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Brief Paper

On the problem of general structural assignments of linear systems through sensor/actuator selection☆

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Abstract

A systematic method is developed for determining an output matrix C for a given matrix pair (A,B) such that the resulting linear system characterized by the matrix triple (A,B,C) has the pre-specified system structural properties, such as the finite and infinite zero structure and the invertibility structures. Since the matrix C describes the locations of the sensors, the procedure of choosing C is often referred to as sensor selection. The method developed in this paper for sensor selection can be applied to the dual problem of actuator selection, where, for a given matrix pair (A,C), a matrix B is to be determined such that the resulting matrix triple (A,B,C) has the pre-specified structural properties.

Keywords: Linear systems; System structures; Invariant indices; Zero placements; Structural assignment

1. Introduction and problem statement

As it is well known in the literature, the structural properties of linear systems, such as the finite and infinite zero structures and the invertibility structures, have played very important roles in many linear systems and control areas (see e.g., robust and H_∞ control, Chen, 2000; H_2 optimal control, Saberi, Sannuti, & Chen 1995; and control with saturation, Lin, 1998). We believe that one of the major difficulties in applying the useful multivariable control synthesis techniques, e.g., such as H_2 and H_∞ control techniques, to actual design is the inadequate study of the linkage between control performance and design implementation involving hardware selection, e.g., appropriate sensors suitable for robustness and performance. This linkage provides a foundation upon which trade-offs can be incorporated at the preliminary design stage. Thus, one can introduce careful control design considerations into the overall engineering design process in an early stage. This is what motivated the work to be reported in this paper. Our objective is to study the flexibility in assigning structural properties to a given linear system, and to identify sets of sensors which would yield desirable structural properties.

It is appropriate to trace a short history of the development of the techniques related to structural assignments of linear systems. To the best of our knowledge, most results in the open literature are related to invariant zero or transmission zero (i.e., finite zero structure) assignments (see for example, Emami-Naeini & Dooren, 1982; Karcanias, Laios, & Ginnakopoulos, 1988; Kouvaritakis & MacFarlane, 1976; Patel, 1978; Vardulakis, 1980; Syrmos & Lewis, 1993). We note that all the results reported in the literature so far, including the ones mentioned above, deal solely with the assignments of the finite zeros. The infinite zero structure and other structures such as invertibility structures of the resulting system are either fixed or of not much concern. Only recently had Chen and Zheng (1995) proposed a technique, which is capable of assigning both finite and infinite zero structures simultaneously. However, up to date, to the best of our knowledge, there still does not exist any method that deals with the assignment of complete system structures, including finite and infinite zero structures and invertibility structures. We propose in this paper a technique which is capable of assigning all these structural properties. More specifically, we consider a linear system characterized by

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the following state space equation:

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input. The problem of structural assignments or sensor selection is to find a measurement output,

$$y = Cx, \tag{2}$$

such that the resulting system characterized by the matrix triple (A, B, C) would have the pre-specified desired structural properties, including finite and infinite zero structures and invertibility structures. We note that this technique can be applied to solve the dual problem of actuator selection, i.e., to find a matrix B provided that matrices A and C are given such that the resulting system again characterized by the triple (A, B, C) would have the pre-specified desired structural properties.

Throughout the paper, X' denotes the transpose of X, and I_k denotes the identity matrix of dimension $k \times k$. With a slight abuse of notation, I_k with $k \leqslant 0$ is treated as an empty matrix. Also, $\star$ denotes some constant matrix which is of less interest in the context. A set of complex scalars, $\mathscr{W}$, is said to be self-conjugate if, for any $w \in \mathscr{W}$, its complex conjugate $\bar{w} \in \mathscr{W}$.

2. Background materials

In this section, we recall two structural decomposition techniques of linear systems, i.e., the controllability structural decomposition for a matrix pair (A, B), which was discovered by Luenberger (1967) and Brunovsky (1970), and the special coordinate basis decomposition for a matrix triple (A, B, C), which was introduced by Sannuti and Saberi (1987). Both decompositions will be instrumental and extensively used in the development of the results reported in the coming sections.

Theorem 2.1 (CSD). *Consider a pair of constant matrices (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that B is of full rank. Then, there exist non-singular state and input transformations T_s and T_i such that $(\tilde{A}, \tilde{B}) := (T_s^{-1}AT_s, T_s^{-1}BT_i)$* has the following form:

$$\left(\begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{k_1-1} & \cdots & 0 & 0 \\ \star & \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{k_m-1} \\ \star & \star & \star & \cdots & \star & \star \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix} \right), \tag{3}$$

where $k_i > 0$, $i = 1, \ldots, m$, A_0 is of dimension $n_0 := n - \sum_{i=1}^m k_i$ and its eigenvalues are the uncontrollable modes of the pair (A, B). Moreover, the set of integers, $\mathscr{C}(A, B) := \{n_0, k_1, \ldots, k_m\}$, is referred to as the controllability index of (A, B).

Next, consider a linear system Σ characterized by (A, B, C) with a transfer function, $H(s) = C(sI - A)^{-1}B$, or in the state space form,

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output, respectively. Without loss of generality, we assume that both B and C are of full rank.

Theorem 2.2 (SCB). *Consider the linear system Σ of (4). There exist* (i) *coordinate free non-negative integers n_a, n_b, n_c, n_d, $m_d \leqslant m$ and q_i, $i = 1, \ldots, m_d$, and* (ii) *non-singular state, output and input transformations Γ_s, Γ_0 and Γ_i which take Σ into a special coordinate basis that displays explicitly both the finite and infinite zero structures of Σ. The special coordinate basis is described by*

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_0 \tilde{y}, \quad u = \Gamma_i \tilde{u}, \tag{5}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_d \\ y_b \end{pmatrix},$$

$$y_d = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_d} \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_d \\ u_c \end{pmatrix}, \quad u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix},$$

$$\dot{x}_a = A_{aa}x_a + L_{ad}y_d + L_{ab}y_b, \tag{6}$$

$$\dot{x}_b = A_{bb}x_b + L_{bd}y_d, \quad y_b = C_b x_b, \tag{7}$$

$$\dot{x}_c = A_{cc}x_c + B_c E_{cb} x_b + L_{cd} y_d + B_c E_{ca} x_a + B_c u_c \tag{8}$$

and for each $i = 1, \ldots, m_d$,

$$\dot{x}_i = A_{q_i} x_i + L_{id} y_d + B_{q_i} \left[u_i + E_{ia}x_a + E_{ib}x_b + E_{ic}x_c + \sum_{j=1}^{m_d} E_{ij}x_j \right], \tag{9}$$

$$y_i = C_{q_i} x_i, \quad y_d = C_d x_d. \tag{10}$$

Here the states x_a, x_b, x_c and *x_d are, respectively, of dimensions n_a, n_b, n_c and $n_d = \sum_{i=1}^{m_d} q_i$, while x_i is of dimension q_i for each $i = 1, \ldots, m_d$. The control vectors u_d and u_c are, respectively, of dimensions m_d and $m_c = m - m_d$ while the output vectors y_d and y_b are, respectively, of dimensions*

$p_d = m_d$ and $p_b = p - p_d$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the following form:

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_{q_i} = [1, 0, \dots, 0]. \tag{11}$$

Assuming that x_i, $i = 1, 2, \dots, m_d$, are arranged such that $q_i \leqslant q_{i+1}$, the matrix L_{id} has the particular form

$$L_{id} = \begin{bmatrix} L_{i1} & L_{i2} & \cdots & L_{ii-1} & 0 & \cdots & 0 \end{bmatrix}. \tag{12}$$

The last row of each L_{id} is identically zero. Moreover, (A_{cc}, B_c) is controllable and (A_{bb}, C_b) is observable.

We can rewrite the special coordinate basis of the triple (A, B, C) given by Theorem 2.2 in a more compact form

$$\tilde{A} = \Gamma_s^{-1} A \Gamma_s = \begin{bmatrix} A_{aa} & L_{ab}C_b & 0 & L_{ad}C_d \\ 0 & A_{bb} & 0 & L_{bd}C_d \\ B_cE_{ca} & B_cE_{cb} & A_{cc} & L_{cd}C_d \\ B_dE_{da} & B_dE_{db} & B_dE_{dc} & A_{dd} \end{bmatrix} \tag{13}$$

and

$$\tilde{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix},$$

$$\tilde{C} = \Gamma_0^{-1} C \Gamma_s = \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned} A_{dd} &= A_{dd}^* + B_dE_{dd} + L_{dd}C_d, \\ &= \text{blkdiag}\{A_{q_1}, \dots, A_{q_{m_d}}\} + B_dE_{dd} + L_{dd}C_d \end{aligned} \tag{15}$$

and all the sub-matrices $A_{dd}^*, B_d, E_{da}, E_{db}$ and E_{dd} are defined in an obvious way.

In what follows, we state some important properties of the special coordinate basis which are pertinent to our present work. The proofs of these properties can be found in Chen (2000).

Property 2.1. Σ is observable (detectable) if and only if the pair $(A_{\text{obs}}, C_{\text{obs}})$ is observable (detectable), where

$$A_{\text{obs}} := \begin{bmatrix} A_{aa} & 0 \\ B_cE_{ca} & A_{cc} \end{bmatrix}, \quad C_{\text{obs}} := [E_{da}\ E_{dc}]. \tag{16}$$

Also, define

$$A_{\text{con}} := \begin{bmatrix} A_{aa} & L_{ab}C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{\text{con}} := \begin{bmatrix} L_{ad} \\ L_{bd} \end{bmatrix}. \tag{17}$$

Similarly, Σ is controllable (stabilizable) if and only if the pair $(A_{\text{con}}, B_{\text{con}})$ is controllable (stabilizable).

The invariant zeros of a system Σ characterized by (A, B, C) can be defined via the Smith canonical form of the (Rosenbrock) system matrix of Σ (see e.g., Rosenbrock, 1970; MacFarlane & Karcanias, 1976). The special coordinate basis of Theorem 2.2 shows explicitly the invariant zeros of Σ.

Property 2.2. Invariant zeros of Σ are the eigenvalues of A_{aa}.

In order to display various multiplicities of invariant zeros, let X_a be a non-singular transformation matrix such that A_{aa} can be transformed into a Jordan canonical form, i.e.,

$$X_a^{-1} A_{aa} X_a = J = \text{blkdiag}\{J_1, J_2, \dots, J_k\}, \tag{18}$$

where J_i, $i = 1, 2, \dots, k$, are some $n_i \times n_i$ Jordan blocks:

$$J_i = \text{diag}\{\alpha_i, \alpha_i, \dots, \alpha_i\} + \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}. \tag{19}$$

For any given $\alpha \in \lambda(A_{aa})$, let there be τ_α Jordan blocks of A_{aa} associated with α. Let $n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}$ be the dimensions of these Jordan blocks. Then we say α is an invariant zero of Σ with multiplicity structure $S_\alpha^\star(\Sigma)$ (see also Saberi, Chen, & Sannuti, 1991),

$$S_\alpha^\star(\Sigma) = \{n_{\alpha,1}, n_{\alpha,2}, \dots, n_{\alpha,\tau_\alpha}\}. \tag{20}$$

The geometric multiplicity of α is then simply given by τ_α, and the algebraic multiplicity of α is given by $\sum_{i=1}^{\tau_\alpha} n_{\alpha,i}$. Here we should note that the invariant zeros together with their structures of Σ are related to the structural invariant indices list $\mathscr{I}_1(\Sigma)$ of Morse (1973).

The special coordinate basis can also reveal the infinite zero structure of Σ. We note that the infinite zero structure of Σ can be defined either in association with root-locus theory or as Smith–McMillan zeros of the transfer function at infinity. For the sake of simplicity, we only consider the infinite zeros from the point of view of Smith–McMillan theory here. To define the zero structure of $H(s)$ at infinity, one can use the familiar Smith–McMillan description of the zero structure at finite frequencies of a general not necessarily square but strictly proper transfer function matrix $H(s)$. Namely, a rational matrix $H(s)$ possesses an infinite zero of order k when $H(1/z)$ has a finite zero of precisely that order at $z = 0$ (see e.g., Rosenbrock, 1970). The number of zeros at infinity together with their orders indeed define an infinite zero structure. Owens (1978) related the orders of the infinite zeros of the root-loci of a square system with a non-singular transfer function matrix to $\mathscr{C}^*$ structural invariant indices list $\mathscr{I}_4$ of Morse (1973). The special coordinate basis of Theorem 2.2 explicitly shows the infinite zero structure of Σ.

Property 2.3. The infinite zero structure of Σ is given by

$$S_\infty^\star(\Sigma) = \{q_1, q_2, \ldots, q_{m_d}\}. \tag{21}$$

That is, each q_i corresponds to an infinite zero of Σ of order q_i. Note that for a single-input–single-output system Σ, we have $S_\infty^\star(\Sigma) = \{q_1\}$, where q_1 is the *relative degree* of the given system Σ.

The special coordinate basis can also exhibit the invertibility structure of a given system Σ. The formal definitions of right invertibility and left invertibility of a linear system can be found in Moylan (1977). Basically, for the usual case when B and C are of maximal rank, the system Σ or equivalently $H(s)$ is said to be left invertible if there exists a rational matrix function, say $L(s)$, such that $L(s)H(s) = I_m$. Σ or $H(s)$ is said to be right invertible if there exists a rational matrix function, say $R(s)$, such that $H(s)R(s) = I_p$. Σ is invertible if it is both left and right invertible, and Σ is degenerate if it is neither left nor right invertible.

Property 2.4. System Σ is right invertible if and only if x_b (and hence y_b) are non-existent, left invertible if and only if x_c (and hence u_c) are non-existent, and invertible if and only if both x_b and x_c are non-existent. Moreover, Σ is degenerate if and only if both x_b and x_c are present.

The special coordinate basis can also be modified to obtain the structural invariant indices lists $\mathscr{I}_2$ and $\mathscr{I}_3$ of Morse (1973) of the given system Σ. In order to display $\mathscr{I}_2(\Sigma)$, we let X_c and X_i be non-singular matrices such that the controllable pair (A_{cc}, B_c) is transformed into the controllability structural decomposition (see Theorem 2.1), i.e.,

$$X_c^{-1} A_{cc} X_c = \begin{bmatrix} 0 & I_{\ell_1 - 1} & \cdots & 0 & 0 \\ \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{\ell_{m_c} - 1} \\ \star & \star & \cdots & \star & \star \end{bmatrix},$$

$$X_c^{-1} B_c X_i = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}, \tag{22}$$

where $\star$s denote constant scalars or row vectors. Then we have

$$\mathscr{I}_2(\Sigma) = \{\ell_1, \ldots, \ell_{m_c}\}, \tag{23}$$

which is also called the controllability index of (A_{cc}, B_c). Similarly, we have

$$\mathscr{I}_3(\Sigma) = \{\mu_1, \ldots, \mu_{p_b}\}, \tag{24}$$

where $\{\mu_1, \ldots, \mu_{p_b}\}$ is the controllability index of the controllable pair (A'_{bb}, C'_b).

3. Structural assignments of linear systems

Having familiarized with all the structural properties of linear systems, i.e., the finite zero and infinite zero structures as well as the invertibility structures, we are now ready to present the main results of this paper. We first have the following theorem.

Theorem 3.1. *Consider the linear system (1). Assume that B is of full column rank, the controllability index of (A,B) is given by $\mathscr{C}(A,B) = \{n_0, k_1, \ldots, k_m\}$, and the uncontrollable modes of (A,B), if any, are given by $\Delta = \{u_1, \ldots, u_{n_0}\}$. Let*

$$\Lambda_2 := \{\ell_1, \ell_2, \ldots, \ell_{m_c}\} \subset \mathscr{C}^* =: \{k_1, k_2, \ldots, k_m\}, \tag{25}$$

$$\mathscr{C}^* \backslash \Lambda_2 := \{\omega_1, \omega_2, \ldots, \omega_{m_d}\},$$

$$m_d = m - m_c, \quad \omega_1 \leqslant \omega_2 \leqslant \cdots \leqslant \omega_{m_d}, \tag{26}$$

$$\Lambda_4 := \{q_1, q_2, \ldots, q_{m_d}\}, \quad q_i \leqslant \omega_i, \;\; i = 1, 2, \ldots, m_d. \tag{27}$$

Moreover, we let a set of complex scalars

$$\Lambda_1 = \Theta \cup \Delta_1 := \{z_1, \ldots, z_{s_1}\} \cup \Delta_1, \tag{28}$$

where Θ is self-conjugate and so is $\Delta_1 \subset \Delta$. For simplicity, we assume that the entries of $\Delta_2 = \Delta \setminus \Delta_1$ are distinct. Furthermore, s_1 is chosen such that

$$s_1 \leqslant n - \sum_{i=1}^{m_c} \ell_i - \sum_{i=1}^{m_d} q_i - n_0. \tag{29}$$

Finally, let

$$\Lambda_3 := \{\mu_1, \mu_2, \ldots, \mu_{p_b}\} \tag{30}$$

be a set of positive integers with $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{p_b}$, which satisfy the following constraint:

$$s_1 + n_0 + \sum_{i=1}^{p_b} \mu_i + \sum_{i=1}^{m_c} \ell_i + \sum_{i=1}^{m_d} q_i = n. \tag{31}$$

Then, there exists a non-empty set $\mathbf{\Omega} \subset \mathbb{R}^{(m_d + p_b) \times n}$ such that for any $C \in \mathbf{\Omega}$, the resulting system characterized by the matrix triple (A, B, C) has the following properties: its invariant zeros are given by Λ_1, and their invariant indices $\mathscr{I}_2 = \Lambda_2$, $\mathscr{I}_3 = \Lambda_3$ and $\mathscr{I}_4 = \Lambda_4$, or equivalently, the infinite zero structure of the triple (A, B, C) is given by Λ_4, and its invertibility structures are, respectively, given by Λ_2 and Λ_3. Fig. 1 *summarizes in a graphical form the above general structural assignment.*

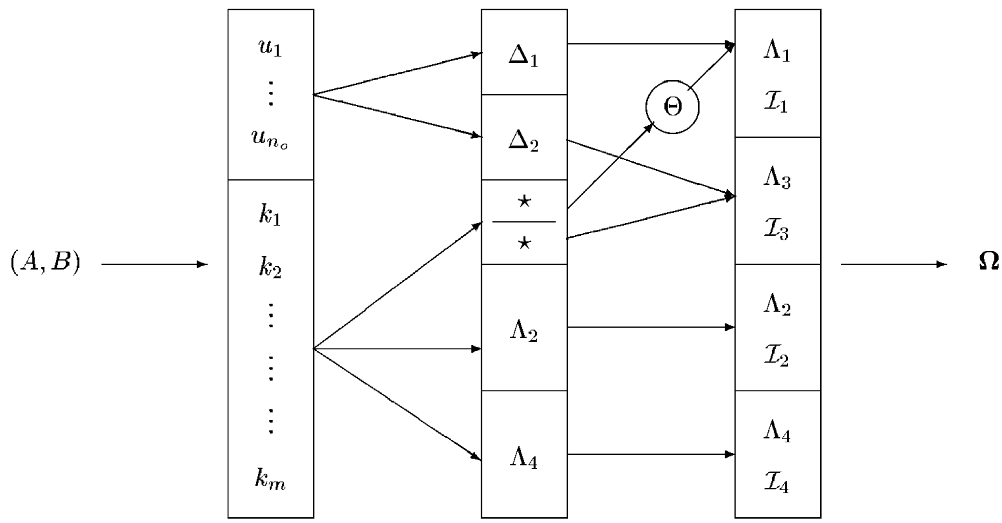

Fig. 1. Graphical summary of the general structural assignment.

Proof. We will give a constructive proof that would yield a desired set $\mathbf{\Omega}$. We first introduce the following key lemma, which is crucial to the proof of Theorem 3.1.

Lemma 3.1. *Consider a linear system $\tilde{\Sigma}$ characterized by a matrix triple $(\tilde{A}, \tilde{B}, \tilde{C})$. We assume that it is already in the form of the special coordinate basis of Theorem* 2.2 *or in the compact form of* (13) *and* (14). *Let*

$$\breve{A} := \begin{bmatrix} A_{aa} & M_{ab} & 0 & M_{ad} \\ 0 & A_{bb} & 0 & L_{bd}C_d \\ B_cE_{ca} & B_cE_{cb} & A_{cc} & M_{cd} \\ B_dE_{da} & B_dE_{db} & B_dE_{dc} & A_{dd} \end{bmatrix}, \tag{32}$$

where M_{ab}, M_{ad} and M_{cd} are arbitrary matrices of appropriate dimensions. Then, the triple $(\breve{A}, \tilde{B}, \tilde{C})$ has the same structural invariant indices $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{I}_3$ and $\mathscr{I}_4$ as those of $\tilde{\Sigma}$.

Proof. It is omitted due to space limitation.

Now, we are ready to give a proof to Theorem 3.1. It follows from Theorem 2.1 that there exist non-singular state and input transformations T_0 and T_i such that the transformed pair,

$$(A_1, B_1) := (T_0^{-1}AT_0, T_0^{-1}BT_i), \tag{33}$$

is in the CSD form of (3) with its controllability index being as $\mathscr{C}(A,B) = \{n_0, k_1, \ldots, k_m\}$. In view of the properties of the special coordinate basis, it is simple to see that each input channel in B_1 could either be assigned to the state variables associated with x_c or x_d of the resulting system. However, if we assign a particular input channel to be a member of x_c of the desired system, we will have to assign the whole block associated with this particular channel to it. This is because of the following reasons: (1) the whole block is completely controllable by the input channel, and (2) both dynamics of x_a and x_b cannot be controlled by input channels associated with x_c. On the other hand, there is no such a constraint for the structure associated with x_d, i.e., the infinite zero structure.

Let Λ_2 and Λ_4 be given, respectively, as in (25) and (27), and let $n_c = \sum_{i=1}^{m_c} \ell_i$ and $n_d = \sum_{i=1}^{m_d} q_i$. It is simple to verify that there exist permutation transformations P_1 and P_{i1} such that

$$A_2 = P_1^{-1}A_1P_1 = \begin{bmatrix} A_0 & 0 & 0 \\ B_c \cdot \bigstar & A_{cc} & B_c \cdot \bigstar \\ \tilde{B}_d \cdot \bigstar & \tilde{B}_d \cdot \bigstar & A_* \end{bmatrix},$$

$$B_2 = P_1^{-1}B_1P_{i1} = \begin{bmatrix} 0 & 0 \\ B_c & 0 \\ 0 & \tilde{B}_d \end{bmatrix}, \tag{34}$$

where

$$A_{cc} := \begin{bmatrix} 0 & I_{\ell_1 - 1} & \ldots & 0 & 0 \\ \bigstar & \bigstar & \ldots & \bigstar & \bigstar \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & I_{\ell_{m_c} - 1} \\ \bigstar & \bigstar & \ldots & \bigstar & \bigstar \end{bmatrix},$$

$$B_c = \begin{bmatrix} 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ 0 & \ldots & 1 \end{bmatrix} \tag{35}$$

and

$$A_* := \begin{bmatrix} 0 & I_{\omega_1-q_1-1} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{q_1-1} & \dots & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \dots & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & I_{\omega_{m_d}-q_{m_d}-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_{q_{m_d}-1} \\ \star & \star & \star & \star & \dots & \star & \star & \star & \star \end{bmatrix}, \quad \tilde{B}_d = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}. \tag{36}$$

Next, it is simple to see that there exists another pair of permutation matrices P_2 and P_{i2} such that the transformed pair $(A_3, B_3) := (P_2^{-1}A_2P_2, P_2^{-1}B_2P_{i2})$ has the following form:

$$A_3 = \begin{bmatrix} A_0 & 0 & 0 & 0 \\ 0 & A_{ab}^* & 0 & \star \\ B_c\cdot\star & B_c\cdot\star & A_{cc} & B_c\cdot\star \\ B_d\cdot\star & B_d\cdot\star & B_d\cdot\star & A_{dd}^* + B_d\cdot\star \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix}, \tag{37}$$

where

$$A_{dd}^* = \begin{bmatrix} 0 & I_{q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{q_{m_d}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix} \tag{38}$$

and

$$A_{ab}^* = \begin{bmatrix} 0 & I_{\omega_1-q_1-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I_{\omega_{m_d}-q_{m_d}-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{39}$$

Let us define

$$C_d = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{40}$$

which is in conformity with the structures of A_{dd}^* and B_d in (38), and we further define

$$C_3 = [\,0 \quad 0 \quad 0 \quad C_d\,], \tag{41}$$

which is in conformity with structures of A_3 and B_3 in (37). Following the proof of Lemma 3.1, we can show that there exists a non-singular state transformation T_3 such that

$$A_4 = T_3^{-1}A_3T_3 = \begin{bmatrix} A_{ab} & 0 & L_{abd}C_d \\ B_c\cdot\star & A_{cc} & L_{cd}C_d \\ B_d\cdot\star & B_d\cdot\star & A_{dd}^* + B_d\cdot\star \end{bmatrix},$$

$$B_4 = T_3^{-1}B_3 = \begin{bmatrix} 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix} \tag{42}$$

and

$$C_4 = C_3T_3 = C_3 = [\,0 \quad 0 \quad C_d\,], \tag{43}$$

where

$$A_{ab} = \begin{bmatrix} A_0 & 0 \\ 0 & A_{ab}^* \end{bmatrix}, \quad L_{abd} = \begin{bmatrix} 0 \\ L_{abd}^* \end{bmatrix}. \tag{44}$$

In view of the properties of the special coordinate basis, it is simple to see that the triple (A_4, B_4, C_4) is in the form of the SCB with its structural invariant indices $\mathscr{I}_2 = \Lambda_2$ and $\mathscr{I}_4 = \Lambda_4$, $\mathscr{I}_3$ being empty and its invariant zeros being $\lambda(A_{ab})$.

Next, we define a new output matrix,

$$\check{C}_4 := C_4 + [\,K_c \quad 0 \quad 0\,] = [\,K_c \quad 0 \quad C_d\,], \tag{45}$$

where $K_c = [K_{c1} \;\; K_{c2}]$ is partitioned in conformity with A_{ab} and L_{abd} in (44) with K_{c1} being an arbitrary matrix with appropriate dimension and K_{c2} being chosen such that $\Theta \subset \lambda(A_{ab}^* - L_{abd}^* K_{c2})$, and the remaining eigenvalues of $A_{ab}^* - L_{abd}^* K_{c2}$ are real and distinct. Moreover, these remaining eigenvalues of $A_{ab}^* - L_{abd}^* K_{c2}$ are distinct from the entries of Δ_2. This can be done because the pair (A_{ab}^*, L_{abd}^*) is completely controllable. Using the result of Chen, Saberi, and Sannuti (1992), we can show that there exists a state transformation T_4 such that

$$A_5 = T_4^{-1} A_4 T_4 = \begin{bmatrix} A_{ab} - L_{abd}K_c & 0 & \tilde{L}_{abd}C_d \\ B_c \cdot \star & A_{cc} & L_{cd}C_d \\ B_d \cdot \star & B_d \cdot \star & A_{dd} + B_d \cdot \star \end{bmatrix},$$

$$B_5 = T_4^{-1} B_4 = \begin{bmatrix} 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix} \tag{46}$$

and

$$C_5 := \check{C}_4 T_4 = [\,0 \quad 0 \quad C_d\,]. \tag{47}$$

Again, the triple (A_5, B_5, C_5) is in the form of SCB and has the same structural indices $\mathscr{I}_2$, $\mathscr{I}_3$ and $\mathscr{I}_4$ as the triple (A_4, B_4, C_4). Moreover, its invariant zeros are given by the eigenvalues of matrix $A_{ab} - L_{abd}K_c$, in which matrix $A_{ab} - L_{abd}K_c$ can be rewritten as

$$A_{ab} - L_{abd}K_c = \begin{bmatrix} A_0 & 0 \\ -L_{abd}^* K_{c1} & A_{ab}^* - L_{abd}^* K_{c2} \end{bmatrix}. \tag{48}$$

We next find a transformation T_{ab} such that $A_{ab} - L_{abd}K_c$ is transformed into the following form:

$$\tilde{A}_{ab} = T_{ab}^{-1}(A_{ab} - L_{abd}K_c)T_{ab} = \begin{bmatrix} A_{aa} & M_{ab} \\ 0 & A_{bb} \end{bmatrix}, \tag{49}$$

where $\lambda(A_{aa}) = \Lambda_1 = \Delta_1 \cup \Theta$ with Θ being given in (28), and A_{bb} being a diagonal matrix. Let

$$T_5 = \begin{bmatrix} T_{ab} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{50}$$

Then, we have

$$A_6 = T_5^{-1} A_5 T_5 = \begin{bmatrix} A_{aa} & M_{ab} & 0 & L_{ad}C_d \\ 0 & A_{bb} & 0 & L_{bd}C_d \\ B_c \cdot \star & B_c \cdot \star & A_{cc} & L_{cd}C_d \\ B_d \cdot \star & B_d \cdot \star & B_d \cdot \star & A_{dd} + B_d \cdot \star \end{bmatrix},$$

$$B_6 = T_5^{-1} B_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c \\ B_d & 0 \end{bmatrix} \tag{51}$$

and

$$C_6 := C_5 T_5 = [0 \quad 0 \quad 0 \quad C_d]. \tag{52}$$

The remaining task is to assign the structural invariant indices $\mathscr{I}_3$ to coincide with the given set $\Lambda_3 = \{\mu_1, \ldots, \mu_{p_b}\}$, which can be done by choosing the following output matrix:

$$\tilde{C}_6 = \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix}$$

with

$$C_b = \begin{bmatrix} C_{b1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{bp_b} \end{bmatrix}, \tag{53}$$

where C_{bi}, $i = 1, \ldots, p_b$, is a $1 \times \mu_i$ vector with all its entries being nonzero. Utilizing the result of Lemma 3.1 one more time, we can show that the triple characterized by $(A_6, B_6, \tilde{C}_6)$ has its invariant zeros at $\lambda(A_{aa})$, and its structural invariant indices $\mathscr{I}_2 = \Lambda_2$, $\mathscr{I}_3 = \Lambda_3$ and $\mathscr{I}_4 = \Lambda_4$, respectively. Let $p = m_d + p_b$. We finally obtain the desired set,

$$\mathbf{\Omega} = \left\{ \Gamma_0 \begin{bmatrix} 0 & 0 & 0 & C_d \\ 0 & C_b & 0 & 0 \end{bmatrix} (T_0 P_1 P_2 T_3 T_4 T_5)^{-1} \,\middle|\, \Gamma_0 \in \mathbb{R}^{p \times p} \text{ is non-singular} \right\}. \tag{54}$$

This completes the proof of Theorem 3.1. □

Remark 3.1. We have the following interesting special cases:

1. If Λ_2 and Λ_3 are chosen to be empty sets, and $\Lambda_4 = \{1, 1, \ldots, 1\}$, the result of Theorem 3.1 will yield a square invertible system (A, B, C) with m infinite zeros of order one. This is corresponding to the result reported in Syrmos (1993).
2. If Λ_2 and Λ_3 are chosen to be empty sets, and Λ_4 is appropriately selected, then the result of Theorem 3.1 will yield again a square invertible system (A, B, C) with appropriate finite and infinite zero structures. Such a result was reported earlier in Chen and Zheng (1995).
3. If Λ_2 is set to be empty, then the resulting system will be left invertible. Similarly, if Λ_3 is set to be empty, the resulting system will be right invertible.

Remark 3.2. It was shown in Chen and Zheng (1995) that the set $\mathbf{\Omega}$ is complete for single-input–single-output systems. In general, we should note that $\mathbf{\Omega}$ is not necessarily complete.

Remark 3.3. We note that if the entries of Λ_2 are not distinct, then the assignment of Λ_3 will be slightly more complicated. We would have to utilize the technique of the real Jordan canonical form (see e.g., Chen, 2000) to assign Λ_3 in accordance with the real Jordan block structure of the part of A_0 assigned to Λ_3. We leave this as an exercise to interested readers.

4. Conclusions

We have proposed a systematic method for constructing a family of output matrices for a given matrix pair (A, B). Any matrix C from this family will result in a linear system (A, B, C) that has the pre-specified structural properties. This method is also applicable to the dual problem of actuator selection, in which the pair (A, C) is given and the input matrix B is constructed such that the resulting linear system (A, B, C) has the pre-specified desired structural properties.

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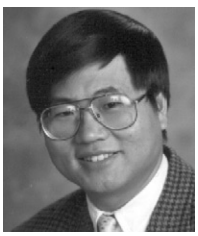

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