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Brief Paper

Robust and perfect tracking of discrete-time systems☆

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Abstract

We study in this paper the problem of robust and perfect tracking for discrete-time linear multivariable systems. By robust and perfect tracking, we mean the ability of a controller to track a given reference signal with arbitrarily small settling time in the face of external disturbances and initial conditions. A set of necessary and sufficient conditions under which the proposed problem is solvable is obtained and, under these conditions, constructive algorithms are given that yield the required solutions. As is general to discrete-time systems, the solvability conditions to the above problem are quite restrictive. To relax these conditions, we propose an almost perfect tracking scheme, which is capable of tracking references precisely after certain initial steps.

Keywords: Robust control; Tracking; Disturbance rejection; Discrete-time systems; Control Systems

1. Introduction to the problem

The problem of robust and perfect tracking (RPT) is to design a controller such that the resulting closed-loop system is asymptotically stable and the controlled output perfectly tracks a given reference signal in the presence of any initial conditions and external disturbances. By robust and perfect tracking, we mean the ability of a controller to track a given reference signal with arbitrarily small settling time in the face of external disturbances and initial conditions. The subject of continuous-time systems has been fully examined in a recent work by Liu, Chen, and Lin (2001). The main objective of this paper is to study such a problem for discrete-time systems. To be more specific, we present in this paper the robust and perfect tracking problem for the following discrete-time system,

$$\Sigma: \begin{cases} x(k+1) = Ax(k) + Bu(k) + Ew(k), \quad x(0) = x_0, \\ y(k) = C_1 x(k) + D_1 w(k), \\ h(k) = C_2 x(k) + D_2 u(k) + D_{22} w(k), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance, $y \in \mathbb{R}^p$ is the measurement output, and $h \in \mathbb{R}^\ell$ is the output to be controlled. We also assume that the pair (A, B) is stabilizable and the pair (A, C_1) is detectable. For future references, we define Σ_P and Σ_Q to be the subsystems characterized by the matrix quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1), respectively. Given the external disturbance $w \in l_p$, $p \in [1, \infty]$, and any reference signal vector $r \in \mathbb{R}^\ell$, the RPT problem for the discrete-time system (1) is to find a parameterized dynamic measurement feedback control law of the following form:

$$\begin{aligned} v(k+1) &= A_{\rm cmp}(\varepsilon)v(k) + B_{\rm cmp}(\varepsilon)y(k) + G(\varepsilon)r(k), \\ u(k) &= C_{\rm cmp}(\varepsilon)v(k) + D_{\rm cmp}(\varepsilon)y(k) + H(\varepsilon)r(k), \end{aligned} \tag{2}$$

such that when (2) is applied to (1):

(i) There exists an $\varepsilon^* > 0$ such that the resulting closed-loop system with $r = 0$ and $w = 0$ is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$; and
(ii) Let $h(k, \varepsilon)$ be the closed-loop controlled output response and let $e(k, \varepsilon)$ be the resulting tracking error, i.e., $e(k, \varepsilon) := h(k, \varepsilon) - r(k)$. Then, for any $x_0 \in \mathbb{R}^n$, $\|e\|_p \to 0$ as $\varepsilon \to 0$.

In this paper, we will derive a set of necessary and sufficient conditions under which the proposed robust

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and perfect tracking problem has a solution, and under these conditions, develop algorithms for the construction of feedback laws that solve the proposed problem. It turns out that the solvability conditions for the above proposed RPT problem are quite restrictive compared to its counterpart in the continuous-time case (see Liu et al., 2001). In particular, the discrete-time problem has a quite restrictive constraint on the infinite zero structure of the given system, while the infinite zero structure can be arbitrary in its continuous-time counterpart. To relax these conditions, we will introduce a modified problem, which can be solved for a much larger class of discrete-time systems. This modified formulation will yield internally stabilizing control laws that are capable of tracking reference signals with some delays. If we know the reference signal a few steps ahead, the modified tracking control scheme will then track the reference precisely after certain steps, provided that the given plant satisfies a new set of more relaxed conditions. We note that this is a unique feature of the discrete-time problem. It is impossible to achieve such a performance for continuous-time systems.

Throughout this paper, the following notation will be used: $X^\dagger$ denotes the pseudo inverse of X; $\mathbb{C}^\bullet$, $\mathbb{C}^\circ$ and $\mathbb{C}^\otimes$ are, respectively, the open unit disc, the unit circle and the set of complex numbers outside the unit circle on the complex plane; $\mathrm{Ker}(X)$ is the kernel of X; $\mathrm{Im}(X)$ is the image of X; Given a linear system Σ_* characterized by a matrix quadruple (A,B,C,D), we define (see e.g., Chen, 2000):

(i) $\mathscr{V}^\bullet(\Sigma_*)$ is the maximal subspace of $\mathbb{R}^n$ for which there exists an F such that $\mathscr{V}^\bullet$ is $(A+BF)$-invariant and is contained in $\mathrm{Ker}(C+DF)$, and the eigenvalues of $(A+BF)\,|\,\mathscr{V}^\bullet$ are contained in $\mathbb{C}^\bullet$.
(ii) $\mathscr{S}^\bullet(\Sigma_*)$ is the minimal subspace of $\mathbb{R}^n$ for which there exists a K such that $\mathscr{S}^\bullet$ is $(A+KC)$-invariant and contains $\mathrm{Im}(B+KD)$, and the eigenvalues of the map which is induced by $(A+KC)$ on the factor space $\mathbb{R}^n/\mathscr{S}^\bullet$ are contained in $\mathbb{C}^\bullet$.

2. Solvability conditions and solutions

The following theorem gives a set of necessary and sufficient conditions under which the proposed RPT problem is solvable for the given plant (1). We will show the sufficiency of these conditions by explicitly constructing the required control laws.

Theorem 1. *Consider the given system* (1) *with its external disturbance* $w\in l_p$, $p\in[1,\infty]$, *and its initial condition* $x(0)=x_0$. *Then, for any reference signal* $r(k)$, *the proposed robust and perfect problem is solvable by the control law of* (2) *if and only if the following conditions are satisfied*: (1) (A,B) *is stabilizable and* (A,C_1) *is detectable*; (2) $D_{22}+D_2SD_1=0$, *where* $S=-(D_2'D_2)^\dagger D_2'\,D_{22}D_1'(D_1D_1')^\dagger$; (3) Σ_P *is right invertible and of minimum phase with no infinite zeros*; *and* (4) $\mathrm{Ker}(C_2+D_2SC_1)\supset C_1^{-1}\{\mathrm{Im}(D_1)\}:=\{\zeta\,|\,C_1\zeta\in\mathrm{Im}(D_1)\}$.

Proof. We first show that Conditions 1–4 in the theorem are necessary. Let us consider the case when $r(k)\equiv 0$. Then, the proposed robust and perfect tracking problem reduces to the perfect regulation problem. We can reformulate the perfect regulation problem for the given system (1) as the well studied almost disturbance decoupling problem for the following system:

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k)+[E\quad I]\tilde{w}(k), \quad x(0)=0,\\ y(k) &= C_1x(k)+[D_1\quad 0]\tilde{w}(k), \\ h(k) &= C_2x(k)+D_2u(k)+[D_{22}\quad 0]\tilde{w}(k). \end{aligned} \tag{3}$$

For easy reference, we let $\tilde{\Sigma}_\mathrm{Q}$ be characterized $(A,[E\ \ I],C_1,[D_1\ \ 0])$. Following the results of the discrete-time almost disturbance decoupling problem (see Chen, 2000), we can show that if the almost disturbance decoupling problem for the above system is solvable, we have: (i) (A,B) is stabilizable and (A,C_1) is detectable; (ii) $D_{22}+D_2SD_1=0$, where $S=-(D_2'D_2)^\dagger D_2'D_{22}D_1'(D_1D_1')^\dagger$; (iii) $\mathrm{Im}([E+BSD_1\ \ I])\subset\mathscr{V}^\bullet(\Sigma_\mathrm{P})+B\,\mathrm{Ker}(D_2)$; (iv) $\mathrm{Ker}(C_2+D_2SC_1)\supset\mathscr{S}^\bullet(\tilde{\Sigma}_\mathrm{Q})\cap C_1^{-1}\{\mathrm{Im}(D_1)\}$; and finally (v) $\mathscr{S}^\bullet(\tilde{\Sigma}_\mathrm{Q})\subset\mathscr{V}^\bullet(\Sigma_\mathrm{P})$.

Next, it is easy to see that $\mathscr{S}^\bullet(\tilde{\Sigma}_\mathrm{Q})=\mathbb{R}^n$ and hence Condition (v) implies that $\mathscr{V}^\bullet(\Sigma_\mathrm{P})=\mathbb{R}^n$, or equivalently, Σ_P is right invertible with no infinite zeros and no invariant zeros in $\mathbb{C}^\otimes$. Furthermore, Condition (iv) reduces to the condition $\mathrm{Ker}(C_2+D_2SC_1)\supset C_1^{-1}\{\mathrm{Im}(D_1)\}$. Thus, it remains to be shown that if the proposed RPT problem is solvable, the subsystem Σ_P must be of minimum phase. In what follows, we proceed to show such a fact:

First, we note that the second condition, i.e., $D_{22}+D_2SD_1=0$, implies that if we apply a pre-output feedback law, $u(k)=Sy(k)$, to System (1), the resulting new system will have a zero direct feed-through term from w to h. Hence, without loss of any generality, we hereafter assume that $D_{22}=0$ throughout the rest of the paper.

Next, we show that if the robust and perfect tracking problem is solvable for general nonzero reference $r(k)$, Σ_P must be of minimum phase, i.e., Σ_P cannot have any invariant zeros on the unit circle. In fact, this condition must hold even for the case when $w=0$ and $x_0=0$, i.e., for the robust and perfect tracking of the following system:

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k),\\ y(k) &= C_1x(k), \\ e(k) &= C_2x(k)+D_2u(k)-r(k)=h(k)-r(k). \end{aligned} \tag{4}$$

Now, if we treat r as an external disturbance, then the above problem is again equivalent to the well-known almost disturbance decoupling problem with measurement feedback and with internal stability for the following system:

$$x(k+1) = Ax(k) + Bu(k),$$
$$\tilde{y}(k) = \begin{pmatrix} C_1 x(k) \\ r(k) \end{pmatrix}, \tag{5}$$
$$e(k) = C_2 x(k) + D_2 u(k) - r(k).$$

Without loss of generality, we assume that the quadruple (A, B, C_2, D_2) has been transformed into the form of the special coordinate basis of Sannuti and Saberi (1987) (see also Chen, 2000), i.e., we have

$$x = \begin{pmatrix} x_a^- \\ x_a^0 \\ x_c \end{pmatrix}, \quad h = h_0, \quad r = r_0, \quad e = e_0 = h_0 - r_0, \tag{6}$$
$$u = \begin{pmatrix} u_0 \\ u_c \end{pmatrix}$$

and

$$x_a^-(k+1) = A_{aa}^- x_a^-(k) + B_{0a}^- h_0(k), \tag{7}$$
$$x_a^0(k+1) = A_{aa}^0 x_a^0(k) + B_{0a}^0 h_0(k), \tag{8}$$
$$x_c(k+1) = A_{cc} x_c(k) + B_{0c} h_0(k) + B_c[E_{ca}^- x_a^-(k) + E_{ca}^0 x_a^0(k)] + B_c u_c(k), \tag{9}$$
$$e_0(k) = C_{2,0a}^- x_a^-(k) + C_{2,0a}^0 x_a^0(k) + C_{2,0c} x_c(k) + u_0(k) - r_0(k), \tag{10}$$

where $\lambda(A_{aa}^-)$ and $\lambda(A_{aa}^0)$ are, respectively, the invariant zeros of (A, B, C_2, D_2) in $\mathbb{C}^-$ and $\mathbb{C}^0$, and (A_{cc}, B_c) is controllable. In order to bring the subsystem from u to e into the standard form of the special coordinate basis, we need to change h_0 in (7)–(9) to $e_0 + r_0$, i.e.,

$$x_a^-(k+1) = A_{aa}^- x_a^-(k) + B_{0a}^- e_0(k) + B_{0a}^- r_0(k), \tag{11}$$
$$x_a^0(k+1) = A_{aa}^0 x_a^0(k) + B_{0a}^0 e_0(k) + B_{0a}^0 r_0(k), \tag{12}$$
$$x_c(k+1) = A_{cc} x_c(k) + B_{0c} e_0(k) + B_c[E_{ca}^- x_a^-(k) + E_{ca}^0 x_a^0(k)] + B_c u_c(k) + B_{0c} r_0(k). \tag{13}$$

It is easy to see that the subsystem from the controlled input, i.e., $(u_0' \ \ u_c')'$, to the error output, i.e., e_0, is now in the standard form of the special coordinate basis. It then follows from the result of Chen (2000) (i.e., Proposition 12.2.1) that if the almost disturbance decoupling problem with measurement feedback and with internal stability for system (5) is solvable, there must exist a nonzero vector ξ such that $\xi^{\mathrm{H}}(\lambda I - A_{aa}^0) = 0$ and $\xi^{\mathrm{H}} B_{0a}^0 = 0$, which implies that the matrix pair (A_{aa}^0, B_{0a}^0) is not completely controllable. Following the property of the special coordinate basis (see Chen, 2000), the uncontrollability of (A_{aa}^0, B_{0a}^0) implies the unstabilizability of the pair (A, B), which is obviously a contradiction. Hence, x_a^0 must be nonexistent and Σ_{P} must be of minimum phase. This proves the necessary part.

We note that for the case when $D_1 = 0$, the direct feedthrough term D_{22} must be a zero matrix as well, and the last condition, i.e., Condition (iv), of Theorem 1 reduces to $\mathrm{Ker}(C_2) \supset \mathrm{Ker}(C_1)$.

We will show the sufficiency of those conditions in Theorem 1 by explicitly constructing controllers which solve the proposed robust and perfect tracking problem under Conditions 1–4 of Theorem 1. This will be done in the next subsections. It turns out that the control laws, which solve the robust and perfect tracking for the given plant (1) under the solvability of Theorem 1, need not be parameterized by any tuning parameter. Thus, (2) can be replaced by

$$\begin{aligned} v(k+1) &= A_{\mathrm{cmp}} v(k) + B_{\mathrm{cmp}} y(k) + Gr(k), \\ u(k) &= C_{\mathrm{cmp}} v(k) + D_{\mathrm{cmp}} y(k) + Hr(k) \end{aligned} \tag{14}$$

and furthermore, the resulting tracking error $e(k)$ can be made identically zero for all $k \geqslant 0$.

2.1. Solutions to state feedback case

When all states of the plant are measured for feedback, i.e., $C_1 = I$ and $D_1 = 0$, the problem can be solved by a static control law. In fact, the conditions in Theorem 1 are reduced to: (i) (A, B) is stabilizable; (ii) $D_{22} = 0$; and (iii) Σ_{P} is right invertible and of minimum phase with no infinite zeros. We construct in this subsection a state feedback control law,

$$u = Fx + Hr \tag{15}$$

which solves the robust and perfect tracking (RPT) problem for (1) under the above conditions.

Step S.1: This step is to transform the subsystem from u to h of the given system (1) into the special coordinate basis, i.e., to find nonsingular state, input and output transformations Γ_{s}, Γ_{i} and Γ_{o} to put it into the structural form (see Chen, 2000, for details), i.e.,

$$\Gamma_{\mathrm{s}}^{-1}(A - B_0 C_{2,0})\Gamma_{\mathrm{s}} = \begin{bmatrix} A_{aa}^- & 0 \\ B_c E_{ca}^- & A_{cc} \end{bmatrix}, \tag{16}$$

$$\Gamma_{\mathrm{s}}^{-1} B \Gamma_{\mathrm{i}} = \Gamma_{\mathrm{s}}^{-1}[B_0 \quad B_1]\Gamma_{\mathrm{i}} = \begin{bmatrix} B_{0a}^- & 0 \\ B_{0c} & B_c \end{bmatrix}, \tag{17}$$

$$\Gamma_{\mathrm{o}}^{-1} D_2 \Gamma_{\mathrm{i}} = [I_{m_0} \quad 0], \tag{18}$$

$$\Gamma_{\mathrm{o}}^{-1} C_2 \Gamma_{\mathrm{s}} = \Gamma_{\mathrm{o}}^{-1} C_{2,0} \Gamma_{\mathrm{s}} = [C_{2,0a}^- \quad C_{2,0c}], \tag{19}$$

where $\lambda(A_{aa}^-)$ are the invariant zeros of (A, B, C_2, D_2) and (A_{cc}, B_c) is controllable.

Step S.2: Choose an appropriate F_c such that $A_{cc}^c = A_{cc} - B_c F_c$ is stable.

Step S.3: Finally, we let

$$F = -\Gamma_{\rm i}\begin{bmatrix} C_{2,0a}^- & C_{2,0c} \\ E_{ca}^- & F_c \end{bmatrix}\Gamma_{\rm s}^{-1} \quad \text{and} \quad H = \Gamma_{\rm i}\begin{bmatrix} I \\ 0 \end{bmatrix}\Gamma_{\rm o}^{-1}. \tag{20}$$

Theorem 2. *Consider the discrete-time system* (1) *with any external disturbance* $w(k)$ *and any initial condition* $x(0)$. *Assume that all its states are measured for feedback. If* $D_{22}=0$ *and* $\Sigma_{\rm P}$ *is stabilizable, right invertible and of minimum phase with no infinite zeros, then, for any reference signal* $r(k)$, *the proposed robust and perfect tracking is solved by the control law of* (15) *with* F *and* H *as given in* (20).

Proof. It follows from some simple calculations. □

2.2. Solutions to measurement feedback case

Without loss of generality, we assume throughout this subsection that matrix $D_{22}=0$. It turns out that, for discrete-time systems, the full order observer based control law is not capable of achieving the robust and perfect tracking performance, because there is a delay of one step in the observer itself. Thus, we will focus on the construction of a reduced order measurement feedback control law to solve the RPT problem. For simplicity of presentation, we assume that matrices C_1 and D_1 have already been transformed into the following forms:

$$C_1 = \begin{bmatrix} 0 & C_{1,02} \\ I_\kappa & 0 \end{bmatrix} \quad \text{and} \quad D_1 = \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix}, \tag{21}$$

where $D_{1,0}$ is of full row rank. We first partition the following system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + [E \quad I_n]\tilde{w}(k), \\ y(k) &= C_1 x(k) + [D_1 \quad 0]\tilde{w}(k) \end{aligned} \tag{22}$$

in conformity with the structures of C_1 and D_1 in (21), i.e.,

$$\begin{pmatrix} \delta(x_1) \\ \delta(x_2) \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} E_1 & I_\kappa & 0 \\ E_2 & 0 & I_{n-\kappa} \end{bmatrix}\tilde{w}, \tag{23}$$

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{bmatrix} 0 & C_{1,02} \\ I_\kappa & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} D_{1,0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\tilde{w},$$

where $\delta(x_1) = x_1(k+1)$ and $\delta(x_2) = x_2(k+1)$. Obviously, $y_1 = x_1$ is directly available and hence need not be estimated. Next, let $\Sigma_{\rm QR}$ be characterized by

$$(A_{\rm R}, E_{\rm R}, C_{\rm R}, D_{\rm R}) = \left(A_{22},\ [E_2 \quad 0 \quad I_{n-\kappa}],\ \begin{bmatrix} C_{1,02} \\ A_{12} \end{bmatrix},\ \begin{bmatrix} D_{1,0} & 0 & 0 \\ E_1 & I_\kappa & 0 \end{bmatrix}\right). \tag{24}$$

It is straightforward to verify that $\Sigma_{\rm QR}$ is right invertible with no finite and infinite zeros. Moreover, $(A_{\rm R}, C_{\rm R})$ is detectable if and only if (A, C_1) is detectable. We are ready to present the following algorithm.

Step R.1: For the given system (1), we again assume that all the state variables of (1) are measurable and then follow Steps S.1–S.3 of the algorithm of the previous subsection to construct gain matrices F and H. We also partition F in conformity with x_1 and x_2 of (23) as follows:

$$F = [F_1 \quad F_2]. \tag{25}$$

Step R.2: Let $K_{\rm R}$ be an appropriate dimensional constant matrix such that the eigenvalues of

$$A_{\rm R} + K_{\rm R} C_{\rm R} = A_{22} + [K_{\rm R0} \quad K_{\rm R1}]\begin{bmatrix} C_{1,02} \\ A_{12} \end{bmatrix} \tag{26}$$

are all in $\mathbb{C}^{\odot}$. This can be done because $(A_{\rm R}, C_{\rm R})$ is detectable.

Step R.3: Let $G = (B_2 + K_{\rm R1}B_1)H$, and

$$G_{\rm R} = [-K_{\rm R0},\quad A_{21} + K_{\rm R1}A_{11} - (A_{\rm R} + K_{\rm R}C_{\rm R})K_{\rm R1}], \tag{27}$$

$$\begin{aligned} A_{\rm cmp} &= A_{\rm R} + B_2F_2 + K_{\rm R}C_{\rm R} + K_{\rm R1}B_1F_2, \\ B_{\rm cmp} &= G_{\rm R} + (B_2 + K_{\rm R1}B_1)[0,\quad F_1 - F_2K_{\rm R1}], \\ C_{\rm cmp} &= F_2, \\ D_{\rm cmp} &= [0,\quad F_1 - F_2K_{\rm R1}]. \end{aligned} \tag{28}$$

Step R.4: Finally, we obtain the following reduced order measurement feedback control law:

$$\begin{aligned} v(k+1) &= A_{\rm cmp}v(k) + B_{\rm cmp}y(k) + Gr(k), \\ u(k) &= C_{\rm cmp}v(k) + D_{\rm cmp}y(k) + Hr(k). \end{aligned} \tag{29}$$

Theorem 3. *Consider the given system* (1) *with any external disturbance* $w(k)$ *and any initial condition* $x(0)$. *If Conditions* 1–4 *of Theorem* 1 *are satisfied, then, for any reference signal* $r(k)$, *the proposed RPT is solved by the reduced order measurement feedback control laws of* (29).

Proof. Due to space limitation, the proof is omitted (see Chen, 2000 for details).

The sufficiency of Theorem 1 is obvious now in view of the result of Theorem 3. The proof of Theorem 1 is thus completed. □

3. An almost perfect tracking problem

As has been seen in the previous section, the solvability conditions for the robust and perfect tracking problem are generally too strong. We introduce now a modified problem, the almost perfect tracking problem, which can be solved for a much larger class of discrete-time systems with any infinite zero structure. This modified formulation will yield internally stabilizing control laws that are capable of tracking reference signal $r(k)$ with some delays. If we know the reference signal a few steps ahead, the modified tracking control scheme will then track the reference precisely after certain steps.

For simplicity, we consider in this section the discrete-time system (1) without external disturbances, i.e.,

$$\Sigma: \begin{cases} x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \\ y(k) = C_1 x(k), \\ h(k) = C_2 x(k) + D_2 u(k). \end{cases} \tag{30}$$

Let us first consider the case that the reference $r(k) \in \mathbb{R}^\ell$ to be tracked is a known vector sequence, which implies that $r(k+d)$, $0 \leqslant d \leqslant \kappa_d$, is known for some integer $\kappa_d \geqslant 0$. This is a quite reasonable assumption in most practical situations when one wants to track references such as step functions, ramp functions and sinusoidal functions. We will later deal with the case when $r(k+d)$, $d > 0$, is unknown. We are ready to formally define the almost perfect tracking problem. Given the discrete-time system (30) with initial condition $x(0) = x_0$ and the reference $r(k)$ with $r(k+d)$, $0 \leqslant d \leqslant \kappa_d$, being known for a nonnegative integer κ_d, the (κ_d,κ_0) almost perfect tracking problem, where κ_0 is another nonnegative integer, is to find a dynamic measurement feedback control law of the following form:

$$\begin{aligned} v(k+1) &= A_{\text{cmp}} v(k) + B_{\text{cmp}} y(k) \\ &\quad + G_0 r(k) + \cdots + G_{\kappa_d} r(k+\kappa_d), \\ u(k) &= C_{\text{cmp}} v(k) + D_{\text{cmp}} y(k) \\ &\quad + H_0 r(k) + \cdots + H_{\kappa_d} r(k+\kappa_d) \end{aligned} \tag{31}$$

such that when (31) is applied to (30), (1) the resulting closed-loop system is internally stable; and (2) for any initial condition $x_0 \in \mathbb{R}^n$, the resulting tracking error $e(k) \equiv 0$ or $h(k) \equiv r(k)$, for all $k \geqslant \kappa_0$.

Theorem 4. *Consider the discrete-time plant* (30) *with* $x(0) = x_0$, *and with* (i) (A,B) *being stabilizable and* (A,C_1) *being observable*; *and* (ii) Σ_P *being right invertible and of minimum phase. Let the infinite zero structure of* Σ_P *be given as* $S_\infty^\star(\Sigma_\text{P}) = \{q_1,\ldots,q_{m_d}\}$, *with* $q_1 \leqslant \cdots \leqslant q_{m_d}$, *and let the controllability index of* (A',C_1') *be* $\mathscr{C} = \{k_1,\ldots,k_p\}$, with $k_1 \leqslant \cdots \leqslant k_p$. *Then, the* (κ_d,κ_0) *almost perfect tracking problem is solvable for any reference with* $\kappa_d = q_{m_d}$ *and* $\kappa_0 = q_{m_d} + k_p - 1$.

Proof. We prove this theorem by explicitly constructing the required control law. Let us first construct the special coordinate basis of Σ_P. It follows from Sannuti and Saberi (1987) (see also Chen, 2000) that there exist nonsingular state, output and input transformations Γ_s, Γ_o and Γ_i, which will take Σ_P into the standard format of the special coordinate basis, i.e.,

$$x = \Gamma_\text{s}\tilde{x}, \quad h = \Gamma_\text{o}\tilde{h}, \quad u = \Gamma_\text{i}\tilde{u}, \quad r = \Gamma_\text{o}\tilde{r}, \tag{32}$$

$$\tilde{x} = \begin{pmatrix} x_a^- \\ x_c \\ x_d \end{pmatrix}, \quad \tilde{h} = \begin{pmatrix} h_0 \\ h_d \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \quad \tilde{r} = \begin{pmatrix} r_0 \\ r_d \end{pmatrix},$$

$$x_d = \begin{pmatrix} x_1 \\ \vdots \\ x_{m_d} \end{pmatrix}, \quad x_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{iq_i} \end{pmatrix}, \quad h_d = \begin{pmatrix} h_1 \\ \vdots \\ h_{m_d} \end{pmatrix},$$

$$r_d = \begin{pmatrix} r_1 \\ \vdots \\ r_{m_d} \end{pmatrix}, \quad u_d = \begin{pmatrix} u_1 \\ \vdots \\ u_{m_d} \end{pmatrix}$$

and

$$\delta(x_a^-) = A_{aa}^- x_a^- + B_{0a}^- h_0 + L_{ad}^- h_d, \tag{33}$$

$$\delta(x_c) = A_{cc} x_c + B_{0c} h_0 + L_{cd} h_d + B_c E_{ca}^- x_a^- + B_c u_c, \tag{34}$$

$$h_0 = C_{2,0a}^- x_a^- + C_{2,0c} x_c + C_{2,0d} x_d + u_0, \quad u_0 \in \mathbb{R}^{m_0} \tag{35}$$

and for each $i = 1,\ldots,m_d$, $x_i \in \mathbb{R}^{q_i}$ and

$$\begin{aligned} \delta(x_i) &= A_{q_i} x_i + L_{i0} h_0 + L_{id} h_d \\ &\quad + B_{q_i}\left[u_i + E_{ia}^- x_a^- + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} x_j\right], \end{aligned} \tag{36}$$

$$h_i = C_{q_i} x_i = x_{i1}, \quad h_d = C_d x_d, \tag{37}$$

where $\delta(\star) = \star(k+1)$, the triple $(A_{q_i}, B_{q_i}, C_{q_i})$ has the special structure as given in (2.4.16) of Chen (2000). Furthermore, L_{id} has the following special format:

$$L_{id} = [L_{i1} \quad L_{i2} \quad \cdots \quad L_{ii-1} \quad 0 \quad \cdots \quad 0]$$

with its last row always being identically zero. Next, we partition L_{i0} and L_{id}, $i = 1,\ldots,m_d$, as follows:

$$L_{i0} = \begin{bmatrix} L_{i0,1} \\ \vdots \\ L_{i0,q_i} \end{bmatrix}, \qquad L_{id} = \begin{bmatrix} L_{id,1} \\ \vdots \\ L_{id,q_i} \end{bmatrix} \tag{38}$$

and define a new controlled output

$$\tilde{h}_{\rm n}(k)=\begin{bmatrix} h_0(k) \\ h_1(k+q_1)-\sum_{j=1}^{q_1}[L_{10,j} \quad L_{1d,j}]\tilde{h}(k+q_1-j) \\ \vdots \\ h_{m_d}(k+q_{m_d})-\sum_{j=1}^{q_{m_d}}[L_{m_d0,j} \quad L_{m_dd,j}]\tilde{h}(k+q_{m_d}-j) \end{bmatrix}. \tag{39}$$

Then, it is straightforward to verify that $\tilde{y}_{\rm n}$ can be expressed as

$$\tilde{h}_{\rm n}(k)=\check{C}_2\tilde{x}(k)+\check{D}_2\tilde{u}(k) \tag{40}$$

with

$$\check{C}_2=\begin{bmatrix} C_{2,0a}^- & C_{2,0c} & C_{2,0d} \\ E_{da}^- & E_{dc} & E_{dd} \end{bmatrix} \tag{41}$$

and

$$\check{D}_2=\begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & I_{m_d} & 0 \end{bmatrix},$$

where

$$E_{da}^-=\begin{bmatrix} E_{1a}^- \\ \vdots \\ E_{m_da}^- \end{bmatrix},\; E_{dc}=\begin{bmatrix} E_{1c} \\ \vdots \\ E_{m_dc} \end{bmatrix},\; E_{dd}=\begin{bmatrix} E_{11} & \cdots & E_{1m_d} \\ \vdots & \ddots & \vdots \\ E_{m_d1} & \cdots & E_{m_dm_d} \end{bmatrix}. \tag{42}$$

Let $\tilde{A}=\Gamma_{\rm s}^{-1}A\Gamma_{\rm s}$ and $\tilde{B}=\Gamma_{\rm s}^{-1}B\Gamma_{\rm i}$, and let $\check{\Sigma}_{\rm P}$ be characterized by $(\tilde{A},\tilde{B},\check{C}_2,\check{D}_2)$. It is simple to show that the auxiliary system $\check{\Sigma}_{\rm P}$ is right invertible and of minimum phase with no infinite zeros.

We first assume that $C_1=I$ and follow Steps S.1–S.3 of the previous section to obtain a state feedback control law

$$\tilde{u}(k)=\tilde{F}\tilde{x}(k)+\tilde{H}\tilde{r}_{\rm n}(k), \tag{43}$$

where

$$\tilde{r}_{\rm n}(k)=\begin{bmatrix} r_0(k) \\ r_1(k+q_1)-\sum_{j=1}^{q_1}[L_{10,j} \quad L_{1d,j}]\tilde{r}(k+q_1-j) \\ \vdots \\ r_{m_d}(k+q_{m_d})-\sum_{j=1}^{q_{m_d}}[L_{m_d0,j} \quad L_{m_dd,j}]\tilde{r}(k+q_{m_d}-j) \end{bmatrix} \tag{44}$$

which has the following properties: (i) $\tilde{A}+\tilde{B}\tilde{F}$ is asymptotically stable, and (ii) the resulting $\tilde{h}_{\rm n}(k)=\tilde{r}_{\rm n}(k)$. This implies that the actual controlled output h is capable of precisely tracking the given reference $r(k)$ after q_{m_d} steps. Rewriting (43), we obtain

$$\begin{aligned} u(k)&=\Gamma_{\rm i}\tilde{u}(k)=\Gamma_{\rm i}[\tilde{F}\tilde{x}(k)+\tilde{H}\tilde{r}_{\rm n}(k)] \\ &=\Gamma_{\rm i}[\tilde{F}\tilde{x}(k)+\tilde{H}L_0\tilde{r}(k)+\tilde{H}L_1\tilde{r}(k+1) \\ &\quad+\cdots+\tilde{H}L_{m_d}\tilde{r}(k+q_{m_d})] \end{aligned} \tag{45}$$

for some $L_0,L_1,\dots,L_{m_d}$. Let $F=\Gamma_{\rm i}\tilde{F}\Gamma_{\rm s}^{-1}$, and $H_j=\Gamma_{\rm i}\tilde{H}L_j\Gamma_{\rm o}^{-1}$, for $j=0,1,\dots,m_d$. We have

$$\begin{aligned} u(k)&=Fx(k)+H_0r(k)+H_1r(k+1) \\ &\quad+\cdots+H_{m_d}r(k+q_{m_d}). \end{aligned} \tag{46}$$

Next, we proceed to construct a reduced order measurement feedback controller. We follow Steps R.1–R.3 of the previous section to obtain matrices B_1, B_2, F_1, F_2 and gain matrices $A_{\rm cmp}$, $B_{\rm cmp}$, $C_{\rm cmp}$ and $D_{\rm cmp}$ as given in (28). Note that the pair (A,C_1) is observable and (A',C_1') has a controllability index $\{k_1,\dots,k_p\}$, it is easy to show that $(A_{\rm R},C_{\rm R})$ is also observable and the controllability index of $(A_{\rm R}',C_{\rm R}')$ is given by $\{k_1-1,\dots,k_p-1\}$. It follows from Theorem 2.3.1 of Chen (2000) that there exists a gain matrix $K_{\rm R}$ such that $A_{\rm R}+K_{\rm R}C_{\rm R}$ has all its eigenvalues at the origin and

$$(A_{\rm R}+K_{\rm R}C_{\rm R})^{k_p-2}\equiv 0. \tag{47}$$

We thus choose such a $K_{\rm R}$ in constructing gain matrices $A_{\rm cmp}$, $B_{\rm cmp}$, $C_{\rm cmp}$ and $D_{\rm cmp}$. The reduced order measurement feedback law is then given by

$$\begin{aligned} v(k+1)&=A_{\rm cmp}v(k)+B_{\rm cmp}y(k)+\sum_{j=0}^{m_d}G_jr(k+j), \\ u(k)&=C_{\rm cmp}v(k)+D_{\rm cmp}y(k)+\sum_{j=0}^{m_d}H_jr(k+j), \end{aligned} \tag{48}$$

where $G_j=(B_2+K_{\rm R}B_1)H_j$, $i=0,1,\dots,m_d$. Let $x_s(k)=x_2(k)-v(k)+K_{\rm R}x_1(k)$. It is straightforward to verify that the closed-loop system comprising the given system (30) and the reduced order measurement feedback control law of (48) can be rewritten as follows:

$$x_s(k+1)=(A_{\rm R}+K_{\rm R}C_{\rm R})x_s(k), \tag{49}$$

$$x(k+1)=(A+BF)x(k)-BF_2x_s(k)+\sum_{j=0}^{m_d}BH_jr(k+j), \tag{50}$$

$$h(k)=(C_2+D_2F)x(k)-D_2F_2x_s(k)+\sum_{j=0}^{m_d}D_2H_jr(k+j). \tag{51}$$

Thus, it is easy to see that the closed-loop system is asymptotically stable as $A+BF$ and $A_{\rm R}+K_{\rm R}C_{\rm R}$ have eigenvalues inside the unit circle. Clearly, for any initial condition, (49) implies that $x_s(k)=0$ for all $k\geqslant k_p-1$. Hence, for $k\geqslant k_p-1$, (50) and (51) reduce to

$$\begin{aligned} x(k+1)&=(A+BF)x(k)+\sum_{j=0}^{m_d}BH_jr(k+j), \\ h(k)&=(C_2+D_2F)x(k)+\sum_{j=0}^{m_d}D_2H_jr(k+j) \end{aligned} \tag{52}$$

which are precisely the same as the closed-loop dynamics under the state feedback law. If we treat $x(k_p-1)$ as a new initial condition to (52), it will take another q_{m_d} steps for h to precisely track the reference r. Thus, we have $h(k)=r(k)$ for all $k \geqslant q_{m_d}+k_p-1$. Hence, the (κ_d,κ_0) almost perfect tracking problem is solved by the control law (31) with $\kappa_d=q_{m_d}$ and $\kappa_0=q_{m_d}+k_p-1$.

Remark 5. Consider the given plant (30) which has all properties as stated in Theorem 4. Then, the (κ_d,κ_0) almost perfect tracking problem is solvable by a full order measurement feedback controller of the form (31) with $\kappa_d=q_{m_d}$ and $\kappa_0=q_{m_d}+k_p$. It can be shown that the required solution is given by

$$v(k+1)=A_{\text{cmp}}v(k)+\sum_{j=0}^{m_d} BH_j r(k+j)-Ky(k),$$

$$u(k)=Fv(k)+\sum_{j=0}^{m_d} H_j r(k+j), \tag{53}$$

where $A_{\text{cmp}}=A+BF+KC_1$, K being chosen such that $(A+KC_1)^{k_p-1}=0$.

Remark 6. For simplicity, we consider Σ_{P} to be a single output system with a relative degree q_1. Clearly, if the reference $r(k+d)$ is unknown for all $d>0$, then the full order output feedback controller (53) with $r(k+\bigstar)$ being replaced by $r(k)$ will be capable of tracking the reference with a delay of q_1 steps after q_1+k_p initial steps. Similarly, under the same situation, the reduced order output feedback controller (48) with $r(k+\bigstar)$ being replaced by $r(k)$ will track the reference with a delay of q_1 steps after q_1+k_p-1 initial steps.

4. Conclusions

We have studied the problem of robust and perfect tracking for discrete-time linear time-invariant multivariable systems. A set of necessary and sufficient conditions under which the proposed problem is solvable is obtained and, under these conditions, constructive algorithms are given that yield required solutions. We have also proposed in this paper an almost perfect tracking scheme, which can track references precisely after certain initial steps.

References

Chen, B. M. (2000). *Robust and H_∞ control*. New York: Springer.

Liu, K., Chen, B. M., & Lin, Z. (2001). On the problem of robust and perfect tracking for linear systems with external disturbances. *International Journal of Control*, *74*, 158–174.

Sannuti, P., & Saberi, A. (1987). A special coordinate basis of multivariable linear systems—Finite and infinite zero structure, squaring down and decoupling. *International Journal of Control*, *45*, 1655–1704.

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