Composite Nonlinear Feedback Control for Linear Systems With Input Saturation: Theory and an Application

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***Abstract*—We study in this paper the theory and applications of a nonlinear control technique, i.e., the so-called composite nonlinear feedback control, for a class of linear systems with actuator nonlinearities. It consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part. It is shown that the proposed technique is capable of beating the well-known time-optimal control in the asymptotic tracking situations. The application of such a new technique to an actual hard disk drive servo system shows that it outperforms the conventional method by more than 30%. The technique can be applied to design servo systems that deal with "point-and-shoot" fast targeting.**

***Index Terms*—Actuator saturation, control applications, hard disk drives, nonlinear control, servo systems.**

I. Introduction

Every physical system in our life has nonlinearities and very little can be done to overcome them. Many practical systems are sufficiently nonlinear so that important features of their performance may be completely overlooked if they are analyzed and designed through linear techniques (see, e.g., [12]). For example, in the computer hard disk drive (HDD) servo systems, major nonlinearities are friction, high frequency mechanical resonance and actuator saturation nonlinearities. Among all these, the actuator saturation could be the most significant nonlinearity in designing an HDD servo system. When the actuator is saturated, the performance of the control system designed will seriously deteriorate.

Traditionally, when dealing with "point-and-shoot" fast-targeting for systems with actuator saturation, one would naturally think of using the well known time optimal control (TOC) (known also as the bang-bang control), which uses maximum acceleration and maximum deceleration for a predetermined time period. Unfortunately, it is well known that the classical TOC is not robust with respect to the system uncertainties and measurement noises. It can hardly be used in any real situation. As such, Workman [21] proposed a modification of this technique, i.e., the so-called proximate time-optimal servomechanism (PTOS), to overcome such a drawback. The PTOS essentially uses maximum acceleration where it is practical to do so. When the error is small, it switches to a linear control law. The overall performance, i.e., the tracking time, is thus discounted. However, it is fairly robust with respect to system uncertainties and noises.

TOC is surely time-optimal for a point-to-point target tracking. However, in most practical situations, it is more appropriate to consider asymptotic tracking instead, i.e., to track the system within a certain neighborhood of the target reference before the system output essentially settles down to the desired point. We will show later by a simple example that the TOC is not time-optimal at all in the asymptotic tracking situation. This observation motivates us to search for a better technique. Inspired by a recent work of Lin *et al.* [17], which was introduced to improve the tracking performance under state feedback laws for a class of second order systems subject to actuator saturation, we have developed in this paper a nonlinear control technique, the so-called composite nonlinear feedback (CNF) control, to a more general class of systems with measurement feedback.

Since the initiation of CNF in [17] for second order systems, there has been efforts to generalize it to more general systems. For example, Turner *et al.* [19] extended the results of [17] to higher order and multiple input systems. This extension was made under a restrictive assumption on the system that excludes many systems including those originally considered in [17]. The restrictiveness of the assumption of [19] will be discussed later in details. Also, as in [17], only state feedback is considered in [19].

The CNF control consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time not exceeding the actuator limits for the desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part.

We will show by an example that such a technique could yield a better performance compared to that of the time-optimal control in asymptotic tracking. It is noted that the new control scheme can be utilized to design servo systems that deal with asymptotic target tracking or "point-and-shoot" fast targeting. In

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this paper, we will apply the technique to design a servo system for a hard disk drive. Actual implementation results will be presented and compared with those obtained from the conventional approach. Again, one will see that there is a big improvement in the new design.

The paper is organized as follows. In Section II, the theory of the composite nonlinear feedback control is developed. Three different cases, i.e., the state feedback, the full order measurement feedback, and the reduced-order measurement cases, are considered with all detailed derivations and proofs. In Section III, we show by an example that the proposed CNF control could yield a better performance compared to that of the time-optimal control. The application of the CNF technique to an actual HDD servo system will be presented in Section IV. Both simulation and implementation results will also be given and compared with those of the conventional PTOS approach. The results show that the CNF control improves the performance by more than 30%. Finally, we draw some concluding remarks and open problems in Section V.

II. Composite Nonlinear Feedback Control

We present in this section the CNF control technique for the following three different situations: 1) the state feedback case, 2) the full order measurement feedback case, and 3) the reduced-order measurement feedback case. We will present rigorous and complete proofs for all results derived. More specifically, we consider a linear system Σ with an amplitude-constrained actuator characterized by

$$\begin{cases} \dot{x} = Ax + B\,\mathrm{sat}(u), \quad x(0) = x_0 \\ y = C_1 x \\ h = C_2 x \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}^p$ and $h \in \mathbb{R}$ are, respectively, the state, control input, measurement output and controlled output of Σ. A, B, C_1 and C_2 are appropriate dimensional constant matrices, and sat: $\mathbb{R} \to \mathbb{R}$ represents the actuator saturation defined as

$$\mathrm{sat}(u) = \mathrm{sgn}(u) \min\{u_{\max}, |u|\} \tag{2}$$

with $u_{\max}$ being the saturation level of the input. The following assumptions on the system matrices are required:

1) (A, B) is stabilizable;
2) (A, C_1) is detectable;
3) (A, B, C_2) is invertible and has no zeros at $s = 0$.

The objective here is to design a CNF control law that will cause the output to track a step input rapidly without experiencing large overshoot and without the adverse actuator saturation effects. This will be done through the design of a linear feedback law with a small closed-loop damping ratio and a nonlinear feedback law through an appropriate Lyapunov function to cause the closed-loop system to be highly damped as system output approaches the command input to reduce the overshoot. As mentioned earlier, we separate the CNF controller design into three distinct situations: 1) the state feedback case, 2) the full-order measurement feedback case, and 3) the reduced-order measurement feedback case.

A. State Feedback Case

In this section, we follow the idea of the work of Lin *et al.* [17] to develop a composite nonlinear feedback control technique for the case when all the states of the plant Σ are measurable, i.e., $y = x$. We have the following step-by-step design procedure.

Step S.1: Design a linear feedback law

$$u_L = Fx + Gr \tag{3}$$

where r is a step command input and F is chosen such that 1) $A + BF$ is an asymptotically stable matrix, and 2) the closed-loop system $C_2(sI - A - BF)^{-1}B$ has certain desired properties, e.g., having a small damping ratio. We note that such an F can be designed using methods such as the H_2 and H_∞ optimization approaches, as well as the robust and perfect tracking technique given in [2]. Furthermore, G is a scalar and is given by

$$G = -\left[C_2(A + BF)^{-1}B\right]^{-1} \tag{4}$$

and r is a step command input. Here, we note that G is well defined because $A + BF$ is stable, and the triple (A, B, C_2) is invertible and has no invariant zeros at $s = 0$.

Step S.2: Next, we compute

$$H := \left[1 - F(A + BF)^{-1}B\right] G \tag{5}$$

and

$$x_e := G_e r := -(A + BF)^{-1}BGr. \tag{6}$$

Note that the definitions of H, G_e and x_e would become transparent later in our derivation in (12) and (13). Given a positive–definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$(A + BF)'P + P(A + BF) = -W \tag{7}$$

for $P > 0$. Note that such a P exists since $A + BF$ is asymptotically stable. Then, the nonlinear feedback control law $u_N(t)$ is given by

$$u_N = \rho(r, y)B'P(x - x_e) \tag{8}$$

where $\rho(r, y)$ is any nonpositive function locally Lipschitz in y, which is used to change the system closed-loop damping ratio as the output approaches the step command input. The choices of W and $\rho(r, y)$ will be discussed later.

Step S.3:The linear and nonlinear feedback laws derived in the previous steps are now combined to form a CNF controller

$$u = u_L + u_N = Fx + Gr + \rho(r, y)B'P(x - x_e). \tag{9}$$

The following theorem shows that the closed-loop system comprising the given plant in (1) with $y = x$ and the CNF control law of (9) is asymptotically stable. It also determines the magnitude of r that can be tracked by such a control law without exceeding the control limit.

Theorem 1: Consider the given system in (1), the linear control law of (3) and the composite nonlinear feedback control law of (9). For any $\delta \in (0, 1)$, let $c_\delta > 0$ be the largest positive scalar satisfying the following condition:

$$|Fx| \le u_{\max}(1 - \delta) \qquad \forall\, x \in \boldsymbol{X}_\delta := \left\{x: x'Px \le c_\delta\right\}. \tag{10}$$

Then, the linear control law of (3) is capable of driving the system controlled output $h(t)$ to track asymptotically a step command input r, provided that the initial state x_0 and r satisfy

$$\tilde{x}_0 := (x_0 - x_e) \in \boldsymbol{X}_\delta, |Hr| \leq \delta u_{\max}. \tag{11}$$

Furthermore, for any nonpositive function $\rho(r, y)$, locally Lipschitz in y, the composite nonlinear feedback law in (9) is capable of driving the system controlled output $h(t)$ to track asymptotically the step command input of amplitude r, provided that the initial state x_0 and r satisfy (11).

Proof: Let $\tilde{x} = x - x_e$. It is simple to verify that the linear control law of (3) can be rewritten as

$$\begin{aligned} u_L(t) =& F\tilde{x}(t) + \left[1 - F(A+BF)^{-1}B\right] Gr \\ =& F\tilde{x}(t) + Hr. \end{aligned}$$

Hence, for all $\tilde{x} \in \boldsymbol{X}_\delta$ and, provided that $|Hr| \leq \delta u_{\max}$, $|F\tilde{x} + Hr| \leq u_{\max}$ and the closed-loop system is linear and is given by

$$\dot{\tilde{x}} = (A+BF)\tilde{x} + Ax_e + BHr. \tag{12}$$

Noting that

$$\begin{aligned} Ax_e + BHr =& \left\{B[1 - F(A+BF)^{-1}B]G \right. \\ & \left. - A(A+BF)^{-1}BG\right\} r \\ =& \left\{[I - BF(A+BF)^{-1}]BG \right. \\ & \left. - A(A+BF)^{-1}BG\right\} r \\ =& \left\{I - BF(A+BF)^{-1} \right. \\ & \left. - A(A+BF)^{-1}\right\} BGr \\ =& 0 \end{aligned} \tag{13}$$

the closed-loop system in (12) can then be simplified as

$$\dot{\tilde{x}} = (A+BF)\tilde{x}. \tag{14}$$

Similarly, the closed-loop system comprising the given plant in (1) and the CNF control law of (9) can be expressed as

$$\dot{\tilde{x}} = (A+BF)\tilde{x} + Bw \tag{15}$$

where

$$w = \operatorname{sat}(F\tilde{x} + Hr + u_N) - F\tilde{x} - Hr. \tag{16}$$

Clearly, for the given x_0 satisfying (11), we have $\tilde{x}_0 = (x_0 - x_e) \in \boldsymbol{X}_\delta$. We note that (15) is reduced to (14) if $\rho = 0$. Thus, we can prove the results, respectively, under the linear control and the composite nonlinear feedback control in one shot. The rest of the proof follows pretty closely to those for the second order systems given in [17].

Next, we define a Lyapunov function $V = \tilde{x}'P\tilde{x}$, and evaluate the derivative of V along the trajectories of the closed-loop system in (15), i.e.,

$$\begin{aligned} \dot{V} =& \dot{\tilde{x}}'P\tilde{x} + \tilde{x}'P\dot{\tilde{x}} \\ =& \tilde{x}'(A+BF)'P\tilde{x} + \tilde{x}'P(A+BF)\tilde{x} + 2\tilde{x}'PBw \\ =& -\tilde{x}'W\tilde{x} + 2\tilde{x}'PBw. \end{aligned} \tag{17}$$

Note that for all

$$\tilde{x} \in \boldsymbol{X}_\delta = \{\tilde{x}: \tilde{x}'P\tilde{x} \leq c_\delta\} \Rightarrow |F\tilde{x}| \leq u_{\max}(1-\delta). \tag{18}$$

We next calculate $\dot{V}$ for three different values of saturation function.

Case 1) If $|F\tilde{x} + Hr + u_N| \leq u_{\max}$, then $w = u_N = \rho B'P\tilde{x}$ and, thus

$$\dot{V} = -\tilde{x}'W\tilde{x} + 2\rho\tilde{x}'PBB'P\tilde{x} \leq -\tilde{x}'W\tilde{x}. \tag{19}$$

Case 2) If $F\tilde{x} + Hr + u_N > u_{\max}$, and by construction $|F\tilde{x} + Hr| \leq u_{\max}$, we have

$$0 < w = u_{\max} - F\tilde{x} - Hr < u_N = \rho B'P\tilde{x} \tag{20}$$

which implies that $\tilde{x}'PB < 0$ and hence $\dot{V} = -\tilde{x}'W\tilde{x} + 2\tilde{x}'PBw \leq -\tilde{x}'W\tilde{x}$.

Case 3) Finally, if $F\tilde{x} + Hr + u_N < -u_{\max}$, we have

$$\rho B'P\tilde{x} = u_N < w = -u_{\max} - F\tilde{x} - Hr < 0 \tag{21}$$

implying $\tilde{x}'PB > 0$ and hence $\dot{V} \leq -\tilde{x}'W\tilde{x}$.

In conclusion, we have shown that

$$\dot{V} \leq -\tilde{x}'W\tilde{x} \quad \tilde{x} \in \boldsymbol{X}_\delta \tag{22}$$

which implies that $\boldsymbol{X}_\delta$ is an invariant set of the closed-loop system in (15). Noting that $W > 0$, all trajectories of (15) starting from inside $\boldsymbol{X}_\delta$ will converge to the origin. This, in turn, indicates that, for all initial states x_0 and the step command input of amplitude r that satisfy (11)

$$\lim_{t\to\infty} x(t) = x_e. \tag{23}$$

Therefore

$$\lim_{t\to\infty} h(t) = C_2 x_e = -C_2(A+BF)^{-1}BGr = r. \tag{24}$$

This completes the proof of Theorem 1.

The following remarks are in order.

Remark 1: Theorem 1 shows that the additional nonlinear feedback control law u_N, as given by (8), does not affect the ability of the closed-loop system to track the command input. Any command input that can be asymptotically tracked by the linear feedback law of (3) can also be asymptotically tracked by the CNF control law in (9). However, this additional term u_N in the CNF control law can be used to improve the performance of the overall closed-loop system. This is the key property of the CNF control technique.

Remark 2: Note that for the case when $x_0 = 0$, any step command of amplitude r can be asymptotically tracked, provided that

$$|r| \leq \left[c_\delta (G_e'PG_e)^{-1}\right]^{1/2}, \qquad |Hr| \leq \delta \cdot u_{\max}. \tag{25}$$

Note that c_δ is the parameter given earlier in the definition of $\boldsymbol{X}_\delta$ in (10). Clearly, the trackable amplitudes of reference inputs by the linear feedback control law can be increased by increasing δ and/or decreasing $G_e'PG_e$ through the choice of W.

Lastly, we note that Turner *et al.* [19] have recently extended the idea of [17] to systems with multiple control inputs and multiple controlled outputs. Again, their result is only applicable to the state feedback case. Assuming that the dynamic equation of the given system is transformed into the following form:

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} \operatorname{sat}(u) \tag{26}$$

where $\bar{B}$ is nonsingular, Turner *et al.* [19] have solved the problem under a rather strange condition, i.e., A_{11} is nonsingular. Such a condition cannot be guaranteed for a simple double-integrator system considered later in Section III

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{sat}(u). \tag{27}$$

In order to overcome such a difficulty, the authors then suggest to perturb the elements of A_{11} to ensure nonsingularity. We note that such a perturbation will not only introduce numerical instability to the problem, but also produce high gain in the control input and bias in the steady–state. It is our belief that the nonsingularity of A_{11} is not necessary.

B. Full-Order Measurement Feedback Case

The assumption that all the states of Σ are measurable is, in general, not practical. For example, in HDD servo systems that we are going to study in Section IV, the velocity of the actuator is generally not measurable. Thus, it is important to develop a technique that uses only measurement information. In what follows, we proceed to develop a CNF control system design using only measurement feedback. We first focus on the full order measurement feedback case, in which the dynamical order of the controller is equal to the order of the given system.

Step F.1: We first construct a linear full order measurement feedback control law:

$$\Sigma_F : \begin{cases} \dot{x}_v = (A + KC_1)x_v - Ky + B\mathrm{sat}(u_L) \\ u_L = F(x_v - x_e) + Hr \end{cases} \tag{28}$$

where r is the reference input and $x_v \in \mathbb{R}^n$ is the state of the controller. As usual, F and K are gain matrices and are designed such that $A+BF$ and $A+KC_1$ are asymptotically stable and the resulting closed-loop system has the desired properties. Finally, G, H and x_e are as defined in (4)–(6).

Step F.2: Given a positive–definite matrix $W \in \mathbb{R}^{n\times n}$, solve the Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W \tag{29}$$

for $P > 0$. As in the state feedback case, the linear control law of (28) obtained in the above step is to be combined with a nonlinear control law to form the following CNF controller:

$$\begin{cases} \dot{x}_v = (A + KC_1)x_v - Ky + B\mathrm{sat}(u) \\ u = F(x_v - x_e) + Hr + \rho(r,y)B'P(x_v - x_e) \end{cases} \tag{30}$$

where $\rho(r,y)$ is a nonpositive scalar function, locally Lipschitz in y, and is to be chosen to improve the performance of the closed-loop system.

It turns out that, for the measurement feedback case, the choice of $\rho(r,y)$, the nonpositive scalar function, is not totally free. It is subject to certain constraints. We have the following theorem.

Theorem 2: Consider the given system in (1), the linear measurement feedback control law of (28), and the composite nonlinear measurement feedback control law of (30). Given a positive–definite matrix $W_Q \in \mathbb{R}^{n\times n}$ with

$$W_Q > F'B'PW^{-1}PBF \tag{31}$$

let $Q > 0$ be the solution to the Lyapunov equation

$$(A + KC_1)'Q + Q(A + KC_1) = -W_Q. \tag{32}$$

Note that such a Q exists as $A+KC_1$ are asymptotically stable, and, for any $\delta \in (0,1)$, let c_δ be the largest positive scalar such that for all

$$\begin{pmatrix} x \\ x_v \end{pmatrix} \in \boldsymbol{X}_{F\delta} := \left\{ \begin{pmatrix} x \\ x_v \end{pmatrix} : \begin{pmatrix} x \\ x_v \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} x \\ x_v \end{pmatrix} \le c_\delta \right\} \tag{33}$$

the following property holds:

$$\left| [F \quad F] \begin{pmatrix} x \\ x_v \end{pmatrix} \right| \le u_{\max}(1-\delta). \tag{34}$$

Then, the linear measurement feedback control law in (28) will drive the system controlled output $h(t)$ to track asymptotically a step command input of amplitude r from an initial state x_0, provided that x_0, $x_{v0} = x_v(0)$ and r satisfy

$$|Hr| \le \delta \cdot u_{\max}, \begin{pmatrix} x_0 - x_e \\ x_{v0} - x_0 \end{pmatrix} \in \boldsymbol{X}_{F\delta}. \tag{35}$$

Furthermore, there exists a scalar $\rho^* > 0$ such that for any nonpositive function $\rho(r,y)$, locally Lipschitz in y and $|\rho(r,y)| \le \rho^*$, the CNF control law of (30) will drive the system controlled output $h(t)$ to track asymptotically the step command input of amplitude r from an initial state x_0, provided that x_0, x_{v0} and r satisfy (35).

Proof: For simplicity, we drop r and h in $\rho(r,y)$ throughout the proof of this theorem. Let $\tilde{x} = x - x_e$ and $\tilde{x}_v = x_v - x$. The linear control law of (28) can be written as

$$\dot{\tilde{x}}_v = (A + KC_1)\tilde{x}_v \quad u_L = [F \quad F] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr. \tag{36}$$

Hence, for all states

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \in \boldsymbol{X}_{F\delta} \Rightarrow \left| [F \quad F] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| \le u_{\max}(1-\delta) \tag{37}$$

and for any r satisfying

$$|Hr| \le \delta \cdot u_{\max} \tag{38}$$

we have

$$\begin{aligned} |u_L| &= \left| [F \quad F] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr \right| \\ &\le \left| [F \quad F] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} \right| + |Hr| = u_{\max}. \end{aligned} \tag{39}$$

Thus, for all $\tilde{x}$ and $\tilde{x}_v$ satisfying the condition as given in (37), the closed-loop system comprising the given plant and the linear control law of (28) can be rewritten as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix}. \tag{40}$$

Similarly, the closed-loop system with the CNF control law of (30) can be expressed as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + KC_1 \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{41}$$

where

$$w = \text{sat}\left[[F \quad F]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + Hr + \rho\,[B'P \quad B'P]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}\right] - [F \quad F]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} - Hr. \quad (42)$$

Clearly, for the given x_0 and x_{v0} satisfying (35), we have

$$\begin{pmatrix}\tilde{x}(0)\\ \tilde{x}_v(0)\end{pmatrix} \in \boldsymbol{X}_{F\delta}. \quad (43)$$

We note that (40) and (41) are identical when $\rho = 0$. Again, the results of Theorem 2 for both the linear and the nonlinear feedback case can be proved in one shot.

Next, we define a Lyapunov function

$$V = \begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}'\begin{bmatrix}P & 0\\ 0 & Q\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} \quad (44)$$

and evaluate the derivative of V along the trajectories of the closed-loop system in (41), i.e.,

$$\dot{V} = \begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}'\begin{bmatrix}-W & PBF\\ F'B'P & -W_Q\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + 2\tilde{x}'PBw. \quad (45)$$

Note that for all

$$\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} \in \boldsymbol{X}_{F\delta} \Rightarrow \left|[F \quad F]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}\right| \le u_{\max}(1-\delta). \quad (46)$$

Again, as done in the state feedback case, let us find the above derivative of V for three different cases.

Case 1) If

$$\left|[F \quad F]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + Hr + \rho\,[B'P \quad B'P]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}\right| \le u_{\max} \quad (47)$$

then

$$w = \rho\,[B'P \quad B'P]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} \quad (48)$$

which implies

$$\begin{aligned}\dot{V} &= \begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}'\begin{bmatrix}-W & PB(F+\rho B'P)\\ (F+\rho B'P)'B'P & -W_Q\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + 2\rho\tilde{x}'PBB'P\tilde{x}\\ &\le \begin{pmatrix}\hat{x}\\ \tilde{x}_v\end{pmatrix}'\begin{bmatrix}-W & 0\\ 0 & -\tilde{W}_Q\end{bmatrix}\begin{pmatrix}\hat{x}\\ \tilde{x}_v\end{pmatrix} \quad (49)\end{aligned}$$

where

$$\hat{x} = \tilde{x} - W^{-1}PB(F+\rho B'P)\tilde{x}_v \quad (50)$$

$$\tilde{W}_Q = W_Q - (F+\rho B'P)'B'PW^{-1} \times PB(F+\rho B'P). \quad (51)$$

Noting (31), i.e., $W_Q > F'B'PW^{-1}PBF$, and $\rho(r,y)$ is locally Lipschitz, it is clear that there exists a $\rho_1^* > 0$ such that for any scalar function satisfying $|\rho(r,y)| \le \rho_1^*$ we have $\tilde{W}_Q > 0$ and, hence, $\dot{V} \le 0$.

Case 2) If

$$[F \quad F]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + Hr + \rho\,[B'P \quad B'P]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} > u_{\max} \quad (52)$$

then for the trajectories inside $\boldsymbol{X}_{F\delta}$

$$\left|[F \quad F]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + Hr\right| \le u_{\max} \quad (53)$$

which implies that

$$0 < w < \rho\,[B'P \quad B'P]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}. \quad (54)$$

Next, let us express

$$w = q\rho\,[B'P \quad B'P]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} \quad (55)$$

for an appropriate positive piecewise continuous function $q(t)$, bounded by 1 for all t. In this case, the derivative of V becomes

$$\begin{aligned}\dot{V} &= \begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix}'\begin{bmatrix}-W & PB(F+q\rho B'P)\\ (F+q\rho B'P)'B'P & -W_Q\end{bmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + 2q\rho\tilde{x}'PBB'P\tilde{x}\\ &\le \begin{pmatrix}\hat{x}_+\\ \tilde{x}_v\end{pmatrix}'\begin{bmatrix}-W & 0\\ 0 & -\tilde{W}_{Q+}\end{bmatrix}\begin{pmatrix}\hat{x}_+\\ \tilde{x}_v\end{pmatrix} \quad (56)\end{aligned}$$

where

$$\hat{x}_+ = \tilde{x} - W^{-1}PB(F+q\rho B'P)\tilde{x}_v \quad (57)$$

$$\tilde{W}_{Q+} = W_Q - (F+q\rho B'P)'B'PW^{-1} \times PB(F+q\rho B'P). \quad (58)$$

Again, noting (31), it can be shown that there exists a ρ_2^* such that for any $\rho(r,y)$ satisfying $|\rho(r,y)| \le \rho_2^*$ we have $\tilde{W}_{Q+} > 0$ and, hence, $\dot{V} \le 0$.

Case 3) Similarly, for the case when

$$[F \quad F]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} + Hr + \rho\,[B'P \quad B'P]\begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} < -u_{\max} \quad (59)$$

we can show that there exists a $\rho_3^* > 0$ such that for any $\rho(r,y)$ satisfying $|\rho(r,y)| \le \rho_3^*$, we have $\dot{V} \le 0$ for all the trajectories in $\boldsymbol{X}_{F\delta}$.

Finally, let $\rho^* = \min\{\rho_1^*, \rho_2^*, \rho_3^*\}$. Then, we have for any nonpositive scalar function $\rho(r,y)$ satisfying $|\rho(r,y)| \le \rho^*$,

$$\dot{V} \le 0 \qquad \forall \begin{pmatrix}\tilde{x}\\ \tilde{x}_v\end{pmatrix} \in \boldsymbol{X}_{F\delta}. \quad (60)$$

Thus, $\boldsymbol{X}_{F\delta}$ is an invariant set of the closed-loop system in (41), and all trajectories starting from $\boldsymbol{X}_{F\delta}$ will remain inside and asymptotically converge to the origin. This, in turn, indicates that, for the initial state of the given system x_0, the initial state of the controller x_{v0}, and step command input r that satisfy (35)

$$\lim_{t\to\infty}\tilde{x}_v(t) = 0 \qquad \lim_{t\to\infty}x(t) = x_e \quad (61)$$

and, hence,

$$\lim_{t\to\infty}h(t) = \lim_{t\to\infty}C_2x(t) = C_2x_e = r. \quad (62)$$

This completes the proof of Theorem 2.

C. Reduced-Order Measurement Feedback Case

For the given system in (1), it is clear that there are p states of the system measurable if C_1 is of maximal rank. Thus, in general, it is not necessary to estimate these measurable states in measurement feedback laws. As such, we will design a dynamic controller that has a dynamical order less than that of the given plant. We now proceed to construct such a control law under the CNF control framework.

For simplicity of presentation, we assume that C_1 is already in the form

$$C_1 = [\, I_p \quad 0 \,]. \tag{63}$$

Then, the system in (1) can be rewritten as

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \operatorname{sat}(u) \\ y = [\, I_p \quad 0 \,] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ h = C_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \end{cases} \tag{64}$$

where the original state x is partitioned into two parts, x_1 and x_2 with $y \equiv x_1$. Thus, we will only need to estimate x_2 in the reduced order measurement feedback design. Next, we let F be chosen such that i) $A + BF$ is asymptotically stable, and ii) $C_2(sI - A - BF)^{-1}B$ has desired properties, and let K_R be chosen such that $A_{22} + K_R A_{12}$ is asymptotically stable. Here, we note that it was shown [1] that (A_{22}, A_{12}) is detectable if and only if (A, C_1) is detectable. Thus, there exists a stabilizing K_R. Again, such F and K_R can be designed using an appropriate control technique. We then partition F in conformity with x_1 and x_2

$$F = [\, F_1 \quad F_2 \,]. \tag{65}$$

Also, let G, H and x_e be as given in (4)–(6). The reduced-order CNF controller is given by

$$\dot{x}_v = (A_{22} + K_R A_{12})x_v + (B_2 + K_R B_1)\operatorname{sat}(u) + \Big[A_{21} + K_R A_{11} - (A_{22} + K_R A_{12})K_R\Big]y \tag{66}$$

and

$$u = F\left[\begin{pmatrix} y \\ x_v - K_R y \end{pmatrix} - x_e\right] + Hr + \rho(r,y)B'P\left[\begin{pmatrix} y \\ x_v - K_R y \end{pmatrix} - x_e\right] \tag{67}$$

where $\rho(r,y)$ is a nonpositive scalar function locally Lipschitz in y subject to certain constraints to be discussed later.

Next, given a positive–definite matrix $W \in \mathbb{R}^{n\times n}$, let $P > 0$ be the solution to the Lyapunov equation

$$(A + BF)'P + P(A + BF) = -W. \tag{68}$$

Given another positive–define matrix $W_R \in \mathbb{R}^{(n-p)\times(n-p)}$ with

$$W_R > F_2' B' P W^{-1} P B F_2 \tag{69}$$

let $Q_R > 0$ be the solution to the Lyapunov equation

$$(A_{22} + K_R A_{12})'Q_R + Q_R(A_{22} + K_R A_{12}) = -W_R. \tag{70}$$

Note that such P and Q_R exist as $A+BF$ and $A_{22}+K_R A_{12}$ are asymptotically stable. For any $\delta \in (0,1)$, let c_δ be the largest positive scalar such that for all

$$\begin{pmatrix} x \\ x_v \end{pmatrix} \in \boldsymbol{X}_{R\delta} := \left\{ \begin{pmatrix} x \\ x_v \end{pmatrix} : \begin{pmatrix} x \\ x_v \end{pmatrix}' \begin{bmatrix} P & 0 \\ 0 & Q_R \end{bmatrix} \begin{pmatrix} x \\ x_v \end{pmatrix} \le c_\delta \right\} \tag{71}$$

the following property holds:

$$\left| [\, F \quad F_2 \,] \begin{pmatrix} x \\ x_v \end{pmatrix} \right| \le u_{\max}(1 - \delta). \tag{72}$$

We have the following theorem.

Theorem 3: Consider the given system in (1). Then, there exists a scalar $\rho^* > 0$ such that for any nonpositive function $\rho(r,y)$, locally Lipschitz in y and $|\rho(r,y)| \le \rho^*$, the reduced-order CNF law given by (66) and (67) will drive the system controlled output $h(t)$ to track asymptotically the step command input of amplitude r from an initial state x_0, provided that x_0, x_{v0} and r satisfy

$$\begin{pmatrix} x_0 - x_e \\ x_{v0} - x_{20} - K_R x_{10} \end{pmatrix} \in \boldsymbol{X}_{R\delta}, |Hr| \le \delta \cdot u_{\max}. \tag{73}$$

Proof: Let $\tilde{x} = x - x_e$ and $\tilde{x}_v = x_v - x_2 - K_R x_1$. Then, the closed-loop system comprising the given plant in (1) and the reduced-order CNF control law of (66) and (67) can be expressed as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_v \end{pmatrix} = \begin{bmatrix} A + BF & BF_2 \\ 0 & A_{22} + K_R A_{12} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w \tag{74}$$

where

$$w = \operatorname{sat}\left\{ [\, F \quad F_2 \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} + Hr + \rho(r,y)B'P \times \left[\tilde{x} + \begin{pmatrix} 0 \\ \tilde{x}_v \end{pmatrix}\right]\right\} - [\, F \quad F_2 \,] \begin{pmatrix} \tilde{x} \\ \tilde{x}_v \end{pmatrix} - Hr. \tag{75}$$

The rest of the proof follows along similar lines to the reasoning given in the full order measurement feedback case.

D. Selecting W and the Nonlinear Gain $\rho(r,y)$

The freedom to choose the function $\rho(r,y)$ is used to tune the control laws so as to improve the performance of the closed-loop system as the controlled output h approaches the set point. Since the main purpose of adding the nonlinear part to the CNF controllers is to speed up the settling time, or equivalently to contribute a significant value to the control input when the tracking error, $r - h$, is small. The nonlinear part, in general, will be in action when the control signal is far away from its saturation level and, thus, it will not cause the control input to hit its limits. Under such a circumstance, it is straightforward to verify that the closed-loop system comprising the given plant in (1) and the three different types of control law can be expressed as

$$\dot{\tilde{x}} = (A + BF)\tilde{x} + \rho(r,y)BB'P\tilde{x}. \tag{76}$$

We note that the additional term $\rho(r,y)$ does not affect the stability of the estimators. It is now clear that eigenvalues of the closed-loop system in (76) can be changed by the function

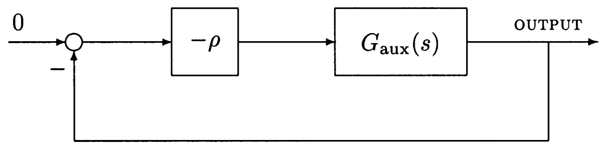

Fig. 1. Interpretation of the nonlinear function $\rho(r, y)$.

$\rho(r, y)$. In what follows, we proceed to interpret the closed-loop system of (76) using the classical feedback control concept as given in Fig. 1, where the auxiliary system $G_{\mathrm{aux}}(s)$ is defined as

$$\begin{aligned} G_{\mathrm{aux}}(s) :=& C_{\mathrm{aux}}(sI - A_{\mathrm{aux}})^{-1} B_{\mathrm{aux}} \\ :=& B'P(sI - A - BF)^{-1} B. \end{aligned} \tag{77}$$

$G_{\mathrm{aux}}(s)$ has the following interesting properties.

Theorem 4: The auxiliary system $G_{\mathrm{aux}}(s)$ defined in (77) is stable and invertible with a relative degree equal to 1, and is of minimum phase with $n-1$ stable invariant zeros.

Proof: First of all, it is obvious to see that $G_{\mathrm{aux}}(s)$ is stable since $A + BF$ is a stable matrix. Next, since $P > 0$ and $B \neq 0$, we have

$$C_{\mathrm{aux}} B_{\mathrm{aux}} = B'PB > 0 \tag{78}$$

which implies that $G_{\mathrm{aux}}(s)$ is invertible and has a relative degree equal to 1 (or an infinite zero of order 1). Furthermore, $G_{\mathrm{aux}}(s)$ has $n-1$ invariant zeros, as it is a single-input–single-output system.

The last property of $G_{\mathrm{aux}}(s)$, i.e., the invariant zeros of $G_{\mathrm{aux}}(s)$ are stable and, hence, it is of minimum phase, can be shown by using the well-known classical root-locus theory. Observing the block diagram in Fig. 1, it follows from the classical feedback control theory (see, e.g., [6]) that the poles of the closed-loop system of (76), which are of course the functions of the tuning parameter ρ, will start from the open-loop poles, i.e., the eigenvalues of $A + BF$, when $\rho = 0$, and end up at the open-loop zeros (including the zero at the infinity) as $|\rho| \to \infty$. It then follows from the proof of Theorem 1 that the closed-loop system will remain asymptotically stable for any nonpositive ρ, which implies that all the invariant zeros of the open-loop system, i.e., $G_{\mathrm{aux}}(s)$, must be stable. This completes the proof of Theorem 4.

It is now clear from Theorem 4 and its proof that the invariant zeros of $G_{\mathrm{aux}}(s)$ play an important role in selecting the poles of the closed-loop system of (76). The poles of the closed-loop system approach the locations of the invariant zeros of $G_{\mathrm{aux}}(s)$ as $|\rho|$ becomes larger and larger. We would like to note that there is freedom in pre-selecting the locations of these invariant zeros. This can actually be done by selecting an appropriate $W > 0$ in (7). In general, we should select the invariant zeros of $G_{\mathrm{aux}}(s)$, which are corresponding to the closed-loop poles for larger $|\rho|$, such that the dominated ones have a large damping ratio, which in turn will yield a smaller overshoot. The following procedure can be used as a guideline for the selection of such a W.

1) Given the pair $(A_{\mathrm{aux}}, B_{\mathrm{aux}})$ and the desired locations of the invariant zeros of $G_{\mathrm{aux}}(s)$, we follow the result of [4] on finite and infinite zero assignment to obtain an appropriate matrix C_{aux} such that the resulting $(A_{\mathrm{aux}}, B_{\mathrm{aux}}, C_{\mathrm{aux}})$ has the desired relative degree and invariant zeros.
2) Solve $C_{\mathrm{aux}} = B'P$ for a $P = P' > 0$. In general, the solution is nonunique as there are $n(n+1)/2$ elements in P available for selection. However, if the solution does not exist, we go back to the previous step to reselect the invariant zeros.
3) Calculate W using (7) and check if W is positive definite. If W is not positive definite, we go back to the previous step to choose another solution of P or go to the first step to reselect the invariant zeros.

Generally, the aforementioned procedure would yield a desired result. The selection of the nonlinear function $\rho(r, y)$ is relatively simple once the desired invariant zeros of $G_{\mathrm{aux}}(s)$ are obtained. We usually choose ρ as a function of the tracking error, i.e., $r - h$, which in most practical situations is known and available for feedback. The following choice of ρ, an exponential function, is modified from the one suggested in [17]:

$$\rho(r, h) = -\frac{\beta}{1 - e^{-1}} \left(e^{-|1-(h-h_0)/(r-h_0)|} - e^{-1} \right) \tag{79}$$

where $h_0 = h(0)$ and $\beta \geq 0$ is a tuning parameter. This function $\rho(r, h)$ changes from 0 to $-\beta$ as the tracking error approaches zero. At the initial stage, when the controlled output h is far away from the final set point, $|1 - (h - h_0)/(r - h_0)|$ closes to 1, which implies that $\rho(r, h)$ is small and the effect of the nonlinear part on the overall system is very limited. When the controlled output h approaches the set point, $|1 - (h - h_0)/(r - h_0)|$ closes to zero and $\rho(r, h) \approx -\beta$, and the nonlinear control law will become effective. In general, the parameter β should be chosen such that the poles of $A + BF - \beta BB'P$ are in the desired locations, e.g., the dominated poles should have a large damping ratio. Finally, we note that the choice of $\rho(r, h)$ is nonunique. Any smooth function would work so long as it has similar properties of that given in (79).

III. Beating the Time Optimal Control

Can we design a control system that would beat the performance of the TOC? Obviously, the answer to this question is no if it is required to have a precise point-to-point tracking, i.e., to track a target reference precisely from a given initial point. However, surprisingly, the answer would be yes if we consider an asymptotic tracking situation, i.e., if we consider the settling time to be the total time that the controlled system output takes to get from its initial position to reach a predetermined neighborhood of the target reference before the system output settles down to the desired point. The reason that we are interested in this issue is that asymptotic tracking is widely used in almost all practical situations.

In what follows, we will show the above observation in an example. Let us consider a system characterized a double integrator, i.e.,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{sat}(u) \quad y = x \\ h &= [1 \quad 0]\, x \end{aligned} \tag{80}$$

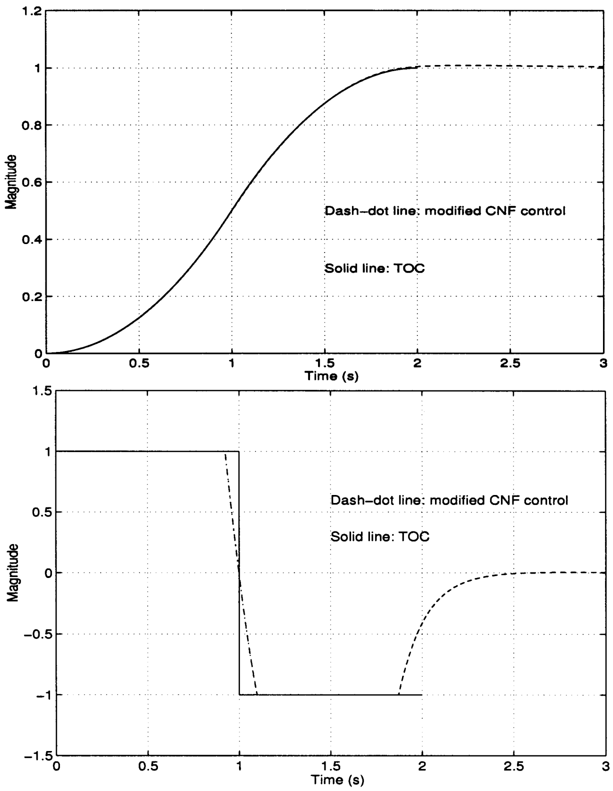

Fig. 2. Responses and control signals of the TOC and CNF control.

where as usual x is the state, u is the input, and y and h are, respectively, the measurement and controlled outputs. Moreover, we assume that

$$\mathrm{sat}(u) = \mathrm{sgn}(u) \cdot \min\{1, |u|\}. \tag{81}$$

Let the initial state $x(0) = 0$ and the target reference $r = 1$. Then, it is simple to compute that the minimum time required for the controlled output to reach precisely the target reference under the TOC is exactly 2 s. Let us now consider an asymptotic tracking situation instead. As is commonly accepted in the literature (see, e.g., [6]), we define the settling time to be the total time that it takes for the control output h to enter the $\pm 1\%$ region of the target reference. The following control law, obtained from the CNF control technique, would give a faster settling time than that of the TOC:

$$u = [-6.5 \quad -1]\, x + 6.5r - \left(e^{-|1-h|} - 0.367\,88\right) \times [1.4481 \quad 10.8609] \left(x - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right). \tag{82}$$

Fig. 2 shows the resulting controlled output responses and the control signals of the TOC and the CNF control. The resulting output response of the CNF control has an overshoot less than 1%. However, if we zoom in on the output responses (see Fig. 3), we will see that the CNF control clearly has a faster settling time than that of the TOC when it enters the target region, i.e., $0.99 \le h \le 1.01$. It can be computed that the CNF control has a settling time of 1.8453 s whereas the TOC has a settling time of 1.8586 s. Although the difference is not much, since we have not tried to optimize the solution of the CNF control, it is, however, significant enough to address one interesting issue: there are control laws that can achieve a faster settling time than that of the TOC in asymptotic tracking situations. It can also be shown that, no matter how small the target region is, say $1 \pm \varepsilon$ for any small $\varepsilon > 0$, we can always find a suitable control law that beats the TOC in settling time. Nonetheless, we believe that it would be interesting to carry out some further studies in this subject.

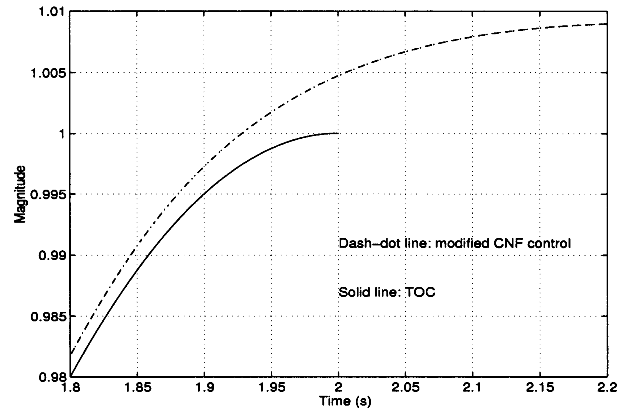

Fig. 3. Controlled output responses around the target reference.

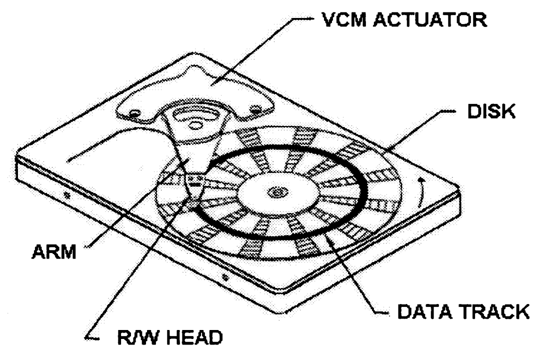

Fig. 4. A typical HDD with a VCM actuator.

IV. An Application

In this section, we apply the theory of CNF control to design a reduced order control law for an HDD servo system. The two main functions of the head positioning servomechanism in disk drives are track seeking and track following. Track seeking moves the read/write (R/W) head from the present track to a specified destination track in minimum time using a bounded control effort. Track following maintains the head as close as possible to the destination track center while information is being read from or written to the disk. Fig. 4 shows a typical hard disk drive with a voice-coil motor (VCM) actuator servo system. On the surface of a disk, there are thousands of data tracks. A magnetic head is supported by a suspension and a carriage, and it is suspended several micro inches above the disk surface. The VCM actuator initiates the carriage and moves the head on a desired track.

Current hard disk drives use a combination of classical control techniques, such as proximate time optimal control technique in the tracking seeking stage, and lead-lag compensators or PID compensators in the track following stage, plus some notch filters to reduce the effects of high frequency resonant

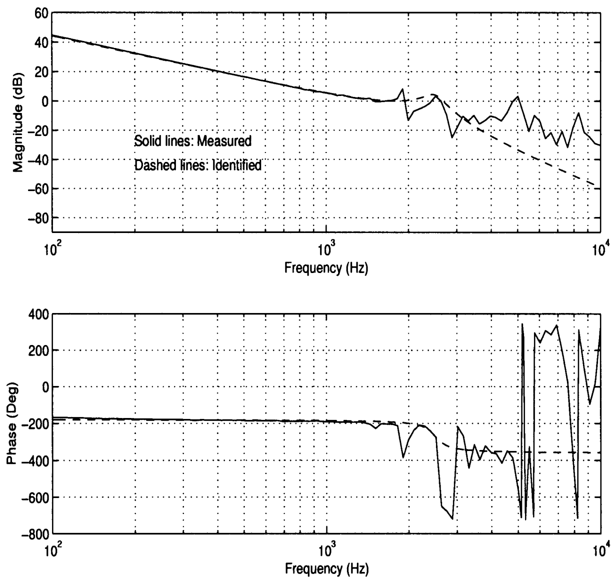

Fig. 5. Frequency response of the HDD.

modes (see, e.g., [6]–[8], [10], [13], [14], [22], and references cited therein). These classical methods can no longer meet the demand for hard disk drives of higher performance. Thus, many control approaches have been tried, such as the linear quadratic Gaussian (LQG) with loop transfer recovery (LTR) approach (see, e.g., [9] and [20]), H_∞ control approach (see, e.g., [2], [3], [11], [15], and [16]), and adaptive control (see, e.g., [18] and [21]), and so on. Although much work has been done to date, more studies need to be conducted to achieve better performance. In what follows, we proceed to design a complete servo system for a commercially available hard disk drive, namely, a Maxtor HDD (Model 51536U3). We will present the model of the HDD first and then utilize the CNF approach to design an appropriate control law. The simulation and actual implementation results will be also given and compared with those of the conventional PTOS approach.

A. Modeling of the HDD

The mechanical part of the plant, that is, the controlled object, consists of the VCM, the carriage, the suspension, and the heads. The controlled variable is the relative head position. The control input u is a voltage to a current amplifier for the VCM and the measurement output y is the head position in tracks. The frequency response characteristics of the HDD servo system from u to y is shown as a solid line in Fig. 5. It is quite conventional to approximate the dynamics of the VCM actuator by a second-order state–space model as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ a \end{bmatrix} u \tag{83}$$

where $x = [y \quad v]'$ is the state vector with y and v are the position (in μm) and the velocity of the R/W head (in μm/s), u is the actuator input (in volts) and is bounded as $|u(t)| \le u_{\max}$, and $a = K_t/J_a$ is the acceleration constant, with K_t being the torque constant and J_a being the moment of inertia of the actuator mass. Thus, the transfer function from u to y of the VCM model can be written as

$$G_{v1}(s) = \frac{a}{s^2}. \tag{84}$$

The response characteristics shows that the servo system has many mechanical resonance modes over 2 kHz. In general, it is difficult to model these high frequency flexible modes exactly. However, if we consider only the dominant resonance frequency, a more realistic model for the VCM actuator can be represented as follows:

$$G_v(s) = \frac{a}{s^2}\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \tag{85}$$

where ω_n corresponds to the resonance frequency and ζ be the associated damping coefficient. To design and implement the proposed controller, an actual HDD was taken and the model was identified through frequency response test (see Fig. 5). Using these measured data from the actual system and the algorithm of [5], we obtained a fourth-order model for the actuator

$$G_v(s) = \frac{6.4013 \times 10^7}{s^2}\left(\frac{2.467 \times 10^8}{s^2 + 2.513 \times 10^3 s + 2.467 \times 10^8}\right) \tag{86}$$

where the output is in micrometer and the input is in V with $u_{\max} = 3$ V. This model will be used throughout the rest of this paper.

B. HDD Servo System Design

The HDD servomechanism model considered is a double integrator with the dominant resonance mode as shown in (86). However, in the design stage, we consider only the double integrator model, i.e.,

$$\begin{pmatrix} \dot{y} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ 6.4013 \times 10^7 \end{bmatrix} u. \tag{87}$$

The measurement output and the controlled output for this system turn out to be identical, i.e., $h = y$. It is simple to verify that the three conditions for the CNF design are fully satisfied. We now carry on to design a CNF controller for this system. For this particular application, the design procedure can be simplified as follows.

1) Find a state feedback gain matrix F using an appropriate method such that $A+BF$ is asymptotically stable and the overall closed-loop system has a quick rising time with its resulting control input not exceeding the saturation level.
2) Compute G_e, x_e, H, and G.
3) Choose an appropriate matrix W and solve (29) for $P > 0$. In fact, for a second order system, it is simple to observe from the proof of Theorem 4 that for any choice of W, the poles of the closed-loop system will always approach to two negative real scalars with one moving toward $-\infty$.
4) Select the function ρ as in (79) with an appropriate β such that the resulting closed-loop system has small overshoot in the time domain response.

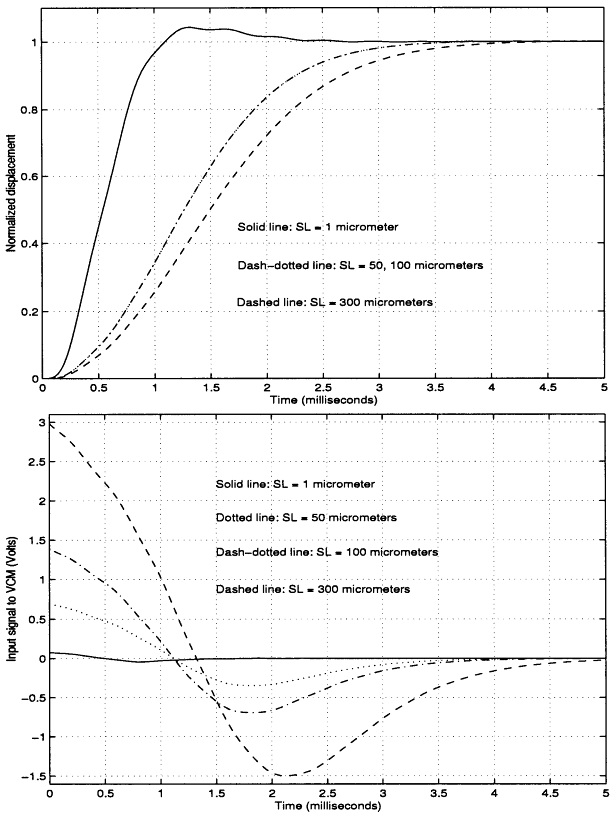

Fig. 6. Simulation result: normalized responses under the CNF control.

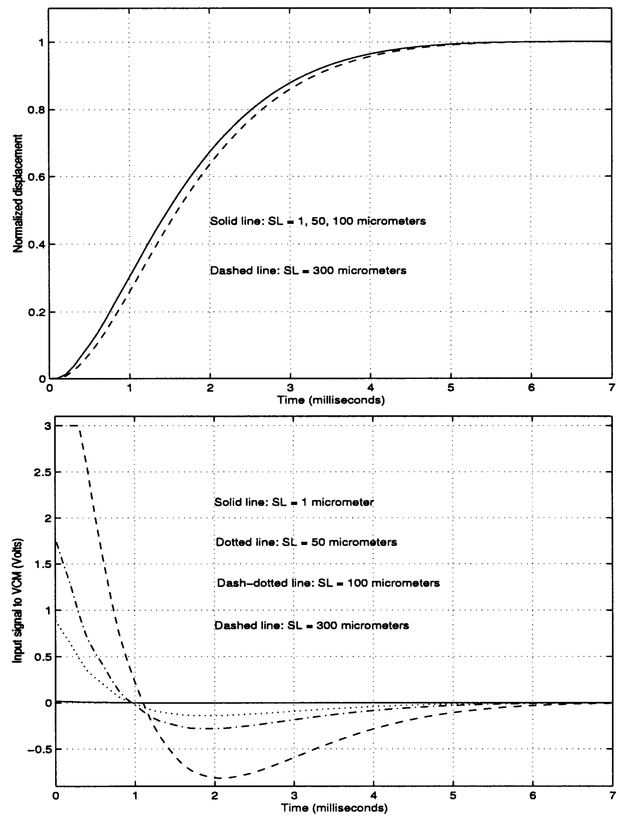

Fig. 7. Simulation result: normalized responses under the PTOS control.

Using the robust and perfect tracking design technique given in [2], we obtain the following parameterized state feedback gain $F(\varepsilon)$ for the HDD system:

$$F(\varepsilon) = -\frac{1}{6.4013 \times 10^7}\begin{bmatrix} \frac{4\pi^2 f^2}{\varepsilon^2} & \frac{4\pi f \zeta}{\varepsilon} \end{bmatrix}. \tag{88}$$

The eigenvalues of the closed-loop system matrix $A + BF(\varepsilon)$ are placed at $(-\zeta \pm j\sqrt{1-\zeta^2})2\pi f/\varepsilon$. We note that such a gain with $\zeta = 0.3$, $f = 350$ and $\varepsilon = 1$ is roughly corresponding to the normal working frequency of the HDD. The nonlinear part of the CNF control law is selected as follows:

$$\rho(r,y) = -1.5820\beta\left(e^{-|1-y/r|} - 0.3679\right). \tag{89}$$

To implement the control law to the actual system for which the velocity is not measurable, we use a reduced order CNF control law with $K_R = -4000$. The complete CNF control is then given by

$$\dot{x}_v = -4000x_v - 1.6 \times 10^7 y + 6.4013 \times 10^7 \mathrm{sat}(u) \tag{90}$$

and

$$u = \kappa_2 x_v + (\kappa_1 + 4000\kappa_2)y - \kappa_1 r + \rho(r,y)\Big[\kappa_3 x_v + (4000\kappa_3 - \kappa_1)y + \kappa_1 r\Big] \tag{91}$$

where

$$\kappa_1 = \frac{-0.0755^2}{\varepsilon} \quad \kappa_2 = \frac{-2.0613 \times 10^{-5}}{\varepsilon} \tag{92}$$

and

$$\kappa_3 = \frac{5.7257 \times 10^{-5}}{\varepsilon}. \tag{93}$$

Note that these parameters β and ε can be adjusted accordingly with respect to the amplitude of the target reference. After few iterations, we find that β and ε can roughly be approximated, respectively, as

$$\varepsilon(r) = \begin{cases} 0.0594r + 1.0805, & 1 \le r 20\ \mu\mathrm{m} \\ 0.0019r + 2.2062, & 20 \le r \le 300\ \mu\mathrm{m} \end{cases} \tag{94}$$

and

$$\beta(r) = \begin{cases} 0.0515r + 1.0756, & 1 \le r < 20\ \mu\mathrm{m} \\ 1.5642, & 20 \le r \le 300\ \mu\mathrm{m} \end{cases}. \tag{95}$$

To compare our design with the conventional PTOS approach, we follow the procedure given in [21] to find an implementable PTOS controller for the given HDD plant. The PTOS control law is given by

$$u_p = u_{\max} \cdot \mathrm{sat}\left(\frac{k_2[f(e) - v]}{u_{\max}}\right) \tag{96}$$

where $e = r - y$ and the function $f(e)$ is defined as

$$f(e) = \begin{cases} \frac{k_1}{k_2}e, & \text{for } |e| \le y_\ell \\ \mathrm{sgn}(e)\left[\sqrt{2u_{\max}a\alpha|e|} - \frac{u_{\max}}{k_2}\right], & \text{for } |e| > y_\ell \end{cases}. \tag{97}$$

The values of various parameters were found to make the resulting closed-loop system implementable up to a seek length of 300 μm. These are given by $a = 6.4013 \times 10^7$, $k_1 = 0.0178$, $k_2 = 2.997 \times 10^{-5}$, $\alpha = 0.62$ and $y_\ell = 168.32$ μm. A velocity estimator with an estimator pole placed at -4000 is used both simulation and implementation.

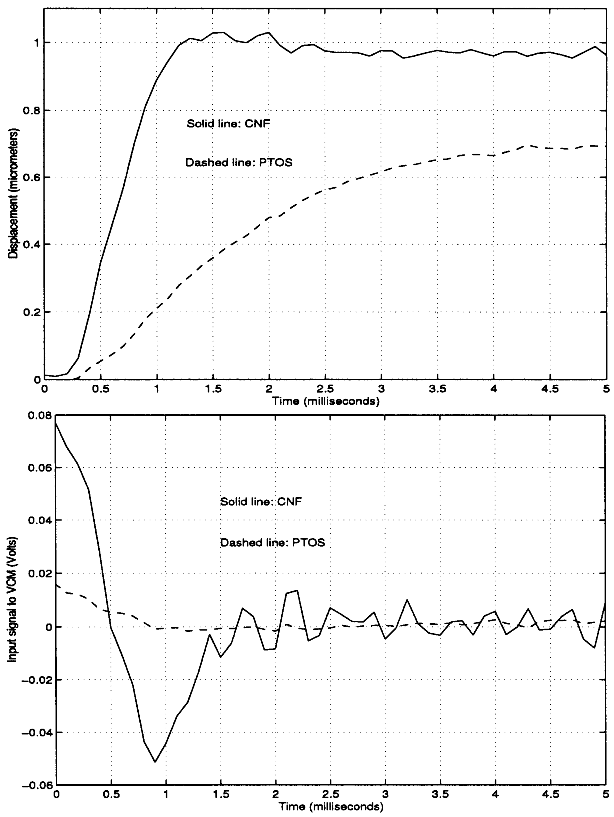

Fig. 8. Experimental result: responses under CNF and PTOS control for $SL = 1\ \mu$m.

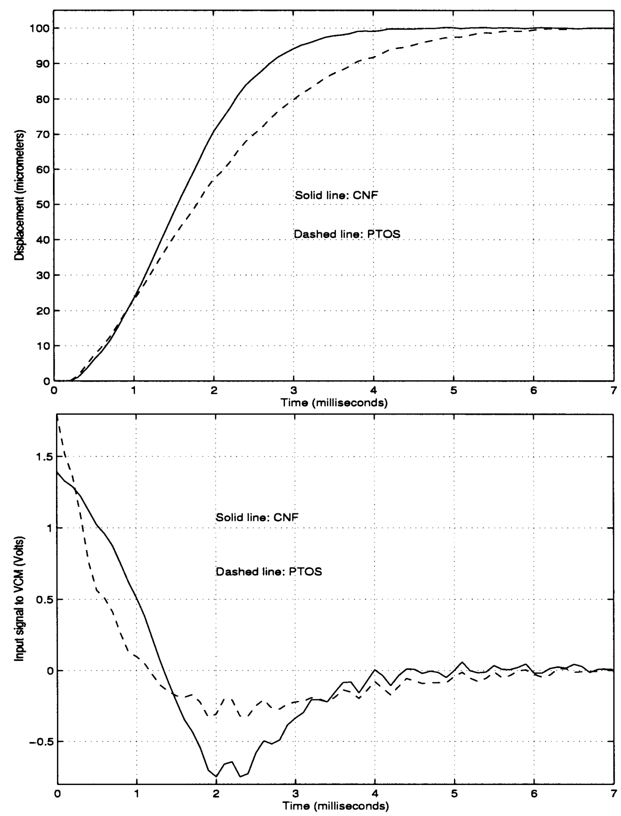

Fig. 9. Experimental result: responses under CNF and PTOS control for $SL = 100\ \mu$m.

C. Simulation and Implementation Results

Our simulation is carried out using Simulink and the results for various seek lengths (SL) using the proposed CNF and the PTOS controllers are respectively shown in Figs. 6 and 7. Both control laws are also implemented on the actual HDD system using a sampling frequency of 10 kHz. The R/W head position was measured using a laser Doppler vibrometer. The implementation results for $\mathrm{SL} = 1$, 100 and 300 μm are, respectively, shown in Figs. 8–10. For an easy comparison, the results are summarized in Table I. We note that we have included the implementation result for $\mathrm{SL} = 50\ \mu$m in Table I. The detailed graphics for this case are omitted as they are almost identical to those for $\mathrm{SL} = 100\ \mu$m. Also, note that the settling time in HDD servo systems is traditionally defined as the total time that take the R/W head to reach the $\pm 0.05\ \mu$m of the target reference. The HDD can start reading or writing data within $\pm 5\%$ of the track width. The results clearly show that the proposed CNF control out perform the conventional PTOS by more than 30% in settling time.

V. Concluding Remarks

We have studied in this paper the theory and an application of a new nonlinear control technique, the composite nonlinear feedback control, for a class of linear systems with actuator nonlinearities. The simulation and implementation results show that the new technique has out performed the conventional method by more than 30%. Furthermore, it has also been shown by an

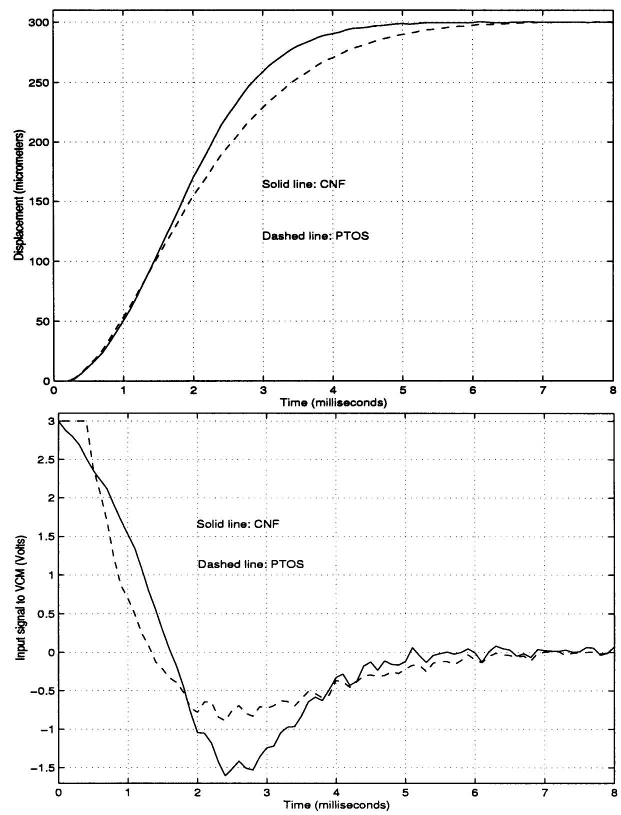

Fig. 10. Experimental result: responses under CNF and PTOS control for $SL = 300\ \mu$m.

TABLE I
SETTLING TIME AND PERCENTAGE OF IMPROVEMENT FROM SIMULATION AND EXPERIMENTAL RESULTS

SEEK LENGTH (μm)	SETTLING TIME (ms)			
	Simulation		Implementation	
	PTOS	CNF	PTOS	CNF
1	3.75	1.0	– –	1.2
50	5.45	3.7	6.5	4.5
100	5.45	3.7	6.5	4.5
300	5.65	4.4	6.7	5.3

SEEK LENGTH (μm)	OVERALL IMPROVEMENT (%)	
	Simulation	Implementation
1	74	—
50	32	31
100	32	31
300	22	21

example that the CNF control is capable of beating the well known time-optimal control (or bang-bang control) in asymptotic tracking. As mentioned earlier, it would be interesting, although it is pretty hard, to carry out a systematic study on how to derive a time-optimal control law in the asymptotic tracking situations. Another direction of future research is to extend our results to systems with multiple control inputs and multiple controlled output with measurement feedback.

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