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Symbolic realization of asymptotic time-scale and eigenstructure assignment design method in multivariable control

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This paper reports on a symbolic realization of the asymptotic time-scale and eigenstructure assignment (ATEA) state feedback design technique for multivariable control. The resulting state feedback laws are parameterized in a scalar ϵ. Under these state feedback laws, the closed-loop system possesses a pre-specified time-scale and its eigenstructure approaches a pre-specified one, as the value of the parameter ϵ approaches zero. By appropriately specifying the time-scale and the eigenstructure, the feedback laws can be obtained to solve various control problems, such as the H_2 and H_∞ suboptimal control, and almost disturbance decoupling problems. We present, in this paper, the software implementation of the ATEA design algorithm using the MATLAB symbolic programming technique. Our m-functions are capable of returning a result, which is explicitly expressed in terms of a symbolic variable epsilon, which represents ϵ. The controller design for a piezoelectric bimorph actuator is used to illustrate how the symbolic realization works.

1. Introduction

The asymptotic time-scale and eigenstructure assignment (ATEA) is one of the major applications of the structural decomposition approach in linear systems theory (Chen *et al*. 2004). The concept of ATEA was originally proposed in Saberi and Sannuti (1989, 1990b) and was further developed in Chen (1991), Saberi *et al*. (1993), Lin (1998) and Chen *et al*. (2004). The ATEA algorithm is decentralized in nature and is in fact rooted in the concept of singular perturbation methods (Kokotovic *et al*. 1986).

More specifically, the main idea behind the ATEA algorithm can be described as follows. The given linear system characterized by a matrix quadruple (A, B, C, D) is first transformed into the form of the special coordinate basis (SCB) (Sannuti and Saberi 1987, Saberi and Sannuti 1989). On the SCB, the system is decomposed into a networked of subsystems, each of which captures some inherent structure of the original system. By exploring the intricate structures of each of these subsystems and the interconnections that exist among them, feedback gain matrices, explicitly parameterized in a scalar, say ϵ, are constructed for each of these subsystems in such a way that, when composed together to form an overall state feedback gain for the system, they result in a closed-loop system with a pre-specified time-scale and engenstructure. The procedure can also been utilized to construct observer gains, which lead to appropriate time-scale and eigenstructure of the resulting error dynamics. By appropriately specifying the time-scale and the eigenstructre, the feedback laws of both state feedback type and output feedback type can be obtained that solve a wide variety of control problems, such as the H_2 and H_∞ suboptimal control problems (Lin *et al*. 1998a, b, Chen 2000, Saberi *et al*. 1995), LTR (Chen 1991, Saberi *et al*. 1993), almost disturbance decoupling problems (Ozcetin *et al*.

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1993a, b, Chen 2000, Lin and Chen 2000), and constrained control problems (Lin 1998).

Among the many distinct features of the ATEA algorithm based control design methods is the ease of the symbolic computation of the feedback laws. The feedback gains for the subsystems are parameterized in a scalar ϵ and given in the form of polynomial matrices in $1/\epsilon$. The construction of these gain matrices only involves the computation of the coefficients of the polynomials and thus, in essence avoiding the direct symbolic computation. The direct symbolic computation is necessary only in the last steps of the algorithm when various feedback gains, polynomial matrices, are composed together to form the overall feedback gain for the original system.

The objective of this paper is to describe the AETA algorithm and its software implementation in detail and to show how the ATEA algorithm has been developed in such a way that facilitates the symbolic computation of the resulting feedback gains. We will also use simple applications to illustrate how the symbolic computation of ATEA based state feedback laws leads to feedback laws that are explicitly parameterized in the design parameter. We will however not describe in detail the wide variety of applications of the ATEA algorithm that have been reported in the literature.

The ATEA algorithm is implemented by using the Symbolic Math toolboxes on the MATLAB platform. The Symbolic Math Toolboxes incorporate symbolic computation into the numeric environment of MATLAB. These toolboxes supplement MATLAB numeric and graphical facilities with several other types of mathematical computation, such as calculus, linear algebra, simplification, solution of equations, special mathematical function, variable-precision arithmetic and transforms. The computational engine underlying the toolboxes is the kernel of Maple, a system developed primarily at the University of Waterloo, Canada and, more recently, at the Eidgenössiche Technische Hochschule, Zürich, Switzerland (The Math Work Inc. 2004).

The remainder of this paper is organized as follows. In §2, we describe in detail the ATEA algorithm and show how it is utilized to solve the H_2 and H_∞ suboptimal control problems as well as the problem of almost disturbances decoupling. In §3, we describe the symbolic implementation of the ATEA algorithm, which the algorithm itself renders very straightforward. Section 4 contains a simple numerical example and the feedback design for a piezoelectric bimorph actuator to demonstrate the ATEA based approach to control design. Section 5 concludes the paper.

Throughout this paper, the following notation will be used: X' denotes the transpose of matrix X; 0 denotes a scalar zero or a zero matrix of appropriate dimensions; I denotes an identity matrix of appropriate dimensions; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$ and $\mathbb{C}^-$ denote the entire complex plane and the open left-half complex plane respectively; and finally, $\lambda(X)$ denotes the set of eigenvalues of a real square matrix X.

2. The ATEA algorithm

In this section, we describe the technique of the ATEA design for continuous-time systems. We will also describe, as examples of its application, how the ATEA algorithm can be utilized in solving the H_2 and H_∞ suboptimal control problems as well as the almost disturbance decoupling problem.

Consider a continuous-time linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of the system, respectively. Without loss of generality, we assume that (A, B) is stabilizable, and both $[B^T, D^T]$ and $[C, D]$ are of row full rank. For simplicity, we also assume that the given system has no invariant zeros on the imaginary axis. Detailed treatments of systems with imaginary invariant zeros involve the concept of low gain feedback and slow time-scale it induces, which can be found in (Chen 1991, Saberi *et al.* 1993).

2.1 *The ATEA algorithm*

What follows is a step-by-step presentation of the ATEA algorithm. The properties of ATEA algorithm will be summarized in a theorem after the presentation of the algorithm itself.

Step 1 Transform Σ into the structural decomposition or the special coordinate basis form (Sannuti and Saberi 1987, Saberi and Sannuti 1989). That is, compute nonsingular state, input and output transformations Γ_s, Γ_i and Γ_o that transform the given system Σ into the special coordinate basis, which can be put in the following compact form:

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\ 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\ B_c E_{ca}^- & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd} \end{bmatrix} + \begin{bmatrix} B_{0a}^- \\ B_{0a}^+ \\ B_{0b} \\ B_{0c} \\ B_{0d} \end{bmatrix} \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \end{bmatrix}, \quad (2)$$

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix}, \quad (3)$$

$$\tilde{C} = \Gamma_o^{-1} C \Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\ 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & C_b & 0 & 0 \end{bmatrix}, \quad (4)$$

$$\tilde{D} = \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (5)$$

where (A_{bb}, C_b) is observable, (A_{cc}, B_c) is controllable, and in particular,

$$A_{dd} = A_{dd}^* + B_d E_{dd} + L_{dd} C_d,$$

for some constant matrices L_{dd} and E_{dd} of appropriate dimensions, and

$$A_{dd}^* = \text{blkdiag}\left\{A_{q_1}, A_{q_2}, \ldots, A_{q_{m_d}}\right\}, \quad (6)$$

$$B_d = \text{blkdiag}\left\{B_{q_1}, B_{q_2}, \ldots, B_{q_{m_d}}\right\},$$

$$C_d = \text{blkdiag}\left\{C_{q_1}, C_{q_2}, \ldots, C_{q_{m_d}}\right\}, \quad (7)$$

with $(A_{q_i}, B_{q_i}, C_{q_i})$ being defined as

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1,\ 0,\ \ldots,\ 0].$$

Next, we define

$$A_{ss} = \begin{bmatrix} A_{aa}^+ & L_{ab}^+ C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{0s} = \begin{bmatrix} B_{0a}^+ \\ B_{0b} \end{bmatrix}, \quad L_{sd} = \begin{bmatrix} L_{ad}^+ \\ L_{bd} \end{bmatrix}, \quad (8)$$

and

$$B_s = \begin{bmatrix} B_{0s} & L_{sd} \end{bmatrix}. \quad (9)$$

Step 2 Let F_s be chosen such that

$$\lambda\left(A_{ss}^c\right) = \lambda(A_{ss} + B_s F_s) \subset \mathbb{C}^-, \quad (10)$$

and partition F_s in conformity with (8) and (9) as

$$F_s = \begin{bmatrix} F_{s0} \\ F_{s1} \end{bmatrix} = \begin{bmatrix} F_{a0}^+ & F_{b0} \\ F_{a1}^+ & F_{b1} \end{bmatrix}. \quad (11)$$

It follows from the property of the special coordinate basis that the pair (A_{ss}, B_s) is controllable provided that the pair (A, B) is stabilizable. Then, we further partition $F_{s1} = \begin{bmatrix} F_{a1}^+ & F_{b1} \end{bmatrix}$ as

$$F_{s1} = \begin{bmatrix} F_{a1}^+ & F_{b1} \end{bmatrix} = \begin{bmatrix} F_{a11}^+ & F_{b11} \\ F_{a12}^+ & F_{b12} \\ \vdots & \vdots \\ F_{a1m_d}^+ & F_{b1m_d} \end{bmatrix},$$

where F_{a1i}^+ and F_{b1i} are of dimensions $1 \times n_a^+$ and $1 \times n_b$, respectively.

Step 3 Let F_c be any arbitrary $m_c \times n_c$ matrix subject to the constraint that

$$A_{cc}^c = A_{cc} + B_c F_c \quad (12)$$

is a stable matrix. Note that the existence of such an F_c is guaranteed by the property that (A_{cc}, B_c) is controllable.

Step 4 This step makes use of the fast subsystems, $i = 1, 2, \ldots, m_d$, represented by (A_{dd}, B_d, C_d). Let

$$\Lambda_i = \{\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iq_i}\}, \quad i = 1, 2, \ldots, m_d,$$

be the sets of q_i elements, all in $\mathbb{C}^-$, which are closed under complex conjugation, where q_i and m_d are given in (6) and (7). Then, we let $\Lambda_d := \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{m_d}$. For $i = 1, 2, \ldots, m_d$, we define

$$p_i(s) := \prod_{j=1}^{q_i}(s - \lambda_{ij}) = s^{q_i} + F_{i1}s^{q_i-1} + \cdots + F_{iq_i-1}s + F_{iq_i}, \tag{13}$$

and a sub-gain matrix parameterized by tuning parameter, ϵ,

$$\tilde{F}_i(\epsilon) := \frac{1}{\epsilon^{q_i}}\left[F_{iq_i},\ \epsilon F_{iq_i-1},\ \ldots,\ \epsilon^{q_i-1}F_{i1}\right]. \tag{14}$$

Step 5 In this step, various gains calculated in Steps 2–4 are put together to form a composite state feedback gain for the given system Σ. Let

$$\begin{aligned}\tilde{F}^+_{a1}(\epsilon) :&= \begin{bmatrix} F^+_{a11}F_{1q_1}/\epsilon^{q_1} \\ F^+_{a12}F_{2q_2}/\epsilon^{q_2} \\ \vdots \\ F^+_{a1m_d}F_{m_dq_{m_d}}/\epsilon^{q_{m_d}} \end{bmatrix},\\ \tilde{F}_{b1}(\epsilon) :&= \begin{bmatrix} F_{b11}F_{1q_1}/\epsilon^{q_1} \\ F_{b12}F_{2q_2}/\epsilon^{q_2} \\ \vdots \\ F_{b1m_d}F_{m_dq_{m_d}}/\epsilon^{q_{m_d}} \end{bmatrix},\end{aligned} \tag{15}$$

and

$$\tilde{F}_{s1}(\epsilon) = \left[\tilde{F}^+_{a1}(\epsilon) \quad \tilde{F}_{b1}(\epsilon)\right].$$

Then, the ATEA state feedback gain is given by

$$F(\epsilon) = \Gamma_i\left(\tilde{F}(\epsilon) - \tilde{F}_0\right)\Gamma_s^{-1}, \tag{16}$$

where

$$\tilde{F}(\epsilon) = \begin{bmatrix} 0 & F^+_{a0} & F_{b0} & 0 & 0 \\ 0 & \tilde{F}^+_{a1}(\epsilon) & \tilde{F}_{b1}(\epsilon) & 0 & -\tilde{F}_d(\epsilon) \\ 0 & 0 & 0 & F_c & 0 \end{bmatrix},$$

$$\tilde{F}_0 = \begin{bmatrix} C^-_{0a} & C^+_{0a} & C_{0b} & C_{0c} & C_{0d} \\ E^-_{da} & E^+_{da} & E_{db} & E_{dc} & E_{dd} \\ E^-_{ca} & E^+_{ca} & 0 & 0 & 0 \end{bmatrix},$$

and where

$$\tilde{F}_d(\epsilon) = \mathrm{diag}\left\{\tilde{F}_1(\epsilon),\ \tilde{F}_2(\epsilon),\ \ldots,\ \tilde{F}_{m_d}(\epsilon)\right\}.$$

This completes the ATEA algorithm.

The following theorem, recapitulated from Chen (2000), captures some key properties of the closed-loop system under an ATEA based state feedback law.

Theorem 1: *Consider the given system Σ of* (1). *The ATEA state feedback law $u = F(\epsilon)x$ with $F(\epsilon)$ given by* (16) *has the following properties*:

1. *There exists a scalar $\epsilon^* > 0$ such that for every $\epsilon \in (0, \epsilon^*]$, the closed-loop system is asymptotically stable. Moreover, as $\epsilon \to 0$, the closed-loop poles are given by*

$$\lambda(A^-_{aa}), \quad \lambda(A^c_{cc}), \quad \lambda(A^c_{ss}) + 0(\epsilon), \quad \frac{\Lambda_d}{\epsilon} + 0(1).$$

There are a total number of n_d closed-loop poles, which have infinite negative real parts as $\epsilon \to 0$.

2. *Let*

$$C_s = \Gamma_o\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_b \end{bmatrix}, \quad D_s = \Gamma_o\begin{bmatrix} I_{m_0} & 0 \\ 0 & I_{m_d} \\ 0 & 0 \end{bmatrix}.$$

Then, we have

$$\begin{aligned} H(s,\epsilon) &:= [C + DF(\epsilon)][sI - A - BF(\epsilon)]^{-1} \\ &\to \begin{bmatrix} 0 & H_s(s) & 0 & 0 \end{bmatrix}\Gamma_s^{-1}, \end{aligned}$$

pointwise in s as $\epsilon \to 0$, where

$$H_s(s) = (C_s + D_sF_s)(sI - A_{ss} - B_sF_s)^{-1}.$$

2.2 *H_2 suboptimal control, H_∞ control and almost disturbance decoupling*

In what follows, we will demonstrate how, by appropriately choosing the sub-feedback gain matrix F_s in Step 2, the ATEA algorithm can be utilized to solve the H_2 and H_∞ suboptimal control problems as well as the almost disturbance decoupling problem.

To be specific, we consider a continuous-time system Σ with a state-space description

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = x, \\ h = Cx + Du, \end{cases} \tag{17}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance input, $y = x$ is the measurement output, and $h \in \mathbb{R}^p$ is the controlled output of Σ. We assume that (A, B) is stabilizable and (A, B, C, D) has no invariant zeros on the imaginary axis. Then, the standard optimization problem is to find a control law

$$u = Fx,$$

such that when it is applied to the given system (17), the resulting closed-loop system is internally stable, i.e., $\lambda(A+BF) \subset \mathbb{C}^-$, and a certain norm of the resulting closed-loop transfer function from the disturbance input w to the controlled output h, i.e.,

$$H_{hw}(s) = (C + DF)(sI - A - BF)^{-1}E,$$

is minimized. The optimization problems do not always possess a solution. A practical approach is to address the so-called suboptimal control problem, where the goal of control design is to meet a pre-specified norm requirement on the closed-loop transfer function. Let

$$\gamma_2^* := \inf \left\{ \|H_{hw}\|_2 \,\middle|\, u = Fx \text{ internally stabilizes } \Sigma \right\}.$$

Then, the H_2 suboptimal control problem with state feedback is, for any given $\gamma > \gamma_2^*$, to design a stabilizing feedback law $u = F(\gamma)x$, under which the H_2 norm of the closed-loop transfer function $H_{hw}(s)$ is less than or equal to γ.

Similarly, let

$$\gamma_\infty^* := \inf \left\{ \|H_{hw}\|_\infty \,\middle|\, u = Fx \text{ internally stabilizes } \Sigma \right\}.$$

Then, the H_∞ suboptimal control problem with state feedback is, for any given $\gamma > \gamma_\infty^*$, to design a stabilizing feedback law $u = F(\gamma)x$, under which the H_∞ norm of the closed-loop transfer function $H_{hw}(s)$ is less than or equal to γ.

Finally, the almost disturbance decoupling problem (either in H_2 sense or in H_∞ sense) is, for any a priori given arbitrarily small $\gamma > 0$, to find a stabilizing feedback control law $u = F(\gamma)x$ such that the H_2 or H_∞ norm of the closed-loop system transfer function $H_{hw}(s)$ is less than or equal to γ.

The following theorem summarizes the ATEA based solutions to the H_2 and H_∞ suboptimal control problems as well as the almost disturbance decoupling problem. In the theorem statement, we recall that Γ_s, Γ_i and Γ_o are the nonsingular state, input and output transformations that transform the matrix quadruple (A, B, C, D) into the special coordinate basis as in (2)–(5). Also, let

$$\tilde{E} := \Gamma_s^{-1} E = \begin{bmatrix} E_a^- \\ E_a^+ \\ E_b \\ E_c \\ E_d \end{bmatrix},$$

and

$$E_s := \begin{bmatrix} E_a^+ \\ E_b \end{bmatrix}.$$

Theorem 2: *Consider the continuous-time system Σ characterized by* (17). *The ATEA algorithm leads to the solution of the H_2 and H_∞ suboptimal control problems as well as the almost disturbance decoupling problem for Σ. More specifically, we have*

1. *If the sub-feedback gain matrix F_s in step 2 is chosen to be*

$$F_s = -(D_s^T D_s)^{-1}(B_s^T P_s + D_s^T C_s), \tag{18}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$P_s A_{ss} + A_{ss}^T P_s + C_s^T C_s - (P_s B_s + C_s^T D_s)(D_s^T D_s)^{-1} (B_s^T P_s + D_s^T C_s) = 0, \tag{19}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the following property:

$$\|H_{hw}\|_2 = \left\| [C + DF(\epsilon)][sI - A - BF(\epsilon)]^{-1} E \right\|_2 \to \gamma_2^*,$$

as $\epsilon \to 0$, i.e., the corresponding ATEA state feedback law solves the H_2 suboptimal control problem for Σ. Furthermore,

$$\gamma_2^* = \sqrt{\text{trace } (E_s^T P_s E_s)}.$$

2. *Given a scalar $\gamma > \gamma_\infty^* \geq 0$, if F_s in step 2 is chosen to be*

$$F_s = -(D_s^T D_s)^{-1}(B_s^T P_s + D_s^T C_s), \tag{20}$$

where $P_s > 0$ is a solution of the algebraic Riccati equation

$$P_s A_{ss} + A_{ss}^T P_s + C_s^T C_s + P_s E_s E_s^T P_s / \gamma^2 - (P_s B_s + C_s^T D_s)(D_s^T D_s)^{-1}(B_s^T P_s + D_s^T C_s) = 0, \tag{21}$$

then the resulting closed-loop transfer function from w to h under the corresponding ATEA state feedback law has the following property:

$$\|H_{hw}\|_\infty = \left\|[C + DF(\epsilon)][sI - A - BF(\epsilon)]^{-1}E\right\|_\infty < \gamma,$$

for sufficiently small ϵ, *i.e., the corresponding ATEA state feedback law solves the* H_∞ *suboptimal control problem for* Σ.

3. *If* $E_s = 0$, *which has been shown in Chen et al.* (2004) *to be the necessary and sufficient condition for the solvability of the almost disturbance decoupling problem for* Σ, *then the ATEA state feedback law with any arbitrarily chosen* F_s (*subject to the constraint on the stability of* A^c_{ss}) *has a resulting closed-loop transfer function* $H_{hw}(s, \epsilon)$ *with*

$$H_{hw}(s, \epsilon) \to 0, \quad \textit{pointwise in s as } \epsilon \to 0,$$

i.e., any ATEA state feedback control law solves the disturbance decoupling problem for Σ.

3. Software implementation of the ATEA algorithm

With the Symbolic Math Toolboxes on MATLAB, users can easily combine numeric and symbolic computation into a single environment. The Symbolic Toolbox defines a new MATLAB data type called symbolic object, by using the command `sym`, to represent a symbolic variable, expression, and matrix. Internally, a symbolic object is a data structure that stores a string representation of the symbol.

Symbolic computations not only improve the accuracy of the results, but also provide explicit expressions. With the aid of symbolic objects, computations need only be done once for a class controller. It is useful for both mathematical analysis and engineering online tuning (Chetty and Dabke 1999).

In the implementation of the ATEA algorithm, the state feedback gain of the H_2/H_∞ suboptimal control problems and almost disturbance decoupling problem are returned in term of a symbolic object `epsilon`, which relates to ϵ in the algorithm in § 2.1. Symbolic expression enables engineers to easily analyse which state feedback gain is sensitive to the choice of the time-scale. By tuning `epsilon` (using the symbolic substitution command `subs`) one can specify the appropriate time-scale, and thus obtain desirable feedback laws corresponding to different design methods.

As pointed out earlier, one of the key features of the ATEA algorithm is the ease in its symbolic implementation. It is clear from the description of the ATEA algorithm, the first three steps of ATEA algorithm involve only numeric operations, symbolic operations involving the tuning parameter ϵ (`epsilon`) are conducted only in step 4 and step 5.

The software implementation of the ATEA algorithm is a part of the beta version of *Linear Systems Toolkit* (Lin *et al*. 2004) that we recently released. This toolkit is available at `http://linearsystemskit.net`.

In this toolkit, four ATEA based design algorithms have been implemented. These are:

- the ATEA algorithm

$$\texttt{F} = \texttt{atea(A, B, C, D[, option])}$$

- the ATEA based H_2 suboptimal control design

$$\texttt{F} = \texttt{h2state(A, B, C, D, E[, option])}$$

- the ATEA based H_∞ suboptimal control design

$$\texttt{F} = \texttt{h8state(A, B, C, D, E, gamma[, option])}$$

- almost disturbance decoupling by ATEA based feedback law

$$\texttt{F} = \texttt{addps(A, B, C, D, E[, option])}$$

These functions could either produce the numerical values of the feedback gain matrix for a pre-specified value of the design parameter ϵ or return the gain matrix as a polynomial matrix in the design parameter $1/\epsilon$. One can use the `option` in the command line to choose the form of output. In the event of an omission of the `option` or a choice of `option` = 0, these functions will ask the user to enter a value for `epsilon` and return a numerical gain matrix. Otherwise, if `option` = 1, these functions will return the resulting matrix as a polynomial matrix in $1/\epsilon$ (i.e., `1/epsilon`). Among these four functions, `atea` is the core. Other three functions can be implemented by calling the `atea` function.

3.1 *Implementation of* `atea`

The flow chart of the function `atea` is showed in figure 1. The implementations of the components in the flow chart are carried out as follows.

3.1.1. SCB of (`A, B, C, D`). Find nonsingular state, input and output transformations to transform $\Sigma(A, B, C, D)$ into the SCB form, i.e., (2)–(5). The SCB algorithm is based on a numerically stable algorithm recently reported in Chu *et al*. 2002, together with an enhanced procedure reported in Chen *et al*. (2004).

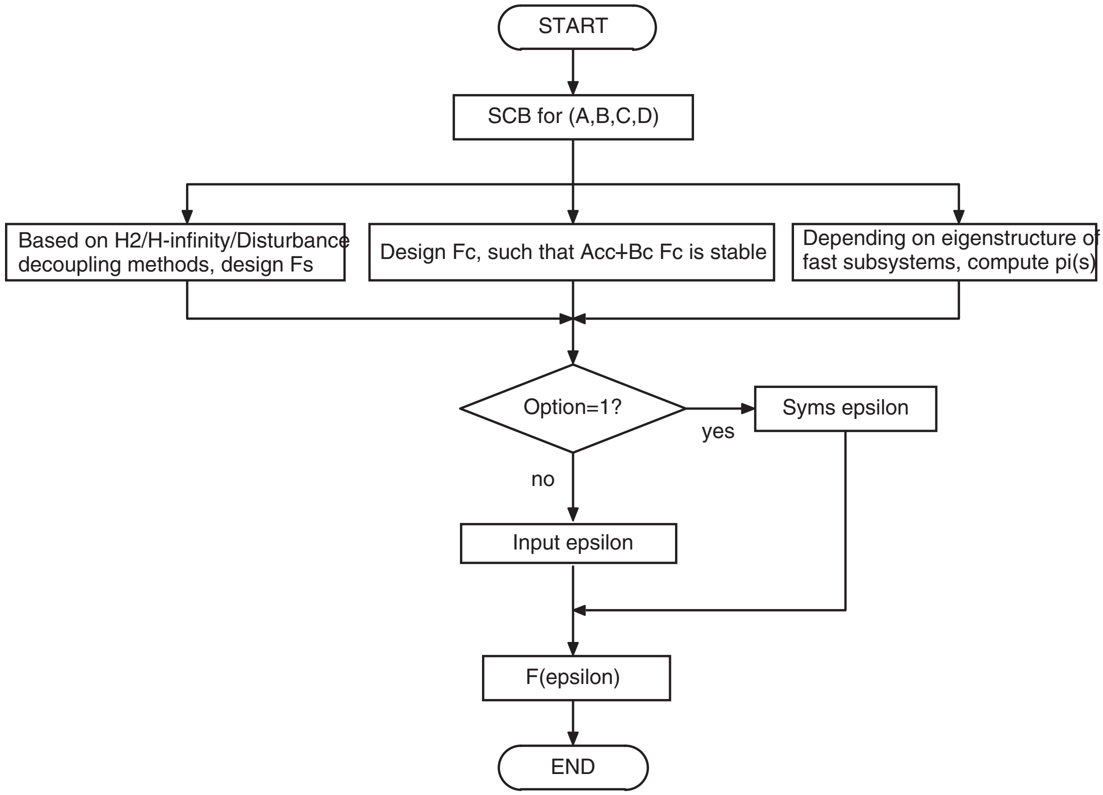

Figure 1. Program flow chart of `atea algorithm`.

The transformation is conducted by using the m-function, `scb`, in the Toolkit (Lin *et al*. 2004). The syntax is

$$[\mathtt{AA}, \mathtt{BB}, \mathtt{CC}, \mathtt{DD}, \mathtt{Gs}, \mathtt{Go}, \mathtt{Gi}, \mathtt{dims}, \mathtt{lv}, \mathtt{rv}, \mathtt{qv}, \mathtt{m0}] = \mathtt{scb}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{tol});$$

The output (`AA,BB,CC,DD`) corresponds to $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, `Gs, Go` and `Gi` are $\tilde{\Gamma}_s$, $\tilde{\Gamma}_o$ and $\tilde{\Gamma}_i$ respectively, and `qv` is the vector $\{m_1, m_2, \ldots, m_d\}$.

3.1.2. Computation of `Fs`. Define `Ass, Bs` as in (8) and (9), then compute `Fs` in (11) according to different design methods.

For the general ATEA design approach, if the user choose to input eigenvalue of `Ass+Bs*Fs`, the function `place` is used to compute an `Fs`. Otherwise, the code generates an `Fs` such that `Ass+Bs*Fs` is stable.

3.1.3. Computation of `Fc`. Choose an `Fc` such that `Acc+Bc*Fc` is stable. The user can input the desired poles of `Acc+Bc*Fc`. The m-function `place` is then called to find an `Fc`.

3.1.4. Assignment of eigenstructure of fast subsystems. Select the desired eigenvalues of fast subsystems in (10), and compute coefficients in $p_i(s)$ of (13).

3.1.5. Computation of state feedback gain. According to the value of `option`, decide whether to compute state feedback gain `Fepsilon` of (16) in the symbolic form or in the numeric form.

If `option=1`, construct a symbolic object to represent the tuning parameter ϵ, by using the command

$$\mathtt{epsilon} = \mathtt{sym}('\mathtt{epsilon}')$$
(or equivalently, `symsepsilon`).

`S = sym(A)` constructs an object `S`, of class 'sym', from A. If the input argument is a string, the result is a symbolic number or variable. If the input argument is a numeric scalar or matrix, the result is a symbolic representation of the given numeric values. `x = sym('x')` creates the symbolic variable with name `'x'` and stores the result in `x`.

Compute various gains $\tilde{F}_i(\epsilon)$, $\tilde{F}^{+}_{a1}(\epsilon)$ and $\tilde{F}_{b1}(\epsilon)$ in (14)–(15). Note that all of these gains are polynomial matrices in symbolic object (`1/epsilon`). Thus, the state feedback gain of (16) is a polynomial matrix in symbolic object (1/epsilon). The actual code of this part is given below,

```
if option==1
  syms epsilon
  tFd=sym([ ]);
  tFa1=sym(zeros(md,nap));
  tFb1=sym(zeros(md,nb));
  tF=sym([ ]);
  tF0=sym([ ]);
else
  disp(' ')
 epsilon=input('Enter the value of epsilon:
epsilon = ');
  tFd=[ ];
  tFa1=zeros(md,nap);
  tFb1=zeros(md,nb);
  tF=[ ];
  tF0=[ ];
end

for kk=1:md
  for j=1:qv(kk)
    tFi(kk,j)=Ft(kk,qv(kk)-j+1)/
      epsilon^(qv(kk)-j+1);
  end
end

%STEP ATEA-C.5
for kk=1:md
  if size(Fa1p,2)~=0
    tFa1(kk,:)=Fa1p(kk,:)*tFi(kk,1);
  end
  if size(Fb1,2)~=0
    tFb1(kk,:)= Fb1(kk,:)*tFi(kk,1);
  end
 tFd=blkdiag(tFd,tFi(kk,1:qv(kk)));
end
tFs1=[tFa1 tFb1];

if m0~=0
  tF=[zeros(m0,nan),Fs0,zeros(m0,nc+nd)];
  tF0=CC(1:m0,:);
end
if md~=0
  tF=[tF;zeros(md,nan),tFs1,zeros(md,nc),
    -tFd];
  tF0=[tF0;Bd'*AA(n-nd+1:n,:)];
end
if mc~=0
  tF=[tF;zeros(mc,nan+nap+nb),
      Fc,zeros(mc,nd)];
  tF0=[tF0;Bc'*AA(n-nc-nd+1:n-nd,
       1:nan+nap),zeros(mc,nb+nc+nd)];
end

Fepsilon=Gi*(tF-tF0)*inv(Gs);
dig=16;
Fepsilon=vpa(Fepsilon,dig);
```

The code returns a state feedback gain with tuning parameter epsilon.

If option $=0$, the user is asked to input a value for epsilon, then the code returns a numerical gain directly.

Remark 3.1: In current codes, we only set the tuning parameter ϵ(epsilon) as a symbolic object. In fact, to have more freedom in control design, F_s in (11), F_c in (12) and $F_{i1}, F_{i2}, \ldots, F_{iq_i}$ in (13) can also be set as symbolic objects. But in this case, the controller design will become much more complicated.

3.2 *Implementation of* h2state, h8state *and* addps

The only difference between the above three functions and atea is in the selection of Fs.

For the H_2 design approach (i.e., h2state), Fs is obtained by (18) through solving algebraic Riccati equation (19).

For the H_∞ design approach (i.e., h8state), Fs is obtained by (20) through solving algebraic Riccati equation (21).

For almost disturbance decoupling problem (i.e., addps), check the value of Es first. If Es $=0$, choose an Fs such that Ass+Bs*Fs is stable. Otherwise, the almost disturbance decoupling problem is not solvable.

After the gain matrix, in term of the tuning parameter ϵ, is returned, the user might use other functions in the Symbolic Math Toolbox to analyse the closed-loop system.

The function subs can be used to compute the gain in numerical form for a given value of epsilon. The command subs(S,new) replaces the default symbolic variable in S with the numerical value new. The command subs(S,old,new) replaces the symbolic variable old in the symbolic expression S with a symbolic or numeric variable or expression new. For example, the command subs(F,0.5) returns the feedback matrix $F(0.5)$.

In MATLAB, by default, the Symbolic Math Toolboxes uses variable precision floating point arithmetic with 32 decimal digit accuracy. Computation precision can be changed by using the function vpa (variable precision arithmetic) or digits. The command vpa(A) uses variable-precision arithmetic to compute each element of A to d decimal digits of accuracy, where d is the current setting of digits. Each element of the result is a symbolic expression. The command vpa(A,d) uses d digits, instead of the current setting of digits. The function vpa can also be used to display results in a compact form for ease in debugging the code.

The Symbolic Math Toolboxes also provides functions to create graphs from symbolic expressions. For example, `ezmesh(f,domain,n)` plots the symbolic function `f` over the specified `domain` divided by an `n`-by-`n` grid, where `domain` can be either a 4-by-1 vector `[xmin, xmax, ymin, ymax]` or a 2-by-1 vector `[min, max]`.

More details on the use of Symbolic Math Toolboxes can be found in The Math Work Inc. (2004).

4. Examples

Example 1: Consider a given system (17) with

$$A=\begin{bmatrix}0&0&-1&0&-1\\0&1&0&2&-1\\1&1&3&2&1\\0&0&1&0&1\\0&0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}-1&0\\0&0\\1&0\\0&0\\-1&1\end{bmatrix},$$

$$E=\begin{bmatrix}0&1\\0&0\\1&0\\0&1\\0&0\end{bmatrix},$$

and

$$C=\begin{bmatrix}0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix},\quad D=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}.$$

By using state, output and input transformations,

$$\Gamma_{\mathrm{s}}=\begin{bmatrix}1&0&-1&0&0\\0&1&0&-1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&-1&0&1\end{bmatrix},\quad \Gamma_{\mathrm{o}}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},$$

$$\Gamma_{\mathrm{i}}=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

the given system Σ is transformed into the form of the special coordinate basis

$$\tilde{A}=\begin{bmatrix}1&1&1&1&0\\0&1&1&1&0\\1&1&1&1&1\\0&0&0&0&1\\1&1&1&1&1\end{bmatrix},\quad \tilde{B}=\begin{bmatrix}0&0\\0&0\\1&0\\0&0\\0&1\end{bmatrix},$$

$$\tilde{E}=\begin{bmatrix}1&1\\0&1\\0&0\\1&0\\0&1\end{bmatrix},$$

$$\tilde{C}=\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\0&1&0&0&0\end{bmatrix},\quad \tilde{D}=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}.$$

It is left invertible and has one unstable invariant zeros at $s=1$ and two infinite zeros of orders 1 and 2, respectively. Moreover, we have

$$A_{\mathrm{ss}}=\begin{bmatrix}1&1\\0&1\end{bmatrix},\quad B_{\mathrm{s}}=\begin{bmatrix}1&1\\1&1\end{bmatrix},\quad E_{\mathrm{s}}=\begin{bmatrix}1&1\\0&1\end{bmatrix},$$

and

$$C_{\mathrm{s}}=\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\quad D_{\mathrm{s}}=\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}.$$

Since $E_{\mathrm{s}}\neq 0$, the disturbance decoupling problem for the given system is not solvable. We will thus focus on solving the H_2 and H_∞ suboptimal control problems for the system. Following the construction procedures of the ATEA algorithm in the previous section, we obtain a state feedback

$$F(\epsilon)=-\begin{bmatrix}\frac{F_{\mathrm{s}11}}{\epsilon}+1 & \frac{F_{\mathrm{s}12}}{\epsilon}+1 & \frac{F_{\mathrm{s}11}+1}{\epsilon}+3 & \frac{F_{\mathrm{s}12}}{\epsilon}+2 & 1\\ \frac{2F_{\mathrm{s}21}}{\epsilon^2}+1 & \frac{2F_{\mathrm{s}22}}{\epsilon^2}+1 & \frac{2F_{\mathrm{s}21}}{\epsilon^2}+\frac{2}{\epsilon}+3 & \frac{2F_{\mathrm{s}22}+2}{\epsilon^2}+2 & \frac{2}{\epsilon}+1\end{bmatrix},\tag{22}$$

where

$$F_{\mathrm{s}}=\begin{bmatrix}F_{\mathrm{s}11}&F_{\mathrm{s}12}\\F_{\mathrm{s}21}&F_{\mathrm{s}22}\end{bmatrix}\tag{23}$$

is to be selected to solve either the H_2 or H_∞ control problem. The closed-loop eigenvalues of $A+BF$ are asymptotically placed at $\lambda(A_{ss}+B_sF_s)$, $-1/\epsilon$ and $-1/\epsilon \pm j/\epsilon$, respectively.

1. *H_2 Control*. Solving the H_2 algebraic Riccati equation of (19), we get

$$P_s = \begin{bmatrix} 7.4641 & -4.7321 \\ -4.7321 & 4.3660 \end{bmatrix},$$

which gives a sub-feedback gain,

$$F_s = \begin{bmatrix} -2.7321 & 0.3660 \\ -2.7321 & 0.3660 \end{bmatrix},$$

and $\gamma_2^* = \sqrt{\mathrm{trace}\,(E_s' P_s E_s)} = 3.1353$. Thus, it follows from (22) and (23) that the H_2 suboptimal control law is given by $u = F(\epsilon)x$, with

$$F(\epsilon) = -\begin{bmatrix} \frac{2.7321}{\epsilon}+1 & -\frac{0.3660}{\epsilon}+1 & \frac{3.7321}{\epsilon}+3 & -\frac{0.3660}{\epsilon}+2 & 1 \\ \frac{5.4641}{\epsilon^2}+1 & \frac{-0.7321}{\epsilon^2}+1 & \frac{5.4641}{\epsilon^2}+\frac{2}{\epsilon}+3 & \frac{1.2679}{\epsilon^2}+2 & \frac{2}{\epsilon}+1 \end{bmatrix}.$$

The diary of the execution of the function `h2state` is shown below:

```
F = h2state(A,B,C,D,E,1);
```

```
This program will guide your through the step-
by-step procedure of the Asymptotic Time-scale
and Eigenstructure Assignment (ATEA) Design...

gamma_2_star =
   3.1353

Eigenstructure assignment for fast subsystems,
x_{d}, ......
   1). Specify your own structures; or
   2). Let me do it for you.

   Select your option (1 or 2): 1

Enter desired eigenvalues for each fast sub-
system. The actual closed-loop eigenvalues will
be placed at [the given eigenvalues/epsilon]...

   Fast Subsystem No: 1, q_1 = 1
   Enter 1 eigenvalues in row vector: -1
   Fast Subsystem No: 2, q_2 = 2
   Enter 2 eigenvalues in row vector: [-1+j -1-j]

   Fepsilon=vpa(F,5)
```

$$\mathtt{Fepsilon} = \begin{bmatrix} -\frac{2.7321}{\mathtt{epsilon}}-1 & \frac{0.36603}{\mathtt{epsilon}}-1 & -\frac{3.7321}{\mathtt{epsilon}}-3 & \frac{0.36603}{\mathtt{epsilon}}-2 & -1 \\ -\frac{5.4641}{\mathtt{epsilon}^2}-1 & \frac{0.73205}{\mathtt{epsilon}^2}-1 & -\frac{5.4641}{\mathtt{epsilon}^2}-3-\frac{2}{\mathtt{epsilon}} & -\frac{1.2679}{\mathtt{epsilon}^2}-2 & -\frac{2}{\mathtt{epsilon}}-1 \end{bmatrix}$$

```
f=subs(F,0.1)
```

$$\mathtt{f} = \begin{matrix} -28.3205 & 2.6603 & -40.3205 & 1.6603 & -1.0000 \\ -547.4102 & 72.2051 & -569.4102 & -128.7949 & -21.0000 \end{matrix}$$

Figure 2 shows the values of the H_2-norm of the resulting closed-loop system versus ϵ. Clearly, it shows that the H_2-norm of the resulting closed-loop system tends to γ_2^* as $\epsilon \to 0$.

2. *H_∞ Control*. It follows from Chen (2002) that

$$\gamma_\infty^* = 2.0090,$$

and for any $\gamma > \gamma_\infty^*$, we can find the sub-feedback gain F_s. For example, let $\gamma_\infty = 3$,

$$F_s = \begin{bmatrix} -5.0036 & 1.7210 \\ -5.0036 & 1.7210 \end{bmatrix},$$

thus,

$$F(\epsilon) = -\begin{bmatrix} \frac{5.0036}{\epsilon}+1 & -\frac{1.7210}{\epsilon}+1 & \frac{6.0036}{\epsilon}+3 & -\frac{1.7210}{\epsilon}+2 & 1 \\ \frac{10.0073}{\epsilon^2}+1 & -\frac{3.4419}{\epsilon^2}+1 & \frac{10.0073}{\epsilon^2}+\frac{2}{\epsilon}+3 & -\frac{1.4419}{\epsilon^2}+2 & \frac{2}{\epsilon}+1 \end{bmatrix}$$

is an H_∞ γ-suboptimal controller for sufficiently small ϵ. For illustration, we plot the maximum singular values of the transfer function of the resulting closed-loop system for a few different pairs of γ and ϵ in figure 3. The results indeed confirm our claim.

The diary of the execution of the function `h8state` is shown below.

```
F = h8state(A,B,C,D,E,0.5,1);
```

```
This program will guide your through the step-
by-step procedure of the Asymptotic Time-scale
and Eigenstructure Assignment (ATEA) Design ...

gm8_star =
  2.0090
gamma =
  0.5000

Enter the value of gamma, which has to be larger
than gm8_star; gamma = 3

Eigenstructure assignment for fast subsystems,
x_{d}, ......
  1). Specify your own structures; or
```

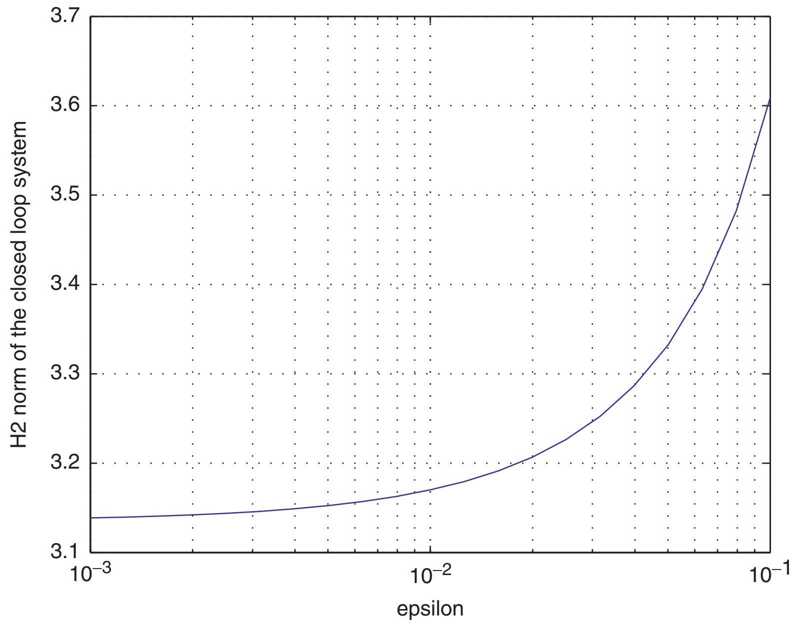

Figure 2. The H_2-norm of the closed-loop system transfer function.

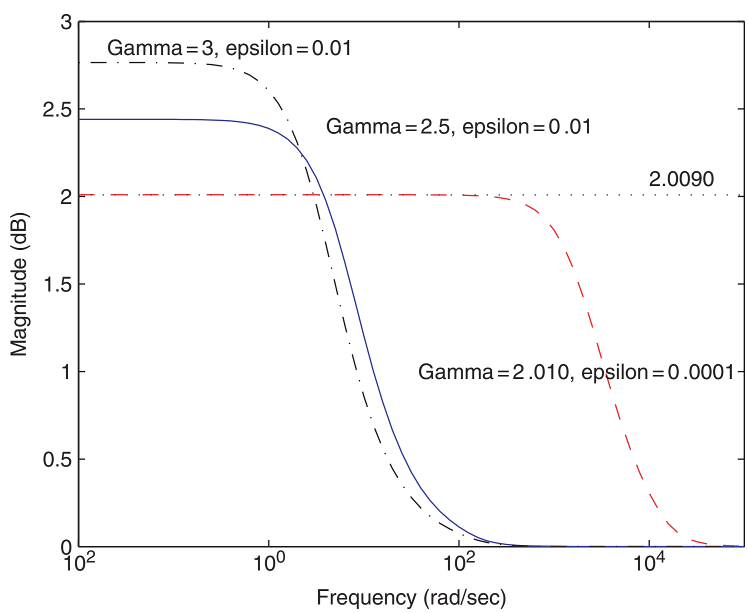

Figure 3. The maximum singular values of the closed-loop system transfer function.

```
  2). Let me do it for you.

Select your option (1 or 2): 1
Enter desired eigenvalues for each fast sub-
system. The actual closed-loop eigenvalues will
be placed at [ the given eigenvalues/epsilon ]...

Fast Subsystem No: 1, q_1 = 1
Enter 1 eigenvalues in row vector: -1
Fast Subsystem No: 2, q_2 = 2
Enter 2 eigenvalues in row vector: [-1+j -1-j]

Fepsilon=vpa(F,5)
```

$$\mathrm{Fepsilon} = \begin{bmatrix} -\frac{5.0036}{\mathrm{epsilon}} - 1 & \frac{1.7210}{\mathrm{epsilon}} - 1 & -\frac{6.0036}{\mathrm{epsilon}} - 3 & \frac{1.7210}{\mathrm{epsilon}} - 2 & -1 \\ -\frac{10.007}{\mathrm{epsilon}^2} - 1 & \frac{3.4419}{\mathrm{epsilon}^2} - 1 & -\frac{10.007}{\mathrm{epsilon}^2} - 3 - \frac{2}{\mathrm{epsilon}} & \frac{1.4419}{\mathrm{epsilon}^2} - 2 & -\frac{2}{\mathrm{epsilon}} - 1 \end{bmatrix}$$

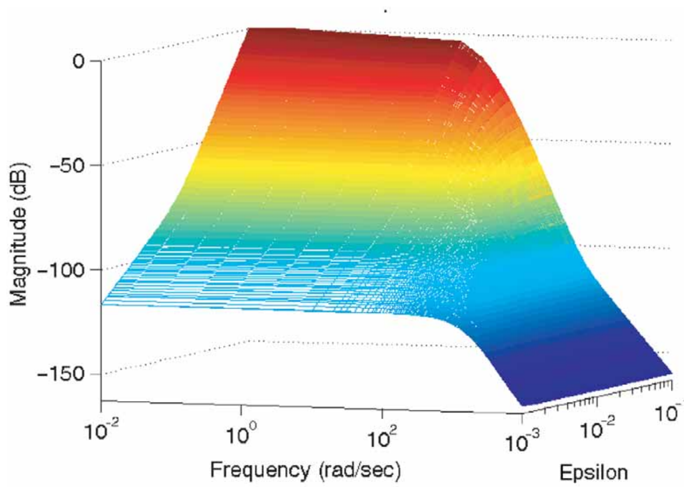

Figure 4. The maximum singular values of the closed-loop system transfer function.

Example 2: We revisit the state feedback design for a piezoelectric bimorph actuator (Chen 2000). The actuator is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -k/m & -b/m & -k/m & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ k(d-k_1)/m \\ k_1 k_2 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ -k/m & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = 0,$$

with $m = 0.01595$ kg, $b = 1.169$ Ns/m, $k = 4385$ N/m, $d = 8.209 \times 10^{-7}$ m/V, $k_1 = 3.5382 \times 10^{-7}$, $k_2 = -0.9597$. The input u is the voltage that generates excitation forces to the actuator system. The output to be controlled y is the displacement of the actuator. The working range of the displacement of this actuator is within $\pm 1\,\mu$m. Our objective is to design a feedback controller that meets the following specifications:

- The steady state tracking errors of the displacement is less than 1% for any input reference signal with a frequency range of 0 to 30 Hz, and
- The control input signal u does not exceed 112.5 volts because of the physical limitations on the piezoelectric materials.

The special coordinate basis of (A, B, C, D) is the following,

$$\tilde{A} = \begin{bmatrix} -0.96385 & -3.8585\times10^{-3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2.7492\times10^{5} & -1.1006\times10^{3} & 1.1418\times10^{3} & -2.7492\times10^{5} & -73.287 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{D} = 0.$$

It is obvious that the system (A, B, C, D) is invertible and of minimum phase with one invariant zero at -0.96385.

It also has one infinite zero of order 4. Thus, E_s is empty. Following Theorem 2, the disturbance decoupling problem for the actuator is solvable.

Let $\Lambda_1 = \{-1, -2, -3, -4\}$. With the aid of Symbolic Math Toolboxes, we obtain the state feedback gain as following,

$$F(\epsilon) = \left[1.2234 \times 10^5 - \frac{15.5756}{\epsilon^2} \quad 32.6160 - \frac{4.4502}{\epsilon} \quad 1.2234 \times 10^5 \quad -\frac{22.2509}{\epsilon^3} \quad -\frac{10.6804}{\epsilon^4}\right],$$

where ϵ is the tuning parameter that can be adjusted to achieve disturbance decoupling. Figure 4 shows that $H_{hw}(s, \epsilon)$ does indeed approach zero pointwise in s as ϵ goes to zero.

Because the feedback controller is explicitly parameterized in a tuning parameter ϵ, it can be easily adjusted to meet other design specifications without repeating the design process.

By tuning the parameter ϵ and simulating the overall design, we found that the maximum peak values of the control signal u are independent of the frequencies of the reference signals. They are only dependent on the initial error between the displacement y and the reference. Let us consider the worst case, i.e., the magnitude of the initial error is 1μm, we are able to obtain a clear relationship between the tuning parameter ϵ and the maximum peak of u. We also found that the tracking error is independent of initial errors. It only depends on the frequency of the reference signal, the larger the frequency, the larger the tracking error. Again, we obtain a simple and linear relationship between the tuning parameter ϵ and the maximum frequency that a reference signal such that the corresponding tracking error is no larger than 1%. With these relationships, we can obtain a tuning parameter ϵ to meet both the two control specifications. The interested reader is referred to (Ozcetin *et al.* 1993a, b, Chen 2000, Lin and Chen 2000) for detail.

The diary of the execution of the function addps we discussed above is shown below:

```
F = addps(A, B, C, D, E, 1); Fepsilon = vpa(F, 5)
Hs = (C + D * F) * inv(j * c * eye(5) - A - B * F) * E;
Hs8 = 10 * log10((abs(Hs(1)) * abs(Hs(1))
      + abs(Hs(2)) * abs(Hs(2))));
ezmesh(Hs8, [0.01, 10000, 0.0010.1], 100);
```

```
This program will guide your through the step-by-step procedure of the Asymptotic Time-scale and Eigenstructure Assignment (ATEA) Design ...
Eigenstructure assignment for fast subsystems, x_{d}, ......
 1). Specify your own structures; or
 2). Let me do it for you.
Select your option (1 or 2): 1

Enter desired eigenvalues for each fast subsystem. The actual closed-loop eigenvalues will be placed at [ the given eigenvalues / epsilon ] ...

Fast Subsystem No: 1, q_1 = 4
Enter 4 eigenvalues in row vector: [-1 -2 -3 -4]
```

$$\texttt{Fepsilon} = \left[1.12234e6 - \frac{15.576}{\texttt{epsilon}^2} \quad 32.616 - \frac{4.4502}{\texttt{epsilon}} \quad 0.12234e6 \quad -\frac{22.251}{\texttt{epsilon}^3} \quad \frac{10.680}{\texttt{epsilon}^4}\right]$$

5. Conclusions

In this paper, we have presented the ATEA algorithm and shown how the algorithm itself enables a straightforward symbolic computation of the resulting feedback gain matrix as a polynomial matrix in the design parameter. Two examples are given to demonstrate how the ATEA algorithm works and how the symbolic implementation of the ATEA algorithm leads to results accurately and efficiently.

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