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Explicit solvability conditions for the general discrete-time H_∞ almost disturbance decoupling problem with internal stability

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We present in this paper several sets of numerically checkable solvability conditions for the discrete-time H_∞ almost disturbance decoupling problem with internal stability and with either (i) full information feedback, or (ii) full state feedback, or (iii) general measurement feedback. The problem considered here is general in the sense that we allow the subsystems of a given system to have invariant zeros on the unit circle. More importantly, our conditions are to be explicitly expressed in terms of some well-defined geometric subspaces, and, furthermore, these conditions can easily be verified numerically.

1. Introduction

The disturbance decoupling or almost disturbance decoupling problem is to find a compensator, either static or dynamic, to a given system affected by external disturbances, such that in the closed-loop system the disturbances have no influence at all or almost no influence in a certain sense (normally in the sense of H_2- or H_∞-norm) on the controlled output. It is one of the main stimuli in the development of control theory and plays a central role in several important problems such as decentralized control, non-interacting control, model reference tracking control, H_2 optimal control and H_∞ optimal control. The question of when the disturbances can be completely decoupled by feedback control from the to-be-controlled outputs led to the development of geometric control theory. Using the concept of (A, B)-invariant subspace, (C, A, B)-pairs, Wohnam (1979), Schumacher (1980), and others solved the complete disturbance decoupling problem with state or measurement feedback and with internal stability. The almost disturbance decoupling problem was introduced and partially solved by Willems (1981, 1982) and Weiland and Willems (1989) using the concept of almost invariant subspaces of linear systems. In the earlier results, which dealt with the almost disturbance decoupling problem, the stability region was normally restricted to a closed set in the complex plane to avoid the situation when the given plants' subsystems have purely imaginary invariant zeros. More recently, Scherer (1992), has finally overcome this difficulty and derived a set of necessary and sufficient conditions for the solvability of the H_∞ almost disturbance decoupling problem for general continuous-time systems without any pre-assumptions. His conditions are elegantly characterized in terms of geometric subspaces of the subsystems of the given system.

In this paper, we consider the problem of H_∞ almost disturbance decoupling for general discrete-time plants whose subsystems are allowed to have invariant zeros on the unit circle of the complex plane. Also, the stability region of a discrete-time system considered in this paper is defined as usual as the open unit disc. To be more specific, we consider the following standard linear time-invariant discrete-time system Σ characterized by

$$\Sigma : \begin{cases} x(k+1) = A\ x(k) + B\ u(k) + E\ w(k), \\ y(k) \ = C_1\ x(k) + D_1\ w(k), \\ h(k) \ = C_2\ x(k) + D_2\ u(k) + D_{22}\ w(k), \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^\ell$ is the measurement, $w \in \mathbb{R}^q$ is the disturbance and $h \in \mathbb{R}^p$ is the output to be controlled. A, B, E, C_1, D_1, C_2, D_2 and D_{22} are constant matrices of appropriate dimension. For the sake of easy reference in future development, we denote by Σ_P and Σ_Q the subsystems characterized by matrix quadruples (A, B, C_2, D_2) and (A, E, C_1, D_1), respectively. The following dynamic feedback control laws are investigated:

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$$\Sigma_c : \begin{cases} x_c(k+1) = A_{\rm cmp}\, x_c(k) + B_{\rm cmp}\, y(k), \\ u(k) \quad\;\; = C_{\rm cmp}\, x_c(k) + D_{\rm cmp}\, y(k), \end{cases} \tag{2}$$

The controller Σ_c of (2) is said to be internally stabilizing when applied to the system Σ if the following matrix is asymptotically stable:

$$A_{\rm cl} := \begin{bmatrix} A + BD_{\rm cmp}C_1 & BC_{\rm cmp} \\ B_{\rm cmp}C_1 & A_{\rm cmp} \end{bmatrix}, \tag{3}$$

i.e. all its eigenvalues lie inside the open unit disc of the complex plane. Denote by $G_{\rm cl}$ the corresponding closed-loop transfer matrix from the disturbance w to the controlled output h. Then the H_∞-norm of the transfer matrix $G_{\rm cl}$ is given by

$$\|G_{\rm cl}\|_\infty := \sup_{\omega\in[0,2\pi]} \sigma_{\max}[G_{\rm cl}(e^{j\omega})],$$

where $\sigma_{\max}[\cdot]$ denotes the largest singular value. We have the following formal definitions of the solvability of the H_∞ almost disturbance decoupling problems for general discrete-time systems.

Definition 1.1: Consider the given discrete-time system Σ of (1). Then, the problem of H_∞ almost disturbance decoupling with measurement feedback and with internal stability (H_∞-ADDPMS) for Σ is said to be solvable if for any given $\gamma > 0$, there exists a controller of the form (2) such that the resulting closed-loop system is asymptotically stable and the resulting closed-loop transfer matrix $\|G_{\rm cl}\|_\infty < \gamma$. □

We would like to note that for the plants whose subsystems Σ_P and Σ_Q have no invariant zeros on the unit circle, Chen *et al.* (1996) have obtained a solvability condition for the H_∞-ADDPMS. For a plant Σ whose subsystems have unit circle invariant zeros, Saberi *et al.* (1996) have recently shown that the following two statements are equivalent.

(1) The H_∞-ADDPMS is solvable.
(2) The H_2 almost disturbance decoupling problem for Σ with measurement feedback and with internal stability for Σ is solvable; and for all $\epsilon > 0$ and for any invariant zero s_0 on the unit circle of either Σ_P or Σ_Q, there exists a matrix K such that $s_0I - A - BKC_1$ is invertible and

$$\|(C_2 + D_2KC_1)(s_0I - A - BKC_1)^{-1}(E + BKC_1) + D_2KD_1\| < \epsilon. \tag{4}$$

The main problem with the above conditions given by Saberi *et al.* (1996) is that it is very difficult, if not impossible, to verify them, especially the second condition of the second statement, in which one would have to find a gain matrix for each unit circle invariant zero of Σ_P or Σ_Q and for each specific $\epsilon > 0$ such that condition (4) is satisfied. Clearly, the second statement of Saberi *et al.* (1996) is simply reformulating the original H_∞-ADDPMS to another problem, which is even more difficult to solve than the former, in our opinion. The main object of this paper is to derive necessary and sufficient conditions for the solvability of the H_∞-ADDPMS for general discrete-time systems. Our conditions are explicitly expressed in terms of geometric subspaces and are simple to be checked numerically.

The remainder of this paper is organized as follows. In section 2, we will recall the special coordinate basis of linear systems, which is instrumental in the derivation of the main results of the paper. Section 3 gives sets of necessary and sufficient conditions for the solvability of the H_∞-ADDPMS for general discrete-time systems in terms of well-defined geometric subspaces. There are three cases considered in this section, namely, the full information feedback, the full state feedback and the general measurement feedback cases. The proofs of these results are separately given in section 4 for the sake of clarity of presentation. Finally, the concluding remarks are drawn in section 5.

Throughout this paper, the following notations will also be used:

$\mathbb{R}$:= the set of real numbers,
$\mathbb{C}$:= the entire complex plane,
$\mathbb{C}^-$:= the open left-half complex plane,
$\mathbb{C}^+$:= the open right-half complex plane,
$\mathbb{C}^0$:= the imaginary axis in the complex plane,
$\mathbb{C}^\odot$:= the set of complex numbers inside the unit circle,
$\mathbb{C}^\otimes$:= the set of complex numbers outside the unit circle,
$\mathbb{C}^\circ$:= the unit circle in the complex plane,
I := an identity matrix,
$\lambda(X)$:= the set of eigenvalues of a square matrix X,
X' := the transpose of matrix X,
$X^\dagger$:= the pseudo inverse of matrix X,
$X^{\rm H}$:= the complex conjugate transpose of matrix X,
$\mathcal{X}^\perp$:= the orthogonal complementary subspace of subspace $\mathcal{X}$,
${\rm Ker}(X)$:= the kernel of X,
${\rm Im}(X)$:= the image of X,
$C^{-1}\{\mathcal{X}\}$:= $\{x \mid Cx \in \mathcal{X}\}$, where $\mathcal{X}$ is a subspace and C is a constant matrix,
$\dim(\mathcal{X})$:= the dimension of subspace $\mathcal{X}$.

The following geometric subspaces will also be heavily used in the paper to characterize the solvability conditions of the proposed problems and their proof.

Definition 1.2: For a linear system Σ_* characterized by a matrix quadruple (A_*, B_*, C_*, D_*), with $A_* \in \mathbb{R}^{n\times n}$, $B_* \in \mathbb{R}^{n\times m}$, $C_* \in \mathbb{R}^{p\times n}$ and $D_* \in \mathbb{R}^{p\times m}$, we define the weakly unobservable subspaces of Σ_*, $\mathcal{V}^{\mathrm{X}}$, and the strongly controllable subspaces of Σ_*, $\mathcal{S}^{\mathrm{X}}$, as follows.

(1) $\mathcal{V}^{\mathrm{X}}(\Sigma_*)$ is the maximal subspace of $\mathbb{R}^n$ which is $(A_* + B_*F_*)$-invariant and contained in $\mathrm{Ker}(C_* + D_*F_*)$ such that the eigenvalues of $(A_* + B_*F_*)|\mathcal{V}^{\mathrm{X}}$ are contained in $\mathbb{C}^{\mathrm{X}} \subset \mathbb{C}$ for some constant matrix F_*.

(2) $\mathcal{S}^{\mathrm{X}}(\Sigma_*)$ is the minimal $(A_* + K_*C_*)$-invariant subspace of $\mathbb{R}^n$ containing $\mathrm{Im}(B_* + K_*D_*)$ such that the eigenvalues of the map that is induced by $(A_* + K_*C_*)$ on the factor space $\mathbb{R}^n/\mathcal{S}^{\mathrm{X}}$ are contained in $\mathbb{C}^{\mathrm{X}} \subset \mathbb{C}$ for some constant matrix K_*.

We further let $\mathcal{V}^- = \mathcal{V}^{\mathrm{X}}$ and $\mathcal{S}^- = \mathcal{S}^{\mathrm{X}}$, if $\mathbb{C}^{\mathrm{X}} = \mathbb{C}^- \cup \mathbb{C}^0$; $\mathcal{V}^+ = \mathcal{V}^{\mathrm{X}}$ and $\mathcal{S}^+ = \mathcal{S}^{\mathrm{X}}$, if $\mathbb{C}^{\mathrm{X}} = \mathbb{C}^+$; $\mathcal{V}^{\odot} = \mathcal{V}^{\mathrm{X}}$ and $\mathcal{S}^{\odot} = \mathcal{S}^{\mathrm{X}}$, if $\mathbb{C}^{\mathrm{X}} = \mathbb{C}^{\odot} \cup \mathbb{C}^{\mathrm{o}}$; $\mathcal{V}^{\otimes} = \mathcal{V}^{\mathrm{X}}$ and $\mathcal{S}^{\otimes} = \mathcal{S}^{\mathrm{X}}$, if $\mathbb{C}^{\mathrm{X}} = \mathbb{C}^{\otimes}$; and finally $\mathcal{V}^* = \mathcal{V}^{\mathrm{X}}$ and $\mathcal{S}^* = \mathcal{S}^{\mathrm{X}}$, if $\mathbb{C}^{\mathrm{X}} = \mathbb{C}$.

Next, for any $\lambda \in \mathbb{C}$, we define

$$\mathcal{S}_\lambda(\Sigma_*) := \left\{ x \in \mathbb{C}^n \,\middle|\, \exists u \in \mathbb{C}^{n+m} \,:\, \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} u \right\}, \tag{5}$$

and

$$\mathcal{V}_\lambda(\Sigma_*) := \left\{ x \in \mathbb{C}^n \,\middle|\, \exists u \in \mathbb{C}^m \,:\, 0 = \begin{bmatrix} A_* - \lambda I & B_* \\ C_* & D_* \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\}. \tag{6}$$

$\mathcal{V}_\lambda(\Sigma_*)$ and $\mathcal{S}_\lambda(\Sigma_*)$ are associated with the so-called state zero directions of Σ_* if λ is an invariant zero of Σ_*. □

Finally, note that $\mathcal{V}^{\mathrm{X}}(\Sigma_*)$ and $\mathcal{S}^{\mathrm{X}}(\Sigma_*)$ are dual in the sense that $\mathcal{V}^{\mathrm{X}}(\Sigma_*^\star) = \mathcal{S}^{\mathrm{X}}(\Sigma_*)^\perp$, where $\Sigma_*^\star$ is characterized by the quadruple (A_*', C_*', B_*', D_*'). The subspaces $\mathcal{S}_\lambda(\Sigma_*)$ and $\mathcal{V}_\lambda(\Sigma_*)$ are also dual, i.e. $\mathcal{S}_\lambda(\Sigma_*) = \mathcal{V}_{\bar\lambda}(\Sigma_*^\star)^\perp$, where $\bar\lambda$ is the complex conjugate of λ.

2. Background materials

We recall in this section a special coordinate basis (SCB) of a linear time-invariant system introduced by Sannuti and Saberi (1987), and Saberi and Sannuti (1990). Such a special coordinate basis has a distinct feature of explicitly displaying the finite and infinite zero structures as well as the invertibility structure of a given system. Let us consider a linear time-invariant (LTI) system Σ_*, which could be of either continuous-time or discrete-time, characterized by the quadruple (A_*, B_*, C_*, D_*), or in the state space form,

$$\left.\begin{aligned} \delta(x) &= A_*\, x + B_*\, u, \\ y &= C_*\, x + D_*\, u, \end{aligned}\right\} \tag{7}$$

where $\delta(x) = \dot{x}(t)$, if Σ_* is a continuous-time system, or $\delta(x) = x(k+1)$, if Σ_* is a discrete-time system. Similarly, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, the input and the output of Σ_*. They represent $x(t)$, $u(t)$ and $y(t)$, respectively, if the given system is of continuous-time, or they represent $x(k)$, $u(k)$ and $y(k)$, respectively, if Σ_* is of discrete-time. It is simple to verify that there exist non-singular transformations U and V such that

$$UD_*V = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \tag{8}$$

where m_0 is the rank of matrix D_*. In fact, U can be chosen as an orthogonal matrix. Hence hereafter, without loss of generality, it is assumed that the matrix D has the form given on the right hand side of (26). One can now rewrite the system of (25) as,

$$\left.\begin{aligned} \delta(x) &= A_*\, x + \begin{bmatrix} B_{*,0} & B_{*,1} \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} &= \begin{bmatrix} C_{*,0} \\ C_{*,1} \end{bmatrix} x + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \end{aligned}\right\} \tag{9}$$

where the matrices $B_{*,0}$, $B_{*,1}$, $C_{*,0}$ and $C_{*,1}$ have appropriate dimensions. The following theorem combines the results of Sannuti and Saberi (1987), and Saberi and Sannuti (1990).

Theorem 2.1 (SCB): *Given the linear system Σ_* of (7), there exist non-singular state, output and input transformations Γ_s, Γ_o and Γ_i such that*

$$\left.\begin{aligned} u &= \Gamma_i \begin{pmatrix} u_0 \\ u_d \\ u_c \end{pmatrix}, \quad x = \Gamma_s \begin{pmatrix} x_c \\ x_a \\ x_b \\ x_d \end{pmatrix}, \\ x_a &= \begin{pmatrix} x_a^- \\ x_a^0 \\ x_a^+ \end{pmatrix}, \quad y = \Gamma_o \begin{pmatrix} y_0 \\ y_d \\ y_b \end{pmatrix}, \end{aligned}\right\} \tag{10}$$

with $x_c \in \mathbb{R}^{n_c}$, $x_a \in \mathbb{R}^{n_a}$, $x_b \in \mathbb{R}^{n_b}$, $x_d \in \mathbb{R}^{n_d}$, and

$$\tilde{A}_* = \Gamma_s^{-1}(A_* - B_{*,0}C_{*,0})\Gamma_s = \begin{bmatrix} A_{cc} & B_cE_{ca}^- & B_cE_{ca}^0 & B_cE_{ca}^+ & L_{cb}C_b & L_{cd}C_d \\ 0 & A_{aa}^- & 0 & 0 & L_{ab}^-C_b & L_{ad}^-C_d \\ 0 & 0 & A_{aa}^0 & 0 & L_{ab}^0C_b & L_{ad}^0C_d \\ 0 & 0 & 0 & A_{aa}^+ & L_{ab}^+C_b & L_{ad}^+C_d \\ 0 & 0 & 0 & 0 & A_{bb} & L_{bd}C_d \\ B_dE_{dc} & B_dE_{da}^- & B_dE_{da}^0 & B_dE_{da}^+ & B_dE_{db} & A_{dd} \end{bmatrix}, \tag{11}$$

$$\tilde{B}_* = \Gamma_s^{-1}[\,B_{*,0} \quad B_{*,1}\,]\Gamma_i = \begin{bmatrix} B_{0c} & 0 & B_c \\ B_{0a}^- & 0 & 0 \\ B_{0a}^0 & 0 & 0 \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0d} & B_d & 0 \end{bmatrix}, \tag{12}$$

$$\tilde{C}_* = \Gamma_o^{-1}\begin{bmatrix} C_{*,0} \\ C_{*,1} \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0c} & C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0d} \\ 0 & 0 & 0 & 0 & 0 & C_d \\ 0 & 0 & 0 & 0 & C_b & 0 \end{bmatrix}, \tag{13}$$

and

$$\tilde{D}_* = \Gamma_o^{-1}D\Gamma_i = \begin{bmatrix} I_{m_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

where (A_{cc}, B_c) is completely controllable, (A_{bb}, C_b) is completely observable, and (A_{dd}, B_d, C_d) is invertible and free of invariant zeros. Moreover, $\lambda(A_{aa}^-) \subset \mathbb{C}^-$, $\lambda(A_{aa}^0) \subset \mathbb{C}^0$, and $\lambda(A_{aa}^+) \subset \mathbb{C}^+$, if Σ_* is a continuous-time system; or $\lambda(A_{aa}^-) \subset \mathbb{C}^{\odot}$, $\lambda(A_{aa}^0) \subset \mathbb{C}^{\circ}$, $\lambda(A_{aa}^+) \subset \mathbb{C}^{\otimes}$, if Σ_* is a discrete-time system. □

By now it is clear that the special coordinate basis decomposes the state-space into several distinct parts. In fact, the state-space $\mathcal{X}$ of Σ_* is decomposed as

$$\mathcal{X} = \mathcal{X}_c \oplus \mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \oplus \mathcal{X}_d. \tag{15}$$

Here, $\mathcal{X}_a^-$ is related to the stable invariant zeros, i.e. the eigenvalues of A_{aa}^- are the stable invariant zeros of Σ_*. Similarly, $\mathcal{X}_a^0$ and $\mathcal{X}_a^+$ are respectively related to the invariant zeros of Σ_* located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_b$ is related to the right invertibility, i.e. the system is right invertible if and only if $\mathcal{X}_b = \{0\}$, while $\mathcal{X}_c$ is related to left invertibility, i.e. the system is left invertible if and only if $\mathcal{X}_c = \{0\}$. Finally, $\mathcal{X}_d$ is related to zeros of Σ_* at infinity.

The following property shows the interconnections between the special coordinate basis and various invariant geometric subspaces.

Property 2.1

(1) $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_c$ spans $\begin{cases} \mathcal{V}^-(\Sigma_*), & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{V}^{\odot}(\Sigma_*), & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(2) $\mathcal{X}_a^+ \oplus \mathcal{X}_c$ spans $\begin{cases} \mathcal{V}^+(\Sigma_*), & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{V}^{\otimes}(\Sigma_*), & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(3) $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c$ spans $\mathcal{V}^*(\Sigma_*)$.

(4) $\mathcal{X}_a^+ \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ spans $\begin{cases} \mathcal{S}^-(\Sigma_*), & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{S}^{\odot}(\Sigma_*), & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(5) $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ spans $\begin{cases} \mathcal{S}^+(\Sigma_*), & \text{if } \Sigma_* \text{ is of continuous-time,} \\ \mathcal{S}^{\otimes}(\Sigma_*), & \text{if } \Sigma_* \text{ is of discrete-time.} \end{cases}$

(6) $\mathcal{X}_c \oplus \mathcal{X}_d$ spans $\mathcal{S}^*(\Sigma_*)$. □

The $\mathcal{S}_\lambda(\Sigma_*)$ and $\mathcal{V}_\lambda(\Sigma_*)$ can also be easily obtained using the special coordinate basis. We have the following property.

Property 2.2

$$\mathcal{S}_\lambda(\Sigma_*) = \operatorname{Im}\left\{ \Gamma_s \begin{bmatrix} \lambda I - A_{aa} & 0 & 0 & 0 \\ 0 & Y_{b\lambda} & 0 & 0 \\ 0 & 0 & I_{n_c} & 0 \\ 0 & 0 & 0 & I_{n_d} \end{bmatrix} \right\}, \tag{16}$$

where

$$\operatorname{Im}\{Y_{b\lambda}\} = \operatorname{Ker}\left[C_b(A_{bb} + K_bC_b - \lambda I)^{-1}\right], \tag{17}$$

and where K_b is any appropriate matrix subject to the constraint that matrix $A_{bb} + K_bC_b$ has no eigenvalues at λ. We note that such a K_b always exists as (A_{bb}, C_b) is completely observable.

$$\mathcal{V}_\lambda(\Sigma_*) = \operatorname{Im}\left\{ \Gamma_s \begin{bmatrix} X_{a\lambda} & 0 \\ 0 & 0 \\ 0 & X_{c\lambda} \\ 0 & 0 \end{bmatrix} \right\}, \tag{18}$$

where $X_{a\lambda}$ is a matrix whose columns form a basis for the subspace,

$$\left\{ \zeta_a \in \mathbb{C}^{n_a} \,\middle|\, (\lambda I - A_{aa})\zeta_a = 0 \right\}, \tag{19}$$

and

$$X_{c\lambda} := \left(A_{cc} + B_cF_c - \lambda I\right)^{-1} B_c, \tag{20}$$

with F_c being any appropriately dimensional matrix subject to the constraint that $A_{cc} + B_cF_c$ has no eigenvalues at λ. Again, we note that the existence of such an F_c is guaranteed by the controllability of (A_{cc}, B_c). □

Clearly, if $\lambda \notin \lambda(A_{aa}^-) \cup \lambda(A_{aa}^0) \cup \lambda(A_{aa}^+)$, then

$$\mathcal{V}_\lambda(\Sigma_*) \subseteq \mathcal{V}^\times(\Sigma_*), \quad \text{and} \quad \mathcal{S}_\lambda(\Sigma_*) \supseteq \mathcal{S}^\times(\Sigma_*). \tag{21}$$

Lastly, we would conclude this section by noting that software packages that realize the special coordinate basis of Theorem 2.1 can be found in LAS by Chen (1988) and in MATLAB by Lin (1989). The rigorous proofs of the above mentioned properties of the special coordinate basis can be found in Chen (1998).

3. Solvability conditions for the H_∞-ADDPMS

We are now ready to present our main results of this paper. In this section we give the solvability conditions for the general H_∞ almost disturbance decoupling problems with internal stability for the following three cases: the full information feedback, the full state feedback and the measurement feedback. These conditions are characterized in terms of some well-defined geometric subspaces. We also develop a numerical algorithm that will check these conditions without actually computing any geometric subspaces. The proofs of the main results of this section are given in the next section just for clarity of presentation.

Let us first examine the full information case. We have the following result.

Theorem 3.1: *Consider the given discrete-time linear time-invariant system* Σ *of* (1) *with the measurement output being*

$$y = \begin{pmatrix} x \\ w \end{pmatrix}, \quad \text{or} \quad C_1 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 \\ I \end{pmatrix}, \tag{22}$$

i.e., all state variables and disturbances (full information) are measurable and available for feedback. The H_∞ almost disturbance decoupling problem with full information feedback and with internal stability for the given system is solvable if and only if the following conditions are satisfied:

(*a*) (A, B) *is stabilizable.*
(*b*) $\mathrm{Im}(D_{22}) \subset \mathrm{Im}(D_2)$, i.e., $D_{22} + D_2 S = 0$, *where* $S = -(D_2' D_2)^\dagger D_2' D_{22}$.
(*c*) $\mathrm{Im}(E + BS) \subset \{\mathcal{V}^\odot(\Sigma_P) + B\mathrm{Ker}(D_2)\} \cap \{\bigcap_{|\lambda|=1} \mathcal{S}_\lambda(\Sigma_P)\}$. □

The result for the general measurement feedback case is given in the next theorem.

Theorem 3.2: *Consider the given discrete-time linear time-invariant system* Σ *of* (1). *The H_∞ almost disturbance decoupling problem with measurement feedback and with internal stability (H_∞-ADDPMS) for* (1) *is solvable if and only if the following conditions are satisfied:*

(*a*) (A, B) *is stabilizable.*
(*b*) (A, C_1) *is detectable.*
(*c*) $D_{22} + D_2 S D_1 = 0$, *where* $S = -(D_2' D_2)^\dagger D_2' D_{22} D_1' (D_1 D_1')^\dagger$.
(*d*) $\mathrm{Im}(E + BSD_1) \subset \{\mathcal{V}^\odot(\Sigma_P) + B\mathrm{Ker}(D_2)\} \cap \{\bigcap_{|\lambda|=1} \mathcal{S}_\lambda(\Sigma_P)\}$.
(*e*) $\mathrm{Ker}(C_2 + D_2 S C_1) \supset \{\mathcal{S}^\odot(\Sigma_Q) \cap C_1^{-1}\{\mathrm{Im}(D_1)\}\} \cup \{\bigcup_{|\lambda|=1} \mathcal{V}_\lambda(\Sigma_Q)\}$.
(*f*) $\mathcal{S}^\odot(\Sigma_Q) \subset \mathcal{V}^\odot(\Sigma_P)$. □

The following remarks are in order.

Remark 3.1: Note that if Σ_P is of minimum phase and right invertible with no infinite zeros, and Σ_Q is of minimum phase and left invertible with no infinite zeros, then Conditions (*d*) to (*f*) of Theorem 3.2 are automatically satisfied. Hence, the solvability conditions of the H_∞-ADDPMS for such a case reduce to:

(*a*) (A, B) is stabilizable.
(*b*) (A, C_1) is detectable.
(*c*) $D_{22} + D_2 S D_1 = 0$, where $S = -(D_2' D_2)^\dagger D_2' D_{22} D_1' \times (D_1 D_1')^\dagger$.

Remark 3.2: For a special case when all the states of the system (1) are measurable and available for feedback, i.e. $y = x$, it can be easily derived from Theorem 3.2 that the H_∞ almost disturbance decoupling problem with full state feedback and with internal stability for the given system is solvable if and only if the following conditions are satisfied:

(*a*) (A, B) is stabilizable.
(*b*) $D_{22} = 0$.
(*c*) $\mathrm{Im}(E) \subset \mathcal{V}^\odot(\Sigma_P) \cap \{\bigcap_{|\lambda|=1} \mathcal{S}_\lambda(\Sigma_P)\}$. □

As mentioned earlier, the conditions in Theorem 3.2 can actually be verified without computing the geometric subspaces of Σ_P and Σ_Q. This can be done by fully understanding and utilizing the properties of the special coordinate basis of linear systems as given in Theorem 2.1. We have the following algorithm that will verify the solvability conditions given in Theorem 3.2.

Step 0. Let $S = -(D_2' D_2)^\dagger D_2' D_{22} D_1' (D_1 D_1')^\dagger$. If $D_{22} + D_2 S D_1 \neq 0$, the H_∞-ADDPMS for (1) is not solvable and the algorithm stops here. Otherwise, go to the next step.

Step 1. Compute the special coordinate basis of Σ_P, i.e. the quadruple (A, B, C_2, D_2). For easy reference, we append a subscript 'P' to all sub-matrices and transformations in the SCB associated with Σ_P, e.g. Γ_{sP} is the state transformation of the SCB of Σ_P, B_{dP} is replacing the sub-matrix B_d, and A_{aaP}^0 is associated with invariant zero dynamics of Σ_P on the unit circle.

Step 2. Next, we denote the set of eigenvalues of A^0_{aaP} with a non-negative imaginary part as $\{\omega_{P1}, \omega_{P2}, \ldots, \omega_{Pk_P}\}$ and for $i = 1, 2, \ldots, k_P$, choose complex matrices V_{iP}, whose columns form a basis for the eigenspace $\{x \in \mathbb{C}^{n^0_{aP}} \mid x^H(\omega_{Pi}I - A^0_{aaP}) = 0\}$, where n^0_{aP} is the dimension of $\mathcal{X}^0_{aP}$. Then, let

$$V_P := [\,V_{1P}\, V_{2P} \ldots V_{k_P P}]. \tag{23}$$

We also compute $n_{xP} := \dim(\mathcal{X}^+_{aP}) + \dim(\mathcal{X}_{bP}) + \dim(\mathcal{X}_{dP})$, and

$$\Gamma^{-1}_{sP}(E + BSD_1) := \begin{bmatrix} E_{cP} \\ E^-_{aP} \\ E^0_{aP} \\ E^+_{aP} \\ E_{bP} \\ E_{dP} \end{bmatrix}. \tag{24}$$

Step 3. Let $\Sigma^\star_Q$ be the dual system of Σ_Q and be characterized by a quadruple (A', C'_1, E', D'_1). We compute the special coordinate basis of $\Sigma^\star_Q$. Again, for ease of reference, we append a subscript 'Q' to all sub-matrices and transformations in the SCB associated with $\Sigma^\star_Q$, e.g. Γ_{sQ} is the state transformation of the SCB of $\Sigma^\star_Q$, B_{dQ} is replacing the sub-matrix B_d, and A^0_{aaQ} is associated with invariant zero dynamics of $\Sigma^\star_Q$ on the unit circle.

Step 4. Similarly, we denote the set of eigenvalues of A^0_{aaQ} with a non-negative imaginary part as $\{\omega_{Q1}, \omega_{Q2}, \ldots, \omega_{Qk_Q}\}$ and for $i = 1, 2, \ldots, k_Q$, choose complex matrices V_{iQ}, whose columns form a basis for the eigenspace $\{x \in \mathbb{C}^{n^0_{aQ}} \mid x^H(\omega_{Qi}I - A^0_{aaQ}) = 0\}$, where n^0_{aQ} is the dimension of $\mathcal{X}^0_{aQ}$. Then, let

$$V_Q := \left[\, V_{1Q}\; V_{2Q} \cdots V_{k_Q}Q \right]. \tag{25}$$

We next compute $n_{xQ} := \dim(\mathcal{X}^+_{aQ}) + \dim(\mathcal{X}_{bQ}) + \dim(\mathcal{X}_{dQ})$, and

$$\Gamma^{-1}_{sQ}(C_2 + D_2 S C_1)' := \begin{bmatrix} E_{cQ} \\ E^-_{aQ} \\ E^0_{aQ} \\ E^+_{aQ} \\ E_{bQ} \\ E_{dQ} \end{bmatrix}. \tag{26}$$

Step 5. Finally, compute

$$\Gamma^{-1}_{sP}(\Gamma^{-1}_{sQ})' = \begin{bmatrix} \star & \star \\ \star & \Gamma \end{bmatrix}, \tag{27}$$

where Γ is a $n_{xP} \times n_{xQ}$ constant matrix.

The following proposition summarizes the result of the above algorithm. It also gives a set of necessary and sufficient conditions, in terms of sub-matrices associated with the special coordinate bases of Σ_P and Σ_Q, for the solvability of the H_∞-ADDPMS for the general discrete-time system Σ of (1).

Proposition 3.1: *Consider the given discrete-time linear time-invariant system Σ of (1). The H_∞ almost disturbance decoupling problem with measurement feedback and with internal stability (H_∞-ADDPMS) for (1) is solvable if and only if the following conditions are satisfied:*

(*a*) (A, B) *is stabilizable,*
(*b*) (A, C_1) *is detectable,*
(*c*) $D_{22} - D_2(D'_2D_2)^\dagger D'_2 D_{22} D'_1 (D_1 D'_1)^\dagger D_1 = 0.$
(*d*) $V^H_P E^0_{aP} = 0$, $E^+_{aP} = 0$, $E_{bP} = 0$, $\mathrm{Im}(E_{dP}) \subset \mathrm{Im}(B_{dP})$,
(*e*) $V^H_Q E^0_{aQ} = 0$, $E^+_{aQ} = 0$, $E_{bQ} = 0$, $\mathrm{Im}(E_{dQ}) \subset \mathrm{Im}(B_{dQ})$,
(*f*) $\Gamma = 0$.

Note that all the matrices in (d)–(f) are well-defined in Steps 0 *to* 5 *of the algorithm.* □

The above result can be directly verified using the properties of the special coordinate basis and the result of Theorem 3.2 (see also Chapter 7 of Chen (1998) for a similar result for continuous-time systems).

4. Proofs of main results

We will prove the main results of the paper in this section. Our idea is to first transform the H_∞-ADDPMS for the discrete-time system (1) into an equivalent H_∞-ADDPMS for an auxiliary continuous-time system using the well-known inverse bilinear transformation and then identify the mappings of geometric conditions under such a transformation.

4.1. *Proof of Theorem 3.1*

Let us first show the result of Theorem 3.1, i.e. the solvability conditions of the H_∞-ADDPMS for the following full information system,

$$\Sigma_{FI} : \begin{cases} x(k+1) = A\; x(k) + B\; u(k) + E\; w(k), \\ y(k) = \begin{pmatrix} I \\ 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} w(k), \\ h(k) = C_2\; x(k) + D_2\; u(k) + D_{22}\; w(k). \end{cases} \tag{28}$$

We define the following auxiliary continuous-time system,

$$\tilde{\Sigma}_{FI} : \begin{cases} \dot{\tilde{x}} = \tilde{A}\ \tilde{x} + \tilde{B}\ \tilde{u} + \tilde{E}\ \tilde{w}, \\ \tilde{y} = \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{x} + \begin{pmatrix} 0 \\ I \end{pmatrix} \tilde{w}, \\ \tilde{z} = \tilde{C}_2\ \tilde{x} + \tilde{D}_2\ \tilde{u} + \tilde{D}_{22}\ \tilde{w}, \end{cases} \tag{29}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{E}$, $\tilde{C}_2$, $\tilde{D}_2$ and $\tilde{D}_{22}$ are defined as

$$\left.\begin{aligned} \tilde{A} &= (A + BF_0 + I)^{-1}(A + BF_0 - I), \\ \tilde{B} &= \sqrt{2}(A + BF_0 + I)^{-1}B, \\ \tilde{E} &= \sqrt{2}(A + BF_0 + I)^{-1}E, \\ \tilde{C}_2 &= \sqrt{2}(C_2 + D_2F_0)(A + BF_0 + I)^{-1}, \\ \tilde{D}_2 &= D_2 - (C_2 + D_2F_0)(A + BF_0 + I)^{-1}B, \\ \tilde{D}_{22} &= D_{22} - (C_2 + D_2F_0)(A + BF_0 + I)^{-1}E, \end{aligned}\right\} \tag{30}$$

and where F_0 is chosen such that $A + BF_0$ has no eigenvalues at -1. This can always be done provided that (A, B) is stabilizable. For future use, we denote $\tilde{\Sigma}_P$ as the subsystem characterized by $(\tilde{A}, \tilde{B}, \tilde{C}_2, \tilde{D}_2)$. It was shown by Glover (1984) that the infimum of H_∞ optimization for the discrete-time system (46) is equivalent to that of H_∞ optimization for the auxiliary continuous-time system (47). Thus, as a direct consequence, the H_∞-ADDPMS for the discrete-time system (46) is solvable if and only if the H_∞-ADDPMS for the continuous-time system (47) is solvable. Following the results of Scherer (1992), one can show that the H_∞-ADDPMS for (47) is solvable if and only if the following conditions are satisfied.

(a) $(\tilde{A}, \tilde{B})$ is stabilizable.
(b) There exists a matrix $\tilde{S}$ such that $\tilde{D}_{22} + \tilde{D}_2\tilde{S} = 0$.
(c) $\mathrm{Im}(\tilde{E} + \tilde{B}\tilde{S}) \subset \mathcal{S}^+(\tilde{\Sigma}_P) \cap \{\bigcap_{\lambda \in \mathbb{C}^0} \mathcal{S}_\lambda(\tilde{\Sigma}_P)\}$.

It is simple to show that (A, B) is stabilizable if and only if $(\tilde{A}, \tilde{B})$ is stabilizable. Hence, it is sufficient to show Theorem 3.1 by showing that the following two statements are equivalent.

(1) The first statement:
  (a) there exists an S such that $D_{22} + D_2S = 0$;
  (b) $\mathrm{Im}(E + BS) \subset \{\mathcal{V}^{\odot}(\Sigma_P) + B\,\mathrm{Ker}(D_2)\} \cap \{\bigcap_{|\lambda|=1} \mathcal{S}_\lambda(\Sigma_P)\}$.
(2) The second statement:
  (a) there exists an $\tilde{S}$ such that $\tilde{D}_{22} + \tilde{D}_2\tilde{S} = 0$,
  (b) $\mathrm{Im}(\tilde{E} + \tilde{B}\tilde{S}) \subset \mathcal{S}^+(\tilde{\Sigma}_P) \cap \{\bigcap_{\lambda \in \mathbb{C}^0} \mathcal{S}_\lambda(\tilde{\Sigma}_P)\}$.

Statement 1 ⇒ Statement 2: It is without loss of any generality to assume that matrix D_{22} in (46) is equal to 0. Also, by the definitions of the geometric subspaces $\mathcal{V}^X$, $\mathcal{S}^X$, $\mathcal{V}_\lambda$ and $\mathcal{S}_\lambda$, it is simple to verify that they are all invariant under any state feedback, output injection laws, and non-singular input as well as non-singular output transformations. Hereafter, we will assume that the subsystem Σ_P, i.e. the quadruple (A, B, C_2, D_2), is in the form of the special coordinate basis of Theorem 2.1. For easy reference in future development, we further assume that the state space of Σ_P has been decomposed as follows:

$$\mathcal{X} = \mathcal{X}_a^{0*} \oplus \mathcal{X}_a^- \oplus \mathcal{X}_c \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \oplus \mathcal{X}_d \oplus \mathcal{X}_a^{01}, \tag{31}$$

where $\mathcal{X}_a^{01}$ corresponds to the zero dynamics of Σ_P associated with the invariant zero at $z = -1$ and $\mathcal{X}_a^{0*}$ corresponds to the zero dynamics of Σ_P associated with the rest invariant zeros on the unit circle. More specifically, we let

$$A = \begin{bmatrix} A_{aa}^{0*} & 0 & 0 & 0 & L_{ab}^{0*}C_b & L_{ad}^{0*}C_d & 0 \\ 0 & A_{aa}^- & 0 & 0 & L_{ab}^-C_b & L_{ad}^-C_d & 0 \\ B_cE_{ca}^{0*} & B_cE_{ca}^- & A_{cc} & B_cE_{ca}^+ & L_{cb}C_b & L_{cd}C_d & B_cE_{ca}^{01} \\ 0 & 0 & 0 & A_{aa}^+ & L_{ab}^+C_b & L_{ad}^+C_d & 0 \\ 0 & 0 & 0 & 0 & A_{bb} & L_{bd}C_d & 0 \\ B_dE_{da}^{0*} & B_dE_{da}^- & B_dE_{dc} & B_dE_{da}^+ & B_dE_{db} & A_{dd} & B_dE_{da}^{01} \\ 0 & 0 & 0 & 0 & L_{ab}^{01}C_b & L_{ad}^{01}C_d & A_{aa}^{01} \end{bmatrix} + B_0C_{2,0}, \tag{32}$$

$$B = [B_0 \quad B_1] = \begin{bmatrix} B_{0a}^{0*} & 0 & 0 \\ B_{0a}^- & 0 & 0 \\ B_{0c} & 0 & B_c \\ B_{0a}^+ & 0 & 0 \\ B_{0b} & 0 & 0 \\ B_{0d} & B_d & 0 \\ B_{0a}^{01} & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} E_a^{0*} \\ E_a^- \\ E_c \\ E_a^+ \\ E_b \\ E_d \\ E_a^{01} \end{bmatrix}, \tag{33}$$

and

$$C_2 = \begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix} = \begin{bmatrix} C_{0a}^{0*} & C_{0a}^- & C_{0c} & C_{0a}^+ & C_{0b} & C_{0d} & C_{0a}^{01} \\ 0 & 0 & 0 & 0 & 0 & C_d & 0 \\ 0 & 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{34}$$

where A_{aa}^{01} has all its eigenvalues at -1 and A_{aa}^{0*} has all its eigenvalues on the unit circle, but excluding the point -1. Then, the condition in Statement 1(b) is equivalent to

$$E_a^+ = 0, \quad E_b = 0, \quad E_a^{01} = (I + A_{aa}^{01})X_a^{01}, \quad E_d = B_dX_d, \tag{35}$$

for some appropriately dimensional X_a^{01} and X_d, and

$$E_a^{0*} = Y_{aa}^{0*} X_a^{0*}, \tag{36}$$

where Y_{aa}^{0*} is a matrix whose columns span $\cap_{\alpha\in\lambda(A_{aa}^{0*})}\mathrm{Im}(\alpha I - A_{aa}^{0*})$ and X_a^{0*} is an appropriately dimensional matrix.

Let us now choose F_0 as,

$$F_0 = -\begin{bmatrix} C_{0a}^{0*} & C_{0a}^{-} & C_{0c} & C_{0a}^{+} & C_{0b} & C_{0d} & C_{0a}^{01} \\ E_{da}^{0*} & E_{da}^{-} & E_{dc} & E_{da}^{+} & E_{db} & 0 & E_{da}^{01} - \hat{E}_{da}^{01} \\ E_{ca}^{0*} & E_{ca}^{-} & 0 & E_{ca}^{+} & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

Then, we have

$$\hat{A} = A + BF_0 = \begin{bmatrix} A_{aa}^{0*} & 0 & 0 & 0 & L_{ab}^{0*}C_b & L_{ad}^{0*}C_d & 0 \\ 0 & A_{aa}^{-} & 0 & 0 & L_{ab}^{-}C_b & L_{ad}^{-}C_d & 0 \\ 0 & 0 & A_{cc} & 0 & L_{cb}C_b & L_{cd}C_d & 0 \\ 0 & 0 & 0 & A_{aa}^{+} & L_{ab}^{+}C_b & L_{ad}^{+}C_d & 0 \\ 0 & 0 & 0 & 0 & A_{bb} & L_{bd}C_d & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{dd} & B_d\hat{E}_{da}^{01} \\ 0 & 0 & 0 & 0 & L_{ab}^{01}C_b & L_{ad}^{01}C_d & A_{aa}^{01} \end{bmatrix}, \tag{38}$$

and

$$\hat{C}_2 = C_2 + D_2F_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_d & 0 \\ 0 & 0 & 0 & 0 & C_b & 0 & 0 \end{bmatrix}. \tag{39}$$

For simplicity, we further assume that A_{cc}, A_{bb} and A_{dd} have no eigenvalues at -1. Otherwise, some additional pre-state feedback will relocate them to somewhere else. Also, $\hat{E}_{da}^{01}$ is chosen such that $\hat{A}$ has no eigenvalues at -1. Next, it can be computed that

$$(A+BF_0+I)^{-1} = \begin{bmatrix} (I+A_{aa}^{0*})^{-1} & 0 & 0 & 0 & X_{15} & X_{16} & X_{17} \\ 0 & (I+A_{aa}^{-})^{-1} & 0 & 0 & X_{25} & X_{26} & X_{27} \\ 0 & 0 & (I+A_{cc})^{-1} & 0 & X_{35} & X_{36} & X_{37} \\ 0 & 0 & 0 & (I+A_{aa}^{+})^{-1} & X_{45} & X_{46} & X_{47} \\ 0 & 0 & 0 & 0 & X_{55} & X_{56} & X_{57} \\ 0 & 0 & 0 & 0 & X_{65} & X_{66} & X_{67} \\ 0 & 0 & 0 & 0 & X_{75} & X_{76} & X_{77} \end{bmatrix}, \tag{40}$$

where

$$X_{55} = (I+A_{bb})^{-1}\{I - L_{bd}C_d(I+A_{dd})^{-1} \times B_d\hat{E}_{da}^{01}\Delta^{-1}L_{ab}^{01}C_b(I+A_{bb})^{-1}\}, \tag{41}$$

$$X_{56} = -(I+A_{bb})^{-1}L_{bd}\{I + L_{bd}C_d(I+A_{dd})^{-1}B_d\hat{E}_{da}^{01}\Delta^{-1} \times [L_{ad}^{01} - L_{ab}^{01}C_b(I+A_{bb})^{-1}L_{bd}]\} \times C_d(I+A_{dd})^{-1}, \tag{42}$$

$$X_{57} = (I+A_{bb})^{-1}L_{bd}C_d(I+A_{dd})^{-1}B_d\hat{E}_{da}^{01}\Delta^{-1}, \tag{43}$$

$$X_{65} = (I+A_{dd})^{-1}B_d\hat{E}_{da}^{01}\Delta^{-1}L_{ab}^{01}C_b(I+A_{bb})^{-1}, \tag{44}$$

$$X_{66} = (I+A_{dd})^{-1}\{B_d\hat{E}_{da}^{01}\Delta^{-1}[L_{ad}^{01} - L_{ab}^{01}C_b(I+A_{bb})^{-1}L_{bd}] \times C_d(I+A_{dd})^{-1} + I\}, \tag{45}$$

$$X_{67} = -(I+A_{dd})^{-1}B_d\hat{E}_{da}^{01}\Delta^{-1}, \tag{46}$$

$$X_{75} = -\Delta^{-1}L_{ab}^{01}C_b(I+A_{bb})^{-1}, \tag{47}$$

$$X_{76} = \Delta^{-1}[L_{ab}^{01}C_b(I+A_{bb})^{-1}L_{bd} - L_{ad}^{01}]C_d(I+A_{dd})^{-1}, \tag{48}$$

$$X_{77} = \Delta^{-1}, \tag{49}$$

$$X_{15} = -(I + A_{aa}^{0*})^{-1}(L_{ab}^{0*}C_bX_{55} + L_{ad}^{0*}C_dX_{65}), \tag{50}$$

$$X_{16} = -(I + A_{aa}^{0*})^{-1}(L_{ab}^{0*}C_bX_{56} + L_{ad}^{0*}C_dX_{66}), \tag{51}$$

$$X_{17} = -(I + A_{aa}^{0*})^{-1}(L_{ab}^{0*}C_bX_{57} + L_{ad}^{0*}C_dX_{67}), \tag{52}$$

$$X_{25} = -(I + A_{aa}^{-})^{-1}(L_{ab}^{-}C_bX_{55} + L_{ad}^{-}C_dX_{65}), \tag{53}$$

$$X_{26} = -(I + A_{aa}^{-})^{-1}(L_{ab}^{-}C_bX_{56} + L_{ad}^{-}C_dX_{66}), \tag{54}$$

$$X_{27} = -(I + A_{aa}^{-})^{-1}(L_{ab}^{-}C_bX_{57} + L_{ad}^{-}C_dX_{67}), \tag{55}$$

$$X_{35} = -(I + A_{cc})^{-1}(L_{cb}C_bX_{55} + L_{cd}C_dX_{65}), \tag{56}$$

$$X_{36} = -(I + A_{cc})^{-1}(L_{cb}C_bX_{56} + L_{cd}C_dX_{66}), \tag{57}$$

$$X_{37} = -(I + A_{cc})^{-1}(L_{cb}C_bX_{57} + L_{cd}C_dX_{67}), \tag{58}$$

$$X_{45} = -(I + A_{aa}^{+})^{-1}(L_{ab}^{+}C_bX_{55} + L_{ad}^{+}C_dX_{65}), \tag{59}$$

$$X_{46} = -(I + A_{aa}^{+})^{-1}(L_{ab}^{+}C_bX_{56} + L_{ad}^{+}C_dX_{66}), \tag{60}$$

$$X_{47} = -(I + A_{aa}^{+})^{-1}(L_{ab}^{+}C_bX_{57} + L_{ad}^{+}C_dX_{67}), \tag{61}$$

and where

$$\Delta = I + A_{aa}^{01} + \left[L_{ab}^{01}C_b(I + A_{bb})^{-1}L_{bd} - L_{ad}^{01}\right] \times C_d(I + A_{dd})^{-1}B_d\hat{E}_{da}^{01}. \tag{62}$$

Furthermore, we have

$$\tilde{B}=\sqrt{2}\begin{bmatrix}(I+A_{aa}^{0*})^{-1}B_{0a}^{0*}+X_{15}B_{0b}+X_{16}B_{0d}+X_{17}B_{0a}^{01} & X_{16}B_d & 0\\ (I+A_{aa}^{-})^{-1}B_{0a}^{-}+X_{25}B_{0b}+X_{26}B_{0d}+X_{27}B_{0a}^{01} & X_{26}B_d & 0\\ (I+A_{cc})^{-1}B_{0c}+X_{35}B_{0b}+X_{36}B_{0d}+X_{37}B_{0a}^{01} & X_{36}B_d & (I+A_{cc})^{-1}B_c\\ (I+A_{aa}^{+})^{-1}B_{0a}^{+}+X_{45}B_{0b}+X_{46}B_{0d}+X_{47}B_{0a}^{01} & X_{46}B_d & 0\\ X_{55}B_{0b}+X_{56}B_{0d}+X_{57}B_{0a}^{01} & X_{56}B_d & 0\\ X_{65}B_{0b}+X_{66}B_{0d}+X_{67}B_{0a}^{01} & X_{66}B_d & 0\\ X_{75}B_{0b}+X_{76}B_{0d}+X_{77}B_{0a}^{01} & X_{76}B_d & 0\end{bmatrix}, \tag{63}$$

$$\tilde{E}=\sqrt{2}\begin{bmatrix}(I+A_{aa}^{0*})^{-1}Y_{aa}^{0*}X_a^{0*}+X_{16}B_dX_d+X_{17}(I+A_{aa}^{01})X_a^{01}\\ (I+A_{aa}^{-})^{-1}E_a^{-}+X_{26}B_dX_d+X_{27}(I+A_{aa}^{01})X_a^{01}\\ (I+A_{cc})^{-1}E_c+X_{36}B_dX_d+X_{37}(I+A_{aa}^{01})X_a^{01}\\ X_{46}B_dX_d+X_{47}(I+A_{aa}^{01})X_a^{01}\\ X_{56}B_dX_d+X_{57}(I+A_{aa}^{01})X_a^{01}\\ X_{66}B_dX_d+X_{67}(I+A_{aa}^{01})X_a^{01}\\ X_{76}B_dX_d+X_{77}(I+A_{aa}^{01})X_a^{01}\end{bmatrix}, \tag{64}$$

$$\tilde{D}_2=\begin{bmatrix}I & 0 & 0\\ -C_d(X_{65}B_{0b}+X_{66}B_{0d}+X_{67}B_{0a}^{01}) & -C_dX_{66}B_d & 0\\ -C_b(X_{55}B_{0b}+X_{56}B_{0d}+X_{57}B_{0a}^{01}) & -C_bX_{56}B_d & 0\end{bmatrix}, \tag{65}$$

and

$$\tilde{D}_{22}=\begin{bmatrix}0\\ -C_d[X_{66}B_dX_d+X_{67}(I+A_{aa}^{01})X_a^{01}]\\ -C_b[X_{56}B_dX_d+X_{57}(I+A_{aa}^{01})X_a^{01}]\end{bmatrix}. \tag{66}$$

Next, let us define

$$\tilde{S}:=\begin{bmatrix}0\\ -X_d+\hat{E}_{da}^{01}X_a^{01}\\ 0\end{bmatrix}. \tag{67}$$

Noting that

$$I+A_{aa}^{01}=\Delta-\left[L_{ab}^{01}C_b(I+A_{bb})^{-1}L_{bd}-L_{ad}^{01}\right]C_d(I+A_{dd})^{-1}B_d\hat{E}_{da}^{01}, \tag{68}$$

it is straightforward to verify that

$$\tilde{D}_{22}+\tilde{D}_2\tilde{S}=\begin{bmatrix}0\\ -C_d[X_{67}(I+A_{aa}^{01})X_a^{01}+X_{66}B_d\hat{E}_{da}^{01}X_a^{01}]\\ -C_b[X_{57}(I+A_{aa}^{01})X_a^{01}+X_{56}B_d\hat{E}_{da}^{01}X_a^{01}]\end{bmatrix}=0, \tag{69}$$

which shows that Statement 2(a) holds, and

$$\tilde{E}+\tilde{B}\tilde{S}=\sqrt{2}\begin{bmatrix}(I+A_{aa}^{0*})^{-1}Y_{aa}^{0*}X_a^{0*}+X_{16}B_d\hat{E}_{da}^{01}X_a^{01}+X_{17}(I+A_{aa}^{01})X_a^{01}\\ (I+A_{aa}^{-})^{-1}E_a^{-}+X_{26}B_d\hat{E}_{da}^{01}X_a^{01}+X_{27}(I+A_{aa}^{01})X_a^{01}\\ (I+A_{cc})^{-1}E_c+X_{36}B_d\hat{E}_{da}^{01}X_a^{01}+X_{37}(I+A_{aa}^{01})X_a^{01}\\ X_{46}B_d\hat{E}_{da}^{01}X_a^{01}+X_{47}(I+A_{aa}^{01})X_a^{01}\\ X_{56}B_d\hat{E}_{da}^{01}X_a^{01}+X_{57}(I+A_{aa}^{01})X_a^{01}\\ X_{66}B_d\hat{E}_{da}^{01}X_a^{01}+X_{67}(I+A_{aa}^{01})X_a^{01}\\ X_{76}B_d\hat{E}_{da}^{01}X_a^{01}+X_{77}(I+A_{aa}^{01})X_a^{01}\end{bmatrix}=\sqrt{2}\begin{bmatrix}(I+A_{aa}^{0*})^{-1}Y_{aa}^{0*}X_a^{0*}\\ \star\\ \star\\ 0\\ 0\\ 0\\ \star\end{bmatrix}, \tag{70}$$

where ★ are matrices of not much interest. Let the state space of $\tilde{\Sigma}_P$, i.e. the matrix quadruple $(\tilde{A}, \tilde{B}, \tilde{C}_2 \tilde{D}_2)$, be decomposed as follows:

$$\tilde{\mathcal{X}} = \tilde{\mathcal{X}}_a^0 \oplus \tilde{\mathcal{X}}_a^- \oplus \tilde{\mathcal{X}}_c \oplus \tilde{\mathcal{X}}_a^{+*} \oplus \tilde{\mathcal{X}}_b \oplus \tilde{\mathcal{X}}_a^{+1} \oplus \tilde{\mathcal{X}}_d, \quad (71)$$

where $\tilde{\mathcal{X}}_a^0$, $\tilde{\mathcal{X}}_a^-$, $\tilde{\mathcal{X}}_c$, $\tilde{\mathcal{X}}_b$ and $\tilde{\mathcal{X}}_d$ are the usual subspaces defined in the special coordinate basis of $\tilde{\Sigma}_P$, while $\tilde{\mathcal{X}}_a^{+1}$ corresponds to the zero dynamics of $\tilde{\Sigma}_P$ associated with the invariant zero at $s = 1$, and $\tilde{\mathcal{X}}_a^{+*}$ corresponds to the zero dynamics of $\tilde{\Sigma}_P$ associated with the rest of the unstable invariant zeros (excluding the point $s = 1$). It was shown by Chen and Weller (1998) that $\tilde{\mathcal{X}}$ of $\tilde{\Sigma}_P$ and $\mathcal{X}$ of Σ_P are related by

$$\tilde{\mathcal{X}}_a^0 = \mathcal{X}_a^{0*}, \quad \tilde{\mathcal{X}}_a^- = \mathcal{X}_a^-, \quad \tilde{\mathcal{X}}_c = \mathcal{X}_c, \quad \tilde{\mathcal{X}}_a^{+*} = \mathcal{X}_a^+, \quad (72)$$

and

$$\tilde{\mathcal{X}}_b = \mathcal{X}_b, \quad \tilde{\mathcal{X}}_a^{+1} = \mathcal{X}_d, \quad \tilde{\mathcal{X}}_d = \mathcal{X}_a^{01}. \quad (73)$$

Moreover, the zero dynamics of $\tilde{\Sigma}_P$, corresponding to the imaginary axis invariant zeros, are fully characterized by the eigenstructure of the following matrix,

$$\tilde{A}_{aa}^0 := (A_{aa}^{0*} + I)^{-1}(A_{aa}^{0*} - I). \quad (74)$$

Noting (54), one is ready to verify that

$$\mathrm{Im}\{(I + A_{aa}^{0*})^{-1} Y_{aa}^{0*}\} = \bigcap_{\beta \in \lambda(\tilde{A}_{aa}^0)} \mathrm{Im}\{\beta I - \tilde{A}_{aa}^0\}. \quad (75)$$

It is now straightforward to see from (70) and the properties of the special coordinate basis that

$$\mathrm{Im}(\tilde{E} + \tilde{B}\tilde{S}) \subset \mathcal{S}^+(\tilde{\Sigma}_P) \cap \left\{ \bigcap_{\lambda \in \mathbb{C}^0} \mathcal{S}_\lambda(\tilde{\Sigma}_P) \right\}, \quad (76)$$

i.e. Statement 2(b) holds.

Statement 2 ⇒ Statement 1: This follows by reversing the above arguments using the well-known bilinear transformation and the results of Chen and Weller (1998). Thus, it is omitted. This completes the proof of Theorem 3.1. □

4.2. *Proof of Theorem* 3.2

For simplicity of presentation, we assume throughout this proof that matrix A has no eigenvalues at -1. Then, we define the following auxiliary continuous-time system,

$$\tilde{\Sigma} : \begin{cases} \dot{\tilde{x}} = \tilde{A}\, \tilde{x} + \tilde{B}\, \tilde{u} + \tilde{E}\, \tilde{w}, \\ \tilde{y} = \tilde{C}_1\, \tilde{x} + \tilde{D}_1\, \tilde{w}, \\ \tilde{z} = \tilde{C}_2\, \tilde{x} + \tilde{D}_2\, \tilde{u} + \tilde{D}_{22}\, \tilde{w}, \end{cases} \quad (77)$$

where $\tilde{A}, \tilde{B}, \tilde{E}, \tilde{C}_1, \tilde{D}_1, \tilde{C}_2, \tilde{D}_2$ and $\tilde{D}_{22}$ are defined as

$$\left.\begin{aligned} \tilde{A} &= (A+I)^{-1}(A-I), \\ \tilde{B} &= \sqrt{2}(A+I)^{-1}B, \\ \tilde{E} &= \sqrt{2}(A+I)^{-1}E, \\ \tilde{C}_1 &= \sqrt{2}C_1(A+I)^{-1}, \\ \tilde{D}_1 &= D_1 - C_1(A+I)^{-1}E, \\ \tilde{C}_2 &= \sqrt{2}C_2(A+I)^{-1}, \\ \tilde{D}_2 &= D_2 - C_2(A+I)^{-1}B, \\ \tilde{D}_{22} &= D_{22} - C_2(A+I)^{-1}E. \end{aligned}\right\} \quad (78)$$

For easy reference later on, we let $\tilde{\Sigma}_P$ denote the subsystem characterized by $(\tilde{A}, \tilde{E}, C_1, \tilde{D}_1)$ and $\tilde{\Sigma}_Q$ denote the subsystem characterized by $(\tilde{A}, \tilde{E}, \tilde{C}_1, \tilde{D}_1)$, respectively. Following the result of Glover (1984), one can show that the following two statements are equivalent.

(1) The H_∞-ADDPMS for the originally given discrete-time system Σ of (1) is solvable.
(2) The H_∞-ADDPMS for the auxiliary continuous-time system $\tilde{\Sigma}$ of (97) is solvable.

It was shown in Scherer (1992) that the second statement above is also equivalent to the following conditions,

(*a*) $(\tilde{A}, \tilde{B})$ is stabilizable.
(*b*) $(\tilde{A}_1, \tilde{C}_1)$ is detectable.
(*c*) $\tilde{D}_{22} + \tilde{D}_2\tilde{S}\tilde{D}_1 = 0$, where $\tilde{S} = -(\tilde{D}_2'\tilde{D}_2)^\dagger \tilde{D}_2'\tilde{D}_{22}\, (\tilde{D}_1'\tilde{D}_1')^\dagger$.
(*d*) $\mathrm{Im}(\tilde{E} + \tilde{B}\tilde{S}\tilde{D}_1) \subset \mathcal{S}^+(\tilde{\Sigma}_P) \cap \{\bigcap_{\lambda \in \mathbb{C}^0} \mathcal{S}_\lambda(\tilde{\Sigma}_P)\}$.
(*e*) $\mathrm{Ker}(\tilde{C}_2 + \tilde{D}_2\tilde{S}\tilde{C}_1) \supset \mathcal{V}^+(\tilde{\Sigma}_Q) \cup \{\bigcup_{\lambda \in \mathbb{C}^0} \mathcal{V}_\lambda(\tilde{\Sigma}_Q)\}$.
(*f*) $\mathcal{V}^+(\tilde{\Sigma}_Q) \subset \mathcal{S}^+(\tilde{\Sigma}_P)$.

First, it is simple to check that the triple $(\tilde{A}, \tilde{B}, \tilde{C}_1)$ is stabilizable and detectable if and only if the triple (A, B, C) is stabilizable and detectable. Next, following the proof in Subsection 4.1, we have the following equivalent statements:

(1) Statement I:
  (*a*) $D_{22} + D_2SD_1 = 0$, where $S = -(D_2'D_2)^\dagger D_2'D_{22}\, D_1'(D_1D_1')^\dagger$.
  (*b*) $\mathrm{Im}(E + BS) \subset \{\mathcal{V}^\odot(\Sigma_P) + B\,\mathrm{Ker}(D_2)\} \cap \{\bigcap_{|\lambda|=1} \mathcal{S}_\lambda(\Sigma_P)\}$.
(2) Statement II:
  (*a*) $\tilde{D}_{22} + \tilde{D}_2\tilde{S}\tilde{D}_1 = 0$, where $\tilde{S} = -(\tilde{D}_2'\tilde{D}_2)^\dagger \tilde{D}_2'\tilde{D}_{22}\, \tilde{D}_1'(\tilde{D}_1\tilde{D}_1')^\dagger$.
  (*b*) $\mathrm{Im}(\tilde{E} + \tilde{B}\tilde{S}\tilde{D}_1) \subset \mathcal{S}^+(\tilde{\Sigma}_P) \cap \{\bigcap_{\lambda \in \mathbb{C}^0} \mathcal{S}_\lambda(\tilde{\Sigma}_P)\}$.

Dualizing the arguments of Subsection 4.1, we can show that the following two statements are also equivalent:

(1) Statement A:
  (*a*) $D_{22} + D_2SD_1 = 0$, where $S = -(D_2'D_2)^\dagger D_2'D_{22}\, D_1'(D_1D_1')^\dagger$.

(b) $\mathrm{Ker}\,(C_2 + D_2 S C_1) \supset \{\mathcal{S}^{\odot}(\Sigma_Q) \cap C_1^{-1} \{\mathrm{Im}\,(D_1)\}\} \cup \{\bigcup_{|\lambda|=1} \mathcal{V}_\lambda(\Sigma_Q)\}$.

(2) Statement B:

(a) $\tilde{D}_{22} + \tilde{D}_2 \tilde{S} \tilde{D}_1 = 0$, where $\tilde{S} = -(\tilde{D}_2'\tilde{D}_2)^{\dagger}\tilde{D}_2'\tilde{D}_{22}\,\tilde{D}_1'(\tilde{D}_1\tilde{D}_1')^{\dagger}$.

(b) $\mathrm{Ker}\,(\tilde{C}_2 + \tilde{D}_2 \tilde{S} \tilde{C}_1) \supset \mathcal{V}^{+}(\tilde{\Sigma}_Q) \cup \{\bigcup_{\lambda \in \mathbb{C}^0} \mathcal{V}_\lambda(\tilde{\Sigma}_Q)\}$.

Finally, it was shown in Chapter 4 of Chen (1998) that

$$\mathcal{V}^{\odot}(\Sigma_P) = \mathcal{S}^{+}(\tilde{\Sigma}_P), \quad \mathcal{S}^{\odot}(\Sigma_P) = \mathcal{V}^{+}(\tilde{\Sigma}_P), \tag{79}$$

and

$$\mathcal{V}^{\odot}(\Sigma_Q) = \mathcal{S}^{+}(\tilde{\Sigma}_Q), \quad \mathcal{S}^{\odot}(\Sigma_Q) = \mathcal{V}^{+}(\tilde{\Sigma}_Q). \tag{80}$$

Hence, the following two statements are equivalent:

(1) $\mathcal{S}^{\odot}(\Sigma_Q) \subset \mathcal{V}^{\odot}(\Sigma_P)$.
(2) $\mathcal{V}^{+}(\tilde{\Sigma}_Q) \subset \mathcal{S}^{+}(\tilde{\Sigma}_P)$.

Thus, the result of Theorem 3.2 follows. □

5. Conclusions

We have presented sets of necessary and sufficient conditions for the solvability of the H_∞ almost disturbance decoupling problem with internal stability for general discrete-time systems whose two subsystems—i.e. the subsystem from the control input to the output to be controlled and the subsystem from the disturbance input to the measurement output—are allowed to have invariant zeros on the unit circle of the complex plane. These conditions are expressed in terms of some well-defined geometric subspaces and they are numerically checkable. Furthermore, we have also developed an algorithm that verifies these conditions without actually calculating any geometric subspaces at all.

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