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On the problem of robust and perfect tracking for linear systems with external disturbances

KEXIU LIU†, BEN M. CHEN†* and ZONGLI LIN‡

We consider in this paper the robust and perfect tracking (RPT) problem for multivariable linear systems with external disturbances. The problem is to design a proper controller such that the resulting overall closed-loop system is asymptotically stable and the controlled output almost perfectly tracks a given reference signal with an arbitrarily fast settling time in the face of external disturbances and initial conditions. The contribution of this paper is two-fold: (1) We derive a set of necessary and sufficient conditions under which the RPT problem is solvable; and (2) Under these solvability conditions, we develop algorithms for constructing state and output feedback laws, explicitly parameterized in ε, that solve the RPT problem. In our construction of feedback laws, we propose a controller structure which enables us to design a tracking controller without introducing additional integrators regardless of what type the system is.

1. Introduction to the Problem

The tracking problem is one of the most common and important issues in designing a control system. Most results in the literature focus on only issues associated with asymptotic tracking problems, in which tracking errors are made to tend to zero as time progresses towards infinity. However, there are many cases, especially in many practical situations, for which one can design a control system that would yield a much better performance, e.g. faster settling time and smaller overshoot, without additional costs. Being motivated by our experience in designing control systems for gyro-stabilized mirror and hard disk drive systems (see, Goh *et al.* 1999, Siew *et al.* 1999), we propose in this paper a so-called robust and perfect tracking (RPT) problem, which is to design a controller for a given linear time-invariant system such that the resulting closed-loop system is asymptotically stable and the controlled output almost perfectly tracks a given reference signal in the presence of any initial conditions and external disturbances. By almost perfect tracking we mean the ability of a controller to track a given reference signal with *arbitrarily fast settling time* in the face of external disturbances and initial conditions. We also note that the *robustness* referred to in this paper is with respect to the external disturbance. More specifically, we consider in this paper the following multivariable linear time-invariant system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, & x(0) = x_0 \\ y = C_1 x + D_1 w \\ h = C_2 x + D_2 u + D_{22} w \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^q$ is the external disturbance, $y \in \mathbb{R}^p$ is the measurement output, and $h \in \mathbb{R}^l$ is the output to be controlled. We also assume that the pair (A, B) is stabilizable and (A, C_1) is detectable. For future references, we define Σ_{P} and Σ_{Q} to be the subsystems characterized by the matrix quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1), respectively. Given the external disturbance $w \in L_p$, $p \in [1, \infty)$, and any reference signal vector, $r \in \mathbb{R}^l$ with r, $\dot{r}$, ..., $r^{(\kappa-1)}$, $\kappa \geq 1$, being available for feedback, and $r^{(\kappa)}$ being either a vector of delta functions or in L_p, the robust and perfect tracking (RPT) problem for the system (1) is to find a parameterized dynamic measurement and reference feedback control law of the

$$\left.\begin{aligned} \dot{v} &= A_{\mathrm{cmp}}(\varepsilon)v + B_{\mathrm{cmp}}(\varepsilon)y + G_0(\varepsilon)r \\ &\quad + \cdots + G_{\kappa-1}(\varepsilon)r^{(\kappa-1)} \\ u &= C_{\mathrm{cmp}}(\varepsilon)v + D_{\mathrm{cmp}}(\varepsilon)y + H_0(\varepsilon)r \\ &\quad + \cdots + H_{\kappa-1}(\varepsilon)r^{(\kappa-1)} \end{aligned}\right\} \tag{2}$$

such that when (2) is applied to (1), we have:

(1) There exists an $\varepsilon^* > 0$ such that the resulting closed-loop system with $r = 0$ and $w = 0$ is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$; and

(2) Let $h(t, \varepsilon)$ be the closed-loop controlled output response and let $e(t, \varepsilon) := h(t, \varepsilon) - r(t)$. Then, for any initial condition of the state, $x_0 \in \mathbb{R}^n$

$$J_p(x_0, w, r, \varepsilon) := \|e\|_p \leq \alpha\varepsilon\Big(|x_0| + \|w\|_p + \|r^{(k)}\|_p\Big) \tag{3}$$

where α is a positive scalar independent of ε.

Various aspects of the robust and perfect regulation problem were heavily investigated by many researchers in the 1970s and early 1980s. The perfect regulation problem via state feedback was studied by Kwakernaak and Sivan (1972), Francis (1979), Kimura

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* Author for correspondence. e-mail: bmchen@nus.edu.sg
† Department of Electrical and Computer Engineering, National University of Singapore, Singapore 117 576, Singapore.
‡Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22903, USA.

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(1981) and Scherzinger and Davison (1985), and was completely solved by Lin *et al.* (1996) (see also Lin 1999). The solution to the problem of perfect regulation via measurement output feedback for general linear systems has only been reported recently by Chen *et al.* (2000). The problem of almost perfect tracking via state feedback was formulated and solved for square invertible linear systems by Lawrence and Rugh (1991). The robust servo mechanism problem with perfect control was formulated by Davison and his co-workers (see, e.g. Davison and Chow (1977), which mainly dealt with state feedback case, and Davison and Scherzinger (1987) and the references therein). A detailed description of and comparison among these problems are omitted due to the limitation of space.

More recently, Saberi *et al.* (1997) formulated the problem of generalized output regulation, in which the disturbance and command signal to be tracked are modelled as the trajectories of a reference model. The reference model is driven by a reference input r (see, e.g. (4.1) of Saberi *et al.*, 1997). Under the condition that certain linear matrix equalities are solvable, the problem can be formulated and solved as a robust control problem, in which certain operator norm of the transfer matrix from the reference input to the tracking error is made small. Although such a formulation leads to a straightforward solution, it can guarantee the tracking error, e, to have a small *modified* L_p*-norm* only when r is in some L_q functional spaces. The modified L_p-norm of e introduced in Saberi *et al.* (1997) is defined as

$$\|e\|_{p,[T,\infty)} := \left(\int_T^\infty |e|^p \mathrm{d}t\right)^{1/p} \tag{4}$$

where T represents a certain waiting period which should be long enough to ensure that the effect of initial condition becomes small enough (less than some pre-set level). In other words, the formulation of generalized output regulation does not capture the transient performance of the closed-loop system. We would like to note that our formulation is capable of tracking references with any initial condition with arbitrarily fast settling time. It does not require such a waiting period for the initial condition to die out. As it will be seen shortly, our results do not involve solving linear matrix equalities either.

In this paper, we derive a set of necessary and sufficient conditions under which the proposed robust and perfect tracking problem has a solution, and under these conditions, develop algorithms for the construction of parameterized feedback laws that solve the proposed problem. Our algorithm for obtaining the state feedback gain matrix $F(\varepsilon)$ utilizes the concept of asymptotic time-scale and eigenstructure assignment (ATEA) procedure. The concept of ATEA design procedure was originally conceived in Saberi and Sannuti (1989) and was used to solve many control problems, including H_∞ optimal control problems (Chen, 1998), loop transfer recovery (Saberi *et al.* 1993) and H_2 optimal control problems (Saberi *et al.* 1995). However, unlike the previous applications where the observer gain matrices are constructed dually, our constructions of observer gain matrices, both for full order and reduced order measurement feedback cases, are totally different from the procedure for their state feedback counterpart.

The outline of this paper is as follows. In §2, we recall some background materials in linear system theory, which would be instrumental to our current development. Section 3 presents our main results, i.e. the solvability conditions as well as solutions to the proposed robust and perfect tracking problem. We will construct three types of parameterized solutions: the state feedback law, the full order measurement and reference feedback controller, i.e. it has a dynamical order equal to n, and the reduced order one whose dynamical order is less than n. As a by-product, we will show in §4 that the main results of the paper can be extended to track more general type of references. Finally, conclusions are drawn in §5.

Throughout this paper, the following notation will be used: X' denotes the transpose of matrix X; $X^\dagger$ denotes the Moore–Penrose (pseudo) inverse of X; I denotes an identity matrix with appropriate dimensions; $\mathbb{R}$ is the set of all real numbers; $\mathbb{C}$ is the set of all complex numbers; $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$ are respectively the open left-half complex plane, the imaginary axis and the open right-half complex plane; $r^{(\kappa)}(t)$ is the κ-th order derivative of $r(t)$; $|x|$ denotes the Euclidean norm of the vector x; $\|\cdot\|_p$ denotes the L_p-norm with $1 \le p \le \infty$; L_p is the set of all time domain functions whose L_p-norms are finite; $\mathrm{Ker}\,(X)$ is the kernel of X; $\mathrm{Im}\,(X)$ is the image of X; $C^{-1}\{\mathcal{X}\} := \{x \mid Cx \in \mathcal{X}\}$, where $\mathcal{X}$ is a vector space and C is a constant matrix. $\lambda(X)$ is the set of eigenvalues of a real square matrix X; and finally $\sigma_{\max}(X)$ denotes the maximum singular value of matrix X. We also introduce the following geometric subspaces:

(1) $\mathcal{V}^+(\Sigma_*)$ is the maximal subspace of $\mathbb{R}^n$ for which there exists an appropriate dimensional constant matrix F such that $\mathcal{V}^+$ is $(A+BF)$-invariant and is contained in $\mathrm{Ker}(C+DF)$, and the eigenvalues of $(A+BF)|\mathcal{V}^+$ are contained in $\mathbb{C}^+$.

(2) $\mathcal{S}^+(\Sigma_*)$ is the minimal subspace of $\mathbb{R}^n$ for which there exists an appropriate dimensional constant matrix K such that $\mathcal{S}^+$ is $(A+KC)$-invariant and contains $\mathrm{Im}(B+KD)$, and the eigenvalues of the map which is induced by $(A+KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}^+$ are contained in $\mathbb{C}^+$.

We note that these geometric subspaces are related to the L_p-almost controlled invariant subspaces introduced in Willems (1981).

2. Background materials

We recall in this section the special coordinate basis of linear time-invariant systems introduced by Sannuti and Saberi (1987) and Saberi and Sannuti (1990). Such a special coordinate basis is instrumental to the development of our results. Consider a linear time-invariant system Σ_* characterized by the quadruple (A, B, C, D) or in the state space form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \tag{5}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output of Σ_*. It is simple to verify that there exist non-singular transformations U and V such that

$$UDV = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

where m_0 is the rank of matrix D. Thus, without loss of generality, it is assumed that the matrix D has the form given on the right hand side of (6). One can now rewrite the system of (5) as

$$\left.\begin{aligned} \dot{x} &= A x + [B_0 \quad B_1] \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{aligned}\right\} \tag{7}$$

where the matrices B_0, B_1, C_0 and C_1 have appropriate dimensions. For simplicity, we will focus in this section the special coordinate basis for the case when (A, B, C, D) is right invertible and has no invariant zeros in $\mathbb{C}^+$, as this will be good enough for the development of our results in this paper. We have the following theorem.

Theorem 1: *Given the linear system Σ_* of (5), which is right invertible and has no invariant zeros in $\mathbb{C}^+$, there exist:*

(1) *Coordinate free non-negative integers n_a^-, n_a^0, n_c, n_d, $m_d \le m - m_0$ and q_i, $i = 1, \dots, m_d$, and*
(2) *Non-singular state, output and input transformations Γ_s, Γ_o and Γ_i which take the given Σ_* into a special coordinate basis that displays explicitly both the finite and infinite zero structures of Σ_*.*

The special coordinate basis is described by the set of equations

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u} \tag{8}$$

$$\tilde{x} = \begin{pmatrix} x_a \\ x_c \\ x_d \end{pmatrix}, \quad x_a = \begin{pmatrix} x_a^- \\ x_a^0 \end{pmatrix}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix} \tag{9}$$

$$\tilde{y} = \begin{pmatrix} y_0 \\ y_d \end{pmatrix}, \; y_d = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m_d} \end{pmatrix}, \; \tilde{u} = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \; u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix} \tag{10}$$

and

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- y_0 + L_{ad}^- y_d \tag{11}$$

$$\dot{x}_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 y_0 + L_{ad}^0 y_d \tag{12}$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} y_0 + L_{cd} y_d + B_c [E_{ca}^- x_a^- + E_{ca}^0 x_a^0] + B_c u_c \tag{13}$$

$$y_0 = C_{0c} x_c + C_{0a}^- x_a^- + C_{0a}^0 x_a^0 + C_{0d} x_d + u_0 \tag{14}$$

and for each $i = 1, \dots, m_d$

$$\dot{x}_i = A_{q_i} x_i + L_{i0} y_0 + L_{id} y_d + B_{q_i} \left[u_i + E_{ia} x_a + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} x_j \right] \tag{15}$$

$$y_i = C_{q_i} x_i, \; y_d = C_d x_d \tag{16}$$

Here the states x_a^-, x_a^0, x_c and x_d are respectively of dimensions n_a^-, n_a^0, n_c and $n_d = \sum_{i=1}^{m_d} q_i$, while x_i is of dimension q_i for each $i = 1, \dots, m_d$. The control vectors u_0, u_d and u_c are respectively of dimensions m_0, m_d and $m_c = m - m_0 - m_d$ while the output vectors y_0 and y_d are respectively of dimensions $p_0 = m_0$ and $p_d = m_d$. The matrices A_{q_i}, B_{q_i} and C_{q_i} have the form

$$A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1, \; 0, \; \dots, \; 0] \tag{17}$$

Moreover, we have $\lambda(A_{aa}^-) \subset \mathbb{C}^-$ and $\lambda(A_{aa}^0) \subset \mathbb{C}^0$. Also, the pair (A_{cc}, B_c) is controllable.

We can rewrite the special coordinate basis of (A, B, C, D) given by Theorem 1 in a more compact form

$$\tilde{A} = \Gamma_s^{-1}(A - B_0C_0)\Gamma_s = \begin{bmatrix} A_{aa}^- & 0 & 0 & L_{ad}^- C_d \\ 0 & A_{aa}^0 & 0 & L_{ad}^0 C_d \\ B_c E_{ca}^- & B_c E_{ca}^0 & A_{cc} & L_{cd} C_d \\ B_d E_{da}^- & B_d E_{da}^0 & B_d E_{dc} & A_{dd} \end{bmatrix} \quad (18)$$

$$\left. \begin{aligned} \tilde{B} &= \Gamma_s^{-1}[B_0 \quad B_1]\Gamma_i = \begin{bmatrix} B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix} \\ \tilde{D} &= \Gamma_o^{-1} D \Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \right\} \quad (19)$$

$$\tilde{C} = \Gamma_o^{-1}\begin{bmatrix} C_0 \\ C_1 \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a}^- & C_{0a}^0 & C_{0c} & C_{0d} \\ 0 & 0 & 0 & C_d \end{bmatrix} \quad (20)$$

In what follows, we state some important properties of the above special coordinate basis which are pertinent to our present work and will be used throughout this paper. The proofs of these properties can be found in Chen (1998).

Property 1:

(1) *Σ_* is stabilizable if and only if the pair $(A_{aa}^0, [B_{0a}^0 \quad L_{ad}^0])$ is controllable.*

(2) *Invariant zeros of Σ_* are the eigenvalues of A_{aa}, which are the unions of the eigenvalues of A_{aa}^- and A_{aa}^0. Σ_* is said to be minimum phase if all its invariant zeros are in $\mathbb{C}^-$. Thus, for minimum phase Σ_*, we have $n_a^0 = 0$.*

(3) *Σ_* has $m_0 = \mathrm{rank}\ (D)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of Σ_* is given by $S_\infty^\star(\Sigma_*) = \{q_1, q_2, \ldots, q_{m_d}\}$, i.e. each q_i corresponds to an infinite zero of Σ_* of order q_i.*

3. Solvability conditions and solutions to RPT problem

We are now ready to present our main results. We will first derive a set of necessary and sufficient conditions under which the proposed robust and perfect tracking (RPT) problem is solvable for the given plant (1). In fact, we will show the sufficiency of these conditions by explicitly constructing two types of parameterized control laws: one is of full order, i.e. its dynamical order is equal to n, the order of the plant, and the other is of reduced order, i.e., its dynamical order is less than n.

We have the following theorem.

Theorem 2: *Consider the given system (1) with its external disturbance $w \in L_p$, $p \in [1, \infty)$, and its initial condition $x(0) = x_0$. Then, for any reference signal $r(t)$, which has all its i-th order derivatives, $i = 0, 1, \ldots, \kappa - 1$, $\kappa \geq 1$, being available for feedback and $r^{(\kappa)}(t)$ being either a vector of delta functions or in L_p, the proposed robust and perfect tracking (RPT) problem is solvable by the control law of (2) if and only if the following conditions are satisfied:*

(1) *(A, B) is stabilizable and (A, C_1) is detectable;*

(2) *$D_{22} + D_2SD_1 = 0$, where $S = -(D_2'D_2)^\dagger D_2'D_{22} \times D_1'(D_1D_1')^\dagger$;*

(3) *Σ_P, i.e., (A, B, C_2, D_2), is right invertible and of minimum phase;*

(4) $\mathrm{Ker}(C_2 + D_2SC_1) \supset C_1^{-1}\{\mathrm{Im}(D_1)\}$.

Proof: We first show that Conditions 1 to 4 in theorem 2 are necessary. Let us consider the case when $x_0 = 0$ and $r(t) \equiv 0$, which of course has all its derivatives of any order being available. It is simple to see that the proposed robust and perfect tracking problem is reduced to the well-known almost disturbance decoupling problem with measurement feedback for the given system (1) with $x_0 = 0$ (see Willems 1981, 1982, for the original formulation of this problem).

Next, let us consider the case when $r(t) \equiv 0$ and $w(t) \equiv 0$. Our proposed problem is then reduced to the perfect regulation problem with measurement feedback. Following the results of Chen *et al.* (2000) (see also Lin 1999 for the state feedback case), we can reformulate the perfect regulation problem for the system of (1) with $w = 0$ again as an almost disturbance decoupling problem for the following auxiliary system

$$\left. \begin{aligned} \dot{x} &= Ax + Bu + I\hat{w}, \qquad x(0) = 0 \\ y &= C_1 x \\ h &= C_2 x + D_2 u \end{aligned} \right\} \quad (21)$$

where $\hat{w}$ is a delta function. In order to solve the proposed RPT problem for the given system (1) with $r = 0$, we will have to solve simultaneously the almost disturbance decoupling problem for (1) with $x_0 = 0$, and the almost disturbance decoupling problem for (21), by using a single and same control law. Clearly, this is equivalent to solving the almost disturbance decoupling problem for the system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + [E \quad I]\tilde{w}, \qquad x(0)=0 \\ y &= C_1 x + [D_1 \quad 0]\tilde{w} \\ h &= C_2 x + D_2 u + [D_{22} \quad 0]\tilde{w} \end{aligned}\right\} \tag{22}$$

which has the same control input u and the same measurement output y as those of the original system Σ in (1). For easy reference, we let $\tilde{\Sigma}_{\mathrm{Q}}$ be the subsystem characterized by $(A, [E \quad I], C_1, [D_1 \quad 0])$. Following the results of the well-known almost disturbance decoupling problem (see e.g. Chen 1998, Weiland and Willems 1989, and references therein), we can show that if the almost disturbance decoupling problem for the above system is solvable, then the following conditions hold:

(1) (A, B) is stabilizable and (A, C_1) is detectable;

(2) $D_{22} + D_2 S D_1 = 0$, where $S = -(D_2' D_2)^{\dagger} D_2' D_{22} \times D_1' (D_1 D_1')^{\dagger}$;

(3) $\mathrm{Im}([E + BSD_1 \quad I]) \subset \mathcal{S}^{+}(\Sigma_{\mathrm{P}})$;

(4) $\mathrm{Ker}(C_2 + D_2 S C_1) \supset \mathcal{V}^{+}(\tilde{\Sigma}_{\mathrm{Q}})$.

Clearly, item (3) above implies that $\mathcal{S}^{+}(\Sigma_{\mathrm{P}}) = \mathbb{R}^n$, which implies that Σ_{P} is right invertible without invariant zeros in $\mathbb{C}^{+}$. Due to the special form of $\tilde{\Sigma}_{\mathrm{Q}}$, it is simple to show that $\mathcal{V}^{+}(\tilde{\Sigma}_{\mathrm{Q}}) = C_1^{-1}\{\mathrm{Im}(D_1)\}$. Hence, items (3) and (4) are respectively equivalent to: (i) Σ_{P} is right invertible without invariant zeros in $\mathbb{C}^{+}$; and (ii) $\mathrm{Ker}(C_2 + D_2 S C_1) \supset C_1^{-1}\{\mathrm{Im}(D_1)\}$. Thus, it remains to show that if the proposed RPT problem is solvable, the subsystem Σ_{P} must be of minimum phase. In what follows, we proceed to show such a fact.

First, we note that second condition, i.e. $D_{22} + D_2 S D_1 = 0$, implies that if we apply a pre-output feedback law $u = Sy$, to the system (1), the resulting new system will have a direct feedthrough term from w to h equal to 0. Hence, without loss of any generality, we hereafter assume that matrix $D_{22} = 0$ throughout the rest of the proof.

Next, we show that if the robust and perfect tracking problem is solvable for general non-zero reference $r(t)$, Σ_{P} must be of minimum phase, i.e. Σ_{P} cannot have any invariant zeros on the imaginary axis. In fact, this condition must hold even for the case when $w = 0$ and $x_0 = 0$, i.e. for the robust and perfect tracking of the following system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= C_1 x \\ e &= C_2 x + D_2 u - r = h - r \end{aligned}\right\} \tag{23}$$

Now, if we treat r as an external disturbance, then the above problem is again equivalent to the well-known almost disturbance decoupling problem with measurement feedback and with internal stability for the system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ \tilde{y} &= \begin{pmatrix} C_1 x \\ r \\ \vdots \\ r^{(\kappa-1)} \end{pmatrix} \\ e &= C_2 x + D_2 u - r \end{aligned}\right\} \tag{24}$$

Without loss of generality, we assume that the quadruple (A, B, C_2, D_2) has been transformed into the form of the special coordinate basis of Theorem 1, i.e. we have

$$\left.\begin{aligned} & x = \begin{pmatrix} x_a^- \\ x_a^0 \\ x_c \\ x_d \end{pmatrix}, \quad h = \begin{pmatrix} h_0 \\ h_d \end{pmatrix}, \quad r = \begin{pmatrix} r_0 \\ r_d \end{pmatrix} \\ & e = \begin{pmatrix} e_0 \\ e_d \end{pmatrix} = \begin{pmatrix} h_0 - r_0 \\ h_d - r_d \end{pmatrix}, \quad u = \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix} \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} & x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix}, \quad x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{iq_i} \end{pmatrix}, \quad h_d = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{m_d} \end{pmatrix} \\ & r_d = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{m_d} \end{pmatrix}, \quad u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix} \end{aligned}\right\} \tag{26}$$

and

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- h_0 + L_{ad}^- h_d \tag{27}$$

$$\dot{x}_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 h_0 + L_{ad}^0 h_d \tag{28}$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} h_0 + L_{cd} h_d + B_c [E_{ca}^- x_a^- + E_{ca}^0 x_a^0] + B_c u_c \tag{29}$$

$$e_0 = C_{2,0a}^- x_a^- + C_{2,0a}^0 x_a^0 + C_{2,0c} x_c + C_{2,0d} x_d + u_0 - r_0 \tag{30}$$

and for each $i = 1, \dots, m_d$,

$$\dot{x}_i = A_{q_i} x_i + L_{i0} h_0 + L_{id} h_d + B_{q_i}\left[u_i + E_{ia} x_a + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} x_j\right] \quad (31)$$

$$h_i = C_{q_i} x_i = x_{i1}, \quad h_d = C_d x_d \quad (32)$$

and finally,

$$e_i = h_i - r_i = C_{q_i} x_i - r_i, \quad e_d = h_d - r_d = C_d x_d - r_d \quad (33)$$

Let us define a set of new state variables, i.e. for $i = 1, 2, \dots, m_d$, we define

$$\bar{x}_i := \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{iq_i} \end{pmatrix} - \begin{pmatrix} r_i \\ \dot{r}_i \\ \vdots \\ r_i^{(q_i-1)} \end{pmatrix} \quad (34)$$

if $\kappa \geq q_i$, or

$$\bar{x}_i := \begin{pmatrix} x_{i1} \\ \vdots \\ x_{iq} \\ x_{iq+1} \\ \vdots \\ x_{iq_i} \end{pmatrix} - \begin{pmatrix} r_i \\ \vdots \\ r_i^{(\kappa-1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (35)$$

if $\kappa < q_i$. Then, we have

$$e_i = C_{q_i} \bar{x}_i, \quad e_d = C_d \bar{x}_d \quad (36)$$

$$\dot{x}_a^- = A_{aa}^- x_a^- + B_{0a}^- e_0 + L_{ad}^- e_d + [\, B_{0a}^- \quad L_{ad}^- \,] r \quad (37)$$

$$\dot{x}_a^0 = A_{aa}^0 x_a^0 + B_{0a}^0 e_0 + L_{ad}^0 e_d + [\, B_{0a}^0 \quad L_{ad}^0 \,] r \quad (38)$$

$$\dot{x}_c = A_{cc} x_c + B_{0c} e_0 + L_{cd} e_d + B_c [E_{ca}^- x_a^- + E_{ca}^0 x_a^0] + B_c u_c + [\, B_{0c} \quad L_{cd} \,] r \quad (39)$$

$$e_0 = C_{2,0a}^- x_a^- + C_{2,0a}^0 x_a^0 + C_{2,0c} x_c + C_{2,0d} \bar{x}_d + u_0 - r_0 + C_{2,0d}^* r_d \quad (40)$$

for an appropriate dimensional matrix $C_{2,0d}^*$, and for $i = 1, 2, \dots, m_d$

$$\dot{\bar{x}}_i = A_{q_i} \bar{x}_i + L_{i0} e_0 + L_{id} e_d + B_{q_i}\left[u_i + E_{ia} x_a + E_{ic} x_c + \sum_{j=1}^{m_d} E_{ij} \bar{x}_j\right] + E_{q_i} \begin{pmatrix} r \\ \vdots \\ r^{(\kappa-1)} \end{pmatrix} \quad (41)$$

for an appropriate dimensional matrix E_{q_i}. Note that the disturbances r_0 and r_d in (40) can be washed out by the pre-output feedback

$$u_0 = \tilde{u}_0 + r_0 - C_{2,0d}^* r_d \quad (42)$$

Moreover, the subsystem from the controlled input, i.e. $\left(u_0' \;\; u_d' \;\; u_c' \right)'$, to the error output, i.e. $\left(e_0' \;\; e_d \right)'$, is now in the standard form of the special coordinate basis of Theorem 1. It then follows from the result of Chen (1998) (i.e. Proposition 7.2.1) that if the almost disturbance decoupling problem with measurement feedback and with internal stability for the system (24) is solvable, there must exist a non-zero vector v such that

$$v^{\mathrm{H}}(\lambda I - A_{aa}^0) = 0 \quad \text{and} \quad v^{\mathrm{H}}\left[\, B_{0a}^0 \quad L_{ad}^0 \,\right] = 0 \quad (43)$$

which implies that $(A_{aa}^0, \left[\, B_{0a}^0 \;\; L_{ad}^0 \,\right])$ is not completely controllable. Following Property 1 of the special coordinate basis, the uncontrollability of $(A_{aa}^0, \left[\, B_{0a}^0 \;\; L_{ad}^0 \,\right])$ implies the unstabilizability of the pair (A, B), which is obviously a contradiction. Hence, x_a^0 must be non-existent. It then follows from Property 1 of the special coordinate basis that Σ_{P} is of minimum phase. This completes the proof of the necessary part of Theorem 2.

We note that for the case when $D_1 = 0$, then D_{22} must be a zero matrix as well, and the last condition, i.e. item (4), of Theorem 2 is reduced to $\mathrm{Ker}(C_2) \supset \mathrm{Ker}(C_1)$.

We will show the sufficiency of those conditions in Theorem 2 by explicitly constructing parameterized controllers which solve the proposed robust and perfect tracking problem under Conditions (1) to (4) of Theorem 2. This will be done in the following subsequent subsections. First, we have the following corollary that deals with the state feedback case.

Corollary 1: *Consider the given system* (1) *with its external disturbance* $w \in L_p$, $p \in [1, \infty)$, *its initial condition* $x(0) = x_0$. *Assume that all its states are measured for feedback, i.e.* $C_1 = I$ *and* $D_1 = 0$. *Then, for any reference signal* $r(t)$, *which has all its i-th order derivatives,* $i = 1, 2, \dots, \kappa - 1$, $\kappa \geq 1$, *being available for feedback and* $r^{(\kappa)}(t)$ *being either a vector of delta functions or in* L_p, *the proposed robust and perfect tracking (RPT)problem is solvable by the control law of* (2) *if and only if the following conditions are satisfied:* (i)

(A,B) *is stabilizable;* (ii) $D_{22}=0$*; and* (iii) $\Sigma_{\rm P}$, *i.e.* (A,B,C_2,D_2)*, is right invertible and of minimum phase.*

3.1. *Solutions to the state feedback case*

When all states of the plant are measured for feedback, the problem can be solved by a static control law. We construct in this subsection a parameterized state and reference feedback control law

$$u = F(\varepsilon)x + H_0(\varepsilon)r + \cdots + H_{\kappa-1}(\varepsilon)r^{(\kappa-1)} \tag{44}$$

which solves the robust and perfect tracking (RPT) problem for (1) under the conditions given in Corollary 1. It is simple to note that we can re-write the given reference in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r \\ \vdots \\ r^{(\kappa-2)} \\ r^{(\kappa-1)} \end{pmatrix} = \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{pmatrix} r \\ \vdots \\ r^{(\kappa-2)} \\ r^{(\kappa-1)} \end{pmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_\ell \end{bmatrix} r^{(\kappa)} \tag{45}$$

Combining (45) with the given system, we obtain the augmented system

$$\Sigma_{\rm AUG}: \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u + \boldsymbol{E}\boldsymbol{w} \\ \boldsymbol{y} = \boldsymbol{x} \\ e = \boldsymbol{C}_2\boldsymbol{x} + \boldsymbol{D}_2 u \end{cases} \tag{46}$$

where

$$\boldsymbol{w} := \begin{pmatrix} w \\ r^{(\kappa)} \end{pmatrix}, \quad \boldsymbol{x} := \begin{pmatrix} r \\ \vdots \\ r^{(\kappa-2)} \\ r^{(\kappa-1)} \\ x \end{pmatrix} \tag{47}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & I_\ell & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_\ell & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & A \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ B \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & I_\ell \\ E & 0 \end{bmatrix} \tag{48}$$

and

$$\boldsymbol{C}_2 = [-I_\ell \quad 0 \quad 0 \quad \cdots \quad 0 \quad C_2], \quad \boldsymbol{D}_2 = D_2 \tag{49}$$

It is then straightforward to show that the subsystem from u to e in the augmented system (46), i.e. the quadruple $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}_2,\boldsymbol{D}_2)$, is right invertible and has the same infinite zero structure as that of $\Sigma_{\rm P}$. Furthermore, its invariant zeros contain those of $\Sigma_{\rm P}$ and $\ell\times\kappa$ extra ones at $s=0$. We are now ready to present a step-by-step algorithm to construct the required control law of the form (44).

Step S.1. This step is to transform the subsystem from u to e of the augmented system (46) into the special coordinate basis of Theorem 1, i.e. to find non-singular state, input and outputtransformations Γ_s, Γ_i and Γ_o to put it into the structural form of Theorem 1 as well as in a small variation of the compact form of (18) to (20). It can be shown that the compact form of (18) to (20) for the subsystem from u to e of (46) can be written as

$$\left.\begin{aligned} \tilde{A} &= \begin{bmatrix} A_{aa}^0 & 0 & 0 & 0 \\ 0 & A_{aa}^- & 0 & L_{ad}^- C_d \\ B_c E_{ca}^0 & B_c E_{ca}^- & A_{cc} & L_{cd} C_d \\ B_d E_{da}^0 & B_d E_{da}^- & B_d E_{dc} & A_{dd} \end{bmatrix} \\ A_{aa}^0 &= \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}\right\} \tag{50}$$

and

$$\left.\begin{aligned} \tilde{B} &= \begin{bmatrix} 0 & 0 & 0 \\ B_{0a}^- & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0d} & B_d & 0 \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} C_{0a}^0 & C_{0a}^- & C_{0c} & C_{0d} \\ 0 & 0 & 0 & C_d \end{bmatrix} \\ \tilde{D} &= \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{51}$$

Step S.2. Choose an appropriate dimensional matrix F_c such that

$$A_{cc}^c = A_{cc} - B_c F_c \tag{52}$$

is asymptotically stable. The existence of such an F_c is guaranteed by the property that (A_{cc},B_c) is completely controllable.

Step S.3. For each x_i of x_d, which is associated with the infinite zero structure of Σ_P or the subsystem from u to e of (46), we choose an F_i such that

$$p_i(s) = \prod_{j=1}^{q_i}(s-\lambda_{ij}) = s^{q_i} + F_{i1}s^{q_i-1} + \cdots + F_{iq_{i-1}}s + F_{iq_i} \tag{53}$$

with all λ_{ij} being in $\mathbb{C}^-$. Let

$$F_i = \begin{bmatrix} F_{iq_i} & F_{iq_{i-1}} & \cdots & F_{i1} \end{bmatrix}, \quad i = 1, \ldots, m_d \tag{54}$$

Step S.4. Next, we construct

$$\boldsymbol{F}(\varepsilon) = -\Gamma_i \begin{bmatrix} C_{0a}^0 & C_{0a}^- & C_{0c} & C_{0d} \\ E_{da}^0 & E_{da}^- & E_{dc} & E_d + F_d(\varepsilon) \\ E_{ca}^0 & E_{ca}^- & F_c & 0 \end{bmatrix} \Gamma_s^{-1} \tag{55}$$

where

$$E_d = \begin{bmatrix} E_{11} & \cdots & E_{1m_d} \\ \vdots & \ddots & \vdots \\ E_{m_d1} & \cdots & E_{m_dm_d} \end{bmatrix} \tag{56}$$

$$F_d(\varepsilon) = \text{blkdiag}\left\{ \frac{F_1}{\varepsilon^{q_1}} S_1(\varepsilon), \frac{F_2}{\varepsilon^{q_2}} S_2(\varepsilon), \ldots, \frac{F_{m_d}}{\varepsilon^{q_{m_d}}} S_{m_d}(\varepsilon) \right\} \tag{57}$$

and where

$$S_i(\varepsilon) = \text{diag}\{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q_i-1}\}. \tag{58}$$

Step S.5. Finally, we partition

$$\boldsymbol{F}(\varepsilon) = [\, H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon)\,] \tag{59}$$

where $H_i(\varepsilon) \in \mathbb{R}^{m\times\ell}$ and $F(\varepsilon) \in \mathbb{R}^{m\times n}$. This ends the constructive algorithm.

We have the following result.

Theorem 3: *Consider the given system* (1) *with its external disturbance* $w \in L_p$, $p \in [1,\infty)$, *its initial condition* $x(0) = x_0$. *Assume that all its states are measured for feedback, i.e.* $C_1 = I$ *and* $D_1 = 0$. *If Conditions* (1) *to* (3) *of Corollary 2 are satisfied, then, for any reference signal* $r(t)$, *which has all its i-th order derivatives,* $i = 0,1,\ldots,\kappa-1$, $\kappa \geq 1$, *being available for feedback and* $r^{(\kappa)}(t)$ *being either a vector of delta functions or in* L_p, *the proposed robust and perfect tracking (RPT) problem is solved by the control law of* (44) *with* $F(\varepsilon)$ *and* $H_i(\varepsilon)$, $i = 0,1,\ldots,\kappa-1$, *as given in* (59).

Proof: See Appendix A.1. □

3.2. *Solutions to the measurement feedback case*

Without loss of generality, we assume throughout this subsection that $D_{22} = 0$. If it is non-zero, it can always be washed out by the following pre-output feedback, $u = Sy$, with S as given in the second item of Theorem 2.

3.2.1 *Full order measurement and reference feedback.* The following is a step-by-step algorithm for constructing a parameterized full order measurement and reference feedback controller, which solves the robust and perfect tracking problem.

Step F.1. For the given reference $r(t)$ and the given system (1), we first assume that all the state variables of (1) are measurable and follow the procedures of the previous subsection to define an auxiliary system,

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{Ax} + \boldsymbol{B}u + \boldsymbol{Ew} \\ \boldsymbol{y} &= \boldsymbol{x} \\ e &= \boldsymbol{C}_2\boldsymbol{x} + \boldsymbol{D}_2 u \end{aligned}\right\} \tag{60}$$

Then, we follow Steps S.1 to S.5 of the algorithm of the previous subsectionto construct a state feedback gain matrix

$$\boldsymbol{F}(\varepsilon) = [\, H_0(\varepsilon) \quad \cdots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon)\,] \tag{61}$$

Step F.2. Let Σ_{Qa} be characterized by a matrix quadruple

$$(A_{Qa}, E_{Qa}, C_{Qa}, D_{Qa}) := (A, [E \;\; I_n], C_1, [D_1 \;\; 0]) \tag{62}$$

This step is to transform this Σ_{Qa} into the special coordinate basis of Theorem 1. Because of the special structure of the matrix E_{Qa}, it is simple to show that Σ_{Qa} is always right invertible and is free of invariant zeros. Utilize the results of Theorem 1 to find non-singular state, input and output transformation Γ_{sQ}, Γ_{iQ} and Γ_{oQ} such that

$$\Gamma_{sQ}^{-1} A \Gamma_{sQ} = \begin{bmatrix} A_{ccQ} & L_{cdQ} \\ E_{dcQ} & A_{ddQ} \end{bmatrix} + \begin{bmatrix} B_{0cQ} \\ B_{0dQ} \end{bmatrix} \begin{bmatrix} C_{0cQ} & 0 \end{bmatrix} \tag{63}$$

$$\Gamma_{sQ}^{-1} E_{Qa} \Gamma_{iQ} = \begin{bmatrix} B_{0cQ} & 0 & I_{n-k} & 0 \\ B_{0dQ} & I_k & 0 & 0 \end{bmatrix} \tag{64}$$

and

$$\Gamma_{oQ}^{-1} C_1 \Gamma_{sQ} = \begin{bmatrix} C_{0cQ} & 0 \\ 0 & I_k \end{bmatrix}$$
$$\Gamma_{oQ}^{-1} [\,D_1 \quad 0\,] \Gamma_{iQ} = \begin{bmatrix} I_{p-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{65}$$

where $k = \text{p-rank}(D_1)$. It can be verified that the pair (A, C_1) is detectable if and only if the pair

$$\left(A_{ccQ}, \begin{bmatrix} C_{0cQ} \\ E_{dcQ} \end{bmatrix} \right) \tag{66}$$

is detectable.

Step F.3. Let K_{cQ} be an appropriate dimensional constant matrix such that the eigenvalues of the matrix

$$A_{ccQ}^{c} = A_{ccQ} - K_{cQ} \begin{bmatrix} C_{0cQ} \\ E_{dcQ} \end{bmatrix} = A_{ccQ} - \begin{bmatrix} K_{c0Q} & K_{cdQ} \end{bmatrix} \begin{bmatrix} C_{0cQ} \\ E_{dcQ} \end{bmatrix} \tag{67}$$

are all in $\mathbb{C}^-$. Next, we define a parameterized observer gain matrix

$$K(\varepsilon) = -\Gamma_{sQ} \begin{bmatrix} B_{0cQ} + K_{c0Q} & L_{cdQ} + K_{cdQ}/\varepsilon \\ B_{0dQ} & A_{ddQ} + I_k/\varepsilon \end{bmatrix} \Gamma_{oQ}^{-1} \tag{68}$$

Step F.4. Finally, we obtain the following full order measurement and reference feedback control law,

$$\begin{aligned} \dot{v} &= [A + BF(\varepsilon) + K(\varepsilon) C_1] v - K(\varepsilon) y \\ &\quad + BH_0(\varepsilon)\, r + \cdots + BH_{\kappa-1}(\varepsilon) r^{(\kappa-1)} \\ u &= F(\varepsilon)\, v + H_0(\varepsilon)\, r + \cdots + H_{\kappa-1}(\varepsilon)\, r^{(\kappa-1)} \end{aligned} \tag{69}$$

This completes the construction of the full order measurement and reference feedback-controller.

We have the following theorem.

Theorem 4: *Consider the given system* (1) *with its external disturbance* $w \in L_p$, $p \in [1, \infty)$, *its initial condition* $x(0) = x_0$. *If Conditions* (1) *to* (4) *of Theorem 2 are satisfied, then, for any reference signal* $r(t)$, *which has all its i-th order derivatives,* $i = 0, 1, \ldots, \kappa - 1$, $\kappa \geq 1$, *being available for feedback and* $r^{(\kappa)}(t)$ *being either a vector of delta functions or in* L_p, *then the proposed robust and perfect tracking (RPT)problem is solved by the parameterized full order measurement and reference feedback control laws as given in* (69).

Proof: See Appendix A.2. □

3.2.2. *Reduced order measurement and reference feedback.* For simplicity of presentation, we assume that matrices C_1 and D_1 have already been transformed into the forms

$$C_1 = \begin{bmatrix} 0 & C_{1,02} \\ I_k & 0 \end{bmatrix} \quad \text{and} \quad D_1 = \begin{bmatrix} D_{1,0} \\ 0 \end{bmatrix} \tag{70}$$

where $D_{1,0}$ is of full row rank. Before we present a step-by-step algorithm to construct a parameterized reduced order measurement and reference feedback controller, we first partition the system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + [E \quad I_n] \tilde{w} \\ y &= C_1 x \qquad + [D_1 \quad 0] \tilde{w} \end{aligned}\right\} \tag{71}$$

in conformity with the structures of C_1 and D_1 in (70), i.e.

$$\left.\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ &\quad + \begin{bmatrix} E_1 & I_k & 0 \\ E_2 & 0 & I_{n-k} \end{bmatrix} \tilde{w} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} 0 & C_{1,02} \\ I_k & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\quad + \begin{bmatrix} D_{1,0} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tilde{w} \end{aligned}\right\} \tag{72}$$

where

$$\tilde{w} = \begin{pmatrix} w \\ x_0 \cdot \delta(t) \end{pmatrix} \tag{73}$$

Obviously, $y_1 = x_1$ is directly available and hence need not to be estimated. Next, we define Σ_{QR} to be characterized by

$$(A_R, E_R, C_R, D_R) = \left(A_{22}, [E_2 \quad 0 \quad I_{n-k}], \begin{bmatrix} C_{1,02} \\ A_{12} \end{bmatrix}, \begin{bmatrix} D_{1,0} & 0 & 0 \\ E_1 & I_k & 0 \end{bmatrix} \right) \tag{74}$$

It is again straightforward to verify that Σ_{QR} is right invertible with no finite and infinite zeros. Moreover, (A_R, C_R) is detectable if and only if (A, C_1) is detectable. We are ready to present the following algorithm.

Step R.1. For the given reference $r(t)$ and the given system (1), we again assume that all the state variables of (1) are measurable and follow the procedures of the previous subsection to define an auxiliary system

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u + \boldsymbol{E}w \\ \boldsymbol{y} &= \boldsymbol{x} \\ e &= \boldsymbol{C}_2\boldsymbol{x} + \boldsymbol{D}_2 u \end{aligned}\right\} \tag{75}$$

Then, we follow Steps S.1 to S.5 of the algorithm of the previous subsection to construct a state feedback gain matrix

$$\boldsymbol{F}(\varepsilon) = [H_0(\varepsilon) \quad \dots \quad H_{\kappa-1}(\varepsilon) \quad F(\varepsilon)] \tag{76}$$

Let us partition $F(\varepsilon)$ in conformity with x_1 and x_2 of (72) as

$$F(\varepsilon) = [\,F_1(\varepsilon) \quad F_2(\varepsilon)\,] \tag{77}$$

Step R.2. Let $K_{\rm R}$ be an appropriate dimensional constant matrix such that the eigenvalues of

$$A_{\rm R} + K_{\rm R} C_{\rm R} = A_{22} + [\,K_{\rm R0} \quad K_{\rm R1}\,]\begin{bmatrix} C_{1,02} \\ A_{12} \end{bmatrix} \tag{78}$$

are all in $\mathbb{C}^-$. This can be done because $(A_{\rm R}, C_{\rm R})$ is detectable.

Step R.3. Let

$$G_{\rm R}(\varepsilon) = [\,-K_{\rm R0}, \;\; A_{21} + K_{\rm R1}A_{11} - (A_{\rm R} + K_{\rm R}C_{\rm R})K_{\rm R1}\,] \tag{79}$$

and

$$\left.\begin{aligned} A_{\rm cmp}(\varepsilon) &= A_{\rm R} + B_2F_2(\varepsilon) + K_{\rm R}C_{\rm R} + K_{\rm R1}B_1F_2(\varepsilon) \\ B_{\rm cmp}(\varepsilon) &= G_{\rm R}(\varepsilon) + (B_2 + K_{\rm R1}B_1)[\,0, F_1(\varepsilon) - F_2(\varepsilon)K_{\rm R1}\,] \\ C_{\rm cmp}(\varepsilon) &= F_2(\varepsilon) \\ D_{\rm cmp}(\varepsilon) &= [\,0, \;\; F_1(\varepsilon) - F_2(\varepsilon)K_{\rm R1}\,] \end{aligned}\right\} \tag{80}$$

Step R.4. Finally, we obtain the following reduced order measurement and reference feedback control law

$$\left.\begin{aligned} \dot{v} &= A_{\rm cmp}(\varepsilon)\, v + B_{\rm cmp}(\varepsilon)y + G_0(\varepsilon)r + \cdots + G_{\kappa-1}(\varepsilon)r^{(\kappa-1)} \\ u &= C_{\rm cmp}(\varepsilon)\, v + D_{\rm cmp}(\varepsilon)y + H_0(\varepsilon)r + \cdots + H_{\kappa-1}(\varepsilon)r^{(\kappa-1)} \end{aligned}\right\} \tag{81}$$

where for $i = 0, 1, \dots, \kappa - 1$

$$G_i(\varepsilon) = (B_2 + K_{\rm R1}B_1)H_i(\varepsilon) \tag{82}$$

This ends the construction of the reduced order measurement and reference feedback controller.

Theorem 5: *Consider the given system* (1) *with its external disturbance* $w \in L_p$, $p \in [1, \infty)$, *its initial condition* $x(0) = x_0$. *If Conditions* (1) *to* (4) *of Theorem 2 are satisfied, then, for any reference signal* $r(t)$, *which has all its i-th order derivatives,* $i = 0, 1, \dots, \kappa - 1$, $\kappa \geq 1$, *being available for feedback and* $r^{(\kappa)}(t)$ *being either a vector of delta functions or in* L_p, *then the proposed robust and perfect tracking (RPT) problem is solved by the parameterized reduced order measurement and reference feedback control laws of* (81).

Proof: See Appendix A.3. □

By now, the sufficiency of Theorem 2 is obvious in view of the results of Theorems 4 and 5. The proof of Theorem 2 is thus completed. □

4. Robust and perfect tracking for other references

It is very often in practical control system design to track some references such as sinusoidal functions, which are in L_∞. It is obvious that we could not make the L_∞ norm of the tracking error arbitrarily small if there is a mismatch in the initial value of the output to be controlled and that of the reference signal. Another very common situation could be that the references $r(t)$ might have some entries belonging to one set, say L_{p_1}, and some belonging to another set, say L_{p_2}, for some $p_1 \in [1, \infty]$ and $p_2 \in [1, \infty]$. Thus, for this class of references, we will have to modify our original problem formulation a little bit in order to obtain some meaningful results. Again, we consider a linear system as given in (1) with an external disturbance

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_q \end{pmatrix} \tag{83}$$

where $w_i \in L_{p_{w_i}}$, $p_{w_i} \in [1, \infty]$, $i = 1, 2, \dots, q$. We also consider a reference

$$r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_\ell \end{pmatrix} \tag{84}$$

which has the following properties: for $i = 1, 2, \dots, \ell$, we have r_i, $\dot{r}_i$, ..., $r_i^{(\kappa_i - 1)}$, $\kappa_i \geq 1$, being available for feedback, and $r_i^{(\kappa_i)}$ being a delta function or in $L_{p_{r_i}}$ for some $p_{r_i} \in [1, \infty]$. Then, the general robust and perfect tracking (GRPT) problem for this type of references is to find a parameterized dynamic measurement and reference feedback control law of the form

$$\left.\begin{aligned} \dot{v} &= A_{\text{cmp}}(\varepsilon)v + B_{\text{cmp}}(\varepsilon)y + \sum_{i=0}^{\kappa_1-1} G_{1,i}(\varepsilon)r_1^{(i)} + \cdots \\ &\quad + \sum_{i=0}^{\kappa_l-1} G_{\ell,i}(\varepsilon)r_\ell^{(i)} \\ u &= C_{\text{cmp}}(\varepsilon)v + D_{\text{cmp}}(\varepsilon)y + \sum_{i=0}^{\kappa_1-1} H_{1,i}(\varepsilon)r_1^{(i)} + \cdots \\ &\quad + \sum_{i=0}^{\kappa_l-1} H_{\ell,i}(\varepsilon)r_\ell^{i} \end{aligned}\right\} \quad (85)$$

such that when (85) is applied to (1), we have

(1) There exists an $\varepsilon^* > 0$ such that the resulting closed-loop system with $r = 0$ and $w = 0$ is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$; and

(2) The resulting closed-loop error signal e, which is obviously a function of ε, can be decomposed as

$$e = e_{r_1} + \cdots + e_{r_\ell} + e_{w_1} + \cdots + e_{w_q} + e_o \quad (86)$$

and as $\varepsilon \to 0$

$$\tilde{J}(x_0, w, r, \varepsilon) = \sum_{i=1}^{\ell} \|e_{r_i}\|_{p_{r_i}} + \sum_{i=1}^{q} \|e_{w_i}\|_{p_{w_i}} + \|e_o\|_p \to 0 \quad (87)$$

for all $1 \le p < \infty$ and for any $x_0 \in \mathbb{R}^n$. Roughly, e_o is the error due to mismatch in initial conditions of the controlled output and reference, while e_{r_i}, $i = 1, 2, \ldots, \ell$, and e_{w_i}, $i = 1, 2, \ldots, q$, are corresponding to the steady state error.

Theorem 6: *Consider the given system* (1) *with its initial condition* $x(0) = x_0$. *Also, consider the external disturbance* w *with its entries* $w_i \in L_{p_{w_i}}$, $p_{w_i} \in [1,\infty]$, $i = 1,2,\ldots,q$. *Then, for any reference signal* $r(t)$ *of the form* (84) *with* r_i, $\dot{r}_i$, $\ldots$, $r_i^{(\kappa_i-1)}$, $\kappa_i \ge 1$, *being available for feedback, and* $r_i^{(\kappa_i)}$ *being a delta function or in* $L_{p_{r_i}}$, $p_{r_i} \in [1,\infty]$, $i = 1,2,\ldots,\ell$, *the general robust and perfect tracking (GRPT) problem is solvable by the control law of* (85) *if and only if all the same four conditions of Theorem 3.1 hold.*

Proof: The proof of this theorem follows from similar lines of reasoning as those of Theorem 2 with some minor fine tuning. The constructive algorithms of the previous section should be modified as follows:

(1) *State feedback case.* For the state feedback case, one first needs to obtain an augmented system

$$\tilde{\sum}_{\text{AUG}} : \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u + \boldsymbol{E}\boldsymbol{w} \\ \boldsymbol{y} = \boldsymbol{x} \\ e = \boldsymbol{C}_2\boldsymbol{x} + \boldsymbol{D}_2 u \end{cases} \quad (88)$$

with

$$\left.\begin{aligned} & e = h - r, \quad \boldsymbol{w} := \begin{pmatrix} w \\ r^{(\kappa)} \end{pmatrix}, \quad \boldsymbol{x} := \begin{pmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_\ell \\ x \end{pmatrix} \\ & \boldsymbol{r}_i = \begin{pmatrix} r_i \\ \vdots \\ r_i^{(\kappa_i-1)} \end{pmatrix}, \ i = 1, \ldots, \ell \end{aligned}\right\} \quad (89)$$

Then, follow the same procedures as in Steps S.1 to S.4 of the previous section to obtain a gain matrix $\boldsymbol{F}(\varepsilon)$, and partition it as

$$\boldsymbol{F}(\varepsilon) = [H_{1,0}(\varepsilon) \quad \cdots \quad H_{1,\kappa_1-1}(\varepsilon) \quad \cdots \quad H_{l,0}(\varepsilon) \quad \cdots \quad H_{l,\kappa_\ell-1}(\varepsilon) \quad F(\varepsilon)] \quad (90)$$

The state and reference feedback controller is given by

$$u = F(\varepsilon)x + \sum_{i=0}^{\kappa_1-1} H_{1,i}(\varepsilon)r_1^{(i)} + \cdots + \sum_{i=0}^{\kappa_\ell-1} H_{\ell,i}(\varepsilon)r_\ell^{(i)} \quad (91)$$

(2) *Full order measurement and reference feedback case.* One only needs to replace Step F.1 of the algorithm in the previous section with item (1) above to obtain the desired $\boldsymbol{F}(\varepsilon)$. Steps F.2 and F.3 remain unchanged, and the full order measure and reference feedback controller is given by

$$\left.\begin{aligned} \dot{v} &= A_{\text{cmp}}v - K(\varepsilon)y + \sum_{i=0}^{\kappa_1-1} BH_{1,i}(\varepsilon)r_1^{(i)} + \cdots \\ &\quad + \sum_{i=0}^{\kappa_\ell-1} BH_{\ell,i}(\varepsilon)r_\ell^{(i)} \\ u &= F(\varepsilon)\, v + \sum_{i=0}^{\kappa_1-1} H_{1,i}(\varepsilon)r_1^{(i)} + \cdots + \sum_{i=0}^{\kappa_\ell-1} H_{\ell,i}(\varepsilon)r_\ell^{(i)} \end{aligned}\right\} \quad (92)$$

where $A_{\text{cmp}} = A + BF(\varepsilon) + K(\varepsilon)C_1$.

(3) *Reduced order measurement and reference feedback case.* Similarly, one again needs only to replace Step R.1 in the algorithm of the previous section with item (1) above. Steps R.2 and R.3 remain the same, and the reduced order measure and reference feedback control is given in the

form of (85) with $A_{\rm cmp}(\varepsilon)$, $B_{\rm cmp}(\varepsilon)$, $C_{\rm cmp}(\varepsilon)$, $D_{\rm cmp}(\varepsilon)$, being given as in (80), $H_{j,i}(\varepsilon)$, $j = 1, 2, \ldots, \ell$ and $i = 0, 1, \ldots, \kappa_j - 1$, being given as in (90), and

$$\left.\begin{aligned} G_{j,i}(\varepsilon) &= (B_2 + K_{\rm R1}B_1)H_{j,i}(\varepsilon),\\ &\quad j = 1, 2, \ldots, \ell;\ i = 0, 1, \ldots, \kappa_j - 1 \end{aligned}\right\} \tag{93}$$

This completes the proof of Theorem 6 □

5. Concluding Remarks

We have proposed in this paper the robust and perfect tracking (RPT) problem for general linear time-invariant multivariable systems. A set of necessary and sufficient conditions under which the proposed problem is solvable are obtained and, under these conditions, constructive algorithms are given that yield solutions, which are explicitly parameterized in a tuning parameter ε.

Appendix A.1 Proof of Theorem 3

It was mentioned in the constructive algorithm of §3.1 that, following the structural algorithms of Sannuti and Saberi (1987) and Saberi and Sannuti (1990), one can transform the system (46) into the special coordinate basis as given in the compact form of (50) and (51). That is there exist non-singular state, input and output transformation Γ_s, Γ_i and Γ_o such that

$$\left.\begin{aligned} \begin{pmatrix} \boldsymbol{r} \\ x \end{pmatrix} &= \Gamma_s \begin{pmatrix} \boldsymbol{r} \\ x_a^- \\ x_c \\ x_d \end{pmatrix} = \begin{bmatrix} I_{\kappa\times\ell} & 0 \\ \star & \tilde{\Gamma}_s \end{bmatrix} \begin{pmatrix} \boldsymbol{r} \\ x_a^- \\ x_c \\ x_d \end{pmatrix} \\ e &= \Gamma_o \begin{pmatrix} e_0 \\ e_d \end{pmatrix}, \quad u = \Gamma_i \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix} \end{aligned}\right\} \tag{94}$$

$$\left.\begin{aligned} \boldsymbol{r} &= \begin{pmatrix} r \\ \dot{r} \\ \vdots \\ r^{(\kappa-1)} \end{pmatrix}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_d} \end{pmatrix}, \quad x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{iq_i} \end{pmatrix} \\ e_d &= \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{m_d} \end{pmatrix}, \quad u_d = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m_d} \end{pmatrix} \end{aligned}\right\} \tag{95}$$

and

$$\dot{\boldsymbol{r}} = \begin{bmatrix} 0 & I_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_\ell \\ 0 & 0 & \cdots & 0 \end{bmatrix} \boldsymbol{r} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_\ell \end{bmatrix} r^{(\kappa)} \tag{96}$$

$$\dot{x}_a^- = A_{aa}^- x_a^- + L_{ad}^- e_d + B_{0a}^- e_0 + E_a^- w + G_a^- r^{(\kappa)} \tag{97}$$

$$\begin{aligned} \dot{x}_c &= A_{cc}x_c + L_{cd}e_d + B_{0c}e_0 \\ &\quad + B_c\left[u_c + E_{ca}^0\mathbf{r} + E_{ca}^- x_a^-\right] + E_c w + G_c r^{(\kappa)} \end{aligned} \tag{98}$$

$$e_0 = C_{0a}^0\boldsymbol{r} + C_{0a}^- x_a^- + C_{0c}x_c + C_{0d}x_d + u_0 \tag{99}$$

and for each $i = 1, \ldots, m_d$,

$$\begin{aligned} \dot{x}_i &= A_{q_i}x_i + L_{i0}e_0 + L_{id}e_d + B_{q_i}\Bigg[u_i + E_{ia}^0\mathrm{r} + E_{ia}^- x_a^- + \\ &\quad E_{ic}x_c + \sum_{j=1}^{m_d} E_{ij}x_j\Bigg] + E_i w + G_i r^{(\kappa)} \end{aligned} \tag{100}$$

$$e_i = C_{q_i}x_i = x_{i1}, \quad e_d = C_d x_d \tag{101}$$

Now, it is straightforward to see that if $r^{(\kappa)}$ is a vector of delta functions, then the terms $G_a^- r^{(\kappa)}$, $G_c r^{(\kappa)}$ and $G_i r^{(\kappa)}$can be treated as some additional initial conditions added to the original ones of the state variables, x_a^0, x_c and x_d, respectively. If $r^{(\kappa)}$ is in L_p, $p \in [1, \infty)$, it can be treated as an additional disturbance and can be merged with the original disturbance w. Thus, in both cases, we can write (97), (98) and (100) as

$$\dot{x}_a^- = A_{aa}^- x_a^- + L_{ad}^- e_d + B_{0a}^- e_0 + \bar{E}_a^- \bar{w} \tag{102}$$

$$\begin{aligned} \dot{x}_c &= A_{cc}x_c + L_{cd}e_d + B_{0c}e_0 \\ &\quad + B_c\left[u_c + E_{ca}^0\boldsymbol{r} + E_{ca}^- x_a^-\right] + \bar{E}_c\bar{w} \end{aligned} \tag{103}$$

and

$$\begin{aligned} \dot{x}_i &= A_{q_i}x_i + L_{i0}e_0 + L_{id}e_d + B_{q_i}\Bigg[u_i + E_{ia}^0\boldsymbol{r} + E_{ia}^- x_a^- + \\ &\quad E_{ic}x_c + \sum_{j=1}^{m_d} E_{ij}x_j\Bigg] + \bar{E}_i\bar{w} \end{aligned} \tag{104}$$

with $\bar{w} \in L_p$, $p \in [1, \infty)$, and $\bar{E}_a^-$, $\bar{E}_c$ and $\bar{E}_i$ being some appropriate constant matrices, and with a new but again bounded initial condition, say $\bar{x}_0$.

Next, we note that the control law $u = \boldsymbol{F}\boldsymbol{x}$ with the gain matrix $\boldsymbol{F}$ given in (55) can be rewritten as

$$u_0 = -C_{0a}^0 \mathbf{r} - C_{0a}^- x_a^- - C_{0c} x_c - C_{0d} x_d \tag{105}$$

$$u_c = -F_c x_c - E_{ca}^0 \mathbf{r} - E_{ca}^- x_a^- \tag{106}$$

and

$$u_i = -E_{ia}^0 \mathbf{r} - E_{ia}^- x_a^- - E_{ic} x_c - \sum_{j=1}^{m_d} E_{ij} x_j - \frac{F_i}{\varepsilon^{q_i}} S_i(\varepsilon) \bar{x}_i \tag{107}$$

Hence, the closed-loop system comprising the given system and the above control law can be expressed as

$$e_0 = 0 \tag{108}$$

$$\dot{x}_a^- = A_{aa}^- x_a^- + L_{ad}^- e_d + \bar{E}_a^- \bar{w} \tag{109}$$

$$\begin{aligned} \dot{x}_c &= (A_{cc} - B_c F_c) x_c + L_{cd} e_d + \bar{E}_c \bar{w} \\ &= A_{cc}^c x_c + L_{cd} e_d + \bar{E}_c \bar{w} \end{aligned} \tag{110}$$

$$\dot{x}_i = A_{q_i} x_i - B_{qi} \frac{F_i}{\varepsilon^{q_i}} S_i(\varepsilon) x_i + L_{id} e_d + \bar{E}_i \bar{w}, \quad e_i = C_{q_i} x_i \tag{111}$$

Let us define a new state transformation as

$$\left.\begin{aligned} &\tilde{x}_a^- := x_a^-, \;\; \tilde{x}_c := x_c, \;\; \tilde{x}_d := \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{m_d} \end{pmatrix} \\ &\tilde{x}_i := S_i(\varepsilon) x_i, \; i = 1, \ldots, m_d \end{aligned}\right\} \tag{112}$$

Then, we have $e_0 = 0$, and

$$\dot{\tilde{x}}_a^- = A_{aa}^- \tilde{x}_a^- + L_{ad}^- \tilde{e}_d + \bar{E}_a^- \bar{w} \tag{113}$$

$$\dot{\tilde{x}}_c = A_{cc}^c \tilde{x}_c + L_{cd} \tilde{e}_d + \bar{E}_c \bar{w} \tag{114}$$

$$\varepsilon \dot{\tilde{x}}_i = (A_{q_i} - B_{q_i} F_i) \tilde{x}_i + \varepsilon \tilde{L}_{id}(\varepsilon) \tilde{e}_d + \varepsilon \tilde{E}_i(\varepsilon) \bar{w} \tag{115}$$

$$\tilde{e}_i = e_i = C_{q_i} \tilde{x}_i, \quad \tilde{e}_d = e_d = C_d \tilde{x}_d$$

$$\tilde{L}_{id}(\varepsilon) = S_i(\varepsilon) L_{id}, \quad \tilde{E}_i(\varepsilon) = S_i(\varepsilon) \bar{E}_i \tag{116}$$

It is simple to show that, for $\varepsilon \in (0, 1]$

$$|\tilde{L}_{id}(\epsilon)| \le \tilde{l}_d, \quad |\tilde{E}_i(\epsilon)| \le \theta_i, \; i = 1, \ldots, m_d \tag{117}$$

for some positive constant $\tilde{l}_d$ and θ_i, which are independent of ε.

We next construct a Lyapunov function for the closed loop system (113)–(115). We do this by composing Lyapunov functions for the subsystems. For the subsystem of $\tilde{x}_a^-$, we choose a Lyapunov function

$$V_a^-(\tilde{x}_a^-) = (\tilde{x}_a^-)' P_a^- \tilde{x}_a^- \tag{118}$$

where $P_a^- > 0$ is the unique solution to the Lyapunov equation, $(A_{aa}^-)' P_a^- + P_a^- A_{aa}^- = -I$, and for the subsystem of $\tilde{x}_c$, we choose a Lyapunov function, $V_c(\tilde{x}_c) = \tilde{x}_c' P_c \tilde{x}_c$, where $P_c > 0$ is the unique solution to the Lyapunov equation, $(A_{cc}^c)' P_c + P_c A_{cc}^c = -I$. Finally, for the subsystem of $\tilde{x}_d$, we choose a Lyapunov function

$$V_d(\tilde{x}_d) = \sum_{i=1}^{m_d} \tilde{x}_i' P_i \tilde{x}_i \tag{119}$$

where P_i is the unique solution to the Lyapunov equation, $(A_{q_i} - B_{q_i} F_i)' P_i + P_i (A_{q_i} - B_{q_i} F_i) = -I$. Since $A_{q_i} - B_{q_i} F_i$ is asymptotically stable, the existence of P_i is guaranteed. We now choose a Lyapunov function for the closed-loop system (113)–(115) as

$$V(\tilde{x}_a^-, \tilde{x}_c, \tilde{x}_d) = V_a^-(\tilde{x}_a^-) + V_c(\tilde{x}_c) + \alpha_d V_d(\tilde{x}_d) \tag{120}$$

where the value of α_d is to be determined. The derivative of V along the trajectory of the closed-loop system (113)–(115) can be evaluated as

$$\begin{aligned} \dot{V} = &-(\tilde{x}_a^-)' \tilde{x}_a^- + 2(\tilde{x}_a^-)' P_a^- [L_{ad}^- \tilde{e}_d + \bar{E}_a^- \bar{w}] - \tilde{x}_c' \tilde{x}_c \\ &+ 2\tilde{x}_c' P_c [L_{cd} \tilde{e}_d + \bar{E}_c \bar{w}] + \alpha_d \sum_{i=1}^{m_d} \Bigg[-\frac{1}{\varepsilon} \tilde{x}_i{}' \tilde{x}_i \\ &+ 2\tilde{x}_i{}' P_i \tilde{L}_{id}(\varepsilon) \tilde{e}_d + 2\tilde{x}_i{}' P_i \tilde{E}_i(\varepsilon) \bar{w} \Bigg] \end{aligned} \tag{121}$$

It is straightforward to see now that there exist an $\alpha_d > 0$ and $\varepsilon^* \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} \le -\frac{1}{2} |\tilde{x}_a|^2 - \frac{1}{2} |\tilde{x}_c|^2 - \frac{1}{2\varepsilon} |\tilde{x}_d|^2 + \alpha_1 |\bar{w}|^2 \tag{122}$$

for some positive constant α_1, independent of ε. Thus, the closed-loop system in the absence of disturbance w and reference input r is asymptotically stable.

It remains to show that the resulting tracking error e, which is a function of ε, has the property

$$J_p(x_0, w, r, \varepsilon) = \|e\|_p \to 0, \text{ as } \varepsilon \to 0 \tag{123}$$

We first assume that the disturbance $\bar{w}$ is non-existent. It follows from (122) that

$$\dot{V} \le -\alpha_2 V, \tag{124}$$

for some positive scalar α_2, independent of ε. Noting the transformation of (112), we have $|\tilde{x}(0)| \le \alpha_0 |\bar{x}_0|$, for some positive $\alpha_0 > 0$, independent of ε, where $\bar{x}_0$ is the combination of the initial condition of the original system, i.e. x_0, and the additional ones introduced by $r^{(\kappa)}$. Thus

$$|V(0)| \le \alpha_3 |\bar{x}_0|^2 \tag{125}$$

where $\alpha_3 > 0$ and is independent of ε. By the standard comparison theorem, it follows from (124) that

$$V \le V(0)\, \mathrm{e}^{-\alpha_2 t} \tag{126}$$

which together with (125) imply that $V \le \alpha_3\, e^{-\alpha_2 t}|\bar{x}_0|^2$, and thus

$$|\tilde{x}_d(t)| \le \alpha_4\, e^{-\alpha_2 t}|\bar{x}_0| \quad \text{and} \quad |\tilde{e}_d(t)| \le \alpha_5\, e^{-\alpha_2 t}|\bar{x}_0| \tag{127}$$

for some positive scalars α_4 and α_5, independent of ε. Now viewing $\tilde{e}_d$ as an input to the subsystem $\tilde{x}_i$ of (115), one can show that

$$\begin{aligned} &|\tilde{x}_d(t)| \le (\alpha_6\, e^{-\alpha_8 t/\varepsilon} + \alpha_7 \varepsilon\, e^{-\alpha_2 t})|\bar{x}_0| \quad \text{and} \\ &|\tilde{e}_d(t)| \le \beta_1(\alpha_6\, e^{-\alpha_8 t/\varepsilon} + \varepsilon\, e^{-\alpha_2 t})|\bar{x}_0| \end{aligned} \tag{128}$$

for some positive scalars α_6, α_7, α_8 and β_1, which are all independent of ε. Noting that

$$e = \Gamma_o \begin{pmatrix} e_0 \\ e_d \end{pmatrix} \tag{129}$$

where $e_0 = 0$ and $e_d = \tilde{e}_d$, we then have

$$\begin{aligned} |e| &\le |\Gamma_o|\beta_1(e^{-\alpha_8 t/\varepsilon} + \varepsilon\, e^{-\alpha_2 t})|\bar{x}_0| \\ &= \beta_2(e^{-\alpha_8 t/\varepsilon} + \varepsilon\, e^{-\alpha_2 t})|\bar{x}_0| \end{aligned} \tag{130}$$

Thus

$$\begin{aligned} \|e\|_p &\le \left(\int_0^\infty [\beta_2 m_d (e^{-\alpha_8 t/\varepsilon} + \varepsilon e^{-\alpha_2 t})|\bar{x}_0|]^p \mathrm{d}t \right)^{1/p} \\ &\le \beta\varepsilon\, |\bar{x}_0| \to 0 \end{aligned} \tag{131}$$

as $\varepsilon \to 0$, for all $1 \le p < \infty$, where β is a positive scalar independent of ε.

Next, we take into consideration the disturbance $\bar{w} \in L_p$, $p \in [1, \infty)$, but with $\bar{x}_0 = 0$. Noting that $\tilde{e}_d$ in (115) is a part of the state variables of the system and $\varepsilon \tilde{L}_{id}(\varepsilon)$ is negligible compared to $A_{q_i} - B_{q_i} F_i$ for sufficiently small ε, the subsystem (115) can then be approximated as

$$\dot{\tilde{x}}_i \approx \frac{1}{\varepsilon}(A_{q_i} - B_{q_i} F_i)\tilde{x}_i + \tilde{E}_i(\varepsilon)\bar{w} \tag{132}$$

where $\bar{w} \in L_p$. Thus, we have

$$\begin{aligned} |e_i| = |\tilde{e}_i| &\le \int_0^t \left| C_{q_i} \exp\left[-\frac{1}{\varepsilon}(A_{q_i} - B_{q_i} F_i)\tau \right] \tilde{E}_i(\varepsilon)\bar{w}(t-\tau) \right| \mathrm{d}\tau \\ &\le \beta_3 \int_0^\infty e^{-\beta_4 \tau/\varepsilon} |\bar{w}(t-\tau)|\, \mathrm{d}\tau \end{aligned} \tag{133}$$

for some positive scalars β_3 and β_4, independent of ε. Using the well-known Hölder Inequality, i.e.

$$\|fg\|_1 \le \|f\|_p \cdot \|g\|_{p^*}, \quad 1/p + 1/p^* = 1 \tag{134}$$

we have

$$\begin{aligned} |e_i| = |\tilde{e}_i| &\le \beta_3 \int_0^\infty \left[\left(e^{-\beta_4 \tau/\varepsilon}\right)^{1/p} |\bar{w}(t-\tau)| \right] \left(e^{-\beta_4 \tau/\varepsilon}\right)^{1/p^*} \mathrm{d}\tau \\ &\le \beta_3 \left[\int_0^\infty e^{-\beta_4 \tau/\varepsilon} |\bar{w}(t-\tau)|^p \mathrm{d}\tau \right]^{1/p} \left[\int_0^\infty e^{-\beta_4 \tau/\varepsilon} \mathrm{d}\tau \right]^{1/p^*} \\ &= \beta_3 \left(\frac{\varepsilon}{\beta_4} \right)^{1/p^*} \left[\int_0^\infty e^{-\beta_4 \tau/\varepsilon} |\bar{w}(t-\tau)|^p \mathrm{d}\tau \right]^{1/p} \end{aligned} \tag{135}$$

Thus,

$$\|e_i\|_p^p \le \beta_3^p \left(\frac{\varepsilon}{\beta_4} \right)^{p/p^*} \int_0^\infty \left[\int_0^\infty e^{-\beta_4 \tau/\varepsilon} |\bar{w}(t-\tau)|^p \mathrm{d}\tau \right] \mathrm{d}t$$

$$= \beta_3^p \left(\frac{\varepsilon}{\beta_4} \right)^{p/p^*} \int_0^\infty e^{-\beta_4 \tau/\varepsilon} \left[\int_0^\infty |\bar{w}(t-\tau)|^p \mathrm{d}t \right] \mathrm{d}\tau \tag{136}$$

$$= \beta_3^p \left(\frac{\varepsilon}{\beta_4} \right)^{p/p^*} \int_0^\infty e^{-\beta_4 \tau/\varepsilon} \left[\int_0^\infty |\bar{w}(t)|^p \mathrm{d}t \right] \mathrm{d}\tau \tag{137}$$

$$= \beta_3^p \left(\frac{\varepsilon}{\beta_4} \right)^{p/p^*} \|\bar{w}\|_p^p \int_0^\infty e^{-\beta_4 \tau/\varepsilon} \mathrm{d}\tau$$

$$= \beta_3^p \left(\frac{\varepsilon}{\beta_4} \right)^{1+p/p^*} \|\bar{w}\|_p^p \tag{138}$$

Note that we have used the property $\bar{w}(t) = 0$, $t < 0$, to get (137) from (136). We would also like to note that the above proof from (135) to (138) was inspired by similar arguments reported in Desoer and Vidyasagar (1975). It is now clear

$$\|e_i\|_p \le \beta_3 \left(\frac{\varepsilon}{\beta_4} \right)^{1/p + 1/p^*} \|w\|_p = \left(\frac{\beta_3}{\beta_4} \right) \varepsilon \|\bar{w}\|_p \to 0, \text{ as } \varepsilon \to 0 \tag{139}$$

In view of (131) and (139), the robust and perfect tracking problem is then solved. This completes the proof of Theorem 3. □

Appendix A.2. Proof of Theorem 4

First, let us define a new state variable, $x_v = x - v$. Then, it is straightforward to verify that the closed-loop system comprising the given system (1) and the full order measurement and reference feedback control law of (69) can be re-written as

$$\dot{x}_v = [A + K(\varepsilon)C_1]x_v + [E + K(\varepsilon)D_1]w \tag{140}$$

$$\begin{aligned} \dot{x} = {} & [A + BF(\varepsilon)]x - BF(\varepsilon)x_v + BH_0(\varepsilon)r + \cdots \\ & + BH_{\kappa-1}(\varepsilon)r^{(\kappa-1)} + Ew \end{aligned} \tag{141}$$

$$\begin{aligned} h = {} & [C_2 + D_2 F(\varepsilon)]x - D_2 F(\varepsilon)x_v \\ & + D_2 H_0(\varepsilon)r + \cdots + D_2 H_{\kappa-1}(\varepsilon)r^{(\kappa-1)} \end{aligned} \tag{142}$$

It is simple to see now the eigenvalues of the closed-loop system are given by $\lambda\{A+BF(\varepsilon)\}$,which have been shown to be in $\mathbb{C}^-$ in Theorem 3, and $\lambda\{A+K(\varepsilon)C_1\}$, which are equivalent to

$$\lambda\left\{\begin{bmatrix} A_{cc\mathrm{Q}}-K_{c0\mathrm{Q}}C_{0c\mathrm{Q}} & -K_{cd\mathrm{Q}}/\varepsilon \\ E_{dc\mathrm{Q}} & -I_k/\varepsilon \end{bmatrix}\right\} \rightarrow \lambda\left(A^c_{cc\mathrm{Q}}\right)\cup\left\{-\frac{1}{\varepsilon},\ldots,-\frac{1}{\varepsilon}\right\} \quad (143)$$

as $\varepsilon\rightarrow 0$. Thus, the closed-loop system is asymptotically stable for sufficiently small ε, when the external disturbance $w=0$ and reference $r=0$.

Next, we intend to investigate the properties of x_v in the subsystem (146). Let us transform the subsystem (62) into the special coordinate basis of Theorem 1 with nonsingular state, input and output transformations $\Gamma_{s\mathrm{Q}}$, $\Gamma_{i\mathrm{Q}}$ and $\Gamma_{o\mathrm{Q}}$, as given in Step F.2 of §3.2 Also, let

$$x_v=\Gamma_{s\mathrm{Q}}\begin{pmatrix} x_{c\mathrm{Q}} \\ x_{d\mathrm{Q}} \end{pmatrix} \quad (144)$$

Then, we can re-write (140) as

$$\dot{x}_{c\mathrm{Q}}=\left(A_{cc\mathrm{Q}}-K_{c0\mathrm{Q}}C_{0c\mathrm{Q}}\right)x_{c\mathrm{Q}}-\frac{K_{cd\mathrm{Q}}}{\varepsilon}x_{d\mathrm{Q}}+E_{c\mathrm{Q}}w \quad (145)$$

$$\dot{x}_{d\mathrm{Q}}=-\frac{1}{\varepsilon}x_{d\mathrm{Q}}+E_{dc\mathrm{Q}}x_{c\mathrm{Q}}+E_{d\mathrm{Q}}w \quad (146)$$

for some appropriate dimensional matrices $E_{c\mathrm{Q}}$ and $E_{d\mathrm{Q}}$, independent of ε. Now, let $\tilde{x}_{c\mathrm{Q}}=x_{c\mathrm{Q}}-K_{cd\mathrm{Q}}x_{d\mathrm{Q}}$. Thus, (145) and (146) can be re-written as

$$\dot{\tilde{x}}_{c\mathrm{Q}}=A^c_{cc\mathrm{Q}}\tilde{x}_{c\mathrm{Q}}+A^c_{cc\mathrm{Q}}K_{cd\mathrm{Q}}x_{d\mathrm{Q}}+(E_{c\mathrm{Q}}-K_{cd\mathrm{Q}}E_{d\mathrm{Q}})w \quad (147)$$

$$\dot{x}_{d\mathrm{Q}}=\left(-\frac{1}{\varepsilon}I+E_{dc\mathrm{Q}}K_{cd\mathrm{Q}}\right)x_{d\mathrm{Q}}+E_{dc\mathrm{Q}}\tilde{x}_{c\mathrm{Q}}+E_{d\mathrm{Q}}w \quad (148)$$

It is clear to see that as $\varepsilon\rightarrow 0$, the poles of the above system are asymptotically given by $\lambda(A^c_{cc\mathrm{Q}})$ and k repeated ones at $-1/\varepsilon$. This confirms with what we have claimed earlier in (143). Following similar arguments as in (123)–(139), we can show that for any bounded initial condition and for $w\in L_p$, $p\in[1,\infty)$

$$\|\tilde{x}_{c\mathrm{Q}}\|_p\le\beta_c\|w\|_p \quad\text{and}\quad \|x_{d\mathrm{Q}}\|_p\le\beta_d\varepsilon\|w\|_p \quad (149)$$

for some positive scalars β_c and β_d, independent of ε. Thus, there exists a scalar β_v, independent of ε, such that

$$\|x_v\|_p\le\beta_v\|w\|_p \quad (150)$$

Following (65), it is simple to verify that

$$C_1^{-1}\{\mathrm{Im}(D_1)\}=\mathrm{Ker}\left(\Gamma_{o\mathrm{Q}}\begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}\Gamma_{s\mathrm{Q}}^{-1}\right)=\mathrm{Ker}\left(\begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}\Gamma_{s\mathrm{Q}}^{-1}\right) \quad (151)$$

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}\Gamma_{s\mathrm{Q}}^{-1}\right)x_v=\begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}\begin{pmatrix} x_{c\mathrm{Q}} \\ x_{d\mathrm{Q}} \end{pmatrix}=\begin{pmatrix} 0 \\ x_{d\mathrm{Q}} \end{pmatrix} \quad (152)$$

Thus, the last condition of Theorem 2, i.e. $\mathrm{Ker}(C_2)\supset C_1^{-1}\{\mathrm{Im}(D_1)\}$, implies that

$$C_2x_v=Mx_{d\mathrm{Q}} \quad\text{and}\quad \|C_2x_v\|_p\le\beta_m\varepsilon\|w\|_p \quad (153)$$

for some appropriate constant matrix M and positive scalar β_m, independent of ε. In fact, for any appropriate matrix N with $\mathrm{Ker}(N)\supset\mathrm{Ker}(C_2)$, we have

$$\|Nx_v\|_p\le|N|\cdot\beta_n\cdot\varepsilon\cdot\|w\|_p \quad (154)$$

for some positive scalar β_n (independent of ε).

We are now ready to show that the full order measurement and reference feedback control law of (69) solves the RPT problem. It is straightforward to verify that (141) and (142) can be re-written as

$$\left.\begin{aligned} \dot{\boldsymbol{x}}&=(\boldsymbol{A}+\boldsymbol{BF})\boldsymbol{x}-\boldsymbol{B}F(\varepsilon)x_v+\boldsymbol{Ew} \\ e&=(\boldsymbol{C}_2+\boldsymbol{D}_2\boldsymbol{F})\boldsymbol{x}-\boldsymbol{D}_2F(\varepsilon)x_v \end{aligned}\right\} \quad (155)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{C}_2$ and $\boldsymbol{D}_2$ are as defined in (48) and (49). Without loss of any generality, we assume hereafter that the quadruple$(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}_2,\boldsymbol{D}_2)$ is in the form of the special coordinate. Following the same procedures as in (94)–(111), we can transform (155) with some appropriate transformations into the form

$$\dot{\tilde{x}}_a^-=A_{aa}^-\tilde{x}_a^-+L_{ad}^-\tilde{e}_d+\bar{E}_a^-\bar{w}+N_a^-x_v \quad (156)$$

$$\dot{\tilde{x}}_c=A^c_{cc}\tilde{x}_c+L_{cd}\tilde{e}_d+\bar{E}_c\bar{w}+N_cx_v \quad (157)$$

$$\dot{x}_i=A_{q_i}x_i-B_{qi}\frac{F_i}{\varepsilon^{q_i}}S_i(\varepsilon)x_i+L_{id}e_d+\bar{E}_i\bar{w}+N_ix_v-\left[\begin{matrix} 0 & 0 & B_{q_i}\dfrac{F_i}{\varepsilon^{q_i}}S_i(\varepsilon) \end{matrix}\right]x_v \quad (158)$$

$$e_0=-[\,C_{0a}^- \quad C_{0c} \quad C_{0d}\,]x_v,\quad e_i=C_{q_i}x_i \quad (159)$$

for some appropriate dimensional matrices N_a^-, N_c and N_i, which are all independent of ε. First, it is simple to see that $\mathrm{Ker}(-[\,C_{0a}\ C_{0c}\ C_{od}\,])\supset\mathrm{Ker}(C_2)$. In view of (154), we have

$$\|e_0\|_p\rightarrow 0,\quad \text{as } \varepsilon\rightarrow 0 \quad (160)$$

Next, let us define a new state transformation as in (112), i.e.

$$\left.\begin{aligned} &\tilde{x}_a^- := x_a^-,\ \tilde{x}_c := x_c,\ \tilde{x}_d := \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_{m_d} \end{pmatrix} \\ &\tilde{x}_i := S_i(\varepsilon)x_i,\ i = 1,\dots,m_d \end{aligned}\right\} \tag{161}$$

Then

$$\dot{\tilde{x}}_a^- = A_{aa}^-\tilde{x}_a^- + L_{ad}^-\tilde{e}_d + [\bar{E}_a^- \ \ N_a^-]\begin{pmatrix} \bar{w} \\ x_v \end{pmatrix} \tag{162}$$

$$\dot{\tilde{x}}_c = A_{cc}^c\tilde{x}_c + L_{cd}\tilde{e}_d + [\bar{E}_c \ \ N_c]\begin{pmatrix} \bar{w} \\ x_v \end{pmatrix} \tag{163}$$

$$\begin{aligned} \dot{\tilde{x}}_i = {} & \frac{1}{\varepsilon}(A_{q_i} - B_{q_i}F_i)\tilde{x}_i + \tilde{L}_{id}(\varepsilon)\tilde{e}_d + [\,\tilde{E}_i(\varepsilon) \quad \tilde{N}_i(\varepsilon)\,] \\ & \times \begin{pmatrix} \bar{w} \\ x_v \end{pmatrix} - [\,0 \quad 0 \quad \tilde{M}_i(\varepsilon)\,]x_v \end{aligned} \tag{164}$$

$$\tilde{e}_i = e_i = C_{q_i}\tilde{x}_i, \quad \tilde{e}_d = e_d = C_d\tilde{x}_d \tag{165}$$

where

$$\tilde{M}_i(\varepsilon) = S_i(\varepsilon)B_{q_i}\frac{F_i}{\varepsilon^{q_i}}S_i(\varepsilon) \tag{166}$$

$$\tilde{L}_{id}(\varepsilon) = S_i(\varepsilon)L_{id}, \quad \tilde{N}_i(\varepsilon) = S_i(\varepsilon)N_i, \quad \tilde{E}_i(\varepsilon) = S_i(\varepsilon)\bar{E}_i \tag{167}$$

It is clear that the 2-norms of $\tilde{L}_{id}(\varepsilon)$, $\tilde{N}_i(\varepsilon)$ and $\tilde{E}_i(\varepsilon)$ are all bounded, and in view of the special structure of B_{q_i}, $S_i(\varepsilon)$ and F_i of (54), we have

$$\begin{aligned} \tilde{M}_i(\varepsilon) &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \dfrac{F_{iq_i}}{\varepsilon} & F_{iq_i-1} & \cdots & \varepsilon^{q_i-2}F_{i1} \end{bmatrix} \\ &= \frac{F_{iq_i}}{\varepsilon}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_{q_i} \end{bmatrix} + \hat{M}_i(\varepsilon) = \frac{F_{iq_i}}{\varepsilon}\hat{C}_{q_i} + \hat{M}_i(\varepsilon) \end{aligned} \tag{168}$$

where $|\hat{M}_i(\varepsilon)| \le \xi_i$ for some positive scalar ξ_i, independent of ε. Thus, (164) can be re-written as

$$\begin{aligned} \dot{\tilde{x}}_i = {} & \frac{1}{\varepsilon}(A_{q_i} - B_{q_i}F_i)\tilde{x}_i + \tilde{L}_{id}(\varepsilon)\tilde{e}_d \\ & + [\,\tilde{E}_i(\varepsilon) \quad \hat{N}_i(\varepsilon)\,]\begin{pmatrix} \bar{w} \\ x_v \end{pmatrix} - \frac{F_{iq_i}}{\varepsilon}[0 \quad 0 \quad \hat{C}_{q_i}]x_v \end{aligned} \tag{169}$$

for some bounded $\hat{N}_i(\varepsilon)$. It is clear that

$$\mathrm{Ker}([0 \quad 0 \quad \hat{C}_{q_i}]) \supset \mathrm{Ker}(C_2) \tag{170}$$

In view of (154), we have

$$\left\|\frac{F_{iq_i}}{\varepsilon}[0 \quad 0 \quad \hat{C}_{q_i}]x_v\right\|_p \le \eta_i\|w\|_p \tag{171}$$

for some positive scalar η_i (independent of ε). Hence, we can view

$$\begin{pmatrix} \bar{w} \\ x_v \end{pmatrix} \quad \text{and} \quad \frac{F_{iq_i}}{\varepsilon}[0 \quad 0 \quad \hat{C}_{q_i}]x_v \tag{172}$$

as some L_p signals, whose l_p norms are bounded by some ε independent scalars. Then, following the similar procedures as in (118)–(139), it is straightforward to show that

$$\|e_d\|_p \to 0, \quad \text{as } \varepsilon \to 0 \tag{173}$$

In view of (160) and (173), it is clear that the RPT problem is solved by the full order measurement and reference feedback control law (69). □

Appendix A.3. Proof of Theorem 5

We first define a new state variable, $x_s = x_2 - v + K_{R1}x_1$. Again, it is straightforward to verify that the closed-loop system comprising the given system (1) and the control law of (81) can be re-written as

$$\dot{x}_s = (A_R + K_R C_R)x_s + \left(E_2 + K_R\begin{bmatrix} D_{1,0} \\ E_1 \end{bmatrix}\right)w \tag{174}$$

$$\begin{aligned} \dot{x} = {} & [A + BF(\varepsilon)\,]x - BF_2(\varepsilon)x_s + BH_0(\varepsilon)r + \cdots \\ & + BH_{\kappa-1}(\varepsilon)r^{(\kappa-1)} + Ew \end{aligned} \tag{175}$$

$$\begin{aligned} h = {} & [C_2 + D_2F(\varepsilon)]x - D_2F_2(\varepsilon)x_s + D_2H_0(\varepsilon)r + \cdots \\ & + D_2H_{\kappa-1}(\varepsilon)r^{(\kappa-1)} \end{aligned} \tag{176}$$

Thus, it is simple to see that the closed-loop system is asymptotically stable for sufficiently small ε, as the closed-loop poles are given by the eigenvalues of $A + BF(\varepsilon)$ and $A_R + K_R C_R$.

Since $A_R + K_R C_R$ is asymptotically stable, it follows that for any initial condition, $x_s \in L_p$ provided that $w \in L_p$. Next, we re-write

$$BF_2(\varepsilon) = BF(\varepsilon)\begin{pmatrix} 0 \\ x_s \end{pmatrix} \quad \text{and} \quad D_2F_2(\varepsilon) = D_2F(\varepsilon)\begin{pmatrix} 0 \\ x_s \end{pmatrix} \tag{177}$$

It follows from (65) that

$$C_1^{-1}\{\mathrm{Im}(D_1)\} = \mathrm{Ker}\left(\begin{bmatrix} 0 & 0 \\ I_k & 0 \end{bmatrix}\right) \tag{178}$$

and

$$\begin{bmatrix} 0 & 0 \\ I_k & 0 \end{bmatrix}\begin{pmatrix} 0 \\ x_s \end{pmatrix} = 0 \tag{179}$$

Thus, the last condition of Theorem 2, i.e. $\mathrm{Ker}(C_2) \supset C_1^{-1}\{\mathrm{Im}(D_1)\}$, implies that

$$C_2\begin{pmatrix} 0 \\ x_s \end{pmatrix} = 0 \quad \text{and} \quad N\begin{pmatrix} 0 \\ x_s \end{pmatrix} = 0 \tag{180}$$

for any appropriate dimensional matrix N with $\mathrm{Ker}(N) \supset \mathrm{Ker}(C_2)$. Following the same procedures as in (155)–(173), we can show that

$$\|e\|_p \to 0, \quad \text{as } \varepsilon \to 0 \tag{181}$$

Hence, the RPT problem is solved by the reduced order measurement feedback control law (81). □

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