m_d}\left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s+2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d\right)\\
&+\sum_{i=1}^{m_d}(2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta)+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{H}_{id}w)
\end{aligned}
$$

where $\Delta=\tilde{y}_d-F_s\tilde{x}_s$. Using (33)–(35) and (73), we have

$$
PA_{aa}^{+}+(A_{aa}^{+})^{\mathrm{T}}P+P\left[\frac{1}{(1-c)\gamma^2}H_a^{+}(H_a^{+})^{\mathrm{T}}+\frac{1}{c}D_a^{+}(D_a^{+})^{\mathrm{T}}-L_{ad}^{+}(L_{ad}^{+})^{\mathrm{T}}\right]P\leqslant 0
$$

Moreover, since $(G_a^{+})^{\mathrm{T}}G_a^{+}\leqslant I$, there exists a positive real $\varepsilon_0>0$ such that

$$
P\left[cB_s(B_s)^{\mathrm{T}}+\frac{1}{c}D_s(D_s)^{\mathrm{T}}+D_sG(\Delta)B_s^{\mathrm{T}}+B_s(G(\Delta))^{\mathrm{T}}(D_s)^{\mathrm{T}}\right]P\geqslant\varepsilon_0 I
$$

Thus, we have

$$
\begin{aligned}
\dot{V}(\tilde{x}_s,\tilde{x}_d)\leqslant &-\varepsilon_0(x_s)^{\mathrm{T}}x_s-(1-c)y_d^{\mathrm{T}}y_d+(1-c)\gamma^2w^{\mathrm{T}}w\\
&+(1-c)\tilde{y}_d^{\mathrm{T}}\tilde{y}_d+2(\tilde{x}_s)^{\mathrm{T}}PD_sG(\Delta)\tilde{y}_d+c(\tilde{x}_a^{+})^{\mathrm{T}}PB_s\tilde{y}_d\\
&+\sum_{i=1}^{m_d}\left(-\frac{1}{\varepsilon}\tilde{x}_i^{\mathrm{T}}\tilde{x}_i+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{L}_{is}\tilde{x}_s+2\tilde{x}_iP_iS_i(\varepsilon)\bar{L}_{id}\tilde{y}_d\right)\\
&+\sum_{i=1}^{m_d}\left(2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\phi}_{id}(\Delta)+2\tilde{x}_i^{\mathrm{T}}P_iS_i(\varepsilon)\bar{\mathscr{H}}_{id}(x)w\right)
\end{aligned}
$$

Noting that $\bar{L}_{id}$ and $\bar{L}_{is}$ are independent on ε, $\|\bar{\mathscr{H}}_{id}(x)\|$ is bounded for all $x\in\mathbb{R}^n$ and $\bar{\phi}_{id}(\cdot)$ satisfies the linear growth condition (28), for any arbitrary small $\delta_0>0$, there exist positive reals $\varepsilon_1>0$ and $\varepsilon^*>0$ such that

$$
\dot{V}(\tilde{x}_s,\tilde{x}_d)\leqslant-\varepsilon_1\left\|\begin{matrix}\tilde{x}_s\\ \tilde{x}_d\end{matrix}\right\|^2-(1-c)\|y_d\|^2+(1-c)(\gamma^2+\delta_0^2)\|w\|^2
$$

for all $0<\varepsilon\leqslant\varepsilon^*$. Finally, the remainder of the proof can be completed by following the same reasoning of the proof of Theorem 4.1. □

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Remark 5.1
By Theorem 5.2, the achievable L_2-gain can be estimated by solving the following minimization problem on c:

$$\hat{\gamma}^* = \min_{\substack{0<c<1 \\ P_L - P_D/c>0 \\ x^\star(\frac{1}{c}D_a^0(D_a^0)^{\mathrm{T}} - L_{ad}^0(L_{ad}^0)^{\mathrm{T}})x<0}} \max\{\hat{\gamma}_+, \hat{\gamma}_0\} \tag{84}$$

where x is the eigenvector of $-(A_a^0)^{\mathrm{T}}$.

6. AN ILLUSTRATIVE EXAMPLE

Consider the system

$$\dot{x} = Ax + Bu + \Phi(y) + Hw \tag{85}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Cx \tag{86}$$

with

$$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} -1 & -2 \\ -1 & 0 \\ -2 & -1 \\ 0 & -1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi(y) = \begin{bmatrix} y_1 \sin(y_2) \\ y_2 \\ 0 \\ \sin(y_1) \end{bmatrix}$$

System (86) is already in the SCB form with

$$A_{aa}^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_{ad}^+ = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \quad H_a^+ = \begin{bmatrix} -1 & -2 \\ -1 & 0 \end{bmatrix}, \quad \phi_a^+(y) = \begin{bmatrix} y_1 \sin(y_2) \\ y_2 \end{bmatrix}$$

and

$$A_{dd} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_{da} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E_{dd} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

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System (86) has two unstable invariant zeros. It is easy to verify that Assumptions A1–A4 are all satisfied. Moreover, let

$$\phi_a^+(y) = \begin{bmatrix} y_1 \sin(y_2) \\ y_2 \end{bmatrix} = D_a^+ G(y) y := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(y_2) & 0 \\ 0 & 1 \end{bmatrix} y$$

It is clear that $(G(y))^{\mathrm{T}} G(y) \leqslant I_2$. Solving the following Lyapunov equations:

$$P_L (A_{aa}^+)^{\mathrm{T}} + A_{aa}^+ P_L = L_{ad}^+ (L_{ad}^+)^{\mathrm{T}}$$

$$P_D (A_{aa}^+)^{\mathrm{T}} + A_{aa}^+ P_D = D_a^+ (D_a^+)^{\mathrm{T}}$$

$$P_H (A_{aa}^+)^{\mathrm{T}} + A_{aa}^+ P_H = H_a^+ (H_a^+)^{\mathrm{T}}$$

yields

$$P_L = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}, \quad P_D = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad P_H = \begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

H_∞ *control law*: Solving the following minimization problem:

$$\hat{\gamma}_+^* = \min_{\substack{0<c<1 \\ P_L - P_D/c > 0}} \sqrt{\frac{\lambda_{\max}((P_L - P_D/c)^{-1} P_H)}{1-c}}$$

gives $\hat{\gamma}_+^* = 1.1496$ under $c^* = 0.4752$. Let $\gamma = 1.2 > \hat{\gamma}_+^*$ and $c = c^* = 0.4752$, then all the conditions in Theorem 4.1 are satisfied. Let

$$P_h = \left(P_L - \frac{1}{c} P_D - \frac{1}{(1-c)\gamma^2} P_H \right)^{-1} = \begin{bmatrix} 4.1737 & -4.9348 \\ -4.9348 & 9.3290 \end{bmatrix}$$

and

$$F_s = (L_{ad}^+)^{\mathrm{T}} P_h = \begin{bmatrix} F_{s_1} \\ F_{s_2} \end{bmatrix} = \begin{bmatrix} 12.5211 & -14.8045 \\ -5.6959 & 13.7232 \end{bmatrix}$$

Let $\lambda_{11} = -2$ and $\lambda_{21} = -3$, we have

$$\bar{F}_d(\varepsilon) = \begin{bmatrix} 2/\varepsilon & 0 \\ 0 & 3/\varepsilon \end{bmatrix}$$

Then

$$\bar{F}_s(\varepsilon) = \begin{bmatrix} 2F_{s_1}/\varepsilon \\ 3F_{s_2}/\varepsilon \end{bmatrix} = \begin{bmatrix} 25.0423/\varepsilon & -29.6090/\varepsilon \\ -17.0878/\varepsilon & 41.1695/\varepsilon \end{bmatrix}$$

Finally, the H_∞ control law is given by

$$F(\varepsilon) = -\begin{bmatrix} 1+25.0423/\varepsilon & 1-29.6090/\varepsilon & 1+2/\varepsilon & 2 \\ -17.0878/\varepsilon & 1+41.1695/\varepsilon & 1 & 1+3/\varepsilon \end{bmatrix}$$

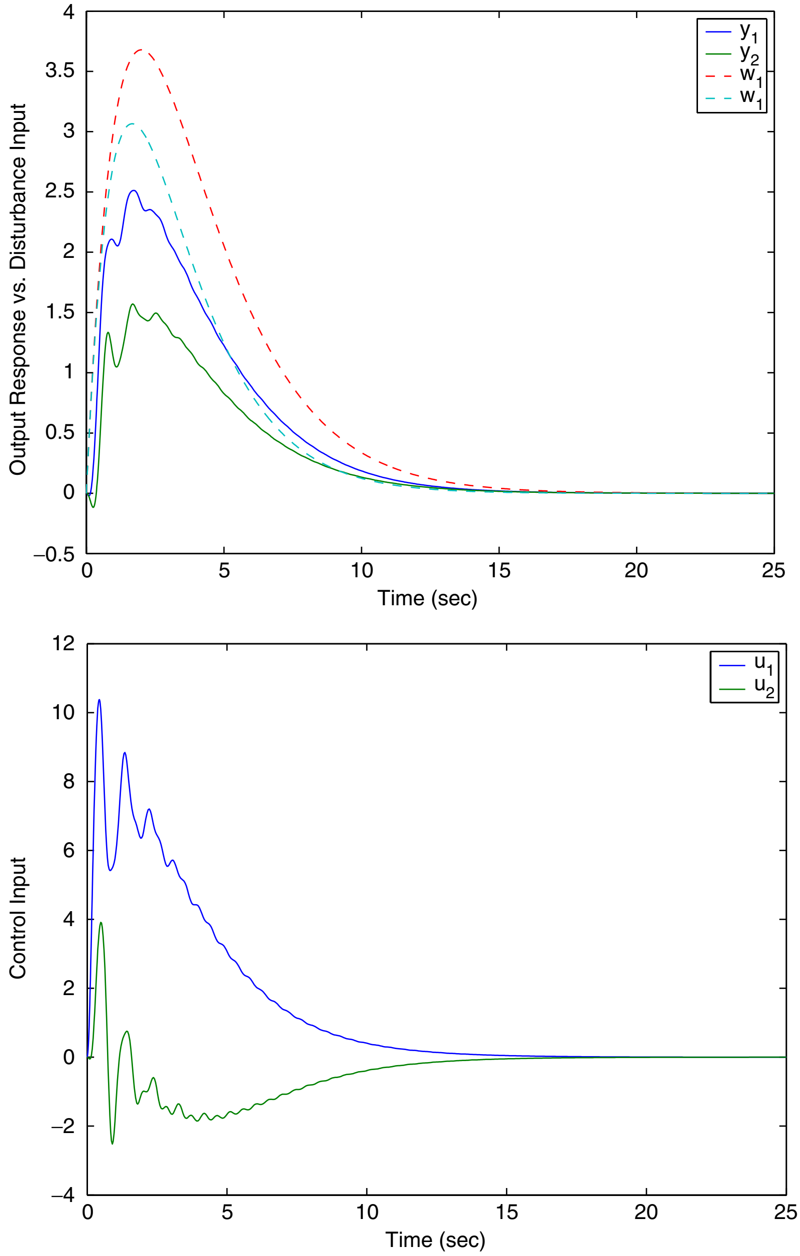

Figure 1. Output response and control input under H_∞ control law.

Let $\varepsilon = 0.8$, initial condition $x(0) = 0$ and the disturbance inputs $w_1 = 5te^{-0.5t}$ and $w_2 = 5te^{-0.6t}$; the simulation result is shown in Figure 1. It is clear that the closed-loop system is asymptotically stable and can reject the disturbance efficiently.

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7. CONCLUSIONS

Global stabilization problem and nonlinear H_∞ control problem of a class of nonminimum phase nonlinear MIMO systems are investigated. The nonminimum phase nonlinear system is globally exponentially stabilized by a linear feedback under the assumption that the nonlinear functions in the system satisfy a group of linear growth conditions. Our method can deal with the nonminimum phase systems with high-order unstable zero dynamics. The designed control law can act as a desired stabilizer for solving the adaptive estimation and rejection problem of the nonminimum phase nonlinear systems (see, e.g. [11] and [12]). Moreover, instead of solving the HJ equations, the nonlinear H_∞ control law is constructed explicitly by solving a set of Lyapunov equations on the unstable zero dynamics. The achievable L_2-gain estimation can be calculated based on the solutions of these Lyapunov equations.

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