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# Improving transient performance in tracking control for linear multivariable discrete-time systems with input saturation

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**Abstract**

In this paper, we present a composite nonlinear feedback (CNF) control technique for linear discrete-time multivariable systems with actuator saturation. The CNF control law serves to improve the transient performance of the closed-loop system by adding an additional nonlinear feedback. The linear feedback can be designed to yield a quick response at the initial stage, then the nonlinear feedback is introduced to smooth out overshoots when the system output approaches the target reference. As such, the resulting closed-loop system typically has very fast transient response and small overshoots. The goal of this work is to complete the theory for general discrete-time systems. The technique is applied to a magnetic-tape-drive servo system design and yields a huge improvement in settling time compared to that of a purely linear controller.


*Keywords:* Nonlinear control; Transient response; Discrete-time systems; Actuator saturation; Tracking control

## 1. Introduction and problem formulation

The problem of reference signal tracking has been a mature subject in the literature. Several excellent textbooks have covered this topic in details (see, for example, [1,7]). Not only the steady state tracking performance but also the transient tracking performance are required in most of the tracking control applications, such as motion control and process control. For the closed-loop transient performance, settling time and overshoot are concerned during the control design procedure. However, it is well known that, in general, quick response results in a large overshoot. Thus, most of the design schemes make a trade-off between these two transient performance indices. In order to improve the transient tracking performance, Lin et al. [14] proposed a so-called composite nonlinear feedback (CNF) control technique in their pioneer work for a class of second order linear systems. The CNF control is a scheme consisting of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part. From the structure of the CNF control law, it is clear that the CNF controller reduces to a linear controller when the gains in the nonlinear feedback law vanish. Therefore, the additional nonlinear feedback make it possible to change the feedback gains to improve the transient performance.

It is worth noting that when dealing with set-point tracking, the so-called reference management approach was proposed in the framework of model predictive control [2] and uncertain linear systems [3]. An improved error governor and a reference governor based on the concept of maximal output admissible sets were adopted to track reference signals inside some constraint set for the output in Gilbert et al. [8] and Gilbert and Tan [9], respectively. In Graettinger and Krogh [10], the authors considered the computation of reference signal constraints for guaranteed tracking performance in supervisory control environment. These ideas were also adopted in Blanchini and Miani [4].

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The merits of the CNF control lie in its simple structure and using linear controller as a basic element which is of especial interest to many researchers and practical engineers as it can be easily implemented. After the work of Lin et al. [14], Turner et al. [19] extended the CNF control technique to higher order and multiple input systems under a restrictive assumption on the system. Recently, Chen et al. [5] have developed a CNF control to a more general class of systems with measurement feedback, and successfully applied the CNF technique to solve a hard disk servo problem. However, along the same line as that of CNF control, very little has been done for linear discrete-time systems except the work of Venkataramanan et al. [20], which is only applicable to linear single-input and single-output (SISO) systems with state feedback. In this paper, we present a CNF control technique for discrete-time multivariable systems with actuator saturation.

To be specific, we consider in this paper the following multi-input and multi-output (MIMO) discrete-time system $\Sigma$ with an amplitude-constrained actuator characterized by

$$\begin{cases} x(k+1) = Ax(k) + B\,\mathrm{sat}(u(k)), & x(0) = x_0, \\ h(k) = C_2 x(k) + D_2\,\mathrm{sat}(u(k)), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $h \in \mathbb{R}^\ell$ are, respectively, the state, control input and controlled output of the given system $\Sigma$. $A$, $B$ and $C_2$ are appropriate dimensional constant matrices, and the saturation function is defined by

$$\mathrm{sat}(u) = \begin{pmatrix} \mathrm{sat}(u_1) \\ \mathrm{sat}(u_2) \\ \vdots \\ \mathrm{sat}(u_m) \end{pmatrix} \tag{2}$$

with

$$\mathrm{sat}(u_i) = \mathrm{sign}(u_i)\min(|u_i|, \bar{u}_i), \tag{3}$$

where $\bar{u}_i$ is the maximum amplitude of the $i$th control channel. We assume that the state variable of the plant is available for feedback. The objective of this paper is to design a state feedback control law for (1) using the CNF approach such that the resulting controlled output will track some desired step references as fast and as smooth as possible. For tracking purpose, the following assumptions on the given system are made:

1. $(A, B)$ is stabilizable; and
2. $(A, B, C_2, D_2)$ is right invertible (and hence $m \geqslant l$) and has no invariant zeros at $z = 1$.

Note that these assumptions are necessary for tracking control of discrete-time systems. We also note that there are many excellent works published in the literature dealing with control problems for systems with saturation nonlinearities, see, for example, [4,13,15,16,18], to name a few. The focus of this work, however, is very different. We aim to design a controller that would improve the transient performance instead. In fact, we can borrow any design from the literature and apply our CNF technique to yield a better transient performance.

The paper is organized as follows. Section 2 deals with the theory of the CNF control. We will address the issue on the selection of design parameters associated with the nonlinear feedback law in Section 3. The technique is then illustrated in a magnetic-tape-drive design example in Section 4, which shows that the proposed design method yields a big improvement in settling time compared to that of conventional linear state feedback design approaches. Finally, we draw some concluding remarks in Section 5.

## 2. CNF controller design

We have the following step-by-step procedure for the design of the CNF control law.

*Step* s.1: Design a linear feedback law,

$$u_{\mathrm{L}}(k) = Fx(k) + Gr, \tag{4}$$

where $r \in \mathbb{R}^l$ contains a set of step references. The state feedback gain matrix $F \in \mathbb{R}^{m\times n}$ is chosen such that the closed-loop system matrix $A + BF$ is asymptotically stable and typically the resulting closed-loop system transfer matrix, i.e., $D_2 + (C_2 + D_2F)(zI - A - BF)^{-1}B$, has certain desired properties, e.g., having a small dominating damping ratio in each channel. We note that the focus of this work is on the improvement of transient performance of the overall closed-loop system over linear control. Thus, it is fair to assume that there exists a linear state feedback control law, which stabilizes the given system. We also note that such an $F$ can be worked out using some well-studied methods such as the LQR, $H_\infty$ and $H_2$ optimization approaches. Furthermore, $G$ is an $m \times l$ constant matrix and is given by

$$G := G_0'(G_0G_0')^{-1}, \tag{5}$$

with $G_0 := D_2 + (C_2 + D_2F)(I - A - BF)^{-1}B$. Here we note that both $G_0$ and $G$ are well defined because $A + BF$ is stable, and $(A, B, C_2, D_2)$ is right invertible and has no invariant zeros at $z = 1$, which implies $(A + BF, B, C_2 + D_2F, D_2)$ is right invertible and has no invariant zeros at $z = 1$ (see e.g., Theorem 3.8.1 of Chen et al. [6]).

*Step* s.2: Next, we compute

$$H := [I + F(I - A - BF)^{-1}B]G \tag{6}$$

and

$$x_{\mathrm{e}} := G_{\mathrm{e}}\,r := (I - A - BF)^{-1}BGr. \tag{7}$$

Note that the definitions of $H$, $G_{\mathrm{e}}$ and $x_{\mathrm{e}}$ would become transparent later in our derivation. Given a positive definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$P = (A + BF)'P(A + BF) + W, \tag{8}$$

for $P > 0$. Such a $P$ exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_{\mathrm{N}}(k)$ is given by

$$u_{\mathrm{N}}(k) = \rho(r, y)B'P(A + BF)(x(k) - x_{\mathrm{e}}), \tag{9}$$

where

$$\rho(r,y)=\mathrm{diag}\{\rho_1,\ldots,\rho_m\}=\begin{bmatrix}\rho_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \rho_m\end{bmatrix}, \tag{10}$$

and $\rho_i=\rho_i(r,y)$, $i=1,2,\ldots,m$, are some non-positive functions, locally Lipschitz in $y$, which are used to change the closed-loop system damping ratios as the outputs approach the targets. In fact, in most of the situations, it is not necessary to restrict $\rho$ to be in a diagonal form. The choice of these nonlinear functions and $W$ will be discussed in Section 3.

*Steps* s.3: The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller:

$$\begin{aligned}u(k)=u_{\mathrm{L}}(k)+u_{\mathrm{N}}(k)&=Fx(k)+Gr\\ &\quad+\rho(r,y)B'P(A+BF)(x(k)-x_{\mathrm{e}}).\end{aligned} \tag{11}$$

This completes the design of the CNF controller for the state feedback case.

For further development, we partition $B\in\mathbb{R}^{n\times m}$, $F\in\mathbb{R}^{m\times n}$ and $H\in\mathbb{R}^{m\times l}$ as follows:

$$B=[B_1\ \cdots\ B_m],\quad F=\begin{bmatrix}F_1\\ \vdots\\ F_m\end{bmatrix},\quad H=\begin{bmatrix}H_1\\ \vdots\\ H_m\end{bmatrix}. \tag{12}$$

The following theorem shows that the closed-loop system comprising the given plant in (1) and the CNF control law of (11) is asymptotically stable. It also determines the magnitudes of the step functions in $r$ that can be tracked by such a control law without exceeding the control limit.

**Theorem 2.1.** *Consider the given system $\Sigma$ in* (1) *with $y=x$, which satisfies Assumptions* 1 *and* 2, *the linear control law of* (4) *and the CNF control law of* (11). *For any $\delta\in(0,1)$, we define*

$$\begin{aligned}\boldsymbol{X}_\delta:=\{v\in\mathbb{R}^n\mid v'Pv\leqslant c_\delta \text{ and } |F_i v|\leqslant(1-\delta)\bar{u}_i,\\ i=1,\ldots,m\},\end{aligned} \tag{13}$$

*where $c_\delta>0$ is the largest positive scalar. Then, the linear control law of* (4) *is capable of driving the system controlled output $h(k)$ to track asymptotically a set of step references, i.e., $r$, provided that the initial state $x_0$ and $r$ satisfy*:

$$\tilde{x}_0:=(x_0-x_{\mathrm{e}})\in\boldsymbol{X}_\delta,\quad |H_i r|\leqslant\delta\bar{u}_i,\ \ i=1,\ldots,m. \tag{14}$$

*Furthermore, for any non-positive function $\rho(r,y)$, locally Lipschitz in $y$, which satisfies*

$$\begin{aligned}&2\rho+\rho B'PB\rho\leqslant 0 \quad \text{or}\\ &-2(B'PB)^{-1}\leqslant\rho\leqslant 0 \ \Leftrightarrow\ \rho^{-1}\leqslant-\tfrac{1}{2}B'PB,\end{aligned} \tag{15}$$

*if $\rho$ is selected to be non-singular, the CNF law in* (11) *is capable of driving the system controlled output $h(k)$ to track asymptotically the step command input of amplitude $r$, provided that the initial state $x_0$ and $r$ satisfy* (14).

**Proof.** Let us first define a new state variable $\tilde{x}(k)=x(k)-x_{\mathrm{e}}$. It is simple to verify that the linear feedback control law of (4) can be rewritten as

$$\begin{aligned}u_{\mathrm{L}}(k)&=F\tilde{x}(k)+[I+F(I-A-BF)^{-1}B]Gr\\ &=F\tilde{x}(k)+Hr,\end{aligned} \tag{16}$$

and hence for all $\tilde{x}(k)\in\boldsymbol{X}_\delta$ and, provided that $|H_i r|\leqslant\delta\bar{u}_i$, $i=1,\ldots,m$, the closed-loop system is linear and is given by

$$x(k+1)=(A+BF)\tilde{x}(k)+Ax_{\mathrm{e}}+BHr. \tag{17}$$

Noting that

$$\begin{aligned}Ax_{\mathrm{e}}+BHr&=\{A(I-A-BF)^{-1}BG\\ &\quad+B[I+F(I-A-BF)^{-1}B]G\}r\\ &=\{A(I-A-BF)^{-1}BG\\ &\quad+[I+BF(I-A-BF)^{-1}]BG\}r\\ &=[A(I-A-BF)^{-1}+I\\ &\quad+BF(I-A-BF)^{-1}]BGr\\ &=(I-A-BF)^{-1}BGr=x_{\mathrm{e}},\end{aligned} \tag{18}$$

the closed-loop system in (17) can then be simplified as

$$\tilde{x}(k+1)=(A+BF)\tilde{x}(k). \tag{19}$$

Similarly, the closed-loop system comprising the given plant in (1) and the CNF control law of (11) can be expressed as

$$\tilde{x}(k+1)=(A+BF)\tilde{x}(k)+Bw(k), \tag{20}$$

where

$$w(k)=\mathrm{sat}(F\tilde{x}(k)+Hr+u_{\mathrm{N}}(k))-F\tilde{x}(k)-Hr. \tag{21}$$

Clearly, for the given $x_0$ satisfying (14), we have $\tilde{x}_0=(x_0-x_{\mathrm{e}})\in\boldsymbol{X}_\delta$. We note that (20) is reduced to (19) if $\rho(r,y)=0$.

Next, we define a Lyapunov function $V(k)=\tilde{x}'(k)P\tilde{x}(k)$ and evaluate the increment of $V(k)$ along the trajectories of the closed-loop system in (20), i.e.,

$$\begin{aligned}\Delta V(k+1)&=\tilde{x}'(k+1)P\tilde{x}(k+1)-\tilde{x}'(k)P\tilde{x}(k)\\ &=\tilde{x}'(k)(A+BF)'P(A+BF)\tilde{x}(k)-\tilde{x}'(k)P\tilde{x}(k)\\ &\quad+2\tilde{x}'(k)(A+BF)'PBw(k)+w'(k)B'PBw(k)\\ &=-\tilde{x}'(k)W\tilde{x}(k)+2\tilde{x}'(k)(A+BF)'PBw(k)\\ &\quad+w'(k)B'PBw(k).\end{aligned} \tag{22}$$

Note that for all

$$\tilde{x}(k)\in\boldsymbol{X}_\delta\ \Rightarrow\ |F_i\tilde{x}(k)|\leqslant(1-\delta)\bar{u}_i,\quad i=1,\ldots,m. \tag{23}$$

In the remainder of this proof, we consider the following different scenarios. For simplicity, we drop the dependent variables of the nonlinear function $\rho$ in the rest of this proof.

*Case* 1: All input channels are unsaturated. It is obvious that we have

$$w(k)=u_{\mathrm{N}}(k)=\rho B'P(A+BF)\tilde{x}(k) \tag{24}$$

and thus

$$\begin{aligned}\Delta V(k+1)&=-\tilde{x}'(k)W\tilde{x}(k)+2\tilde{x}'(k)(A+BF)'PB\rho B'\\ &\quad\times P(A+BF)\tilde{x}(k)\\ &\quad+\tilde{x}'(k)(A+BF)'PB\rho B'PB\rho B'\\ &\quad\times P(A+BF)\tilde{x}(k)\\ &=-\tilde{x}'(k)W\tilde{x}(k)+\tilde{x}'(k)(A+BF)'\\ &\quad\times PB(2\rho+\rho B'PB\rho)B'P(A+BF)\tilde{x}(k).\end{aligned} \tag{25}$$

In view of (15), we have

$$\Delta V(k+1) \leqslant -\tilde{x}'(k)W\tilde{x}(k) < 0. \tag{26}$$

*Case* 2: All input channels are exceeding their upper limits. In this case, we let

$$u_{Ni}(k) = \rho_i B_i' P(A+BF)\tilde{x}(k). \tag{27}$$

Thus, the assumption that all input channels are exceeding their upper limits, i.e.,

$$F_i\tilde{x}(k) + H_i r + u_{Ni}(k) \geqslant \bar{u}_i, \quad i = 1, \ldots, m, \tag{28}$$

implies that

$$u_{Ni}(k) \geqslant \bar{u}_i - F_i\tilde{x}(k) - H_i r, \quad i = 1, \ldots, m \tag{29}$$

and

$$w_i(k) = \bar{u}_i - (F_i\tilde{x}(k) + H_i r). \tag{30}$$

For all $\tilde{x}(k) \in \boldsymbol{X}_\delta$, which implies that (23) holds, and $r$ satisfies (14), we have

$$F_i\tilde{x}(k) + H_i r \leqslant \bar{u}_i, \quad i = 1, \ldots, m. \tag{31}$$

Hence,

$$0 \leqslant w_i(k) \leqslant u_{Ni}(k) \tag{32}$$

and

$$\begin{aligned}
\Delta V(k+1) &= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)[2B'P(A+BF)]\tilde{x}(k) \\
&\quad + w'(k)B'PBw(k) \\
&= -\tilde{x}'(k)W\tilde{x}(k) + \sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}u_{Ni}(k)] \\
&\quad + w'(k)B'PBw(k) \\
&\leqslant -\tilde{x}'(k)W\tilde{x}(k) + \sum_{i=1}^{m} w_i(k)[2\rho_i^{-1}w_i(k)] \\
&\quad + w'(k)B'PBw(k) \\
&= -\tilde{x}'(k)W\tilde{x}(k) + w'(k)(2\rho^{-1})w(k) \\
&\quad + w'(k)B'PBw(k) \\
&= -\tilde{x}'(k)W\tilde{x}(k) + w'(k) \\
&\quad \times (2\rho^{-1} + B'PB)w(k) < 0.
\end{aligned} \tag{33}$$

*Case* 3: All input channels are exceeding their lower limits. For this case, we have

$$\begin{aligned}
&F_i\tilde{x}(k) + H_i r + \rho_i B_i' P(A+BF)\tilde{x}(k) \leqslant -\bar{u}_i, \\
&\quad i = 1, \ldots, m.
\end{aligned} \tag{34}$$

Following similar arguments as in the previous case, we can show that

$$\Delta V(k+1) \leqslant -\tilde{x}'(k)W\tilde{x}(k) < 0. \tag{35}$$

*Case* 4: Some control channels are saturated and some are unsaturated. In view of Cases 1–3, the increment is just a combination of the above three cases. For those unsaturated channels, we have

$$w_i(k) = u_{Ni}(k) = \rho_i B_i' P(A+BF)\tilde{x}(k) \tag{36}$$

and

$$w_i(k)(2\rho_i^{-1})u_{Ni}(k) = w_i(k)(2\rho_i^{-1})w_i(k). \tag{37}$$

On the other hand, for those saturated channels, we have either

$$0 \leqslant w_i(k) = \bar{u}_i(k) - (F_i\tilde{x}(k) + H_i r) \leqslant u_{Ni}(k) \tag{38}$$

or

$$u_{Ni}(k) \leqslant w_i(k) = -\bar{u}_i(k) - (F_i\tilde{x}(k) + H_i r) \leqslant 0. \tag{39}$$

Thus, we have

$$w_i(k)[2\rho_i^{-1}u_{Ni}(k)] \leqslant w_i(k)(2\rho_i^{-1})w_i(k). \tag{40}$$

It is then straightforward to verify that for this case, again, we have

$$\Delta V(k+1) \leqslant -\tilde{x}'(k)W\tilde{x}(k) < 0. \tag{41}$$

In conclusion, we have shown that

$$\Delta V(k+1) \leqslant -\tilde{x}(k)W\tilde{x}(k), \quad \tilde{x}(k) \in \boldsymbol{X}_\delta, \tag{42}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (20). Noting that $W > 0$, all trajectories of (20) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial state $x_0$ and the step command input $r$ that satisfy (14), we have

$$\lim_{k\to\infty} x(k) = x_e, \tag{43}$$

which implies

$$\begin{aligned}
\lim_{k\to\infty} u(k) &= F \lim_{k\to\infty} x(k) + Gr + \rho B'P(A+BF) \\
&\quad \times \left[\lim_{k\to\infty} x(k) - x_e\right] = Fx_e + Gr.
\end{aligned} \tag{44}$$

Hence,

$$\begin{aligned}
\lim_{k\to\infty} h(k) &= C_2 \lim_{k\to\infty} x(k) + D_2 \lim_{k\to\infty} u(k) \\
&= C_2 x_e + D_2(Fx_e + Gr) \\
&= (C_2 + D_2F)x_e + D_2Gr \\
&= (C_2 + D_2F)(I - A - BF)^{-1}BGr + D_2Gr \\
&= [D_2 + (C_2 + D_2F)(I - A - BF)^{-1}B]Gr \\
&= G_0G_0'(G_0G_0')^{-1}r = r.
\end{aligned} \tag{45}$$

This completes the proof of Theorem 2.1. □

## 3. Selection of design parameters $\rho(r, y)$ and $W$

The key component in designing the CNF controllers is the selection of $\rho$ (hereafter we drop the dependent variables of $\rho$ for simplicity) and $W$. The freedom in choosing the nonlinear function $\rho$ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output $h$ approaches the set point. Since the main purpose of adding the nonlinear part to the CNF controller is to speed up the settling time and to reduce the overshoot, or equivalently to contribute a significant value to the control input when the tracking error, $r - h$, is small, it is appropriate for us to select a nonlinear gain matrix such that the nonlinear part will be in action when the control signal is far away from its saturation level, and thus it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the given plant in (1) and the CNF control law (11) can be expressed as

$$\tilde{x}(k+1) = (A + BF)\tilde{x}(k) + B\rho B'P(A + BF)\tilde{x}(k). \tag{46}$$

It is now clear that eigenvalues of the closed-loop system in (46) can be changed by the nonlinear function $\rho$. In fact, for such a situation, it follows from Case 1 in the proof of Theorem 2.1 that the nonlinear gain matrix $\rho$ is not necessary to be in a diagonal form. It is only required to satisfy the following condition

$$-2(B'PB)^{-1} \leqslant \rho \leqslant 0. \tag{47}$$

Assuming that $h$ is available and assuming that $h_i(0) \neq r_i$ (for the trivial case when $h_i(0) = r_i$, there is no need to add any nonlinear gain to the control), we propose the following nonlinear gain matrix

$$\begin{aligned}\rho(r, h) = (B'PB)^{-1/2}&\mathrm{diag}\{\rho_i(r_i, h_i), \ldots, \rho_m(r_m, h_m)\}\\ &\times (B'PB)^{-1/2},\end{aligned} \tag{48}$$

with

$$\begin{aligned}\rho_i(r_i, h_i) = \frac{-\beta_i}{|h_i(0) - r_i|^{\alpha_i}}&||h_i(k) - r_i|^{\alpha_i} - |h_i(0) - r_i|^{\alpha_i}|,\\ 0 \leqslant \beta_i \leqslant 1,&\end{aligned} \tag{49}$$

$i = 1, \ldots, m$, which starts from 0 and gradually increases to a final gain of $-\beta_i$ as $h_i$ approaches to the target reference $r_i$. The parameter $\alpha_i$ is used to determine the speed of change in $\rho_i$. We note that such a nonlinear function matrix $\rho$ indeed satisfies the condition of (48). Unlike the continuous-time counterpart, in which it is possible to use a large nonlinear gain to push the closed-loop eigenvalues far in the left-half plane, the nonlinear gain matrix $\rho$ in the discrete-time case is always bounded to ensure that all the closed-loop eigenvalues remain inside the unit circle.

To examine the behavior of the closed-loop system (46) more explicitly, we define an auxiliary system $G_{\mathrm{aux}}(z)$ characterized by

$$\begin{aligned}G_{\mathrm{aux}}(z) &:= C_{\mathrm{aux}}(zI - A_{\mathrm{aux}})^{-1}B_{\mathrm{aux}}\\ &:= B'P(zI - A - BF)^{-1}B.\end{aligned} \tag{50}$$

Obviously, $G_{\mathrm{aux}}(z)$ is stable. We note that

$$C_{\mathrm{aux}}B_{\mathrm{aux}} = B'PB > 0, \tag{51}$$

which implies $G_{\mathrm{aux}}(z)$ is a square, invertible and uniform rank system with $m$ infinite zeros of order 1 and with $n - m$ invariant zeros. We will show that this auxiliary system is in fact of minimum phase, i.e., all its invariant zeros are stable. We note that for such a system, it follows from the result reported in Chapter 5 of Chen et al. [6] (see also [17]) that there exist non-singular transformations $\Gamma_{\mathrm{s}} \in \mathbb{R}^{n\times n}$, $\Gamma_{\mathrm{i}} \in \mathbb{R}^{m\times m}$ and $\Gamma_{\mathrm{o}} \in \mathbb{R}^{m\times m}$ such that the transformed system has the following special form,

$$\begin{aligned}&(\Gamma_{\mathrm{s}}^{-1}A_{\mathrm{aux}}\Gamma_{\mathrm{s}},\ \Gamma_{\mathrm{s}}^{-1}B_{\mathrm{aux}}\Gamma_{\mathrm{i}},\ \Gamma_{\mathrm{o}}^{-1}C_{\mathrm{aux}}\Gamma_{\mathrm{s}})\\ &\quad = \left(\begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ad}} \\ E_{\mathrm{da}} & A_{\mathrm{dd}} \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \end{bmatrix}, [0\ \ I_m]\right),\end{aligned} \tag{52}$$

where the eigenvalues of $A_{\mathrm{aa}}$ are the invariant zeros of the auxiliary system $G_{\mathrm{aux}}(z)$, $L_{\mathrm{ad}}$, $E_{\mathrm{da}}$ and $A_{\mathrm{dd}}$ are some constant matrices. Next, we proceed to show that all the eigenvalues of $A_{\mathrm{aa}}$ are inside the unit circle and thus $G_{\mathrm{aux}}(z)$ is of minimum phase. We note that at the steady state when $h = r$, the nonlinear function matrix $\rho$ of (48) with $\beta_i = 1$, $i = 1, \ldots, m$, is reduced to $\rho = -(B'PB)^{-1}$ and the closed-loop system of (46) can be expressed as

$$\begin{aligned}\tilde{x}(k+1) &= (A + BF)\tilde{x}(k) - B(B'PB)^{-1}B'P(A + BF)\tilde{x}(k)\\ &= [I - B(B'PB)^{-1}B'P](A + BF)\tilde{x}(k)\\ &= [I - B_{\mathrm{aux}}(C_{\mathrm{aux}}B_{\mathrm{aux}})^{-1}C_{\mathrm{aux}}]A_{\mathrm{aux}}\tilde{x}(k)\\ &= \left[I - \Gamma_{\mathrm{s}}\begin{bmatrix} 0 \\ I \end{bmatrix}\Gamma_{\mathrm{i}}^{-1}\left(\Gamma_{\mathrm{o}}[0\ \ I]\Gamma_{\mathrm{s}}^{-1}\Gamma_{\mathrm{s}}\begin{bmatrix} 0 \\ I \end{bmatrix}\Gamma_{\mathrm{i}}^{-1}\right)^{-1}\right.\\ &\qquad \left.\times\ \Gamma_{\mathrm{o}}[0\ \ I]\Gamma_{\mathrm{s}}^{-1}\right]\Gamma_{\mathrm{s}}\begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ad}} \\ E_{\mathrm{da}} & A_{\mathrm{dd}} \end{bmatrix}\Gamma_{\mathrm{s}}^{-1}\tilde{x}(k)\\ &= \left(\Gamma_{\mathrm{s}}\begin{bmatrix} A_{\mathrm{aa}} & L_{\mathrm{ad}} \\ 0 & 0 \end{bmatrix}\Gamma_{\mathrm{s}}^{-1}\right)\tilde{x}(k).\end{aligned} \tag{53}$$

Clearly, the closed-loop system has $n - m$ eigenvalues at $\lambda(A_{\mathrm{aa}})$ and the rest at 0. Thus, the stability of the closed-loop system with $\rho = -(B'PB)^{-1}$ implies the eigenvalues of $A_{\mathrm{aa}}$ are all inside the unit circle. This shows that $G_{\mathrm{aux}}(z)$ is indeed of minimum phase.

It should be noted that there is freedom in pre-selecting the locations of these invariant zeros by choosing an appropriate $W$ in (8). In general, we should select the invariant zeros of $G_{\mathrm{aux}}(z)$, which are corresponding to the closed-loop poles of (46) for the steady state nonlinear gain matrix, with dominating ones having a large damping ratio, which in turn generally yield a smaller overshoot. The following procedure might be used for such a purpose.

1. Given a set of $n - m$ self-conjugated complex scalars, which should include all the uncontrollable modes, if any, of $(A, B)$, we are to determine an appropriate $W > 0$ such that the resulting auxiliary system $G_{\mathrm{aux}}(z)$ has its invariant zeros placed exactly at the locations given in the set.

Firstly, use the singular value decomposition technique to find a unitary matrix $U \in \mathbb{R}^{n\times n}$ and a non-singular matrix $T_{\mathrm{i}} \in \mathbb{R}^{m\times m}$ such that

$$\tilde{B}_{\mathrm{aux}} = U'B_{\mathrm{aux}}T_{\mathrm{i}} = U'BT_{\mathrm{i}} = \begin{bmatrix} 0 \\ I_m \end{bmatrix} \tag{54}$$

and partition accordingly

$$\tilde{A}_{\mathrm{aux}} = U'A_{\mathrm{aux}}U = U'(A + BF)U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{55}$$

It is straightforward to verify that the stabilizability of $(A, B)$ implies the stabilizability of $(A_{11}, A_{12})$. In fact, their uncontrollable modes, if any, are identical.

Next, for determining an appropriate matrix $P = P' > 0$, we partition it accordingly as follows

$$\tilde{P} = U'PU = \begin{bmatrix} P_{11} & P_{21}' \\ P_{21} & P_{22} \end{bmatrix}. \tag{56}$$

Then, $C_{\text{aux}}$ can be expressed as

$$\begin{aligned} C_{\text{aux}} &= B'P = (T_{\text{i}}^{-1})'[0 \;\; I_m]U'U \begin{bmatrix} P_{11} & P_{21}' \\ P_{21} & P_{22} \end{bmatrix} U' \\ &= (T_{\text{i}}^{-1})'[P_{21} \;\; P_{22}]U' \\ &= [(T_{\text{i}}^{-1})'P_{22}][P_{22}^{-1}P_{21} \;\; I_m]U' \\ &:= T_{\text{o}}[P_{22}^{-1}P_{21} \;\; I_m]U'. \end{aligned} \tag{57}$$

Using the results of Chen et al. [6] (see e.g., Chapters 8 and 9), we can show that the invariant zeros of the auxiliary system $G_{\text{aux}}(z)$ are given by the eigenvalues of $A_{11} - A_{12}P_{22}^{-1}P_{21}$. Since $(A_{11}, A_{12})$ is stabilizable and the given set of complex scalars include all its uncontrollable modes, there exists a constant matrix, say $F_*$, such that $A_{11}$–$A_{12}F_*$ has its eigenvalues placed exactly at the locations given in the set. Obviously, we can select $P_{22}$ and $P_{21}$ such that

$$P_{22}^{-1}P_{21} = F_*. \tag{58}$$

2. Select an appropriate $P_{22} = P_{22}' > 0$, $P_{21} = P_{22}F_*$, and an appropriate $P_{11} = P_{11}' > P_{21}'P_{22}^{-1}P_{21}$ to ensure that

$$P = U \begin{bmatrix} P_{11} & P_{21}' \\ P_{21} & P_{22} \end{bmatrix} U' > 0. \tag{59}$$

3. Compute

$$W = P - (A + BF)'P(A + BF). \tag{60}$$

If $W$ is not positive definite, we go back to Step 2 to choose another solution of $P$ or go to the first step to re-select another set of desired invariant zeros.

Another method for selecting $W$ is based on a trial and error approach by limiting the choice of $W$ to be in a diagonal matrix and adjusting its diagonal weights through simulation. Generally, such an approach would yield a satisfactory result as well. We next illustrate the CNF design together with the detailed selection of $\rho$ and $W$ in a design example in the following section.

## 4. A design example

To illustrate the concept of the discrete-time CNF control, we apply the technique to design a magnetic-tape-drive servo system. The dynamics of the system are given in Franklin et al. [7]. The goal of the control system is to enable commanding the tape to specific positions over the read/write head while maintaining a specified tension in the tape at all times. The time-scaled dynamics of the drive is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 10 \\ 3.315 & -3.315 & -0.5882 & -0.5882 \\ 3.315 & -3.315 & -0.5882 & -0.5882 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 8.533 & 0 \\ 0 & 8.533 \end{bmatrix} \text{sat}(u), \tag{61}$$

where $x = (x_1 \;\; x_2 \;\; \omega_1 \;\; \omega_2)'$ with $x_1$ and $x_2$ being the positions of the tape at capstans (in mm), and $\omega_1$ and $\omega_2$ being angular rates of motors/capstan assemblies (in rad/s); and $u = (i_1 \;\; i_2)'$ with $i_1$ and $i_2$ being electric currents supplied to drive motors (in A). The saturation levels of the actuators are $\bar{i}_1 = \bar{i}_2 = 1$ A. The controlled output of the system is given by

$$h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} \bar{x}(t) \\ T_{\text{e}}(t) \end{pmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -2.113 & 2.113 & 0.375 & 0.375 \end{bmatrix} x(t), \tag{62}$$

where $\bar{x} = (x_1 + x_2)/2$ is the position of the tape over read/write head (in mm), and $T_{\text{e}}$ is the tension in the tape (in N).

The design specifications are as follows: (i) the 1% settling time due to a 1 mm step change in position of the tape head, $\bar{x}$, should be less than 2.5 s for the time-scaled system of (61), which is equivalent to 250 ms for the actual system; (ii) overshoot should be less than 20%; (iii) the tape tension, $T_{\text{e}}$, should be controlled to 2 N with the constraint that $0 < T_{\text{e}} < 4$ N; and (iv) the input current should not exceed 1 A at each drive motor.

As suggested in [7], we follow to select a sampling $T = 0.05$ s to carry out our controller design. The discretized dynamical equation is then given by

$$x(k+1) = \begin{bmatrix} 0.95992 & 0.04008 & -0.48614 & 0.01386 \\ 0.04008 & 0.95992 & -0.01386 & 0.48614 \\ 0.15656 & -0.15656 & 0.93214 & -0.06786 \\ 0.15656 & -0.15656 & -0.06786 & 0.93214 \end{bmatrix} \times x(k) + \begin{bmatrix} -0.10492 & 0.00175 \\ -0.00175 & 0.10492 \\ 0.41482 & -0.01183 \\ -0.01183 & 0.41482 \end{bmatrix} \text{sat}(u(k)). \tag{63}$$

The controlled output is given by

$$h(k) = \begin{pmatrix} h_1(k) \\ h_2(k) \end{pmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ -2.113 & 2.113 & 0.375 & 0.375 \end{bmatrix} x(k). \tag{64}$$

Given a target reference and an initial condition,

$$r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad x_0 = \begin{pmatrix} -0.1 \\ 0.1 \\ 0 \\ 0 \end{pmatrix}, \tag{65}$$

our aim is to design a CNF controller, which would control the controlled output of the system to track the command reference

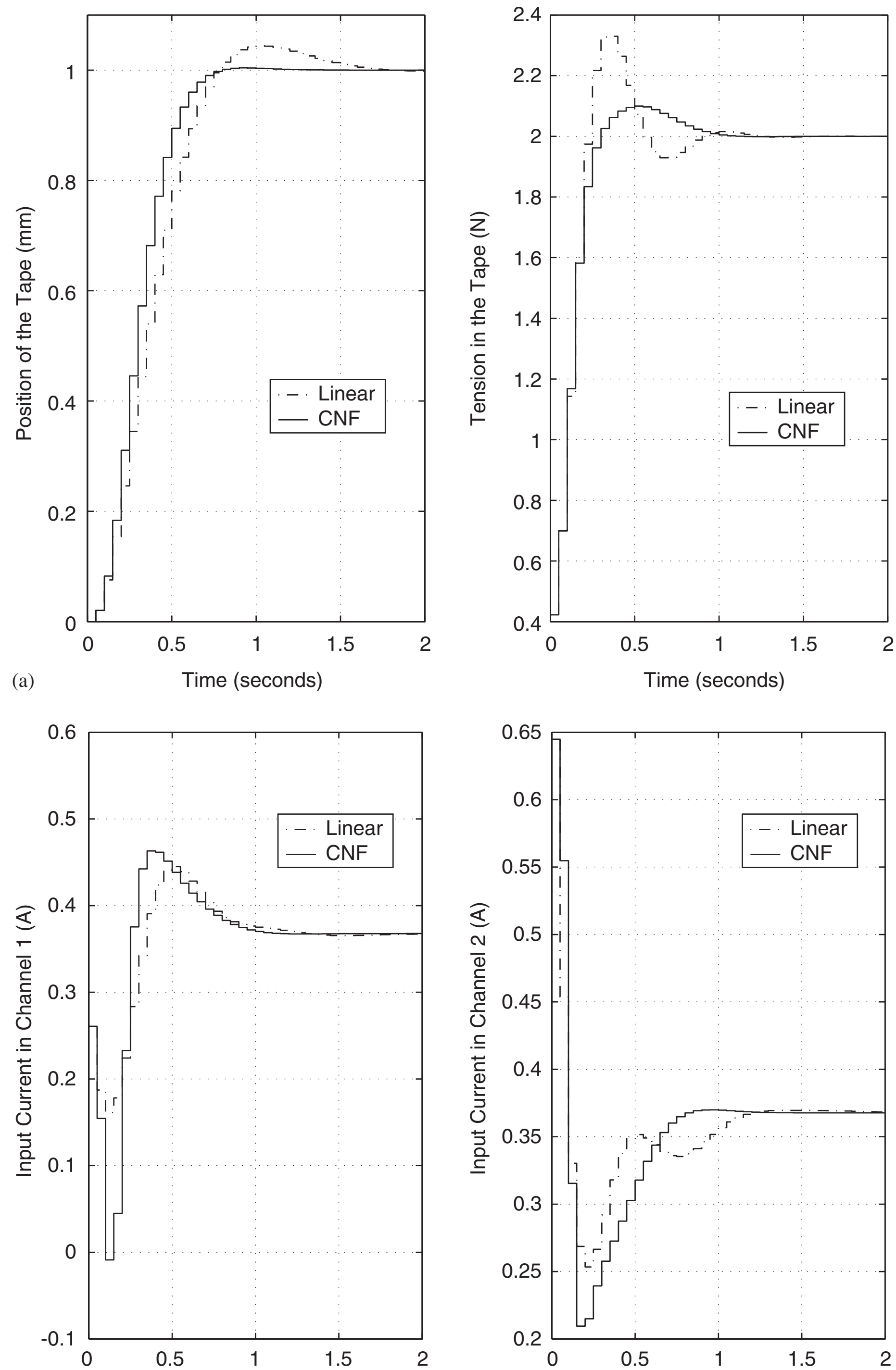


Fig. 1. Simulation results of linear and CNF control with $\alpha_1 = \alpha_2 = 8$, $\beta_1 = 0.4$ and $\beta_2 = 0.15$: (a) controlled output; (b) control input.

as fast as possible and as smooth as possible. For easy comparison, the linear state feedback gain, $F$, is selected precisely the same as that given in [7], i.e.,

$$F = \begin{bmatrix} 0.210 & -0.018 & -0.744 & -0.074 \\ 0.018 & -0.210 & -0.074 & -0.744 \end{bmatrix},$$

and the resulting $G$ and $x_e$ are given by

$$G = \begin{bmatrix} -0.192 & 0.2378 \\ 0.192 & 0.2378 \end{bmatrix}, \quad x_e = \begin{pmatrix} 0.5267 \\ 1.4733 \\ 0 \\ 0 \end{pmatrix}.$$

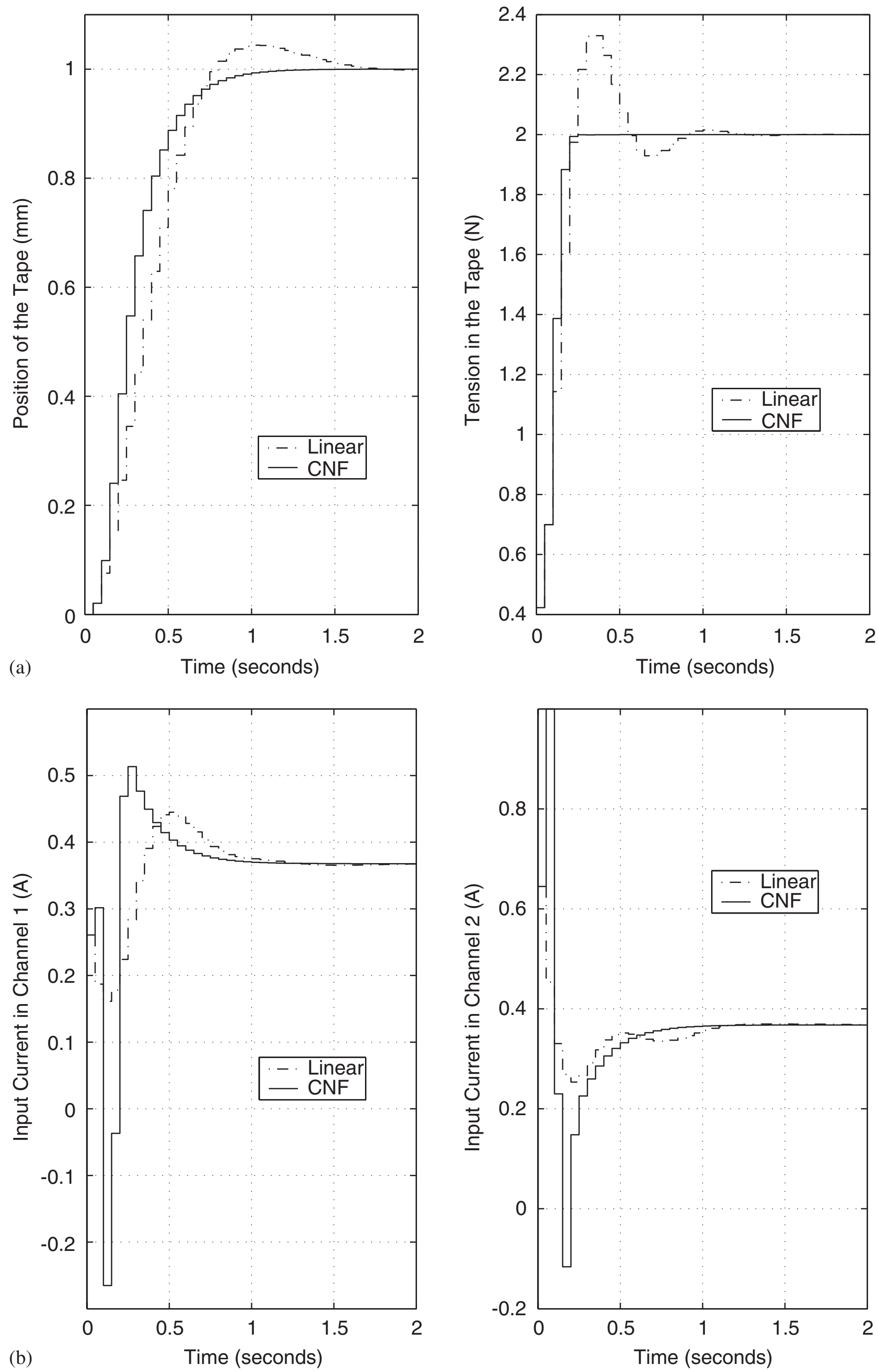


Fig. 2. Simulation results of linear and CNF control with $\alpha_1 = \alpha_2 = 6$ and $\beta_1 = \beta_2 = 1$: (a) controlled output; (b) control input.

Next, choosing $W = I_4$ and solving the Lyapunov equation (8), we obtain

$$P = \begin{bmatrix} 4.6816 & 1.0045 & -3.1240 & 2.1193 \\ 1.0045 & 4.6816 & -2.1193 & 3.1240 \\ -3.1240 & -2.1193 & 6.7399 & -3.9816 \\ 2.1193 & 3.1240 & -3.9816 & 6.7399 \end{bmatrix}.$$

It is then straightforward to verify that the auxiliary system $G_{\text{aux}}(z)$ has two invariant zeros at 0.7192 and 0.7567, respectively. Thus, $G_{\text{aux}}(z)$ is indeed of minimum phase. Next, following the suggestion given in the previous section, we select

$$\rho(r,h) = (B'PB)^{-1/2} \begin{bmatrix} \rho_1(r_1,h_1) & 0 \\ 0 & \rho_2(r_2,h_2) \end{bmatrix} (B'PB)^{-1/2},$$

with

$$\rho_i(r_i, h_i) = \frac{-\beta_i}{|h_i(0) - r_i|^{\alpha_i}} \, ||h_i(k) - r_i|^{\alpha_i} - |h_i(0) - r_i|^{\alpha_i}|, \quad i = 1, 2.$$

We have obtained two sets of simulation results. Fig. 1 shows the results of the linear and CNF control with $\alpha_1 = \alpha_2 = 8$, $\beta_1 = 0.4$ and $\beta_2 = 0.15$. The 1% settling times for the position of the tape under the linear and CNF control are 1.55 and 0.7 s, respectively, resulted in a 55% improvement. The overshoot of this channel is reduced from about 5% under the linear control to almost zero with the CNF control. For the tension of the tape, the overshoot is reduced from 16% under the linear control to less than 5% with the CNF control.

Although the tension of the tape is not critical for this magnetic-tape-drive system so long as it is kept within 0 and 4 N, we present in Fig. 2 the results of the linear and CNF control with $\alpha_1 = \alpha_2 = 6$, $\beta_1 = \beta_2 = 1$, to demonstrate the powerfulness of the CNF control technique. For this case, both the position and the tension of the tape under the CNF control have quite impressively fast settling times (0.95 and 0.2 s, respectively) and have no overshoot at all.

## 5. Conclusion

We have presented a nonlinear tracking control technique which is able to improve transient response of set-point tracking performance for general discrete-time systems with actuator saturation. The CNF control law consists of two parts, a linear component, which has been designed to solve the tracking problem under actuator saturation using any appropriate method in the literature and a nonlinear component, which is used to improve transient performance. The technique has been successfully demonstrated to yield a nice tracking performance in a real application. Also, we note that it is not difficult to extend this method to measurement feedback cases along the lines of [12,11].

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