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# Solvability conditions for disturbance decoupling problems with static measurement feedback

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Two well-known disturbance decoupling problems are considered. A checkable necessary and sufficient condition is derived under which the disturbance decoupling problem with static measurement feedback (DDPKM) is solvable for a class of multi-input and multi-output linear time-invariant systems, which have a left invertible transfer matrix function from the control input to the controlled output. The set of all solutions that solve the DDPKM is constructed and parametrized explicitly. This set is then used to solve the disturbance decoupling problem with static measurement feedback and with internal stability (DDPKMS).

## 1. Introduction and background material

The so-called disturbance decoupling problem, which is quite well-known, has been investigated extensively during the last three decades. It was actually the starting point for the development of a geometric approach to systems theory. It also plays a central role in several important problems such as decentralized control, non-interacting control, model reference tracking control, $H_2$ optimal control and $H_\infty$ optimal control. The problem is to find either a static or dynamic compensator such that the resulting closed-loop transfer matrix function from the disturbance to the controlled output is equal to zero for all frequencies, i.e. there is no effect whatsoever from the disturbance to the controlled output. The problem of disturbance decoupling with state feedback (DDP) was solved by Basile and Marro (1968) and Wonham and Morse (1970) and the problem of disturbance decoupling with dynamic measurement feedback (DDPM) was solved by Akashi and Imai (1979) and Schumacher (1980). Finally, the disturbance decoupling problem with state feedback and internal stability (DDPS) and the disturbance decoupling problem with dynamic measurement feedback and with internal stability (DDPMS) were, respectively, solved by Morse and Wonham (1970), Wonham and Morse (1970), Imai and Akashi (1981), and Willems and Commault (1981). However, to the best of the author's knowledge, the disturbance decoupling problem with static or constant measurement feedback (DDPKM) and the disturbance decoupling problem with static measurement feedback and with internal stability (DDPKMS) still remain unsolved in open literature. The main purpose of this paper is to make an attempt to solve the DDPKM and the DDPKMS for a class of multi-input multi-output linear time-invariant systems.

To be more specific, we consider in this paper the following linear time-invariant

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system $\Sigma$ characterized by

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u} + \mathbf{E}\boldsymbol{w} \\ \boldsymbol{y} &= \mathbf{C}_1\boldsymbol{x} \\ \boldsymbol{z} &= \mathbf{C}_2\boldsymbol{x} + \mathbf{D}_2\boldsymbol{u} \end{aligned}\right\} \tag{1.1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the control input, $\boldsymbol{y} \in \mathbb{R}^l$ is the measurement, $\boldsymbol{w} \in \mathbb{R}^q$ is the disturbance and $\boldsymbol{z} \in \mathbb{R}^p$ is the output to be controlled. $\mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{C}_1, \mathbf{C}_2$ and $\mathbf{D}_2$ are constant matrices of appropriate dimension. Without any loss of generality but for the simplicity of presentation, we assume that the matrices $\mathbf{C}_1, [\mathbf{C}_2\ \ \mathbf{D}_2]$ and $[\mathbf{B}'\ \ \mathbf{D}_2']$ are of maximal rank. We also assume that the quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ is left invertible throughout this paper. The disturbance decoupling problem with static measurement feedback (DDPKM) is to find a static output feedback law

$$\boldsymbol{u} = \mathbf{K}\boldsymbol{y} \tag{1.2}$$

where $\mathbf{K} \in \mathbb{R}^{m\times l}$ such that the resulting closed-loop transfer matrix function from $\boldsymbol{w}$ to $\boldsymbol{z}$ is equal to zero, i.e.

$$\mathbf{H}_{zw}(s) = (\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}\mathbf{C}_1)(s\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}_1)^{-1}\mathbf{E} = 0 \tag{1.3}$$

The disturbance decoupling problem with static measurement feedback and with internal stability (DDPKMS) is to design a control law as in (1.2) such that (1.3) is satisfied and $\mathbf{A} + \mathbf{B}\mathbf{K}\mathbf{C}_1$ is asymptotically stable. The goal of this paper is to derive a checkable solvability condition for the DDPKM for the given system of (1.1). Furthermore, all solutions to the DDPKM for the class of systems that we considered in this paper are explicitly constructed and characterized. These solutions, if they exist, are then used to solve the DDPKMS.

The remainder of this paper is organized as follows. In the rest of this section we recall the special coordinate basis of linear systems, which is instrumental to the derivation of the main results of the paper. Section 2 presents our main results. We first give a checkable solvability condition for the DDPKM for a class of systems, which have a left invertible transfer function from $\boldsymbol{u}$ to $\boldsymbol{z}$. The constructive algorithm that parametrizes all solutions to the DDPKM for this class of systems is then presented. Moreover, a solvability condition for the DDPKMS is also derived in this section. Finally, concluding remarks are made in § 3.

Throughout this paper, $\mathbf{X}'$ denotes the transpose of matrix $\mathbf{X}$, $\ker(\mathbf{X})$ denotes the kernel of $\mathbf{X}$, $\mathrm{Im}(\mathbf{X})$ denotes the image of $\mathbf{X}$, and $\langle \mathbf{A}|\chi\rangle$ denotes the smallest $\mathbf{A}$-invariant subspace containing $\chi$, which itself is a subspace. We will also use the following definitions of geometric subspace.

**Definition 1.1:** Consider a linear time-invariant system characterized by a matrix quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$, with $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C}_2 \in \mathbb{R}^{p\times n}$ and $\mathbf{D}_2 \in \mathbb{R}^{p\times m}$. We define $\mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ to be the maximal subspace of $\mathbb{R}^n$ which is $(\mathbf{A} + \mathbf{B}\mathbf{F})$-invariant and is contained in $\ker(\mathbf{C}_2 + \mathbf{D}_2\mathbf{F})$ for some $\mathbf{F} \in \mathbb{R}^{m\times n}$.

Now, let us recall from Sannuti and Saberi (1987) a theorem that deals with the so-called special coordinate basis for linear systems. Consider a linear system described by

$$\left.\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{z} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_2\mathbf{u} \end{aligned}\right\} \tag{1.4}$$

Without loss of generality, we assume that $\mathbf{D}_2$ is in the form

$$\mathbf{D}_2 = \begin{bmatrix} \mathbf{I}_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \tag{1.5}$$

where $m_0 = \text{rank}(\mathbf{D}_2)$. Hence, the system equations of (1.4) can be partitioned as

$$\left.\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \begin{bmatrix} \mathbf{B}_0 & \mathbf{B}_1 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} &= \begin{bmatrix} \mathbf{C}_{20} \\ \mathbf{C}_{21} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{I}_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{aligned}\right\} \tag{1.6}$$

where $\mathbf{B}_0, \mathbf{B}_1, \mathbf{C}_{20}, \mathbf{C}_{21}$ are matrices with appropriate dimensions. The following theorem is due to Sannuti and Saberi (1987).

**Theorem 1.1:** *Consider the system* (1.4). *Assume that the quadruple* $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ *is left invertible. Then there exist non-singular transformations* $\Gamma_s, \Gamma_i$ *and* $\Gamma_o$ *such that*

$$\mathbf{x} = \Gamma_s \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix}, \quad \mathbf{u} = \Gamma_i \begin{pmatrix} u_0 \\ u_d \end{pmatrix}, \quad \mathbf{z} = \Gamma_o \begin{pmatrix} z_0 \\ z_d \\ z_b \end{pmatrix} \tag{1.7}$$

$$\Gamma_s^{-1}(\mathbf{A} - \mathbf{B}_0\mathbf{C}_{20})\Gamma_s = \begin{bmatrix} A_{aa} & L_{ab}C_b & L_{ad}C_d \\ 0 & A_{bb} & L_{bd}C_d \\ B_b E_{da} & B_d E_{db} & A_{dd} \end{bmatrix} \tag{1.8}$$

$$\Gamma_s^{-1}\begin{bmatrix} B_0 & B_1 \end{bmatrix}\Gamma_i = \begin{bmatrix} B_{a0} & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix} \tag{1.9}$$

$$\Gamma_o^{-1}\begin{bmatrix} \mathbf{C}_{20} \\ \mathbf{C}_{21} \end{bmatrix}\Gamma_s = \begin{bmatrix} C_{0a} & C_{0b} & C_{0d} \\ 0 & 0 & C_d \\ 0 & C_b & 0 \end{bmatrix}, \quad \Gamma_o^{-1}\mathbf{D}_2\Gamma_i = \begin{bmatrix} \mathbf{I}_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{1.10}$$

*where the eigenvalues of* $A_{aa}$ *are the invariant zeros of* $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$*, the pair* $(A_{bb}, C_b)$ *is observable, while the subsystem* $(A_{dd}, B_d, C_d)$ *is invertible and free of invariant zeros with* $B_d$ *being of full column rank and* $C_d$ *being of full row rank. Moreover*

$$\chi_a := \text{Im}\left\{ \Gamma_s \begin{bmatrix} \mathbf{I} \\ 0 \\ 0 \end{bmatrix} \right\} = \mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2) \tag{1.11}$$

*In addition, the quadruple* $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ *is invertible if and only if* $x_b$ *is non-existent.*

The software realization of the above special coordinate basis of linear systems can be found in LAS by Chen (1988) and in MATLAB by Lin (1989).

## 2. Solvability conditions for DDPKM and DDPKMS

We present the main results of the paper in this section. We will first give a checkable necessary and sufficient condition under which the DDPKM is solvable for the class of systems, which have an invertible transfer function from $\boldsymbol{u}$ to $\boldsymbol{z}$, i.e. the quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ is left invertible. All control laws that solve the DDPKM for this class of systems will be constructed and explicitly parametrized as well. Finally, we will derive a solvability condition for the DDPKMS, in which one would need to search a stabilizing control law over all the solutions to the DDPKM. Obviously, this condition is not readily checkable. Thus, we cannot claim that the DDPKMS is totally solved.

First, let us use the result of the special coordinate basis in Theorem 1.1 to find non-singular transformations $\Gamma_s, \Gamma_i$ and $\Gamma_o$. Then we apply these transformations to the system $\Sigma$ in (1.1), i.e. let

$$\mathbf{x} = \Gamma_s \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix}, \quad \boldsymbol{u} = \Gamma_i \begin{pmatrix} u_0 \\ u_d \end{pmatrix}, \quad z = \Gamma_o \begin{pmatrix} z_0 \\ z_d \\ z_b \end{pmatrix} \tag{2.1}$$

which yields the following transformed system $\Sigma_s$

$$\left.\begin{aligned} \begin{pmatrix} \dot{x}_a \\ \dot{x}_b \\ \dot{x}_d \end{pmatrix} &= \left( \begin{bmatrix} A_{aa} & L_{ab}C_b & L_{ad}C_d \\ 0 & A_{bb} & L_{bd}C_d \\ B_dE_{da} & B_dE_{db} & A_{dd} \end{bmatrix} + A_0 \right) \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix} + \begin{bmatrix} B_{a0} & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix} \begin{pmatrix} u_0 \\ u_d \end{pmatrix} + \begin{bmatrix} E_a \\ E_b \\ E_d \end{bmatrix} \mathbf{w} \\ y &= \begin{bmatrix} C_{1a} & C_{1b} & C_{1d} \end{bmatrix} \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix} \\ \begin{pmatrix} z_0 \\ z_b \\ z_d \end{pmatrix} &= \begin{bmatrix} C_{0a} & C_{0b} & C_{0d} \\ 0 & 0 & C_d \\ 0 & C_b & 0 \end{bmatrix} \begin{pmatrix} x_a \\ x_b \\ x_d \end{pmatrix} + \begin{bmatrix} \mathbf{I}_{m_0} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_d \end{pmatrix} \end{aligned}\right\} \tag{2.2}$$

where

$$A_0 := \begin{bmatrix} B_{a0} \\ B_{b0} \\ B_{d0} \end{bmatrix} \begin{bmatrix} C_{0a} & C_{0b} & C_{0d} \end{bmatrix}, \quad \begin{bmatrix} E_a \\ E_b \\ E_d \end{bmatrix} := \Gamma_s^{-1} E, \quad \begin{bmatrix} C_{1a} & C_{1b} & C_{1d} \end{bmatrix} := \mathbf{C}_1 \Gamma_s \tag{2.3}$$

It is straightforward to verify that the DDPKM for (1.1) is equivalent to the DDPKM for (2.2). Next, let $\Gamma_a$ be a non-singular transformation such that

$$\Gamma_a^{-1} A_{aa} \Gamma_a = \begin{bmatrix} A_{aa}^{cc} & A_{aa}^{c\bar{c}} \\ 0 & A_{aa}^{\bar{c}\bar{c}} \end{bmatrix}, \quad \Gamma_a^{-1} E_a = \begin{bmatrix} E_a^c \\ 0 \end{bmatrix}, \quad \Gamma_a^{-1} B_{a0} = \begin{bmatrix} B_{a0}^c \\ B_{a0}^{\bar{c}} \end{bmatrix} \tag{2.4}$$

$$\Gamma_a^{-1} L_{ab} = \begin{bmatrix} L_{ab}^c \\ L_{ab}^{\bar{c}} \end{bmatrix}, \quad \Gamma_a^{-1} L_{ad} = \begin{bmatrix} L_{ad}^c \\ L_{ad}^{\bar{c}} \end{bmatrix} \tag{2.5}$$

$$C_{0a}\Gamma_a = \begin{bmatrix} C_{0a}^c & C_{0a}^{\bar{c}} \end{bmatrix}, \quad E_{da}\Gamma_a = \begin{bmatrix} E_{da}^c & E_{da}^{\bar{c}} \end{bmatrix}, \quad C_{1a}\Gamma_a = \begin{bmatrix} C_{1a}^c & C_{1a}^{\bar{c}} \end{bmatrix} \tag{2.6}$$

where the pair $(A_{aa}^{cc}, E_a^c)$ is completely controllable. We have the following theorem. In fact, the proof of the theorem yields a constructive algorithm that parametrizes all solutions to the DDPKM for the given system.

**Theorem 2.1:** *Consider the linear time-invariant system* $\Sigma$ *of* (1.1) *with* $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ *being left invertible. The disturbance decoupling problem with static measurement feedback (DDPKM) for* $\Sigma$ *is solvable if and only if*

$$E_b = 0, \quad E_d = 0, \quad \ker(C_{1a}^c) \subset \ker\left\{\begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix}\right\} \tag{2.7}$$

*where* $E_b, E_d, E_{da}^c, C_{0a}^c$ *and* $C_{1a}^c$ *are as defined in* (2.2)–(2.6).

**Proof:** Without loss of generality, we will assume that the given system is in the form of (2.2) with $x_a$ being further decomposed into the form as in (2.4) and (2.6).

($\Rightarrow$) If $\ker(C_{1a}^c) \subset \ker\left\{\begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix}\right\}$, then there exists at least one $K \in \mathbb{R}^{m\times l}$ such that

$$\begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} + \mathbf{K}C_{1a}^c = \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} + \begin{bmatrix} K_0 \\ K_d \end{bmatrix} C_{1a}^c = 0 \tag{2.8}$$

In addition, if $E_b = 0$ and $E_d = 0$, it is straightforward to verify that the closed-loop system of $\Sigma$ with a static measurement feedback law $u_d = \mathbf{K}y$ is given by

$$H_{zw}(s) = (\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}\mathbf{C}_1)(s\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}_1)^{-1}\mathbf{E}$$

$$= \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & C_d \\ 0 & 0 & C_b & 0 \end{bmatrix} \begin{bmatrix} (s\mathbf{I} - A_{aa}^{cc})^{-1} & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \begin{bmatrix} E_a^c \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \tag{2.9}$$

where the * are some matrices of not much interest. Clearly, the control law $\boldsymbol{u} = \mathbf{K}y$ with $\mathbf{K}$ that satisfies (2.8) solves the DDPKM for $\Sigma$.

($\Leftarrow$) Conversely, if the DDPKM is solvable for $\Sigma$, i.e. there exists a matrix $\mathbf{K} \in \mathbb{R}^{m\times l}$ such that

$$H_{zw}(s) = (\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}\mathbf{C}_1)(s\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}_1)^{-1}\mathbf{E} = 0 \tag{2.10}$$

First we note that the set of all static measurement feedback laws is a subset of the set of all static state feedback laws. Thus, it follows from Wonham and Morse (1970) or Wonham (1979) that

$$\operatorname{Im}(\mathbf{E}) \subset \mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2) = \chi_a \tag{2.11}$$

It then follows from the property of the special coordinate basis as in Theorem 1.1 that $E_b = 0$ and $E_d = 0$. Next, let us define

$$\mathcal{W} := \langle \mathbf{A} + \mathbf{B}\mathbf{K}\mathbf{C}_1 | \operatorname{Im}(\mathbf{E}) \rangle \tag{2.12}$$

i.e. the smallest $(\mathbf{A} + \mathbf{B}\mathbf{K}\mathbf{C}_1)$-invariant subspace containing $\operatorname{Im}(\mathbf{E})$. Thus, (2.10) implies that $\mathcal{W} \subset \ker(\mathbf{C}_2 + \mathbf{D}_2\mathbf{K}\mathbf{C}_1)$ and by definition

$$\mathcal{W} \subset \mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2) = \chi_a = \operatorname{span}\left\{\begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\} \tag{2.13}$$

Hence, there exists a similarity transformation $\mathbf{T}$ such that

$$\mathbf{T}^{-1}(\mathbf{A}+\mathbf{B}\mathbf{K}\mathbf{C}_1)\mathbf{T} = \begin{bmatrix} A^{cc} & A^{c\bar{c}} \\ 0 & A^{\bar{c}\bar{c}} \end{bmatrix}, \quad \mathbf{T}^{-1}\mathbf{E} = \begin{bmatrix} E^c \\ 0 \end{bmatrix} \tag{2.14}$$

$$(\mathbf{C}_2+\mathbf{D}_2\mathbf{K}\mathbf{C}_1)\mathbf{T} = \begin{bmatrix} 0 & C^{\bar{c}} \end{bmatrix}, \quad \mathcal{W} = \text{span}\left\{\mathbf{T}\begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix}\right\} \tag{2.15}$$

where $(A^{cc}, E^c)$ is completely controllable. It is now straightforward to verify that $\mathbf{T}$ can be chosen as

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_* & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \tag{2.16}$$

where $\mathbf{T}_*$ is of dimension $\dim(\chi_a) \times \dim(\chi_a)$. Let

$$\mathbf{K} = \begin{bmatrix} K_0 \\ K_d \end{bmatrix} \tag{2.17}$$

We note that (2.14)–(2.17) imply that

$$\mathbf{T}_*^{-1}\begin{bmatrix} A_{aa}^{cc} + B_{a0}^c(C_{0a}^c + K_0 C_{1a}^c) & A_{aa}^{c\bar{c}} + B_{a0}^c(C_{0a}^{\bar{c}} + K_0 C_{1a}^{\bar{c}}) \\ B_{a0}^{\bar{c}}(C_{0a}^c + K_0 C_{1a}^c) & A_{aa}^{\bar{c}\bar{c}} + B_{a0}^{\bar{c}}(C_{0a}^{\bar{c}} + K_0 C_{1a}^{\bar{c}}) \end{bmatrix}\mathbf{T}_* = \begin{bmatrix} A^{cc} & A_*^{c\bar{c}} \\ 0 & A_*^{\bar{c}\bar{c}} \end{bmatrix} \tag{2.18}$$

$$\mathbf{T}_*^{-1}\begin{bmatrix} E_a^c \\ 0 \end{bmatrix} = \begin{bmatrix} E^c \\ 0 \end{bmatrix} \tag{2.19}$$

$$\begin{bmatrix} C_{0a}^c + K_0 C_{1a}^c & C_{0a}^{\bar{c}} + K_0 C_{1a}^{\bar{c}} \\ B_d(E_{da}^c + K_d C_{1a}^c) & B_d(E_{da}^{\bar{c}} + K_d C_{1a}^{\bar{c}}) \end{bmatrix}\mathbf{T}_* = \begin{bmatrix} 0 & * \end{bmatrix} \tag{2.20}$$

where again $*$ denotes a matrix of not much interest. Here we note that (2.18)–(2.20) imply that the system characterized by the matrix triple

$$\left(\begin{bmatrix} A_{aa}^{cc} & A_{aa}^{c\bar{c}} \\ 0 & A_{aa}^{\bar{c}\bar{c}} \end{bmatrix}, \begin{bmatrix} E_a^c \\ 0 \end{bmatrix}, \begin{bmatrix} C_{0a}^c + K_0 C_{1a}^c & C_{0a}^{\bar{c}} + K_0 C_{1a}^{\bar{c}} \\ B_d(E_{da}^c + K_d C_{1a}^c) & B_d(E_{da}^{\bar{c}} + K_d C_{1a}^{\bar{c}}) \end{bmatrix}\right) \tag{2.21}$$

does not have any infinite zeros. Then the controllability of the pair $(A_{aa}^{cc}, E_a^c)$ implies that

$$C_{0a}^c + K_0 C_{1a}^c = 0 \quad \text{and} \quad B_d(E_{da}^c + K_d C_{1a}^c) = 0 \tag{2.22}$$

As $B_d$ is of full column rank (see Theorem 1.1), (2.22) is equivalent to

$$C_{0a}^c + K_0 C_{1a}^c = 0 \quad \text{and} \quad E_{da}^c + K_d C_{1a}^c = 0 \tag{2.23}$$

or

$$\begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} + \begin{bmatrix} K_0 \\ K_d \end{bmatrix} C_{1a}^c = 0 \Rightarrow \ker(C_{1a}^c) \subset \ker\left\{\begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix}\right\} \tag{2.24}$$

This completes the proof of Theorem 2.1. □

The following is an interesting and useful proposition, which follows directly from the proof of Theorem 2.1.

**Proposition 2.1:** *If the DDPKM for* $\Sigma$ *of* (1.1) *with the quadruple* $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ *being left invertible is solvable, then all the static measurement gain matrices that solve the*

*DDPKM are characterized by*

$$\mathcal{K} := \left\{ \Gamma_i \mathbf{K} \middle| \mathbf{K} \in \mathbb{R}^{m\times l} \text{ and } \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} + KC_{1a}^c = 0 \right\} \tag{2.25}$$

*where $\Gamma_i$ is defined in Theorem 1.1, and $E_{da}^c$, $C_{0a}^c$ and $C_{1a}^c$ are as defined in* (2.6).

It is interesting to note that if $\mathbf{C}_1 = \mathbf{I}$, i.e. all the states of $\Sigma$ are available for feedback, then $C_{1a}^c$ is always of full column rank, which implies that $\ker(C_{1a}^c) = \{0\}$ and the third condition of Theorem 2.1, i.e.

$$\ker(C_{1a}^c) \subset \ker\left\{ \begin{bmatrix} C_{0a}^c \\ E_{da}^c \end{bmatrix} \right\}$$

is automatically satisfied. Hence, the conditions in Theorem 2.1 are reduced to $E_b = 0$ and $E_d = 0$, which is equivalent to $\text{Im}(\mathbf{E}) \subset \mathcal{V}^*(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$, i.e. the well-known condition for the solvability of the disturbance decoupling problem with static state feedback (DDP).

It is also interesting to point out the disturbance decoupling problem with dynamic output feedback can be converted into a problem of disturbance decoupling with static measurement feedback. To see this, let us consider the given system $\Sigma$ of (1.1) with a general dynamic feedback law

$$\left. \begin{aligned} \dot{\boldsymbol{v}} &= A_{\text{cmp}} \boldsymbol{v} + B_{\text{cmp}} y \\ \boldsymbol{u} &= C_{\text{cmp}} \boldsymbol{v} + D_{\text{cmp}} \boldsymbol{y} \end{aligned} \right\} \tag{2.26}$$

Then it is straightforward to show that the DDP for $\Sigma$ of (1.1) with the dynamic measurement feedback law (2.26) is equivalent to a DPP for the following auxiliary system,

$$\left. \begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{A} \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{B} \end{bmatrix} \tilde{\boldsymbol{u}} + \begin{bmatrix} 0 \\ \mathbf{E} \end{bmatrix} \boldsymbol{w} \\ \tilde{\boldsymbol{y}} &= \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{C}_1 \end{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{z} &= \begin{bmatrix} 0 & \mathbf{C}_2 \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} 0 & D_2 \end{bmatrix} \tilde{\boldsymbol{u}} \end{aligned} \right\} \tag{2.27}$$

where

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{x} \end{pmatrix}, \quad \tilde{\boldsymbol{u}} = \begin{pmatrix} \dot{\boldsymbol{v}} \\ \boldsymbol{u} \end{pmatrix}, \quad \tilde{\boldsymbol{y}} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{y} \end{pmatrix} \tag{2.28}$$

with a static measurement feedback law

$$\tilde{\boldsymbol{u}} = \tilde{\mathbf{K}} \tilde{\boldsymbol{y}} = \begin{bmatrix} A_{\text{cmp}} & B_{\text{cmp}} \\ C_{\text{cmp}} & D_{\text{cmp}} \end{bmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{y} \end{pmatrix} \tag{2.29}$$

Unfortunately, the quadruple

$$\left( \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{A} \end{bmatrix}, \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{B} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{C}_2 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{D}_2 \end{bmatrix} \right) \tag{2.30}$$

is no longer left invertible. The main difficulty is solving the DDPKM for systems with non-left-invertible $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ is that there are too many degrees of freedom to be characterized.

The following theorem deals with the disturbance decoupling problem with static measurement feedback and with internal stability (DDPKMS).

**Theorem 2.2:** *Consider the linear time-invariant system* $\Sigma$ *of* (1.1) *with* $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ *being left invertible. The disturbance decoupling problem with static measurement feedback and with internal stability (DDPKMS) for* $\Sigma$ *is solvable if and only if*

(*a*) *the DDPKM for* $\Sigma$ *is solvable*

(*b*) *the eigenvalues of* $A_{aa}^{cc}$ *are all in the open left-half plane*

(*c*) *there exists at least one* $\mathbf{K} \in \mathcal{K}$, *where* $\mathcal{K}$ *is as defined in* (2.25), such that $\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}_1$ *is asymptotically stable, where*

$$\tilde{\mathbf{A}} = \begin{bmatrix} A_{aa}^{cc} & L_{ab}^{\bar{c}} C_b & L_{ad}^{\bar{c}} C_d \\ 0 & A_{bb} & L_{bd} C_d \\ B_d E_{da}^{\bar{c}} & B_d E_{db} & A_{dd} \end{bmatrix} + \begin{bmatrix} B_{a0}^{\bar{c}} \\ B_{b0} \\ B_{d0} \end{bmatrix} \begin{bmatrix} C_{0a}^{\bar{c}} & C_{0b} & C_{0d} \end{bmatrix} \tag{2.31}$$

$$\tilde{\mathbf{C}}_1 = \begin{bmatrix} C_{1a}^{\bar{c}} & C_{1b} & C_{1d} \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} B_{a0}^{\bar{c}} & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{bmatrix} \Gamma_i^{-1} \tag{2.32}$$

*Here all submatrices are as defined in* (2.1) *to* (2.6).

**Proof:** Again, without loss of any generality, we will assume that the given system is in the form of (2.2) with $x_a$ being further decomposed into the form as in (2.4) and (2.6).

($\Rightarrow$) If the DDPKMS for $\Sigma$ is solvable, then the DDPKM for $\Sigma$ is also solvable. It follows from Proposition 2.1 that all the solutions that solve the DDPKM for $\Sigma$ is given by $\mathcal{K}$ of (2.25). Then it is straightforward to verify that for any $\mathbf{K} \in \mathcal{K}$

$$\mathbf{A} + \mathbf{B}\mathbf{K}\mathbf{C}_1 = \Gamma_s \begin{bmatrix} A_{aa}^{cc} & * \\ 0 & \tilde{\mathbf{A}} + \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}_1 \end{bmatrix} \Gamma_s^{-1} \tag{2.33}$$

The stability of the closed-loop system implies that $A_{aa}^{cc}$ must be a stable matrix, and moreover there must exist at least one $\mathbf{K} \in \mathcal{K}$ such that $\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\mathbf{K}\tilde{\mathbf{C}}_1$ is asymptotically stable.

($\Leftarrow$) The converse part of the theorem follows by simply reversing the above arguments. This completes the proof of Theorem 2.2.

We now present a numerical example that illustrates the results we have obtained in this section.

**Example:** Consider a linear time-invariant system of (1.1) with

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 2 & -3 & -4 & -5 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.34}$$

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.35}$$

$$\mathbf{C}_2 = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.36}$$

It is simple to verify using the software toolboxes of Chen (1988) or Lin (1989) that the quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ is already in the form of the special coordinate basis as in Theorem 1.1. Moreover, $(\mathbf{A}, \mathbf{B}, \mathbf{C}_2, \mathbf{D}_2)$ is left invertible with two invariant zeros at $s = -1$ and $s = -2$, respectively, and two infinite zeros of order 1 and 2. In addition, $E_b = 0$ and $E_d = 0$

$$A_{aa} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad E_a = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{2.37}$$

$$C_{0a} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad E_{da} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad C_{1a} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.38}$$

It is straightforward to see that $(A_{aa}, E_a)$ is completely controllable and

$$\ker \left\{ \begin{bmatrix} C_{0a} \\ E_{da} \end{bmatrix} \right\} = \ker (C_{1a}) \tag{2.39}$$

By Theorem 2.1, the DDPKM for this system is solvable. It follows from Proposition 2.1 that all the static measurement gain matrices that solve the DDPKM for the given system are characterized by

$$\mathcal{K} = \left\{ \begin{bmatrix} -1 & k_0 \\ -1 & k_1 \\ -2 & k_2 \end{bmatrix} \middle| \, k_0 \in \mathbb{R}, k_1 \in \mathbb{R} \text{ and } k_2 \in \mathbb{R} \right\} \tag{2.40}$$

i.e. any $\boldsymbol{u} = \mathbf{K}\boldsymbol{y}$ with $\mathbf{K} \in \mathcal{K}$ solves the DDPKM for the given system and any $\mathbf{K}$ such that $\boldsymbol{u} = \mathbf{K}\boldsymbol{y}$ solves the DDPKM for the given system must belong to $\mathcal{K}$.

Next, it is easy to observe that

$$\tilde{\mathbf{A}} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ -3 & -4 & -5 & 6 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{\mathbf{C}}_1 = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.41}$$

After a few iterations, we find that the following static measurement feedback gain

$$\mathbf{K} = \begin{bmatrix} -1 & 9 \\ -1 & -15 \\ -2 & -20 \end{bmatrix} \tag{2.42}$$

achieves complete disturbance decoupling for $\Sigma$ and guarantees the internal stability of the closed-loop system. The closed-loop poles of $\mathbf{A} + \mathbf{BKC}_1$ are actually given by $-1, -2, -11{\cdot}276, -4{\cdot}8372$, and $-0{\cdot}9434 \pm j1{\cdot}0786$. Hence, the DDPKMS for $\Sigma$ has been solved.

## 3. Conclusion

We have derived the necessary and sufficient condition for the solvability of the well-known disturbance decoupling problem with static measurement feedback

(DDPKM) for a class of systems which have a left invertible transfer function from the control input ***u*** to the control output ***z***. The condition for the DDPKM is computationally checkable. Furthermore, we have also constructed and parametrized the set of all possible solutions to the DDPKM to this class of systems. This set is then used to derive the solvability condition for the DDPKMS. We feel that the results presented in this paper can be generated to general systems which have a non-left-invertible transfer function from ***u*** to ***z***. This will be our future task.

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